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VANISHING OF THE CONTACT HOMOLOGY OF OVERTWISTED CONTACT 3-MANIFOLDS

BY

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(with an appendix by Yakov Eliashberg)

Abstract

We give a proof of, for the case of contact structures defined by global contact 1-forms, a theorem stated by Eliashberg that for any overtwisted contact structure on a closed 3-manifold, its contact homology is 0. A different proof is also outlined in the appendix by Yakov Eliashberg.

1. Introduction

A contact structure ξ on an odd dimensional manifold is a nowhere integrable hyperplane distribution. If ξ is coorientable then it is defined by a global 1-form α, i.e., $\xi := \ker\alpha$. α is called a defining *contact 1-form* of ξ. *In this article all contact structures which we consider are defined by global contact 1-forms, and all manifolds are closed and orientable.* A contact manifold (M,ξ) consists of a $(2n-1)$-dimensional manifold M and a contact structure ξ on M. Write $\xi := \ker\alpha$ where α is a contact 1-form defining ξ, then (M,ξ) has a natural orientation defined by the volume form $\alpha\wedge(d\alpha)^{n-1}$.

A contact structure ξ on a 3-manifold M is *overtwisted* if there exists an embedded disc $D\subset M$ such that $T(\partial D)\subset \xi|_{\partial D}$, $T_zD\not\subset\xi_z$ for all $z\in\partial D$, and $\xi\cap TD$ defines a singular foliation with exactly one singular

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point (which has to be elliptic). It is proved by Eliashberg [3] that every homotopy class of plane distributions of a 3-manifold has a unique (up to a contact isotopy) overtwisted contact representative. On the other hand, many 3-manifolds possess non-overtwisted contact structures, making the classification of contact manifolds a very subtle question.

Based on the introduction of pseudoholomorphic curves in symplectic manifolds by Gromov [10], Eliashberg and Hofer in the mid 90's introduced Contact Homology Theory ([4, 6], see also [1] for the Morse-Bott version) to provide Floer-type invariants $H\Theta(\xi)$, the *contact homology*, for contact manifolds. Here let us describe very briefly the construction of contact homology, which is in fact an algebra. Readers are suggested to consult [1], [4] and [6] for more detail. First of all, each contact 1-form α associates a unique vector field $R = R_\alpha$ transversal to $\xi := \ker\alpha$, R is called the *Reeb* vector field associated to α and is defined by

$$d\alpha(R,\cdot) = 0, \quad \alpha(R) = 1.$$

The *symplectization* $(\mathbb{R}\times M, d(e^t\alpha))$ $(t \in \mathbb{R})$ is equipped with an α-*admissible* (see Section 4.2) almost complex complex structure. The contact homology is a homology whose complex is generated by *good* (see Section 4) periodic Reeb trajectories, and whose boundary operator ∂ is defined by counting in $\mathbb{R} \times M$ one-dimensional moduli of pseudoholomorphic curves of *genus 0* with finite $d\alpha$-energy which converge asymptotically to a single Reeb orbit as $t \to \infty$ and an arbitrary number of Reeb orbits as $t \to -\infty$. The resulting homology depends only on the isotopy class of contact structures ([4, 6]).

The main purpose of this paper is to give a proof to the following theorem stated by Y. Eliashberg [4] in the case where the overtwisted contact structure is defined by a global contact 1-form.

Theorem 1.1. (Eliashberg [4]) *If ξ is an overtwisted contact structure on a 3-manifold M, then $H\Theta(\xi) = 0$.*

Note that in Hofer's proof of Weinstein Conjecture [11] for overtwisted contact 3-manifolds he showed there exists a contractible Reeb orbit in every overtwisted contact 3-manifold, and such a Reeb orbit must be the asymptotic boundary of a pseudoholomorphic plane with finite energy. *If* this pseudoholomorphic plane is the *only* pseudoholomorphic curve that converges to the said Reeb orbit as $t \to \infty$, then the contact homology of the overtwisted contact structure is 0.

In this paper our proof of Theorem 1.1 is based on the classification of overtwisted contact structures by Eliashberg [3], open book representations of contact manifolds (see Theorem 2.1) by Thurston and Winkelnkemper [15] and Giroux [8], a construction of an overtwisted contact structure from a trivial Dehn surgery inspired by Geiges [7], as well as some techniques and conclusions of some S^1-invariant moduli of pseudoholomorphic curves from [16] and finally, the fact that contact homology is independent of the choices of a contact 1-form and an admissible almost complex structure in the construction of $H\Theta(\xi)$.

Here is an outline of the paper: Section 2 consists of some background on open books. In Section 3 we first construct a contact 1-form α from an open book then "twist" it along a contractible Reeb orbit by a trivial Dehn surgery to get a new contact 1-form α'. It is then proved that $\xi' := \ker \alpha'$ is overtwisted and yet is homotopic to ξ as plane distributions. In particular we get a special contractible Reeb orbit t_x which will be proved in Section 5 to satisfy the equation $\partial t_x = \pm 1$. Section 4 consists of brief definitions of contact complex and contact homology, as well as some discussion on cylindrical contact homology. In Section 5 we first show that energy and homotopy constraints severely limit the types of pseudoholomorphic curves asymptote t_x at positive infinity. Such holomorphic curves must be finite energy planes. Then by using methods similar to [16] we prove that, modulo free $\mathbb{R}$-actions, the algebraic number of such holomorphic planes is equal to the algebraic number of certain gradient trajectories of a Morse function, hence is ± 1. Thus $\partial t_x = \pm 1$ and the contact homology of ξ' is 0.

Acknowledgments

After this paper is written we were kindly reminded [5] that a proof to Theorem 1.1 was already known to Eliashberg before his talk at ICM-Berlin 1998 [4], albeit not written. Eliashberg also generously offered us an outline of his proof [5] which we include in the appendix of this paper. This paper was written during the author's stay at Michigan State University. The author is grateful for the hospitality of Michigan State University.

2. Open Books and Contact Structures

For any surface S and any $\psi \in \mathrm{Diff}(S)$ we denote by S_ψ the mapping torus

$$S \times [0,1]/\sim,$$
$$(\psi(x), 0) \sim (x, 1).$$

An *open book decomposition* of a connected closed orientable 3-manifold M consists of a 1-dimensional submanifold B (a link in M), called the *binding*, and a fibration $\pi : M \setminus B \to S^1$ with fibers connected embedded surfaces with boundary B. The fibers are called *pages*. In this paper we assume that there is a tubular neighborhood $B \times D^2$ of B so that π restricts to the normal angular coordinate of $B = B \times \{0\}$ in $B \times D^2$. Then

$$M = \Sigma_\phi \cup_{id} (B \times D^2),$$

where Σ is an orientable surface with boundary $\partial\Sigma \cong B$ and $\phi \in \mathrm{Diff}^+(\Sigma, \partial\Sigma)$ an orientation-preserving diffeomorphism with $\phi = id$ near $\partial\Sigma$. Note that ϕ is unique up to isotopy. The pages of $\pi : M \setminus B \to S^1$ are diffeomorphic to Σ.

We will need the following important result by Thurston and Winkelnkemper [15] and Giroux [9] concerning contact structures on 3-manifolds.

Theorem 2.1. *Each open book associates a unique up to isotopy contact structure and conversely, each contact structure is supported by an open book unique up to* positive stabilizations.

We will not go into the detail here but point out that by applying positive stabilizations several times if necessary we may assume that $\partial\Sigma \cong B$ is *connected.*

With Theorem 2.1 in hand we can start with any open book (Σ, ϕ) and alter the corresponding contact structure ξ by a trivial Dehn surgery to get a new contact structure ξ' homotopic to ξ as plane distribution, as we will do in the next section.

3. Overtwisted Contact Form from Trivial Dehn Surgery

In this section we will construct an overtwisted contact structure $\xi' := \ker \alpha'$ that is homotopic to a given contact structure $\xi := \ker \alpha$ as plane distributions. We sketch the idea of the construction of α' here before going into the detail.

Start with a contact 1-form α (following [15]) associated to a given open book (Σ, ϕ) such that α has a pair of contractible Reeb orbits t_x and t_y associated to a pair of birth-death type of critical points x, y of a smooth function K on Σ, such that x is a saddle point and y is a local minimum point. Following the construction in [7] of contact 1-forms under Dehn surgeries, we apply a trivial Dehn surgery to t_y. In particular, we cut out a tubular neighborhood of t_y and glue it back, identifying the two boundaries by using the gluing matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The resulting 3-manifold is the same (up to an orientation-preserving diffeomorphism) while the contact 1-form α, after being modified near the boundaries, glued back to a new contact 1-form α'. The new contact structure ξ' is then shown to be overtwisted and yet is homotopic to ξ as plane distributions. Now start the construction.

Given an open book (Σ, ϕ) with $\partial\Sigma \cong B$ connected, we can define an associated contact 1-form α on $M = \Sigma_\phi \cup_{id} B \times D^2$ as follows.

Let $F \subset \Sigma$ be a collar of $\partial\Sigma$ such that $\phi|_F = id$. Let (q, p) be coordinates of $F = [q_-, q_+] \times S^1_p$ so that $\partial\Sigma$ is identified with $\{q_+\} \times S^1_p$. We may assume that $0 \ll q_- \ll q_+$. Let (ρ, t) be the polar coordinates of the D^2-factor of $B \times D^2 \cong S^1 \times D^2$ so that $D^2 = \{\rho \leq 1\}$. Let p be the coordinate of S^1. Note that ρ can be viewed as a smooth function of the coordinate q of F, and $\frac{d\rho}{dq} < 0$. Also, S^1_t acts on $F_\phi \cup (B \times D^2)$ via rotations in the t-direction. It fixes B and acts freely on the complement of B in $F_\phi \cup B \times D^2$. Let V denote the orbit space and let

$$\pi_t : F_\phi \cup (B \times D^2) \to V \tag{1}$$

denote the corresponding projection.

Let $K \geq 0$ be a smooth function on M such that

1. $K|_{(\Sigma\setminus F)_\phi} \gg 1$ is a constant;
2. K is S^1_t-invariant on $F_\phi \cup (B \times D^2)$; $\bar{K} := (\pi_t)_* K$ is a Morse function on $int(V)$ the interior of V with $|dK| \sim 0$ on F, $K = 1$ on $\partial\Sigma$;
3. $\bar{K}$ has precisely two critical points on $int(V)$: one saddle point x and one local minimum point y; x and y are of birth-death type, i.e., there is only one gradient trajectory of $\bar{K}$ that connects x and y;
4. $\bar{K}(x) > \bar{K}(y) > 1$ and $\bar{K}(x) \sim \bar{K}(y) \sim 1$;
5. in the interior of F choose a small disc neighborhood

$$D_y = \{(r, \theta) \mid r \leq \delta\}, \quad \delta > 0 \text{ a constant},$$

with center y, here (r, θ) are the polar coordinates of D_y, then $\bar{K}|_{D_y}$ depends only on r;

6. $K|_{B\times D^2}$ depends only on ρ, $\frac{dK}{d\rho} > 0$ on $0 < \rho \leq 1$, $K(\rho) = h(\rho)\rho^2$ for some smooth function h depending only on ρ.

Let β be a 1-form on Σ such that $d\beta$ is a symplectic 2-form on Σ with

$$\beta = qdp \text{ near } \partial\Sigma, \qquad \beta = r^2 d\theta \text{ on } D_y.$$

On $B \times D^2$ we also consider a smooth function $Q > 0$ depending only on ρ such that

$$Q = q \text{ near } \rho = 1, \qquad \frac{dQ}{d\rho}(0) = 0,$$
$$\frac{dQ}{d\rho} > 0 \text{ and } Q\frac{dK}{d\rho} - K\frac{dQ}{d\rho} > 0 \text{ for } \rho > 0.$$

Then

$$\alpha := \begin{cases} (1-t)\beta + t\phi^*\beta + Kdt & \text{on } \Sigma_\phi \\ Q(\rho)dp + K(\rho)dt & \text{on } B \times D^2 \end{cases} \tag{2}$$

is a contact 1-form on M provided that $K|_{\Sigma\setminus F}$ is a large enough constant. Denote $\xi := \ker\alpha$.

We have two contractible simple Reeb orbits of α:

$$t_x := \{x\} \times S^1_t, \quad t_y := \{y\} \times S^1_t.$$

Both t_x and t_y are oriented by the vector field ∂_t.

Now define a contact 1-form α' on M:

$$\alpha' = \begin{cases} \alpha & \text{on } M \setminus (D_y)_\phi, \\ h_1(r)dt + h_2(r)d\theta & \text{on } (D_y)_\phi. \end{cases} \tag{3}$$

where h_1, h_2 are smooth functions of r satisfying

1. $h_1(r) = -1$ and $h_2(r) = -r^2$ for $r < \epsilon$;
2. $h_1(r) = K(r)$ and $h_2(r) = r^2$ for $\delta - \epsilon \leq r \leq \delta$;
3. $h_1'(r)h_2(r) - h_2'(r)h_1(r) > 0$ for $0 < r \leq \delta$;

where $0 < \epsilon \ll \frac{\delta}{2}$ is a constant. The third condition above is to ensure that α' is a contact form on $(D_y)_\phi$. Note that α' has two special Reeb orbits

$$t_x = \{x\} \times S^1_t, \quad \bar{t}_y = \{y\} \times S^1_t$$

The notation $\bar{t}_y$ represents the curve t_y but with the reversed orientation, i.e., the orientation given by $-\partial_t$. Let $\xi' := \ker \alpha'$.

Lemma 3.1. *ξ' is an overtwisted contact structure.*

Proof. Let ℓ be the gradient trajectory of $-\bar{K}$ that goes from x to y. Then $\ell \times S^1_t$ is a homotopy between t_x and t_y. We have

$$\int_{t_y} \alpha' < 0 < \int_{t_x} \alpha'$$

so there exists a point $z \in \ell$ such that $\int_{t_z} \alpha' = 0$. Since α' is S^1_t-invariant on $\ell \times S^1_t$ we conclude that

$$t_z \text{ is a Legendrian curve of } \xi'.$$

Since t_z is homotopic to t_x, t_z is contractible. Moreover, t_z is contained in a tubular neighborhood of the binding B and its winding number with B is

± 1, so t_z is spanned by an embedded disc in a tubular neighborhood of B. We can find an overtwisted disc spanning t_z as follows.

Note that h_1 vanishes at z, and that $\partial_t \pitchfork \xi'$ on $F_\phi \cup (B \times D^2)$ except where $h_1 = 0$. Recall that V is the projection of $F_\phi \cup (B \times D^2)$ via π_t. Let $\gamma \subset V \setminus \{h_1(r) < 0\}$ be an embedded smooth path such that

1. z is an endpoint of γ,
2. $\gamma|_{D_y}$ is transversal to ∂_r, and
3. $\dot{\gamma} \parallel \partial_\rho$ near B.

Define $D_z := \pi_t^{-1}\gamma$. Then D_z is an embedded smooth spanning disc of t_z. Moreover, since $\xi'|_{t_z} = \mathrm{Span}(\partial_t, \partial_r)$, $D_z \pitchfork \xi'$ along t_z by the second condition above, ∂_t is tangent to $D_z \setminus \{\rho = 0\}$ hence $\xi' \cap TD_z$ has only one singular point and the singular point is elliptic. So D_z is an overtwisted disc. $\square$

Lemma 3.2. *ξ' and ξ are homotopic as plane distributions.*

Proof. It is enough to show that α and α' are homotopic as nowhere vanishing 1-forms. Since $\alpha = \alpha'$ on $M \setminus (D_y)_\phi$ it is enough to consider a homotopy supported in $(D_y)_\phi$.

For $s \in [0, 1]$ define

$$\alpha_s := s(1-s)\chi(r)dr + (1-s)\alpha + s\alpha' \quad \text{on } (D_y)_\phi$$

where $\chi(r) \geq 0$ is a smooth function on r such that $\chi(0) = 0 = \chi(\delta)$ and $\chi > 0$ away from $r = 0, \delta$. We have

$$\alpha_s = s(1-s)\chi(r)dr + \big((1-s)K(r) + sh_1(r)\big)dt + \big((1-s)r^2 + sh_2(r)\big)d\theta.$$

It is clear that α_s is nowhere vanishing when s is close to 0 or 1. Also for every s, α_s is nonvanishing on the region where χ is positive and the region where r is close to 1. For $s \neq 0, 1$ we have

$$(1-s)K(r) + sh_1(r) = 0 \Leftrightarrow h_1(r) = \frac{(s-1)K(r)}{s}$$
$$(1-s)r^2 + sh_2(r) = 0 \Leftrightarrow h_2(r) = \frac{(s-1)r^2}{s}$$

The function $g(s) := \frac{s-1}{s}$, $s \in (0,1)$, is an increasing function of s such that $g(s) \to -\infty$ as $s \to 0^+$, and $g(s) \to 0^-$ as $s \to 1^-$.

Recall that near $r = 0$,

$$\frac{h_1(r)}{K(r)} = \frac{-1}{K(r)}, \quad \frac{h_2(r)}{r^2} = \frac{-1}{1}$$

Since $K(r) \neq 1$ for r near 0, for r small enough $(1-s)K(r) + sh_1(r)$ and $(1-s)r^2 + sh_2(r)$ will not vanish simultaneously for any $s \in [0,1]$. Therefore α_s is nowhere vanishing for $s \in [0,1]$. Since $\alpha_0 = \alpha$ and $\alpha_1 = \alpha'$, α and α' are homotopic as nowhere vanishing 1-forms on M. So ξ and ξ' are homotopic as plane distributions. □

4. An Outline of Contact Homology

In this section we give a brief account on definitions of contact complex, contact homology and cylindrical contact homology. Readers are referred to [1, 4, 6] for more detail.

4.1. Contact complex algebra

Let (M, ξ) be a $(2n-1)$-dimensional closed contact manifold with ξ defined by a global contact 1-form α. For a generic choice of α, there are only countably many periodic trajectories (including all positive multiple ones) of the Reeb vector field R_α; and these Reeb orbits are *nondegenerate*, meaning that 1 is not an eigenvalue of their Poincar'e return map. We call such contact 1-forms *regular*.

Definition 4.1. A Reeb orbit is said to be *bad* (see Section 1.2 of [6]) if it is an even multiple of another Reeb orbit whose Poincaré return map has the property that the total multiplicity of its eigenvalues from the interval $(-1,0)$ is odd. A Reeb orbit is *good* if it is not bad.

We denote by $\mathcal{P}_\alpha$ the set of all good Reeb orbits of α. Note that $\mathcal{P}_\alpha$ includes all positive multiple ones as individual elements.

Definition 4.2. Let α be a regular contact 1-form defining the contact structure ξ. The contact complex $\Theta(\alpha)$ is then defined to be the free commutative algebra over $\mathbb{Q}$ (or $\mathbb{R}$, $\mathbb{C}$) generated by all elements of $\mathcal{P}_\alpha$.

Remark 4.1. In [6] $\Theta(\alpha)$ is defined with coefficients in the algebra $\mathbb{C}[H_2(M)][[t]]$. Here we use $\mathbb{Q}$-coefficients for the sake of simplicity.

4.2. Contact homology

An almost complex structure J on the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$ of $(M, \xi = \ker \alpha)$ is said to be *α-admissible* if $J(\partial_t) = R_\alpha$ and $J|_\xi : \xi \to \xi$ on ξ is *$d\alpha$-compatible*, i.e., $d\alpha(v, Jv) > 0$ for all nonzero $v \in \xi$ and $d\alpha(Jv_1, Jv_2) = d\alpha(v_1, v_2)$ for $v_1, v_2 \in \xi$. Note that compatibility property does not depend on the choice of the defining contact 1-form for ξ.

Let γ be a good Reeb orbit. We use the following notations:

1. Υ:= a finite collection of (not necessarily distinct) good Reeb orbits of α. Υ can be empty.

2. $|\Upsilon|$:= the cardinality of Υ.

3. $\tilde{\mathcal{M}}(\Upsilon, \gamma)$:= the moduli space of finite $d\alpha$-energy pseudoholomorphic maps from a $(1+|\Upsilon|)$-punctured sphere into $\mathbb{R} \times M$ with one puncture goes to γ at $t = \infty$ and other punctures go to Υ at $t = -\infty$ (see [6]).

4. $\mathcal{M}(\Upsilon, \gamma)$:= the union of 1-dimensional components of $\tilde{\mathcal{M}}(\Upsilon, \gamma)$.

Note that Υ was treated in [6] as an *ordered* set, yet *here we consider Υ an unordered set.*

Secondly, an admissible almost complex structure J on $\mathbb{R} \times M$ is $\mathbb{R}$-invariant, hence there is a free $\mathbb{R}$-action on $\tilde{\mathcal{M}}(\Upsilon, \gamma)$. For generic α-admissible J, $\mathcal{M}(\Upsilon, \gamma)/\mathbb{R}$ consists of finitely many points. Note that because of energy constraint (7) there are only finitely many choices for Υ such that $\tilde{\mathcal{M}}(\Upsilon, \gamma) \neq \emptyset$. Let κ_γ denote the multiplicity of γ. For $C \in \mathcal{M}(\Upsilon, \gamma)/\mathbb{R}$ we also denote by κ_C the multiplicity of C.

The boundary operator ∂ of the contact complex $(\Theta(\alpha), \partial)$, when applied to γ, is defined by (see [2 , 4, 6] but for a different coefficient ring)

$$\partial\gamma := \sum_{i=0}^{\infty} \partial_i \gamma, \qquad \text{where} \tag{4}$$

$$\partial_i \gamma := \kappa_\gamma \sum_{|\Upsilon|=i} \Big(\sum_{\substack{C \in \mathcal{M}(\Upsilon,\gamma)/\mathbb{R} \\ \dim \mathcal{M}(\Upsilon,\gamma)=1}} \frac{\pm 1}{\kappa_C} \Big) \Upsilon. \tag{5}$$

Here Υ denotes the monomial $\gamma_1\gamma_2\cdots\gamma_{|\Upsilon|}$, and the $\pm$ sign in (5) depends on the orientation of $C \in \mathcal{M}(\Upsilon, \gamma)/\mathbb{R}$.

Note that because the action $\mathcal{A}(\sigma) := \int_\sigma \alpha$ of any Reeb orbit σ of α is bounded from below by a positive number independent of σ, $\partial_i\gamma = 0$ for all i sufficiently large, and the right hand side of (5) consists of only finitely many nonvanishing terms. Then extend ∂ over $\Theta(\alpha)$ according to the Leibnitz rule [6]. $(\Theta(\alpha), \partial)$ is now a differential algebra.

Theorem 4.1.([6]) $\partial^2 = 0$ *for regular contact 1-form* α *and a generic* α*-admissible almost complex structure* J.

Definition 4.3. The contact homology algebra $H\Theta(\alpha, J)$ of the pair (α, J) with J α-admissible is defined to be the quasi-isomorphism class of the differential algebra $(\Theta(\alpha), \partial)$, i.e., $H\Theta(\alpha, J) := \frac{\ker \partial}{\mathrm{im}\partial}$.

$H\Theta(M, \xi) := H\Theta(\alpha, J)$ is independent of the choices of (α, J) hence is an invariant of the contact manifold (M, ξ).

4.3. Cylindrical contact homology

Recall the boundary operator $\partial = \sum_{i=0}^{\infty} \partial_i$. Since $\partial^2 = 0$ we have

$$\partial_1^2 + \partial_0\partial_2 + \partial_2\partial_0 = 0$$

If $\partial_1^2 = 0$ then one can forget the algebraic structure of $\Theta(\alpha)$, simply thinking it as a free module over $\mathbb{Q}$ generated by all good Reeb orbits, and then use $\partial_1 : \Theta \to \Theta$ as the boundary operator to define the *cylindrical contact homology* $HC(\xi) := \frac{\ker \partial_1}{\mathrm{im}\partial_1}$ of (M, ξ) as a vector space. A sufficient (but not necessary) condition for $\partial_1^2 = 0$ to be true is $\partial_0 = 0$. Note that since ∂_0

is defined via counting finite $d\alpha$-energy pseudoholomorphic planes bounding contractible Reeb orbits at $t=\infty$, $\partial_0=0$ holds trivially when there exist no contractible Reeb orbits.

5. Contact Homology of Overtwisted ξ'

Back to the overtwisted contact 3-manifold (M,ξ') and the contact 1-form α' that we constructed in Section 3. From Definition 4.1 it is clear that t_x is *good* hence is a generator of the contact algebra $\Theta(\alpha')$ for the contact homology of ξ'.

From now on Υ denotes a finite collection of (not necessarily distinct) good Reeb orbits of α'. Note that Υ can be empty. The notations $\tilde{\mathcal{M}}(\Upsilon,t_x)$ and $\mathcal{M}(\Upsilon,t_x)$ are as defined in Section 4 (with respect to the overtwisted contact 1-form α'). Here we point out a few things about $\tilde{\mathcal{M}}(\Upsilon,t_x)$ which we will need later.

Given $\tilde{u}\in\tilde{\mathcal{M}}(\Upsilon,t_x)$ we write $\tilde{u}=(a,u)$ according to the splitting $\mathbb{R}\times M$ and let $C\subset M$ denote the image of u. *Assume that* $\Upsilon\neq t_x$, then

$$d\alpha'>0 \text{ on } C \text{ except at finitely many points of } C. \tag{6}$$

Moreover the $d\alpha'$-energy of $\tilde{u}$ is

$$\int_C d\alpha'=\int_{t_x}\alpha'-\int_{\Upsilon}\alpha'>0, \tag{7}$$

unless when $\Upsilon=t_x$. In the exceptional case, $\tilde{\mathcal{M}}(t_x,t_x)=\mathrm{pt}$, the corresponding pseudoholomorphic curve is the trivial cylinder $\mathbb{R}\times t_x$. Since we are only interested in holomorphic curves with positive finite $d\alpha'$-energy, we may assume that Υ does not contain t_x and its positive iterates.

In the following we will study $\tilde{\mathcal{M}}(\Upsilon,t_x)$ and $\mathcal{M}(\Upsilon,t_x)$. First a few more notations.

Recall that $F\subset\Sigma$ is a collar of $\partial\Sigma$ such that $\phi|_F=id$. Denote $N:=F_\phi\cup(B\times D^2)$, then $V=\pi_t N$ (see (1)). Let $L_s:=\{K=s\}\subset N$ denote the s-level set of K. In particular, $L_0=B$. Let $i_s:L_s\hookrightarrow N$ denote the inclusion. L_s is S^1_t-invariant for any s. Note that $\alpha'|_N$ is independent of t, hence we have the following simple

Fact 5.1. *$i_s^*\alpha'$ is a t-independent closed 1-form on L_s for any regular value $s > 0$ of $K|_N$.*

Let $N_x := \{K \leq K|_{t_x}\} \subset N$. Let $C \subset M$ be the image of some $\tilde{u} \in \tilde{\mathcal{M}}(\Upsilon, t_x)$.

Lemma 5.1. $C \subset N_x$.

Proof. Denote $C_s := C \cap L_s$ for $L_s \subset N$. Suppose that $C \not\subset N_x$. Then there exists a level set $L_{s_o} \subset N \setminus N_x$ for some s_o such that $C \pitchfork L_{s_o}$. So $C_{s_o} \subset C$ is a union of finitely many embedded circles which are pairwise disjoint. Note that $[C_{s_o}] = [S_t^1] \in H_1(L_s, \mathbb{Z})$. Let $A \subset C$ denote the domain bounded by C_{s_o} and t_x then we have

$$\int_A d\alpha' = \int_{t_x} K dt - \int_{C_{s_o}} s dt = K(t_x) - s_o < 0,$$

which contradicts with (6). So we conclude that $C \subset N_x$. □

The above lemma implies that, if $\Upsilon \neq \emptyset$ then Υ consists of Reeb orbits of α' in $N_x \setminus t_x$. Note that $N_x \setminus t_x$ consists of two disjoint connected components. We write

$$N_x \setminus t_x = N_y \cup N_B,$$

where N_y is the connected component containing $\bar{t}_y$, and N_B is the connected component containing B.

Note that C intersects with t_x at finitely many points. If $C \cap t_x \neq \emptyset$ then $C \setminus t_x = (C \cap N_y) \cup (C \cap N_B)$ is a union of two disjoint set, which is impossible, so C does not intersect with t_x geometrically. Therefore we must have either $C \subset N_y$ or $C \subset N_B$.

Lemma 5.2. $C \subset N_B$.

Proof. Suppose not. Then $C \subset N_y$ and Υ consists of Reeb orbits in N_y. Note that for $K(t_y) < s < K(t_x)$,

$$L_s \cap N_y \cong S_\theta^1 \times S_t^1.$$

Also recall that

$$\alpha' = \begin{cases} h_1(r)dt + h_2(r)d\theta & \text{on } (D_y)_\phi \\ Kdt + \beta & \text{on } N_y \setminus (D_y)_\phi. \end{cases}$$

Then the Reeb vector field R' of α' is

$$R' = \begin{cases} \big(h_2'(r)\partial_t - h_1'(r)\partial_\theta\big)/(h_1(r)h_2'(r) - h_2(r)h_1'(r)) & \text{on } (D_y)_\phi, \\ \big(\partial_t + Y\big)/(K + \beta(Y)) & \text{on } N_y \setminus (D_y)_\phi, \end{cases}$$

where Y is the vector field in $\pi_t(N_y)$ such that $d\beta(Y, \cdot) = dK$. Parametrize the family of level sets $L_s \cap N_y$ by s. We have $d\beta = gds \wedge d\theta$ for some positive function g. Hence $Y = \frac{-K_s}{g}\partial_\theta$ where $K_s := \frac{dK}{ds} > 0$. Therefore if $\gamma \subset N_y$ is a Reeb orbit of R' then exactly one of the following two cases must be held:

1. γ is a positive iterate of $\bar{t}_y$, hence γ is homotopic to a curve in $N_y \setminus t_y$ that represents the class $-n[S^1_t] \in H_1(S^1_\theta \times S^1_t)$ for some $n \in \mathbb{N}$;
2. γ is not a positive iterate of $\bar{t}_y$, $[\gamma] = -n[S^1_\theta] + m[S^1_t] \in H_1(S^1_\theta \times S^1_t)$ for some $n \in N$, $m \in \mathbb{Z}$.

On the other hand $[t_x] = -[\bar{t}_y] = -[S^1_t] \in H_1(N_y, \mathbb{Z})$. So $\Upsilon \neq \emptyset$ as t_x is not contractible in N_y. Also Υ does not consists of positive iterates of $\bar{t}_y$. Now that t_x is not homologous in $N_y \setminus t_y$ to any nonempty finite collection of orbits of R', C has to intersect with the Reeb orbit $\bar{t}_y$ nontrivially and positively at every point of intersection. Let $U \subset N_y$ be a tiny tubular neighborhood of $\bar{t}_y$ such that $C \pitchfork \partial U$. Let $\sigma := C \pitchfork \partial U$, then $[\sigma] = -n[S^1_\theta] \in H_1(N_y \setminus U)$ for some $n \in \mathbb{N}$. Write $C' := C \setminus U$ and $\partial C' = \partial_+ C' \cup \partial_- C'$, where $\partial_+ C' = t_x$, $\partial_- C' = \Upsilon \cup \sigma$. But $[\Upsilon] = -n'[S^1_\theta] \in H_1(N_y \setminus U, \mathbb{Z})$ for some $n' \in \mathbb{N}$ hence t_x is not homologous in $N_y \setminus U$ to $\Upsilon \cup \sigma$, which implies that C does not exist if $C \subset N_y$. So $C \subset N_B$. □

Lemma 5.3. *$\tilde{\mathcal{M}}(\Upsilon, t_x) = \emptyset$ unless $\Upsilon = \emptyset$.*

Proof. By now we have known that $C \subset N_B$. If $\Upsilon \neq \emptyset$ then Υ consists of Reeb orbits of α' in N_B. It is easy to check that B is a generator of $H_1(N_B, \mathbb{Z}) \cong \mathbb{Z}$ and every Reeb orbit in N_B is homotopic to a positive multiple of B, while t_x is contractible in N_B. So $\Upsilon = \emptyset$. □

Lemma 5.4. *$\tilde{\mathcal{M}}(t_x) = \mathcal{M}(t_x)$.*

Proof. Let $D_x \subset N_B$ be an embedded, S^1_t-invariant spanning disc of t_x such that $D_x \cap B$ is a point. One can show that the *Conley-Zehnder index* $CZ(t_x, D_x)$ of t_x relative to D_x is 2 (see [13] and Section 1.2 of [6]). Note that t_x is not homologically trivial in $M \setminus B$. Otherwise there would exist another surface $S \subset M \setminus B$ with boundary $\partial S = t_x$. Then the closed surface $S \cup D_x$ intersects with B at exactly one point, which is impossible given the fact that B is homologically trivial. Moreover, $H_2(N_B, \mathbb{Z}) = 0$ so the Conley-Zehnder index $CZ(t_x) := CZ(t_x, D_x) = 2$ is independent of the choice of a spanning surface of t_x. By Lemma 5.2, the fact that $H_2(N_B, \mathbb{Z}) = 0$ and the formula for the formal dimensions (see [6] Proposition 1.7.1 for the general formula) of components of $\tilde{\mathcal{M}}(t_x)$ we get that $\tilde{\mathcal{M}}(t_x)$ is of pure dimension with

$$\dim \tilde{\mathcal{M}}(t_x) = CZ(t_x) - 1 = 2 - 1 = 1$$

provided that $\tilde{\mathcal{M}}(t_x) \neq \emptyset$. Hence $\tilde{\mathcal{M}}(t_x) = \mathcal{M}(t_x)$. □

Lemma 5.5. *The algebraic number of* $\mathcal{M}(t_x)/\mathbb{R}$ *is* ± 1.

Proof. By Lemma 5.2, suppose that $\mathcal{M}(t_x) \neq \emptyset$ then the image C in M of any element $\tilde{u} \subset \mathcal{M}(t_x)$ is contained in N_B. Moreover, $C \pitchfork B$ is a single point, C intersects with B positively at their point of intersection. Recall that S^1_t acts on N_B via rotations, fixing B pointwise and acting freely on $N_B \setminus B$. Note that α' is S^1_t-invariant. To study $\mathcal{M}(t_x)$ we consider an α'-admissible almost complex structure J which is also S^1_t-invariant. Let $\mathcal{M}^s(t_x) \subset \mathcal{M}(t_x)$ denote the subset consisting of S^1_t-invariant elements of $\mathcal{M}(t_x)$.

Let i denote the standard complex structure on $\mathbb{C}$. Let $z = s + it$ be the complex coordinate of $\mathbb{C}$. Denote $\tilde{u}_s := \frac{d\tilde{u}}{ds}$, $\tilde{u}_t := \frac{d\tilde{u}}{dt}$. A map $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ is a element of $\mathcal{M}(t_x)$ if $\tilde{u}$ satisfies the d-bar equation

$$\bar{\partial}_J \tilde{u} := \tilde{u}_s + J(\tilde{u})\tilde{u}_t = 0, \tag{8}$$

$u(\{|z| = r\}) \to t_x$ as $r \to \infty$, $a(z) \to \infty$ as $z \to \infty$, and $\int_{u(\mathbb{C})} d\alpha' > 0$ is finite. By applying a reparametrization if necessary we may assume that $u(0) \in B$.

Most of the proof essentially follows the arguments and methods used in Section 7 of [16]. Here we only outline the idea.

Claim 1. $\mathcal{M}^s(t_x)/\mathbb{R}$ consists of a single element.

Like in [16], this is done by showing that there is a 1-1 correspondence between $\mathcal{M}^s(t_x)/\mathbb{R}$ and the trajectories from x to B of some gradient-like vector field with respect to $-\bar{K}$. And there is one and only one such trajectory.

Claim 2. For generic S^1_t-invariant J, the linearized d-bar operator is surjective at any element of $\mathcal{M}^s(t_x)$.

In particular, this implies that an S^1_t-invariant solution to the d-bar equation (8) is an isolated solution.

Note that

$$T_{\mathbb{R}\times B}(R\times M) = (\underline{\mathbb{R}} \oplus \underline{R'}) \oplus \underline{\mathbb{C}}, \quad \xi'|_{\mathbb{R}\times B} = \underline{\mathbb{C}} \tag{9}$$

where $\underline{\mathbb{R}}$, $\underline{R'}$ denote the trivial real line bundles generated by ∂_t and R' the Reeb vector field of α' respectively, and $\underline{\mathbb{C}}$ is the trivial complex line bundle which equals ξ' when restricted to $\mathbb{R}\times B$.

Now that the α'-admissible almost complex structure J is also S^1_t invariant, then

$$J|_{\mathbb{R}\times B} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

according to the decomposition in (9). Let $D_{\tilde{u}}$ denote the linearized d-bar operator $\bar{\partial}$ at $\tilde{u}$. Then for $\eta \in W^{1,2}(\mathbb{C}, \tilde{u}^*T(\mathbb{R}\times M))$,

$$D_{\tilde{u}}\eta = \eta_s + J\eta_t + \nabla_\eta J. \tag{10}$$

In particular, near $z = 0$ (10) is just the perturbation of the standard Cauchy-Riemann equation by a bounded zero order term, hence is surjective on $|z| \leq \epsilon$ for some $\epsilon > 0$.

Now away from $z = 0$ we have $u : \mathbb{C}\setminus\{0\} \cong \mathbb{R}\times S^1 \to N_B \setminus B$. $N_B \setminus B$ has the structure of a trivial S^1 bundle over an annulus. Then following Lemma 7.5 of [16], for generic S^1_t-invariant J, $D_{\tilde{u}}$ is surjective when $|z| \geq \epsilon$. We then conclude that for generic S^1_t-invariant J, $D_{\tilde{u}}$ is surjective at every S^1_t-invariant solution $\tilde{u}$.

Finally we will show that $\mathcal{M}^s(t_x) = \mathcal{M}(t_x)$ essentially. This is done by considering branched covers over N_B.

For each $n \in N$ let $\mathbb{Z}_n \subset S^1$ denote the cyclic subgroup of order n generated by the $\frac{2\pi}{n}$-rotation on S^1_t. The action of $\mathbb{Z}_n$ induces an $n : 1$ branched covering map

$$\Phi_n : N_B \to N_B \quad \text{with } B \text{ the branch set.}$$

Each S^1_t-invariant J induces an infinite sequence of S^1_t-invariant almost complex structures

$$J_n := (\Phi_n)_* J(\Phi_n^{-1})_*$$

Let $\alpha'_n := (\Phi_n)_*\alpha'$. J_n is α'_n-admissible. Note that α'_n is homotopic to α' as contact 1-forms, hence $\xi'_n := \ker \alpha'_n$ is isotopic to ξ' as contact structures on N_B. Also t_x is a Reeb orbit of α'_n.

Claim 3. For n large enough, $\mathcal{M}^s_{J_n}(t_x) = \mathcal{M}_{J_n}(t_x)$.

If not, then there is an infinite sequence subsequence n_i, $n_i \to \infty$ as $i \to \infty$ such that $\mathcal{M}_{J_{n_i}}(t_x) \setminus \mathcal{M}^s_{J_{n_i}}(t_x) \neq \emptyset$. Given $\tilde{v}_i \in \mathcal{M}_{J_{n_i}}(t_x) \setminus \mathcal{M}^s_{J_{n_i}}(t_x)$, $\tilde{v}_i$ lifts via Φ_{n_i} to an $\mathbb{Z}_{n_i}$-invariant element $\tilde{u}_i \in \mathcal{M}_J(t_x) \setminus \mathcal{M}^s_J(t_x)$, $\tilde{u}_i$ is unique up to $\mathbb{R}$-translation. By applying $\mathbb{R}$-translations if necessary we get that a subsequence of $\tilde{u}_i$, also denoted by $\tilde{u}_i$ by the abuse of language, will converge to an S^1_t-invariant solution $\tilde{u} \in \mathcal{M}^s_J(t_x)$ as $i \to \infty$, this contradicts with Claim 2, hence is impossible.

By trading J for J_n for any n large enough we have $\mathcal{M}^s(t_x) = \mathcal{M}(t_x)$. This complete the proof. □

Following the definition of the boundary operator of the contact homology we have the following

Lemma 5.6. $\partial t_x = \pm 1$. *Hence* $H\Theta(M, \xi') = 0$.

Thus finished the proof of Theorem 1.1.

Appendix (by Yakov Eliashberg). Sketch of an Alternative Proof

The following paragraph is a brief description of the argument by Eliashberg of why one can get in the overtwisted case a contact form with an exactly 1 holomorphic plane bounded by one of the orbits.

Consider a model of an overtwisted contact structure which contains a solid torus with a Lutz $2\pi n$-twist. The contact form in this solid torus can be chosen as

$$\alpha = \cos r dz + \sin(nr) d\varphi$$

where $0 \leq \rho \leq 1$, $z \in \mathbb{R}/\mathbb{Z}$. This is not exactly good formula because it is not smooth for $r = 0$ but can be smoothed without any problems. The torus $\rho = \pi/2n$ foliated by horizontal Reeb orbits which bound holomorphic planes. By making n large we can make the action of these orbits arbitrarily small. Now it is easy to see explicitly that there is no other holomorphic planes bounded by these orbits inside the the considered solid torus. Also there are no other holomorphic curves different from the planes for which these orbits can be at the positive end because their action is less than anybody else's action. On the other hand, if there is a plane bounded by these orbits which goes outside than the integral of $d\alpha$ along the piece of this curve inside the solid torus can be made bigger than the action of the orbit. This is, of course, Morse-Bott type form, but by a small perturbation we get 2 orbits out of the whole torus, and one of them has the required properties.

References

1. F. Bourgeois, A Morse-Bott approach to contact homology, thesis, 2002.

2. F. Bourgeois, *Introduction to Contact Homology*, Summer School in Berder: Holomorphic curves and contact topology, June 2003.

3. Y. Eliashberg, Classification of overtwisted contact structures, *Invent. Math.*, **98**(1989), 623-637.

4. Y. Eliashberg, Invariants in contact topology, *Proc. ICM Berlin*, **2**(1998), 327-338.

5. Y. Eliashberg, personal communication, 2004.

6. Y. Eliashberg, A. Givental and H. Hofer, *Introduction to Symplectic Field Theory*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part II, 560-673; arXiv:math.SG/0010059.

7. H. Geiges, *Contact Geometry*, Handbook of Differential Geometry, Vol. 2; arXiv: math.SG/0307242.

8. E. Giroux, preprint, 2001.

9. E. Giroux's and J.-P. Mohsen, Contact structures and symplectic open books, preprint, 2003.

10. M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Inv. Math.*, **82** (1985), 307-347.

11. H. Hofer, Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three, *Invent. Math.* **114**(1993), 515-563.

12. D. Rolfsen, *Knots and Links*, Publish Or Perish, 1976.

13. J. Robbin, D. Salamon, The Maslov index for paths, *Topology*, **32**(1993), 827-844.

14. D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.*, **45**(1992), 1303-1360.

15. W. P. Thurston and H. E. Winkelnkemper, On the existence of contact forms, *Proc. Amer. Math. Soc.*, **52**(1975), Issue 1, 345-347.

16. M.-L. Yau, Contact homology and subcritical Stein-fillable contact manifolds, *Geom. Topol.*, **8**(2004), 1243-1280. Also available at ArXiv: math.SG/0409542.

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