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ON GEODESICS IN SUBRIEMANNIAN GEOMETRY

BY

BERNARD GAVEAU AND PETER GREINER

Abstract

We study a subRiemannian geometry induced by 2 specific vector fields in $\mathbb{R}^3$, and obtain the canonical curve whose tangents provide the missing direction.

1. Introduction

Let

$$X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} \tag{1.1}$$

denote two vector fields in $\mathbb{R}^3$. In this and in a subsequent article we shall consider the following question:

"*How many geodesics induced by X and Y join two given points* (x_0, y_0, t_0) *and* (x, y, t)*?*"

By a geodesic we mean the projection of a bicharacteristic on the base. Bicharacteristic curves

$$\big(x(s), y(s), t(s), \xi(s), \eta(s), \tau(s)\big) \tag{1.2}$$

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are solutions of Hamilton's differential equations. Hamilton's function is

$$H = \frac{1}{2}(\xi + y^2\tau)^2 + \frac{1}{2}\eta^2, \tag{1.3}$$

and the Hamiltonian system of differential equations for the bicharacteristic curves are

$$\dot{x}(s) = H_\xi = \xi + y^2\tau, \tag{1.4}$$
$$\dot{y}(s) = H_\eta = \eta, \tag{1.5}$$
$$\dot{t}(s) = H_\tau = \dot{x}y^2, \tag{1.6}$$
$$\dot{\xi}(s) = -H_x = 0 \;\Rightarrow\; \xi(s) = \xi = \text{constant}, \tag{1.7}$$
$$\dot{\eta}(s) = -H_y = -2\dot{x}y\tau, \tag{1.8}$$
$$\dot{\tau}(s) = -H_t = 0 \;\Rightarrow\; \tau(s) = \tau = \text{constant}, \tag{1.9}$$

where s designates arclength. Given a point $P(x, y, t)$ we introduce the boundary conditions

$$\big(x(0), y(0), t(0)\big) = (x_0, y_0, t_0), \tag{1.10}$$

and

$$\big(x(s_f), y(s_f), t(s_f)\big) = (x, y, t), \tag{1.11}$$

at some final length s_f.

Definition 1.1. $\big(x(s), y(s), t(s)\big)$, the projection of the bicharacteristic curve onto the base is a subRiemannian geodesic which joins $P = P(x, y, t)$ to $P_0 = P(x_0, y_0, t_0)$.

To simplify matters we note that the vector fields (1.1) are translation invariant with respect to x and t, so we may assume $x_0 = 0$, $t_0 = 0$ with no loss of generality. Also, when $y_0 \neq 0$, symmetry considerations permit us to set $y_0 > 0$, see (2.35), (2.36), again with no loss of generality.

Theorem 1.2. $y_0 > 0$. *Every point $P(x, y, t)$, $y > 0$, can be joined to $P(0, y_0, 0)$ by at least one local geodesic. The number of these local geodesics is finite if and only if*

(i) $y \neq y_0$, *or*
(ii) $y = y_0$ *and* $t + y_0^2 x \neq 0$.

Theorem 1.3. $y_0 > 0$. *When* $y = y_0$ *and* $t + y_0^2 x = 0$, *then* $P(x, y_0, t)$ *is joined to* $P(0, y_0, 0)$ *by a discrete infinity of local geodesics.*

Theorem 1.4. $y_0 = 0$. *Every point* $P(x, y, t)$ *is connected to the origin by at least one geodesic. The number of geodesics joining* $P(x, y, t)$ *to the origin is finite if and only if* $y \neq 0$. *When* $y = 0$, *every point of the "canonical submanifold"* $\{(x, 0, 0),\ x \neq 0\}$ *is joined to the origin by a continuous infinity of geodesics, while every point of the complement* $\{(x, 0, t),\ t \neq 0\}$ *is joined to the origin by a discrete infinity of geodesics.*

Due to the length of the proof of Theorems 1.2 and 1.3 we present it in two consecutive articles. The present paper counts the number of geodesics which join points of the $y = y_0$-plane to $(0, y_0, 0)$; in particular it describes the canonical submanifold and proves Theorem 1.2(ii) and Theorem 1.3. A subsequent article counts the number of geodesics which join points $P(x, y, t)$, $y \neq y_0$ to $P(0, y_0, 0)$; in particular it contains the proof of Theorem 1.2(i). Theorem 1.4 was stated and proved in [9].

Theorems 1.2−1.4 are self-explanatory. Still, we need to discuss the idea of a "local geodesic" and the idea of a "canonical submanifold". But first we start with the notion of a subRiemannian geometry.

Suppose we are given m linearly independent vector fields $X_1, \ldots, X_m$ on an n-dimensional manifold M_n; we shall refer to $X_1, \ldots, X_m$ and their linear combinations as horizontal vector fields, and a curve with horizontal tangents will be called a horizontal curve. It is useful to introduce a metric by saying that the X_j-s have length 1 and they are perpendicular to each other. If $m = n$, we have a Riemannian metric, and if we let X_j^* denote the vector field adjoint to X_j with respect to the obvious volume element, then

$$\Delta = -\frac{1}{2} \sum_{j=1}^{n} X_j^* X_j \tag{1.12}$$

is the usual Laplace-Beltrami operator. The Newton potential is

$$N(x, x_0) = \frac{1}{(2-n)|\Omega_n(x_0)| \mathrm{d}(x, x_0)^{n-2}}, \quad n > 2, \tag{1.13}$$

where $|\Omega_n(x_0)|$ is the surface area of the induced unit ball with center x_0, and $d(x, x_0)$ is the Riemannian distance between x and x_0. Then

$$\Delta_x N(x, x_0) = \delta(x - x_0) + O\big(d(x, x_0)^{-n+1}\big), \tag{1.14}$$

in other words, the inverse kernel, or fundamental solution differs from $N(x, x_0)$ by a negligible error.

When $m < n$, we shall assume Hörmander's bracket generating condition [10] which says that $X_1, \ldots, X_m$ and a finite number of their Lie brackets generate all of TM_n. Then Chow's Theorem [5] says that between every two points, locally, there is a piecewise C^1 horizontal curve. This yields a distance and therefore a geometry which we shall call subRiemannian. To see how remarkable Chow's Theorem is, note that given two vector fields $\partial/\partial x$ and $\partial/\partial y$ in $\mathbb{R}_3(x, y, z)$, there is no horizontal curve joining any two points whose z-components differ. Chow's Theorem has been improved all the way up to smooth horizontal curves on smooth manifolds; in particular, nearby points can be connected by subRiemannian geodesics.

We came to these questions by trying to find the analogue of the Newton potential for operators of the form

$$\Delta = \frac{1}{2}\sum_{j=1}^{m} X_j^2 + \cdots\,, \qquad m < n, \tag{1.15}$$

where $\cdots$ stands for lower order terms; we note that the Laplace-Beltrami operator has the form (1.15) with $m = n$.

In Riemannian geometry, given a point C, every point in a sufficiently small neighbourhood of C is connected to C by one, single, unique geodesic. In subRiemannian geometry this is not the case. A number of examples have been worked out where given a point C, one can find points arbitrarily near C which may be joined to C by a finite, more than one, and even by an infinite number of geodesics. The first example was the Heisenberg group studied by Gaveau in [7]. Then Strichartz [11] pointed out that if the first brackets generate TM_n then every point will have at least two cut points arbitrarily near it. A rather complete description of the geometry of Heisenberg groups can be found in [2], and the geometry induced by the Grusin operator is discussed in [4]. We note that the geometry induced by the Grusin operator is not included in our definition of a subRiemannian geometry, although the same concepts will work there. Example (1.1), discussed in [9], and the present article, is the first higher step example, more than one bracket is needed to cover TM_n, where infinite number of geodesics connecting arbitrarily near points have been observed. We believe that this is the case in all subRiemannian geometries.

When $y \neq 0$, X, Y and $[Y, X] = YX - XY = 2y\partial/\partial t$ yield the full tangent space. When $y = 0$, we need X, Y, ($[Y, X] = 0$) and $\big[Y, [Y, X]\big] = 2\partial/\partial t$ to cover the tangent space. So $\mathbb{R}^3$ naturally breaks up into 2 domains, $y > 0$ and $y < 0$, and their boundary $y = 0$; sometime we say that the domain $y > 0$ (or $y < 0$) is step 2, while the set $y = 0$ is step 3. As already mentioned $y \gtrless 0$ can be treated at the same time, so we may assume that $y > 0$. We shall say that a geodesic which joins 2 points in $y > 0$ is local if it stays in the domain $y > 0$. Geodesics that cross the boundary $y = 0$ cannot be localized so we refer to them as nonlocal.

Every point of the line $y = y_0$, $t + y_0^2 x = 0$ is joined to $(0, y_0, 0)$ by a discrete infinity of geodesics. This property provides this line with extra structure and more symmetry. We shall refer to this line as the "canonical submanifold", and speculate that using it as the third coordinate to supplement x and y, or their exponential version, should help our understanding of the structure of the operator (1.15). Δ does not have to look simpler in these coordinates, but it may have geometric meaning; somewhat like spherical coordinates, which make the Laplace operator look more complicated than cartesian coordinates, but they also give more geometric structure. We note that the canonical curve goes into the x-axis as $y_0 \to 0$, see also Theorem 1.4. Thus our "natural coordinates" near $(0, y_0, 0)$ degenerate into the x and y axes at the origin. In particular this suggest that in higher step cases, higher than 2, we shall need to deal with singular coordinates, somewhat like polar coordinates in $\mathbb{R}^2$.

The notion of the canonical submanifold is probably quite general. Its tangent space provides the missing directions not covered by the horizontal vector fields. We state this as follows:

"*We are given m linearly independent vector fields on an n-dimensional manifold M_n. For every point $P_0 \in M_n$ there is an $n - m$ dimensional submanifold S_0, $P_0 \in S_0$, characterized by having all its points connected to P_0 by an infinite number of geodesics.*"

Of course, [2] and [9] suggest that this general statement will have to be refined somewhat.

Chapter 2 discusses Hamiltonian mechanics. Chapters 3–8 prove Theorem 1.2(ii) and Theorem 1.3. Chapter 9 contains a discussion of the behaviour of nonlocal geodesics on the $y = y_0$-plane and we apply these results to obtain parts of Theorem 1.4.

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2. The Geodesics

Hamilton's equations.

We consider 2 vector fields in $\mathbb{R}^3$,

$$X = \frac{\partial}{\partial x} + g(y)\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y}, \tag{2.1}$$

and assume that X, Y and their successive Lie brackets

$$[Y]^n(X) = \Big[Y, \big[Y, \ldots, [Y, X]\big]\Big] = \left(\frac{\partial}{\partial y}\right)^n g(y)\frac{\partial}{\partial t} = g^{(n)}(y)\frac{\partial}{\partial t}$$

cover the tangent space at every point $\vec{r} = (x, y, t)$. This is equivalent to having $g^{(m_y)}(y) \neq 0$ for all $y \in \mathbb{R}$ with some $m_y = 1, 2, \ldots$. A curve $\big(x(s), y(s), t(s)\big)$ is called horizontal if its tangent vector $\big(\dot{x}(s), \dot{y}(s), \dot{t}(s)\big)$ is always a linear combination of X and Y; equivalently

$$\dot{t}(s) = g\big(y(s)\big)\dot{x}(s). \tag{2.2}$$

According to a result of Chow [5] this hypothesis implies that for every 2 points $\vec{r}_0, \vec{r}_1 \in \mathbb{R}^3$ there is a continuous, piecewise C^1 curve $\vec{r}(s)$ which is horizontal and joins $\vec{r}_0$ and $\vec{r}_1$:

$$\vec{r}(0) = \vec{r}_0, \qquad \vec{r}(s_1) = \vec{r}_1.$$

Let $\Gamma(\vec{r}_0, 0 \mid \vec{r}_1, s_1)$ denote the space of such curves, and for $\vec{r}(s) \in \Gamma$ we let

$$E\big(\vec{r}(\cdot)\big) = \frac{1}{2}\int_0^{s_1} \big[\dot{x}(s)^2 + \dot{y}(s)^2\big]\mathrm{d}s \tag{2.3}$$

denote its energy. We are interested in the following variational problem:

"find a curve $\vec{r}(s)$ of minimal energy in $\Gamma(\vec{r}_0, 0 \mid \vec{r}_1, s_1)$".

In other words we are looking for a curve $\vec{r}(s)$ such that

$$\begin{aligned}
&\vec{r}(0) = \vec{r}_0, \qquad \vec{r}(s_1) = \vec{r}_1,\\
&\dot{t}(s) - g\big(y(s)\big)\dot{x}(s) = 0, \qquad 0 \le s \le s_1,\\
&E\big(\vec{r}(\cdot)\big) = \mathrm{Min}\Big\{E\big(\vec{\rho}(\cdot)\big); \vec{\rho}(s) \in \Gamma\Big\}.
\end{aligned}$$

This is a control problem and to find the stationary curves we introduce a Lagrange multiplier $\tau(s)$ into the Lagrangian

$$L = \int_0^{s_1} \Big[\frac{1}{2}\big(\dot{x}(s)^2 + \dot{y}(s)^2\big) + \tau(s)\big(\dot{t}(s) - g\big(y(s)\big)\dot{x}(s)\big)\Big]\mathrm{d}s. \tag{2.4}$$

One has Lagrange's equations

$$\begin{aligned}
&\ddot{x}(s) - \frac{\mathrm{d}}{\mathrm{d}s}\Big(\tau(s)g\big(y(s)\big)\Big) = 0,\\
&\ddot{y}(s) - \tau(s)g'\big(y(s)\big) = 0,\\
&\dot{\tau}(s) = 0,
\end{aligned}$$

and $\tau(s) = \tau$ is a constant. Thus L takes the form

$$L = \int_0^{s_1} \Big[\frac{1}{2}\big(\dot{x}(s)^2 + \dot{y}(s)^2\big) + \tau\big(\dot{t}(s) - g\big(y(s)\big)\dot{x}(s)\big)\Big]\mathrm{d}s.$$

Introducing the associated momenta

$$\xi = \frac{\partial L}{\partial \dot{x}} = \dot{x} - \tau g, \quad \eta = \frac{\partial L}{\partial \dot{y}} = \dot{y}, \quad \tau = \frac{\partial L}{\partial \dot{t}},$$

we obtain the corresponding Hamiltonian

$$H = \frac{1}{2}\Big[\big(\xi + \tau g(y)\big)^2 + \eta^2\Big]. \tag{2.5}$$

Hamilton's equations are

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \eta, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = -\tau g'(y)\big(\xi + \tau g(y)\big), \tag{2.6}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \xi + \tau g(y), \qquad \frac{\mathrm{d}t}{\mathrm{d}s} = g(y)\big(\xi + \tau g(y)\big), \tag{2.7}$$

$$\xi \text{ and } \tau \text{ are constants of motion.} \tag{2.8}$$

Equations (2.6) form a closed system which can be integrated. Once $y(s)$ is known, (2.7) yields $x(s)$ and $t(s)$ by quadrature.

Symmetries and reductions.

(i) We need 3 parameters, ξ, τ, s_f, and fix H,

$$H = \frac{1}{2}, \tag{2.9}$$

which implies that s is arclength.

(ii) Changing (ξ, τ) to $(-\xi, -\tau)$, the trajectory $\big(x(s), y(s), t(s)\big)$ changes to $\big(-x(s), y(s), -t(s)\big)$, see (2.6), (2.7). Consequently, trajectories, or "geodesics", with $\tau \geq 0$ will give us all geodesics, and from now on we shall assume that

$$\tau \geq 0. \tag{2.10}$$

(iii) Solutions of (2.6), (2.7) are invariant with respect to translation along x and t, so it suffices to assume that

$$x_0 = 0, \qquad t_0 = 0. \tag{2.11}$$

From now on all solutions of (2.6)$-$(2.8) will start at $(0, y_0, 0)$.

(iv) $\tau = 0$ implies that $\eta =$ constant, and the trajectories starting from $(0, y_0, 0)$ are given by

$$x(s) = \xi s, \tag{2.12}$$

$$y(s) = y_0 + \eta s, \tag{2.13}$$

$$t(s) = \xi \int_0^s g(y_0 + \eta s')\mathrm{d}s' = \frac{\xi s}{\eta s} \int_0^{\eta s} g(y_0 + s')\mathrm{d}s'. \tag{2.14}$$

The set of points $\big(x(s), y(s), t(s)\big)$ is a surface parametrized by $(\xi s, \eta s)$.

(v) When

$$g(0) = 0 \quad \text{and} \quad g'(0) = 0,$$

the x-axis,

$$x(s) = s, \qquad y(s) = 0, \qquad t(s) = 0 \tag{2.15}$$

is a solution of (2.6)$-$(2.8). In this case $\xi = 1$, $\eta(s) = 0$, and the parameter τ is indeterminate.

Integration of the equations of motion.

With

$$H = \frac{1}{2}, \qquad \eta = \frac{\mathrm{d}y}{\mathrm{d}s}, \tag{2.16}$$

we have

$$\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1 - \big(\xi + \tau g(y)\big)^2, \tag{2.17}$$

or,

$$\mathrm{d}s = \operatorname{sgn}\big(\dot{y}(s)\big) \frac{\mathrm{d}y}{\Big(1 - \big(\xi + \tau g(y)\big)^2\Big)^{1/2}}. \tag{2.18}$$

Thus

$$\big(\xi + \tau g(y)\big)^2 \le 1, \tag{2.19}$$

and we may replace y by the variable α, where

$$\sin\,\alpha = \xi + \tau g(y) = \dot{x}(s), \tag{2.20}$$

$$\cos\,\alpha = \dot{y}(s), \ \text{see (2.17)}. \tag{2.21}$$

This fixes α,

$$-\frac{\pi}{2} \le \alpha < \frac{3\pi}{2}. \tag{2.22}$$

We note that

$$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2} \;\Rightarrow\; \dot{y}(s) \geq 0, \tag{2.23}$$

$$\frac{\pi}{2} \leq \alpha < \frac{3\pi}{2} \;\Rightarrow\; \dot{y}(s) \leq 0. \tag{2.24}$$

Differentiating (2.20) yields

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = \tau g'\big(y(s)\big), \qquad \alpha \neq \pm\frac{\pi}{2}, \tag{2.25}$$

and

$$\mathrm{sgn}(\mathrm{d}\alpha) = \mathrm{sgn}\big(g'(y)\big). \tag{2.26}$$

Finally, (2.7) and (2.25) give us

$$\frac{\mathrm{d}x}{\mathrm{d}\alpha} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \frac{\sin\,\alpha}{\tau g'(y)}, \tag{2.27}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \frac{\sin\,\alpha(\sin\,\alpha - \xi)}{\tau^2 g'(y)}. \tag{2.28}$$

$g(y) = y^n$, $n = 1, 2, \ldots$.

Again we have $\tau \geq 0$.

a) Trajectories with $\tau = 0$. According to (2.6) η is constant. When $\eta \neq 0$, (2.12)–(2.14) give

$$x(s) = \xi s \tag{2.29}$$

$$y(s) = y_0 + \eta s \tag{2.30}$$

$$t(s) = \frac{\xi}{(n+1)\eta}\big[(y_0 + \eta s)^{n+1} - y_0^{n+1}\big]. \tag{2.31}$$

Extracting ξ and η, $\eta \neq 0$, one finds that the trajectories fill the surface

$$t(y - y_0) - \frac{x}{n+1}(y^{n+1} - y_0^{n+1}) = 0. \tag{2.32}$$

When $\eta = 0$ the trajectories are

$$x(s) = \xi s, \qquad y(s) = y_0, \qquad t(s) = \xi y_0^n s, \tag{2.33}$$

$\xi = \pm 1$. This is the straight line

$$t = y_0^n x \tag{2.34}$$

in the $y = y_0$ plane.

b) Trajectories with $\tau > 0$. We shall treat odd n and even n separately.

(i) $n = 2m$. The hamiltonian system is invariant with respect to the symmetry,

$$(x, y, t, \xi, \eta, \tau) \longrightarrow (x, -y, t, \xi, -\eta, \tau), \tag{2.35}$$

so it suffices to find all trajectories with

$$y_0 > 0 \quad \text{and} \quad \dot{y}(0) = \eta(0) \in \mathbb{R}, \tag{2.36}$$

or

$$y_0 = 0 \quad \text{and} \quad \dot{y}(0) = \eta(0) > 0. \tag{2.37}$$

Here (2.25) takes the form

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} = 2m\tau y^{2m-1},$$

so,

$$\operatorname{sgn}(\mathrm{d}\alpha) = \operatorname{sgn}\big(y(\alpha)\big). \tag{2.38}$$

From (2.19),

$$-1 \le \xi + \tau y_0^{2m} \le 1, \tag{2.39}$$

and since $\tau > 0$, we have

$$-\infty < \xi < 1, \qquad \text{if } y_0 > 0, \tag{2.40}$$

$$-1 \le \xi \le 1, \qquad \text{if } y_0 = 0. \tag{2.41}$$

To simplify notation we set

$$\zeta = -\xi, \qquad -1 < \zeta < \infty. \tag{2.42}$$

Then (2.20) takes the form

$$y(\alpha) = \operatorname{sgn}\bigl(y(\alpha)\bigr)\frac{(\zeta + \sin\alpha)^{1/2m}}{\tau^{1/2m}}. \tag{2.43}$$

We choose $\alpha_0 \in \bigl[-\frac{\pi}{2}, \frac{3\pi}{2}\bigr)$, such that

$$y_0 = y(0) = \frac{(\zeta + \sin\alpha_0)^{1/2m}}{\tau^{1/2m}}, \tag{2.44}$$

so

$$\operatorname{sgn}(\cos\alpha_0) = \operatorname{sgn}\bigl(\dot{y}(0)\bigr). \tag{2.45}$$

Integrating (2.27) and (2.28) we obtain

$$x(\alpha) = \frac{1}{n\tau^{1/n}}\int_{\alpha_0}^{\alpha} \operatorname{sgn}\bigl(y(\alpha')\bigr)\frac{\sin\alpha' \mathrm{d}\alpha'}{(\zeta + \sin\alpha')^{1-1/n}}, \tag{2.46}$$

$$t(\alpha) = \frac{1}{n\tau^{1+1/n}}\int_{\alpha_0}^{\alpha} \operatorname{sgn}\bigl(y(\alpha')\bigr)\sin\alpha'(\zeta + \sin\alpha')^{1/n}\mathrm{d}\alpha', \tag{2.47}$$

and then (2.43)–(2.47) yield the trajectories with $y_0 > 0$; note that the y-motion is bounded and $y(\alpha)$ oscillates between two extreme positions. Next we describe our trajectories in terms of ζ.

α) $\zeta > 1$. (2.43) implies that $y(\alpha)$ never vanishes, $y(\alpha) > 0$ so $\mathrm{d}\alpha > 0$, and α always increases starting at α_0.

$y(\alpha)$ is maximum at $\alpha = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$.

$y(\alpha)$ is minimum at $\alpha = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$.

Consequently, $\alpha = (2k+1)\frac{\pi}{2}$ are the turning points of the y-motion.

β) $\zeta = 1$ corresponds to trajectories with the minimal turning points on the $y = 0$ plane.

γ) $-1 < \zeta < 1$. (2.43) requires that $\zeta + \sin\alpha > 0$, therefore

$$\sin^{-1}(-\zeta) \le \alpha \le \pi - \sin^{-1}(-\zeta), \qquad -\frac{\pi}{2} \le \sin^{-1}(-\zeta) \le \frac{\pi}{2}. \tag{2.48}$$

These trajectories are more complicated. α starts at α_0 and increases till $\pi - \sin^{-1}(-\zeta)$, then decreases to $\sin^{-1}(-\zeta)$ and starts increasing again till $\alpha = \alpha_0$. Then the cycle starts again. Assuming $\dot{y}(0) > 0$, y increases till a maximal turning point, then starts decreasing, passes through the $y = y_0$-plane and the $y = 0$-plane till it hits a minimal turning point which is negative, turns around, increases till $y = y_0$ for a full cycle, and then starts again.

When $\dot{y}(0) < 0$, it starts decreasing and does the same cycle; this happens when $\alpha_0 \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$. In either case the turning points of the y-component of the trajectory occur at $\alpha = \frac{\pi}{2}$:

$$y_{\max} = \frac{(1+\zeta)^{1/2m}}{\tau^{1/2m}}, \qquad y_{\min} = -\frac{(1+\zeta)^{1/2m}}{\tau^{1/2m}}, \tag{2.49}$$

and if $\dot{y}(0) > 0$, we have

$$\sin^{-1}(-\zeta) < \alpha_0 < \frac{\pi}{2}. \tag{2.50}$$

δ) $\zeta = -1$, $(\xi = 1)$, $\tau > 0 \Rightarrow y \equiv 0$, and (2.7) yields

$$x(s) = s, \qquad y(s) \equiv 0, \qquad t(s) \equiv 0, \tag{2.51}$$

which is the x-axis.

(ii) $n = 2m+1$, $\tau > 0$. Here the Hamiltonian system is invariant with respect to the symmetry

$$(x, y, t, \xi, \eta, \tau) \;\longrightarrow\; (-x, -y, t, -\xi, -\eta, \tau), \tag{2.52}$$

so we may assume that

$$y_0 > 0, \quad \text{or} \quad y_0 = 0 \quad \text{and } \dot{y}(0) > 0. \tag{2.53}$$

Also, (2.26) implies that

$$\mathrm{d}\alpha > 0, \tag{2.54}$$

and α always increases. Again,

$$-1 \le -\zeta + \tau y_0^{2m+1} \le 1,$$

thus $y_0 \ge 0$, $\tau > 0$ yield

$$-1 \le \zeta < \infty. \tag{2.55}$$

Formulas (2.43), (2.46) and (2.47) still hold if we leave off the factor $\operatorname{sgn}\big(y(\alpha)\big)$, since the odd n-th root takes care of the sign of $y(\alpha)$. $y(\alpha)$ has turning points at $\alpha = (2k+1)\pi/2$, $k = 0, 1, 2, \ldots$. When $\zeta \in [-1, 1]$, $y(\alpha)$ vanishes at $2k\pi + \sin^{-1}(-\zeta)$, $2k\pi + \pi - \sin^{-1}(-\zeta)$, $k = 0, 1, 2, \ldots$.

The discussion of the trajectories in this, the odd n, case is the same as the discussion in the even n case, as long as we remember that here α always increases.

Length and action of a trajectory.

With our convention $H = \frac{1}{2}$, $\dot{x}^2 + \dot{y}^2 = 1$, and then the parameter s is arclength. Also, (2.25) yields

$$\mathrm{d}s = \frac{\mathrm{d}\alpha}{\tau g'(y(\alpha))} = \frac{\mathrm{d}\alpha}{n\tau y(\alpha)^{n-1}}. \tag{2.56}$$

Consequently, the arclength ℓ of the trajectory between α_0 and α is

$$\ell = \frac{1}{n\tau^{1/n}} \int_{\alpha_0}^{\alpha} \frac{\operatorname{sgn}(y(\alpha'))}{(\zeta + \sin\alpha')^{1-1/n}} \mathrm{d}\alpha', \qquad n \text{ even}, \tag{2.57}$$

$$\ell = \frac{1}{n\tau^{1/n}} \int_{\alpha_0}^{\alpha} \frac{\mathrm{d}\alpha'}{(\zeta + \sin\alpha')^{(n-1)/n}}, \qquad n \text{ odd}. \tag{2.58}$$

We note that when n is even, $\zeta + \sin\alpha' > 0$, and so is $\operatorname{sgn}\big(y(\alpha')\big)\mathrm{d}\alpha'$, and if n is odd then $\mathrm{d}\alpha' > 0$ and so is $(\zeta + \sin\alpha')^{(n-1)/n}$. Consequently $\ell > 0$, as expected. The action integral along a trajectory is

$$S = \int_0^s (\xi \mathrm{d}x + \eta \mathrm{d}y + \tau \mathrm{d}t - H \mathrm{d}s). \tag{2.59}$$

According to (2.2), $\mathrm{d}t = g(y)\mathrm{d}x$, so

$$\xi \mathrm{d}x + \eta \mathrm{d}y + \tau \mathrm{d}t = \big(\xi + \tau g(y)\big)\mathrm{d}x + \eta \mathrm{d}y,$$

and the equations of motion yield

$$\xi \mathrm{d}x + \eta \mathrm{d}y + \tau \mathrm{d}t = \Big(\big(\xi + \tau g(y)\big)^2 + \eta^2\Big)\mathrm{d}s = 2H\mathrm{d}s.$$

Consequently the action is half the arclength of the trajectory,

$$S = \frac{1}{2}s. \tag{2.60}$$

$\boldsymbol{g(y) = y^2}$.

A given trajectory $\big(x(\alpha), y(\alpha), t(\alpha)\big)$ depends on three parameters, α_0, $\zeta \geq -1$, $\tau > 0$, and is given by

$$y_0 = \frac{(\zeta + \sin\alpha_0)^{1/2}}{\tau^{1/2}}, \tag{2.61}$$

$$y(\alpha) = \operatorname{sgn}\big(y(\alpha)\big)\frac{(\zeta + \sin\alpha)^{1/2}}{\tau^{1/2}}, \tag{2.62}$$

$$x(\alpha) = \frac{1}{2\tau^{1/2}}\int_{\alpha_0}^{\alpha} \operatorname{sgn}\big(y(\alpha')\big)\frac{\sin\alpha' \mathrm{d}\alpha'}{\sqrt{\zeta + \sin\alpha'}}, \tag{2.63}$$

$$t(\alpha) = \frac{1}{2\tau^{3/2}}\int_{\alpha_0}^{\alpha} \operatorname{sgn}\big(y(\alpha')\big)\sin\alpha'\sqrt{\zeta + \sin\alpha'}\mathrm{d}\alpha'. \tag{2.64}$$

The arclength along this trajectory from $(0, y_0, 0)$ to $\big(x(\alpha), y(\alpha), t(\alpha)\big)$ is

$$\ell(\alpha) = \frac{1}{2\tau^{1/2}}\int_{\alpha_0}^{\alpha} \operatorname{sgn}\big(y(\alpha')\big)\frac{\mathrm{d}\alpha'}{\sqrt{\zeta + \sin\alpha'}}; \tag{2.65}$$

we note that (2.63)–(2.65) are elliptic integrals. Recall that $-1 \leq \zeta < \infty$, and

(i) for $\zeta > 1$, α always increases starting at α_0, and $y(\alpha)$ remains positive when $y_0 > 0$,

(ii) for $|\zeta| < 1$, α oscillates in the interval $\big[\sin^{-1}(-\zeta), \pi - \sin^{-1}(-\zeta)\big]$, $\sin^{-1}(-\zeta) \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. $y(\alpha)$ vanishes at $\sin^{-1}(-\zeta)$ and at $\pi - \sin^{-1}(-\zeta)$.

When $y_0 = 0$, $|\zeta| \leq 1$. Suppose a geodesic starts at $(0, y_0, 0)$, $y_0 > 0$. For reasons which will become clear later, *we shall refer to such geodesics as "local geodesics" when they stay in the half space $y > 0$, otherwise we shall call them "nonlocal geodesics".*

Our aim in this and in a subsequent article is to "count" the number of geodesics which join $(0, y_0, 0)$ to (x, y, t) when $g(y) = y^2$. We shall distinguish three cases:

(a) $y = y_0$, $y_0 \neq 0$,

(b) $y \neq y_0$, $y_0 \neq 0$,
(c) $y_0 = 0$.

The last case was already discussed in detail in [9] using Jacobi's theory of elliptic functions.

3. $g(y) = y^2$, $\zeta > 1$. Preliminary Formulas

In Sections 3–7 we are concerned with the following question:

"*What points* (x, y_0, t) *are connected to* $(0, y_0, 0)$ *by geodesics, and by how many geodesics, if* $\zeta > 1$*?*"

From (2.61) one has

$$y_0 = \frac{(\zeta + \sin\alpha_0)^{1/2}}{\tau^{1/2}} > 0. \tag{3.1}$$

Geodesics with $\zeta > 1$ stay in the half space $y > 0$, so they are local and $\operatorname{sgn}\big(y(\alpha)\big) = 1$, and α increases, always, starting at α_0. So we have

$$y(\alpha) = \frac{(\zeta + \sin\alpha)^{1/2}}{\tau^{1/2}}. \tag{3.2}$$

If $y = y(\alpha) = y_0$, then α must be

$$\alpha = \alpha_0 + 2n\pi, \qquad n = 1, 2, \ldots, \text{ or} \tag{3.3}$$

$$\alpha = \pi - \alpha_0 + 2n\pi, \qquad n = 0, 1, 2, \ldots, \tag{3.4}$$

and (2.63)–(2.65) yield

(i) $\alpha = \alpha_0 + 2n\pi$, $n = 1, 2, \ldots$

$$x(\alpha_0 + 2n\pi) = \frac{n}{2\tau^{1/2}} \int_0^{2\pi} \frac{\sin\alpha \, d\alpha}{(\zeta + \sin\alpha)^{1/2}}, \tag{3.5}$$

$$t(\alpha_0 + 2n\pi) = \frac{n}{2\tau^{3/2}} \int_0^{2\pi} (\zeta + \sin\alpha)^{1/2} \sin\alpha \, d\alpha, \tag{3.6}$$

$$\ell(\alpha_0 + 2n\pi) = \frac{n}{2\tau^{1/2}} \int_0^{2\pi} \frac{d\alpha}{(\zeta + \sin\alpha)^{1/2}}. \tag{3.7}$$

(ii) $\alpha = \pi - \alpha_0 + 2n\pi$, $n = 0, 1, 2, \ldots$ if $\dot{y}(0) > 0$, and $n = 1, 2, \ldots$ if

$\dot{y}(0) < 0$.

$$x(\pi-\alpha_0+2n\pi) = \frac{1}{2\tau^{1/2}}\int_{\alpha_0}^{\pi-\alpha_0}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}} + \frac{n}{2\tau^{1/2}}\int_0^{2\pi}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}, \tag{3.8}$$

$$t(\pi-\alpha_0+2n\pi) = \frac{1}{2\tau^{3/2}}\int_{\alpha_0}^{\pi-\alpha_0}(\zeta+\sin\alpha)^{1/2}\sin\alpha d\alpha + \frac{n}{2\tau^{3/2}}\int_0^{2\pi}(\zeta+\sin\alpha)^{1/2}\sin\alpha d\alpha, \tag{3.9}$$

$$\ell(\pi-\alpha_0+2n\pi) = \frac{1}{2\tau^{1/2}}\int_{\alpha_0}^{\pi-\alpha_0}\frac{d\alpha}{(\zeta+\sin\alpha)^{1/2}} + \frac{n}{2\tau^{1/2}}\int_0^{2\pi}\frac{d\alpha}{(\zeta+\sin\alpha)^{1/2}}. \tag{3.10}$$

We note that

$$\alpha_0 \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \Rightarrow \dot{y}(0) \geq 0, \tag{3.11}$$

$$\alpha_0 \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right) \Rightarrow \dot{y}(0) < 0. \tag{3.12}$$

4. $\alpha_{\mathrm{end}} = \alpha_0 + 2n\pi,\ n = 1, 2, \ldots$

For $\zeta \geq 1$ we define the functions

$$I(\zeta) = \int_0^{2\pi}(\zeta+\sin\alpha)^{1/2}\sin\alpha d\alpha = 2\int_{-\pi/2}^{\pi/2}(\zeta+\sin\alpha)^{1/2}\sin\alpha d\alpha, \tag{4.1}$$

$$J(\zeta) = \int_0^{2\pi}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}} = 2\int_{-\pi/2}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}. \tag{4.2}$$

Then (3.5) and (3.6) take the following form:

$$x(\alpha_0+2n\pi) = \frac{n}{2\tau^{1/2}}J(\zeta), \tag{4.3}$$

$$t(\alpha_0+2n\pi) = \frac{n}{2\tau^{3/2}}I(\zeta). \tag{4.4}$$

With

$$\tau = \frac{\zeta + \sin \alpha_0}{y_0^2} \tag{4.5}$$

we eliminate τ from (4.3) and (4.4):

$$\frac{2x(\alpha_0 + 2n\pi)}{y_0} = \frac{n}{(\zeta + \sin \alpha_0)^{1/2}} J(\zeta), \tag{4.6}$$

$$\frac{2t(\alpha_0 + 2n\pi)}{y_0^3} = \frac{n}{(\zeta + \sin \alpha_0)^{3/2}} I(\zeta) \tag{4.7}$$

which suggests the following notation,

$$\hat{x} = \frac{2x}{y_0}, \qquad \hat{t} = \frac{2t}{y_0^3}. \tag{4.8}$$

We set $n = 1$, and study the map

$$(\alpha_0, \zeta) \longrightarrow \big(\hat{x}(\alpha_0, \zeta), \hat{t}(\alpha_0, \zeta)\big), \qquad \zeta \geq 1, \tag{4.9}$$

with

$$\hat{x}(\alpha_0, \zeta) = \frac{J(\zeta)}{(\zeta + \sin \alpha_0)^{1/2}}, \tag{4.10}$$

$$\hat{t}(\alpha_0, \zeta) = \frac{I(\zeta)}{(\zeta + \sin \alpha_0)^{3/2}}. \tag{4.11}$$

Lemma 4.1. *One has*

$$\frac{\mathrm{d}I}{\mathrm{d}\zeta} = \frac{1}{2} J(\zeta). \tag{4.12}$$

$J(\zeta)$ increases from $J(1) = -\infty$ to $J(\infty) = 0$.

$I(\zeta)$ decreases from $I(1) > 0$ to $I(\infty) = 0$.

Proof. (4.12) is immediate. Also,

$$J(\zeta) = \int_0^\pi \sin \alpha \left(\frac{1}{\sqrt{\zeta + \sin \alpha}} - \frac{1}{\sqrt{\zeta - \sin \alpha}} \right) \mathrm{d}\alpha < 0, \tag{4.13}$$

therefore (4.13) implies that $I(\zeta)$ decreases from $I(1)$ to $I(\infty)$. Since

$$I(\zeta) = \sqrt{\zeta}\int_0^{2\pi}\left(1+\frac{\sin\alpha}{\zeta}\right)^{1/2}\sin\alpha\mathrm{d}\alpha$$
$$= \frac{\pi}{2\sqrt{\zeta}} + \frac{1}{16\zeta^{5/2}}\int_0^{2\pi}\sin^4\alpha\mathrm{d}\alpha + O\left(\frac{1}{\zeta^{9/2}}\right), \tag{4.14}$$

we have $I(\infty) = 0$, and therefore

$$I(\zeta) > 0, \qquad \zeta\in(1,\infty). \tag{4.15}$$

Also,

$$\frac{\mathrm{d}J}{\mathrm{d}\zeta} = -\frac{1}{2}\int_0^{2\pi}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}$$
$$= -\frac{1}{2}\int_0^{\pi}\sin\alpha\left(\frac{1}{(\zeta+\sin\alpha)^{3/2}} - \frac{1}{(\zeta-\sin\alpha)^{3/2}}\right)\mathrm{d}\alpha > 0, \tag{4.16}$$

and $J(\zeta)$ is an increasing function of $\zeta\in(1,\infty)$. With $\alpha = -\pi/2+\alpha'$, $(1+\sin\alpha)^{-1/2}\sim\frac{\sqrt{2}}{\alpha'}$, and then (4.2) yields $J(1) = -\infty$. Also $J(\infty) = 0$, because for large ζ one has

$$J(\zeta) = -\frac{\pi}{2\zeta^{3/2}} - \frac{5}{16\zeta^{7/2}}\int_0^{2\pi}\sin^4\alpha\mathrm{d}\alpha + O\left(\frac{1}{\zeta^{11/2}}\right). \tag{4.17}$$

This proves Lemma 4.1. □

We note that

$$\frac{\hat{x}^3}{\hat{t}} = \frac{J^3}{I}. \tag{4.18}$$

Lemma 4.2. *The function $J(\zeta)^3/I(\zeta)$ increases from $(J^3/I)(1) = -\infty$ to $(J^3/I)(\infty) = 0$.*

Proof. (4.12) yields

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J^3}{I} = \frac{J^2}{I^2}(3J'I - JI') = \frac{J^2}{I^2}\left(3J'I - \frac{1}{2}J^2\right).$$

For large ζ one has

$$J'(\zeta) = -\frac{3\pi}{4\zeta^{5/2}} + O\left(\frac{1}{\zeta^{9/2}}\right), \tag{4.19}$$

and (4.14), (4.17) and (4.19) imply that

$$3J'I - \frac{1}{2}J^2\Big|_{\zeta=\infty} = 0. \tag{4.20}$$

We note that

$$\frac{d}{d\zeta}\left(3J'I - \frac{1}{2}J^2\right) = 3J''I + \frac{1}{2}JJ',$$

and

$$J' = -\frac{1}{2}\int_0^{2\pi} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} > 0,$$

see (4.16). By similar reasoning

$$J'' = \frac{3}{4}\int_0^{2\pi} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{5/2}} < 0, \qquad I > 0, \qquad J < 0,$$

and therefore

$$3J''I + \frac{1}{2}JJ' < 0.$$

Consequently, $3J'I - \frac{1}{2}J^2$ is a decreasing function of ζ which vanishes at $\zeta = \infty$ by (4.20), hence

$$3J'I - \frac{1}{2}J^2 > 0, \qquad \zeta \in [1,\infty).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J^3}{I} > 0,$$

and J^3/I is an increasing function which is $-\infty$ at $\zeta = 1$, see Lemma 4.1, and vanishes at $\zeta = \infty$ in view of (4.14) and (4.17). This completes the proof of Lemma 4.2. □

Lemma 4.3. *The mapping* (4.9),

$$(\alpha_0,\zeta) \longrightarrow \big(\hat{x}(\alpha_0,\zeta), \hat{t}(\alpha_0,\zeta)\big),$$

is one-to-one onto its image from the domain

$$(\alpha_0,\zeta) \in \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big] \times (1,\infty).$$

Proof. Lemma 4.2 shows that

$$\frac{J^3(\zeta)}{I(\zeta)} = \frac{\hat{x}^3}{\hat{t}}$$

has a unique solution $\zeta \in (1, \infty)$ for every $\hat{x}^3/\hat{t} \in (-\infty, 0)$. Given this ζ, the solution $\alpha_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of

$$\hat{x} = \frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}} \tag{4.21}$$

is clearly unique, and we have Lemma 4.3. □

We shall obtain the range of the map (4.9) as the union of the curves $C_1(\alpha_0)$,

$$C_1(\alpha_0) = \big(\hat{x}(\alpha_0, \zeta), \hat{t}(\alpha_0, \zeta)\big), \qquad \zeta \in (1, \infty), \tag{4.22}$$

over $\alpha_0 \in [-\pi/2, \pi/2]$. One has the large ζ expansions of $\hat{x}$ and $\hat{t}$ from (4.14) and (4.17):

$$\begin{aligned}&\hat{x}(\alpha_0, \zeta)\\ &= -\frac{\pi}{2\zeta^2} + \frac{\pi\sin\alpha_0}{4\zeta^3} - \frac{1}{16\zeta^4}\left(5\int_0^{2\pi}\sin^4\alpha d\alpha + 3\pi\sin^2\alpha_0\right) + \cdots,\end{aligned} \tag{4.23}$$

$$\begin{aligned}&\hat{t}(\alpha_0, \zeta)\\ &= \frac{\pi}{2\zeta^2} - \frac{3\pi\sin\alpha_0}{4\zeta^3} + \frac{1}{16\zeta^4}\left(\int_0^{2\pi}\sin^4\alpha d\alpha + 15\pi\sin^2\alpha_0\right) + \cdots,\end{aligned} \tag{4.24}$$

which are convergent power series in ζ^{-1}. Therefore

$$\begin{aligned}&\hat{x}(\alpha_0, \zeta) + \hat{t}(\alpha_0, \zeta)\\ &= -\frac{\pi\sin\alpha_0}{2\zeta^3} + \frac{1}{4\zeta^4}\left(3\pi\sin^2\alpha_0 - \int_0^{2\pi}\sin^4\alpha d\alpha\right) + \cdots.\end{aligned} \tag{4.25}$$

These formulas yield

Lemma 4.4. *The curves $C_1(\alpha_0)$ start at $(0, 0)$ tangent to the line $\hat{x}+\hat{t} =$*

0 when $\zeta = \infty$. More precisely,

$$\alpha_0 \in \left[0, \frac{\pi}{2}\right] : \ C_1(\alpha_0) \text{ starts below the line } \hat{x} + \hat{t} = 0, \tag{i}$$

$$\alpha_0 \in \left[-\frac{\pi}{2}, 0\right) : \ C_1(\alpha_0) \text{ starts above the line } \hat{x} + \hat{t} = 0. \tag{ii}$$

Lemma 4.5. (i) *For a fixed $\alpha_0 \in (-\pi/2, \pi/2]$, $\hat{t}(\alpha_0, \zeta)$ decreases from $I(1)(1+\sin\alpha_0)^{-3/2}$ to $\hat{t}(\alpha_0, \infty)=0$, and $\hat{t}(-\pi/2, \zeta)$ decreases from $\hat{t}(-\pi/2, 1) = \infty$ to $\hat{t}(-\pi/2, \infty) = 0$.*

(ii) *$\hat{x}(\alpha_0, \zeta)$ is an increasing function of ζ, $\alpha_0 \in [-\pi/2, \pi/2]$; $\hat{x}$ increases from $\hat{x}(\alpha_0, 1) = -\infty$ to $\hat{x}(\alpha_0, \infty) = 0$.*

(iii) *For $\alpha_0 \in (-\pi/2, \pi/2]$ the curves $C_1(\alpha_0)$ have a horizontal asymptote given by $\hat{t}(\alpha_0, 1) = I(1)(1 + \sin\alpha_0)^{-3/2}$.*

(iv) *Along $C_1(-\pi/2)$,*

$$\hat{x}\left(-\frac{\pi}{2}, \zeta\right) \sim \frac{\sqrt{2}\log(\zeta - 1)}{(\zeta - 1)^{1/2}} < 0 \ \text{ as } \zeta \to 1^+,$$
$$\hat{t}\left(-\frac{\pi}{2}, \zeta\right) \sim \frac{I(1)}{(\zeta - 1)^{3/2}} > 0 \ \text{ as } \zeta \to 1^+.$$

In particular,

$$\hat{x}\left(-\frac{\pi}{2}, \zeta\right) + \hat{t}\left(-\frac{\pi}{2}, \zeta\right) \sim \frac{I(1)}{(\zeta - 1)^{3/2}} \ \text{ as } \zeta \to 1^+.$$

Proof. (i) (4.13) yields

$$\frac{\partial \hat{t}(\alpha_0, \zeta)}{\partial \zeta} = -\frac{3}{2}\frac{I(\zeta)}{(\zeta + \sin\alpha_0)^{5/2}} + \frac{1}{2}\frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}} < 0, \tag{4.26}$$

as $I > 0$ and $J < 0$, and Lemma 4.1 gives the range of $\hat{t}$.

(ii) Again,

$$\frac{\partial \hat{x}(\alpha_0, \zeta)}{\partial \zeta} = -\frac{1}{2}\frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}} - \frac{1}{2}\frac{1}{(\zeta + \sin\alpha_0)^{1/2}} \int_0^{2\pi} \frac{\sin\alpha d\alpha}{(\zeta + \sin\alpha)^{3/2}}$$
$$> 0, \tag{4.27}$$

and Lemma 4.1 gives the range of $\hat{x}$.

(iii) As ζ decreases from ∞ to 1, $\hat{t}(\alpha_0,\zeta)$ increases to $I(1)(1+\sin\alpha_0)^{-3/2}$ and $\hat{x}(\alpha_0,\zeta)$ decreases to $-\infty$, $\alpha_0\in(-\pi/2,\pi/2]$.

(iv) Also,

$$\hat{x}\left(-\frac{\pi}{2},\zeta\right)=\frac{J(\zeta)}{(\zeta-1)^{1/2}},\qquad \hat{t}\left(-\frac{\pi}{2},\zeta\right)=\frac{I(\zeta)}{(\zeta-1)^{3/2}}.$$

As $\zeta\to 1^+$, $I(\zeta)\to I(1)$ and $J(\zeta)\sim\sqrt{2}\log(\zeta-1)$; indeed, with $\zeta=1+\varepsilon$,

$$\begin{aligned}J(\zeta)&=\int_0^\pi\frac{\sin\alpha d\alpha}{\sqrt{\zeta+\sin\alpha}}-\int_0^\pi\frac{\sin\alpha d\alpha}{\sqrt{\zeta-\sin\alpha}}\\&\sim-\int_0^\pi\frac{\sin\alpha d\alpha}{\sqrt{\zeta-\sin\alpha}}=-2\int_{\pi/2}^\pi\frac{\sin\alpha d\alpha}{\sqrt{\zeta-\sin\alpha}}\\&=-2\int_0^{\pi/2}\frac{\cos\alpha d\alpha}{\sqrt{1-\cos\alpha+\varepsilon}}\sim-2\int_0^\delta\frac{d\alpha}{\sqrt{\alpha^2/2+\varepsilon}}\\&\sim\sqrt{2}\log\varepsilon.\end{aligned}$$

□

Lemma 4.6. (i) *The curves $C_1(\alpha_0)$ do not intersect each other, and $\alpha_0'<\alpha_0$ implies that $C_1(\alpha_0')$ is above $C_1(\alpha_0)$.*

(ii) *When $\alpha_0\in(-\pi/2,0)$, $C_1(\alpha_0)$ crosses the line $\hat{x}+\hat{t}$ at least once.*

(iii) *The curves $C_1(\pm\pi/2)$ do not cross the line $\hat{x}+\hat{t}=0$.*

(iv) *The half line $\hat{x}+\hat{t}=0$, $\hat{t}>0$ is contained in the interior of the image of the mapping (4.9).*

Proof. (i) is a consequence of Lemma 4.3: the mapping $(\alpha_0,\zeta)\to\big(\hat{x}(\alpha_0,\zeta),\hat{t}(\alpha_0,\zeta)\big)$ is one-to-one and onto its image, so the curves $C_1(\alpha_0)$ do not intersect. Along $C_1(\alpha_0)$, $-\pi/2\le\alpha_0\le\pi/2$, $\hat{t}(\alpha_0,\zeta)$ has the limit $I(1)(\zeta+\sin\alpha_0)^{-3/2}$ as $\zeta\to1^+$. This limit function is a decreasing function of α_0, so the horizontal asymptote of $C_1(\alpha_0')$ is above the horizontal asymptote of $C_1(\alpha_0)$. Since these curves do not intersect, $C_1(\alpha_0')$ is above $C_1(\alpha_0)$.

(ii) When $\alpha_0\in(-\pi/2,0)$, $C_1(\alpha_0)$ starts at $(0,0)$ tangent to, and above, the line $\hat{x}+\hat{t}=0$, and $\hat{x}(\alpha_0,\zeta)+\hat{t}(\alpha_0,\zeta)\to-\infty$ as $\zeta\to1^+$, see Lemma 4.5. Consequently, $\hat{x}+\hat{t}$ must vanish along $C_1(\alpha_0)$, at least once.

(iii) We have

$$\hat{x}\left(\pm\frac{\pi}{2},\zeta\right)+\hat{t}\left(\pm\frac{\pi}{2},\zeta\right)=\frac{2}{(\zeta\pm 1)^{3/2}}\int_0^{\pi/2}\sin\alpha\big(h_\pm(\sin\alpha)-h_\pm(-\sin\alpha)\big)\mathrm{d}\alpha,$$

with

$$h_\pm(x)=\sqrt{x+\zeta}+\frac{\zeta\pm 1}{\sqrt{x+\zeta}},\qquad 0\le x\le 1.$$

Then

$$h'_\pm(x)=\frac{x\mp 1}{2(x+\zeta)^{3/2}},$$

so,

$$h'_+(x)\le 0,\qquad h'_-(x)\ge 0,$$

and therefore,

$$h_+(\sin\alpha)-h_+(-\sin\alpha)\le 0,$$

$$h_-(\sin\alpha)-h_-(-\sin\alpha)\ge 0.$$

Hence

$$\hat{x}\left(\frac{\pi}{2},\zeta\right)+\hat{t}\left(\frac{\pi}{2},\zeta\right)<0,$$

$$\hat{x}\left(-\frac{\pi}{2},\zeta\right)+\hat{t}\left(-\frac{\pi}{2},\zeta\right)>0,$$

which implies (iii).

(iv) follows from the fact that the curves $C_1\left(\pm\frac{\pi}{2}\right)$ bound the image of the mapping (4.9), and we have completed the proof of Lemma 4.6. □

Next we consider the periods. A geodesic, with $\zeta>1$, leaves $(0,y_0,0)$ at $\alpha=\alpha_0$, and at $\alpha=\alpha_0+2n\pi$, $n=1,2,\ldots$ returns to the $y=y_0$-plane at $\big(x(\alpha_0+2n\pi),y_0,t(\alpha_0+2n\pi)\big)$. Here

$$\frac{2x(\alpha_0+2n\pi)}{y_0}=\frac{nJ(\zeta)}{(\zeta+\sin\alpha_0)^{1/2}}=n\hat{x}(\alpha_0,\zeta),\tag{4.28}$$

$$\frac{2t(\alpha_0+2n\pi)}{y_0^3}=\frac{nI(\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}=n\hat{t}(\alpha_0,\zeta),\tag{4.29}$$

see (4.6), (4.7), so that this set of points is obtained as a dilation of the mapping (4.9) by a natural number $n = 1, 2, \ldots$. We have the following consequence:

Proposition 4.7. (i) *Every point* (x, y_0, t), $y_0 > 0$, *with* $(\hat{x}, \hat{t})$ *in the domain bounded by* $C_1(\pi/2)$ *and* $C_1(-\pi/2)$, *is connected to* $(0, y_0, 0)$ *by at least one geodesic of the form* (3.5), (3.6) *with* $\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta > 1$ *and* $\alpha_{\rm end} = \alpha_0 + 2n\pi$, $n = 1, 2, \ldots$.

(ii) *If a point* (x, y_0, t), $\hat{x} < 0$, $\hat{t} > 0$, $\hat{x}+\hat{t} \neq 0$, *is connected to* $(0, y_0, 0)$ *by a geodesic of the form discussed in (i), then the number of these connecting geodesics is at most finite.*

(iii) *Every point* (x, y_0, t), $\hat{x} < 0$, $\hat{t} > 0$, $\hat{x} + \hat{t} = 0$, *can be joined to* $(0, y_0, 0)$ *by a discrete infinity of geodesics of the form* (3.5), (3.6) *with* $\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta > 1$, $\alpha_{\rm end} = \alpha_0 + 2n\pi$; *more precisely, for each* $n = 1, 2, \ldots$ *there is* $(\alpha_{0,n}, \zeta_n) \in [-\pi/2, \pi/2] \times (1, \infty)$, *such that*

$$\hat{x} = n\hat{x}(\alpha_{0,n}, \zeta_n), \qquad \hat{t} = n\hat{t}(\alpha_{0,n}, \zeta_n). \tag{4.30}$$

Proof. (i) holds with $n = 0$, see Lemma 4.3. As for (ii), given $(\hat{x}, \hat{t})$, (4.30) has a solution if and only if $(\hat{x}/n, \hat{t}/n)$ is in the domain of the $(\hat{x}, \hat{t})$ plane which is bounded by the curves $C_1(\pi/2)$ and $C_1(-\pi/2)$. As both these curves start tangent to the half line $\hat{x} + \hat{t} = 0$, $\hat{t} > 0$, at $(0, 0)$, the points $(\hat{x}/n, \hat{t}/n)$ leave this region for n sufficiently large, which proves (ii). We note that this half line is the only ray from the origin entirely contained in the region bounded by $C_1(\pi/2)$ and $C_1(-\pi/2)$.

On the other hand, (4.30) has a solution $(\alpha_{0,n}, \zeta_n)$ for all $n = 1, 2, \ldots$ if $\hat{x}/n + \hat{t}/n = 0$, $\hat{t} > 0$. This proves (iii). □

Lemma 4.8. *Let* $(\hat{x}, -\hat{x})$ *denote a point on the line* $\hat{x} + \hat{t} = 0$ *with* $\hat{x} < 0$. *The length of the geodesic joining* $(0, y_0, 0)$ *to* $(x, y_0, -x)$ *after* n *periods, with* $\alpha_{\rm end} = \alpha_0 + 2n\pi$, *is*

$$\ell_n = O(\sqrt{n}), \tag{4.31}$$

as $n \to \infty$.

Proof. From (3.7),

$$\ell_n = \frac{n}{2\tau_n^{1/2}} \int_0^{2\pi} \frac{\mathrm{d}\alpha}{(\zeta_n + \sin\alpha)^{1/2}}, \tag{4.32}$$

where ζ_n and $\tau_n = (\zeta_n + \sin\alpha_{0,n})y_0^{-2}$ are the parameters of the geodesic which, after n periods, joins $(0, y_0, 0)$ and $(x, y_0, -x)$. When n is large ζ_n may be estimated using (4.23); indeed,

$$\frac{\hat{x}}{n} = \hat{x}(\alpha_{0,n}, \zeta_n) \cong -\frac{\pi}{2\zeta_n^2},$$

so that $\zeta_n = O(\sqrt{n})$, $\tau_n = O(\sqrt{n})$, and therefore (4.32) yields (4.31), and Lemma 4.8. □

We shall use R_1 to denote the image of the mapping (4.9):

$$(\alpha_0, \zeta) \longrightarrow \big(\hat{x}(\alpha_0, \zeta), \hat{t}(\alpha_0, \zeta)\big),$$

where

$$\hat{x}(\alpha_0, \zeta) = \frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}}, \qquad \hat{t}(\alpha_0, \zeta) = \frac{I(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}}$$

represent the curves $C_1(\alpha_0)$ as $\zeta \in (1, \infty)$ and $\alpha_0 \in [-\pi/2, \pi/2]$. We summarize the results about R_1 in

Proposition 4.9. (i) *The mapping* (4.9) *is one-to-one onto the region* R_1 *which is the union of the curves* $C_1(\alpha_0)$, $\alpha_0 \in [-\pi/2, \pi/2]$. R_1 *is bounded by* $C_1(\pm\pi/2)$.

(ii) $\zeta \sim \infty$. $C_1(\alpha_0)$ *starts at* $(0, 0)$ *tangent to the line* $\hat{x} + \hat{t} = 0$; $\alpha_0 \in [-\pi/2, 0) \Rightarrow C_1(\alpha_0)$ *starts above* $\hat{x} + \hat{t} = 0$, $\alpha_0 \in [0, \pi/2] \Rightarrow C_1(\alpha_0)$ *starts below* $\hat{x} + \hat{t} = 0$.

(iii) $\zeta \in (1, \infty)$. $\alpha_0 \in [0, \pi/2] \Rightarrow C_1(\alpha_0)$ *is below the line* $\hat{x} + \hat{t} = 0$.

$\alpha_0 \in (-\pi/2, 0) \Rightarrow C_1(\alpha_0)$ *crosses the line* $\hat{x} + \hat{t} = 0$; $C_1(-\pi/2)$ *does not cross* $\hat{x} + \hat{t} = 0$.

All curves $C_1(\alpha_0)$, $\alpha_0 \in (-\pi/2, \pi/2]$ *have a horizontal asymptote at*

$\hat{x} \sim -\infty$. It is

$$\hat{t}(\alpha_0, 1) = \frac{I(1)}{(1+\sin\alpha_0)^{3/2}}.$$

(iv) *R_1 contains the entire half line $\hat{x}+\hat{t}=0$, $\hat{t}>0$.*

(v) *Points (x, y_0, t) of the $y = y_0$-plane with $(\hat{x}, \hat{t}) \in \bigcup_{n=1}^{\infty} nR_1$ can be joined to $(0, y_0, 0)$ by at least one geodesic of the form* (3.5), (3.6) *with $\alpha_0 \in [-\pi/2, \pi/2]$. The number of such geodesics connecting such a point (x, y_0, t) to $(0, y_0, 0)$ is finite if $\hat{x}+\hat{t} \neq 0$, $\hat{t}>0$. When $\hat{x}+\hat{t}=0$, $\hat{t}>0$, (x, y_0, t) is joined to $(0, y_0, 0)$ by a discrete infinity of geodesics with parameters $(\alpha_{0,n}, \zeta_n)$, $n = 1, 2, \ldots$.*

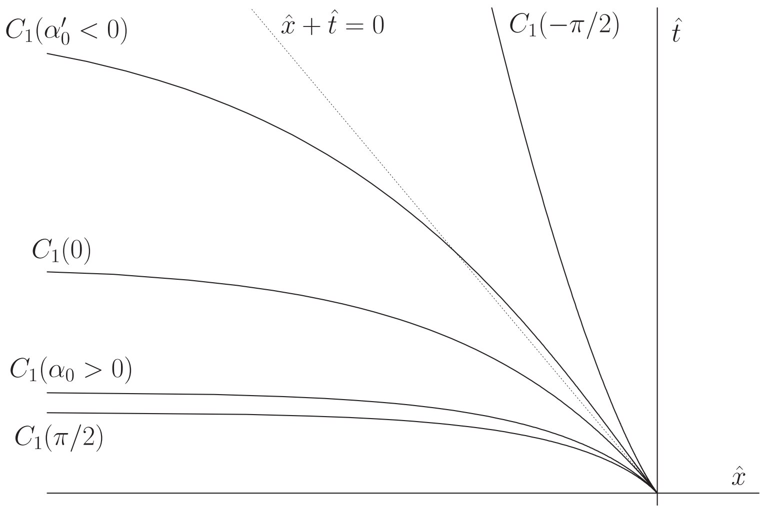

Figure 1. R_1.

Proof. We only need to prove the first statement of (iii). A slight refinement of the proof of Lemma 4.6(iii) will do the trick. In particular,

$$\hat{x}(\alpha_0, \zeta) + \hat{t}(\alpha_0, \zeta) = \frac{2}{(\zeta+\sin\alpha_0)^{3/2}} \int_0^{\pi/2} \sin\alpha \big[h(\sin\alpha) - h(-\sin\alpha)\big]\, \mathrm{d}\alpha,$$

where we set

$$h(x) = \sqrt{\zeta+x} + \frac{\zeta+\sin\alpha_0}{\sqrt{\zeta+x}}, \qquad x \in [0,1].$$

Then

$$h'(x) = \frac{x - \sin\alpha_0}{2(\zeta + x)^{3/2}},$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\big[h(x) - h(-x)\big] &= h'(x) - h'(-x) \\ &= \frac{x - \sin\alpha_0}{2(\zeta + x)^{3/2}} - \frac{x + \sin\alpha_0}{2(\zeta - x)^{3/2}} \\ &< 0, \qquad x \in (0,1),\ \alpha_0 \in [0, \pi/2]. \end{aligned} \tag{4.33}$$

Since $h(0) - h(-0) = 0$, one has $h(\sin\alpha) - h(-\sin\alpha) < 0$ when $\alpha \in (0, \pi]$, therefore

$$\hat{x}(\alpha_0, \zeta) + \hat{t}(\alpha_0, \zeta) < 0, \qquad \alpha_0 \in [0, \pi/2].$$

This proves the first statement of (iii), and we have established Proposition 4.9. □

As for

$$\alpha_0 \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \tag{4.34}$$

we note that

$$C_1(\alpha_0) = C_1(\pi - \alpha_0), \tag{4.35}$$

so the mapping (4.9) with $\alpha_0 \in [\pi/2, 3\pi/2]$ also covers R_1 is a 1-1 manner, and we have

Corollary 4.10. *The results of Proposition* 4.9*, derived when* $\alpha_0 \in [-\pi/2, \pi/2]$*, also hold for* $\alpha_0 \in [\pi/2, 3\pi/2]$*, mutatis mutandis. In particular, the statements made for* $\alpha_0 \in [0, \pi/2]$ *are true when* $\alpha_0 \in [\pi/2, \pi]$*, and the results stated for* $\alpha_0 \in [-\pi/2, 0)$ *hold when* $\alpha_0 \in (\pi, 3\pi/2]$.

5. $\alpha_{\text{end}} = \pi - \alpha_0 + 2n\pi, n = 0, 1, 2, \ldots, \dot{y}(0) > 0$

The geodesics of this chapter are local, start at $(0, y_0, 0)$ with parameters $\alpha_0 \in [-\pi/2, \pi/2]$ and $\zeta > 1$, and return to the $y = y_0$-plane at $\alpha_{\text{end}} = \pi - \alpha_0 + 2n\pi$. The x and t components of the return point are given by formulas (3.8) and (3.9), and may be put in the following form:

$$x(\pi-\alpha_0+2n\pi)=\frac{1}{2\tau^{1/2}}\left(2\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}+nJ(\zeta)\right),$$

$$t(\pi-\alpha_0+2n\pi)=\frac{1}{2\tau^{3/2}}\left(2\int_{\alpha_0}^{\pi/2}\sin\alpha(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha+nI(\zeta)\right),$$

and, eliminating τ, $\tau=y_0^{-2}(\zeta+\sin\alpha_0)$, we have

$$\hat{x}(\pi-\alpha_0+2n\pi)=\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\left(2\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}+nJ(\zeta)\right), \quad (5.1)$$

$$\hat{t}(\pi-\alpha_0+2n\pi) =\frac{1}{(\zeta+\sin\alpha_0)^{3/2}}\left(2\int_{\alpha_0}^{\pi/2}\sin\alpha(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha+nI(\zeta)\right), \quad (5.2)$$

with $n=0,1,2,\ldots$. In the rest of chapter 5 we shall assume that $n=0$.

The mapping $(\alpha_0,\zeta)\to(\hat{x},\hat{t})$ when $n=0$.

$(\alpha_0,\zeta)\in[-\pi/2,\pi/2]\times(1,\infty)$, and we set

$$(\hat{x},\hat{t})=\big(\Phi(\alpha_0,\zeta),\Psi(\alpha_0,\zeta)\big), \quad (5.3)$$

$$\Phi(\alpha_0,\zeta)=\frac{J(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}}, \quad (5.4)$$

$$\Psi(\alpha_0,\zeta)=\frac{I(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}, \quad (5.5)$$

$$J(\alpha_0,\zeta)=2\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}, \quad (5.6)$$

$$I(\alpha_0,\zeta)=2\int_{\alpha_0}^{\pi/2}\sin\alpha(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha. \quad (5.7)$$

We note that

$$J(\zeta)=J\Big(-\frac{\pi}{2},\zeta\Big),\qquad I(\zeta)=I\Big(-\frac{\pi}{2},\zeta\Big). \quad (5.8)$$

Lemma 5.1. *Fix $\zeta > 1$. Then*

$$J(\alpha_0, \zeta) > 0, \qquad \alpha_0 \in [0, \pi/2),$$

$$J\Big(\frac{\pi}{2}, \zeta\Big) = 0, \qquad J\Big(-\frac{\pi}{2}, \zeta\Big) < 0, \qquad J\Big(-\frac{\pi}{2}, 1\Big) = -\infty.$$

$J(\alpha_0, \zeta)$ is an increasing function of α_0 when $\alpha_0 < 0$, and a decreasing function of α_0 when $\alpha_0 > 0$. It has a unique zero $\tilde{\alpha}_0(\zeta)$. $\tilde{\alpha}_0(\zeta) \in (-\pi/2, 0)$, and

$$\tilde{\alpha}_0(\zeta) \sim -\frac{\pi}{2} + \frac{\pi}{4\zeta} + o\Big(\frac{1}{\zeta}\Big), \qquad \zeta \to \infty. \tag{5.9}$$

Proof. Clearly, $J(\alpha_0, \zeta) > 0$ if $\alpha_0 \in [0, \pi/2)$ and $J(\pi/2, \zeta) = 0$. Also,

$$J(-\pi/2, \zeta) = 2\int_0^{\pi/2} \sin\alpha \left(\frac{1}{(\zeta + \sin\alpha)^{1/2}} - \frac{1}{(\zeta - \sin\alpha)^{1/2}}\right) \mathrm{d}\alpha < 0.$$

Since

$$\frac{\partial J(\alpha_0, \zeta)}{\partial \alpha_0} = -\frac{2\sin\alpha_0}{(\zeta + \sin\alpha_0)^{1/2}},$$

$J(\alpha_0, \zeta)$ is increasing if $\alpha_0 < 0$, and decreasing when $\alpha_0 > 0$, so it has a unique root $\tilde{\alpha}_0$ in $(-\pi/2, \pi/2)$ which is in $(-\pi/2, 0)$. If $\alpha = -\pi/2 + \varepsilon$, then $1 + \sin\alpha \sim \varepsilon^2/2$, $\varepsilon \sim 0$, and $J(-\pi/2, 1) = -\infty$. Finally, when ζ is large,

$$\int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}} = \frac{1}{\sqrt{\zeta}}\left(\cos\alpha_0 - \frac{1}{4\zeta}\Big(\frac{\pi}{2} - \alpha_0\Big) - \frac{1}{8\zeta}\sin 2\alpha_0 + O\Big(\frac{1}{\zeta^2}\Big)\right),$$

so when $\alpha_0 = \tilde{\alpha}_0(\zeta)$,

$$0 = \cos\tilde{\alpha}_0(\zeta) - \frac{1}{4\zeta}\left(\frac{\pi}{2} - \tilde{\alpha}_0(\zeta) + \frac{1}{2}\sin 2\tilde{\alpha}_0(\zeta)\right) + O\Big(\frac{1}{\zeta^2}\Big).$$

Therefore

$$\tilde{\alpha}_0(\zeta) \longrightarrow -\frac{\pi}{2} + \varepsilon(\zeta), \qquad \lim_{\zeta\to\infty} \varepsilon(\zeta) = 0,$$

$$0 = \sin\varepsilon(\zeta) - \frac{1}{4\zeta}\big(\pi + o(1)\big) + O\Big(\frac{1}{\zeta^2}\Big)$$

which yields (5.9), and we have derived Lemma 5.1. $\square$

Lemma 5.2. *One has*

(i) $\Phi\Big(\frac{\pi}{2},\zeta\Big)=0$, $\Phi\Big(-\frac{\pi}{2},\zeta\Big)<0$, $\zeta>1$ and $\Phi\Big(-\frac{\pi}{2},1\Big)=-\infty$,
(ii) $\partial\Phi/\partial\alpha_0$ *has a unique zero at* $\alpha_0^*(\zeta)$,

$$\tilde{\alpha}_0(\zeta)<\alpha_0^*(\zeta)<0, \tag{5.10}$$

where Φ *takes its maximum value* $\Phi\big(\alpha_0^*(\zeta),\zeta\big)>0$. Φ *is an increasing function of* $\alpha_0\in(-\pi/2,\alpha_0^*)$, *and a decreasing function of* $\alpha_0\in(\alpha_0^*,\pi/2]$.

Proof. (i) is self-evident. As for (ii) we start with

$$\begin{aligned}\frac{\partial\Phi}{\partial\alpha_0}&=-\frac{1}{2}\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}(\Phi+4\tan\alpha_0)\\&=-\frac{\cos\alpha_0}{2(\zeta+\sin\alpha_0)^{3/2}}\big(J(\alpha_0,\zeta)+4\tan\alpha_0(\zeta+\sin\alpha_0)^{1/2}\big).\end{aligned} \tag{5.11}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\big(\tan\alpha_0(\zeta+\sin\alpha_0)^{1/2}\big)=\frac{\zeta+\sin\alpha_0\Big(1+\frac{1}{2}\cos^2\alpha_0\Big)}{\cos^2\alpha_0(\zeta+\sin\alpha_0)^{1/2}},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\left(\sin\alpha_0\Big(1+\frac{1}{2}\cos^2\alpha_0\Big)\right)=\frac{3}{2}\cos^3\alpha_0>0$$

on $(-\pi/2,\pi/2)$. Therefore

$$\sin\alpha_0\Big(1+\frac{1}{2}\cos^2\alpha_0\Big)>-1,$$

and $\tan\alpha_0(\zeta+\sin\alpha_0)^{1/2}$ is an increasing function of α_0 on $[-\pi/2,\pi/2]$ with a zero at $\alpha_0=0$. $J(\alpha_0,\zeta)\geq 0$ on $[0,\pi/2]$, and <0 on $\big(-\pi/2,\tilde{\alpha}_0(\zeta)\big)$, it increases in $\big(\tilde{\alpha}_0(\zeta),0\big)$ from $J\big(\tilde{\alpha}_0(\zeta),\zeta\big)=0$. Therefore $J(\alpha_0,\zeta)+4\tan\alpha_0(\zeta+\sin\alpha_0)^{1/2}$ increases in the interval $\big(\tilde{\alpha}_0(\zeta),0\big)$ from negative to positive values with a unique zero $\alpha_0^*(\zeta)\in\big(\tilde{\alpha}_0(\zeta),0\big)$. This proves (ii) and Lemma 5.2. □

In particular,

$$\Phi\big(\alpha_0^*(\zeta),\zeta\big)+4\tan\alpha_0^*(\zeta)=0. \tag{5.12}$$

Lemma 5.3. *One has*

$$\frac{\mathrm{d}\alpha_0^*(\zeta)}{\mathrm{d}\zeta} > 0. \tag{5.13}$$

Proof. Differentiating (5.12) with respect to ζ we obtain

$$\frac{\mathrm{d}\alpha_0^*}{\mathrm{d}\zeta} = -\frac{1}{4}\cos^2\alpha_0^*(\zeta)\frac{\partial\Phi}{\partial\zeta}\big(\alpha_0^*(\zeta),\zeta\big). \tag{5.14}$$

From (5.4),

$$\frac{\partial\Phi}{\partial\zeta} = -\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\left[\frac{\Phi}{2(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right], \tag{5.15}$$

so (5.12) yields

$$\begin{aligned}&\frac{\partial\Phi}{\partial\zeta}\big(\alpha_0^*(\zeta),\zeta\big)\\ &= \frac{1}{(\zeta+\sin\alpha_0^*(\zeta))^{1/2}}\left[\frac{2\tan\alpha_0^*(\zeta)}{(\zeta+\sin\alpha_0^*(\zeta))^{1/2}} - \int_{\alpha_0^*(\zeta)}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right].\end{aligned} \tag{5.16}$$

Integrating by parts we have

$$\int_{\alpha_0}^{0}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} = \frac{2\tan\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}} + 2\int_{\alpha_0}^{0}\frac{1}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}. \tag{5.17}$$

Consequently,

$$\begin{aligned}&\frac{\partial\Phi}{\partial\zeta}\big(\alpha_0^*(\zeta),\zeta\big)\\ &= -\frac{1}{(\zeta+\sin\alpha_0^*(\zeta))^{1/2}}\left(\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} + \int_{\alpha_0^*(\zeta)}^{0}\frac{2}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}\right)\\ &< 0,\end{aligned}$$

and therefore (5.14) implies (5.13). □

Lemma 5.4. *When* $\alpha_0 \geq \alpha_0^*(\zeta)$*, one has*

(i) $\dfrac{\partial\Phi}{\partial\zeta} < 0,$ (ii) $\dfrac{\partial^2\Phi}{\partial\alpha_0\partial\zeta} > 0.$

Proof. When $\alpha_0 \geq 0$, (5.15) implies (i). When $\alpha_0 < 0$, we may use

(5.17) to rewrite (5.15):

$$\begin{aligned}\frac{\partial\Phi}{\partial\zeta} =& -\frac{1}{2}\frac{\Phi+4\tan\alpha_0}{\zeta+\sin\alpha_0}\\ &-\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\left[\int_{\alpha_0}^{0}\frac{2}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}+\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right],\end{aligned}$$

and then (5.11) yields

$$\begin{aligned}\frac{\partial\Phi}{\partial\zeta} =& \frac{1}{\cos\alpha_0}\frac{\partial\Phi}{\partial\alpha_0}\\ &-\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\left[\int_{\alpha_0}^{0}\frac{2}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}+\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right].\end{aligned}$$

The square bracket is always positive and $\partial\Phi/\partial\alpha_0 < 0$ when $\alpha_0 > \alpha_0^*$, consequently $\partial\Phi/\partial\zeta < 0$ if $\alpha_0 > \alpha_0^*$ and we have derived (i). As for (ii), we differentiate the square bracket in (5.15):

$$\begin{aligned}&\frac{\partial}{\partial\alpha_0}\left[\frac{1}{2}\frac{\Phi}{(\zeta+\sin\alpha_0)^{1/2}}+\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right]\\ &=\frac{1}{(\zeta+\sin\alpha_0)^{3/2}}\left[\frac{1}{2}(\zeta+\sin\alpha_0)\frac{\partial\Phi}{\partial\alpha_0}-\frac{1}{4}\cos\alpha_0(\Phi+4\tan\alpha_0)\right].\end{aligned}$$

Then (5.11) yields

$$\frac{\partial^2\Phi}{\partial\alpha_0\partial\zeta} = -\frac{1}{2}\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}\left(\frac{\partial\Phi}{\partial\zeta}+\frac{2}{\cos\alpha_0}\frac{\partial\Phi}{\partial\alpha_0}\right). \tag{5.18}$$

When $\alpha_0 > \alpha_0^*$ both Φ_ζ and Φ_{α_0} are negative and therefore $\Phi_{\alpha_0\zeta} > 0$ which proves (ii) and Lemma 5.4. □

Lemma 5.5. Φ *has the following convergent power series expansion in* ζ^{-1} *when* ζ *is large:*

$$\begin{aligned}\Phi(\alpha_0,\zeta) =& \frac{2\cos\alpha_0}{\zeta}-\frac{1}{2\zeta^2}\left(\frac{\pi}{2}-\alpha_0+\frac{3}{2}\sin 2\alpha_0\right)\\ &+\frac{1}{4\zeta^3}\left(\left(\frac{\pi}{2}-\alpha_0\right)\sin\alpha_0+\frac{1}{8}\cos\alpha_0(2+5\sin^2\alpha_0)\right)\\ &+O\Big(\frac{1}{\zeta^4}\Big),\qquad \alpha_0\neq\frac{\pi}{2}.\end{aligned} \tag{5.19}$$

Remark. In view of (5.11), (5.15) and (5.18), $\partial\Phi/\partial\zeta > 0$ and is a decreasing function of α_0 in a neighbourhood of $\alpha_0 = -\pi/2$.

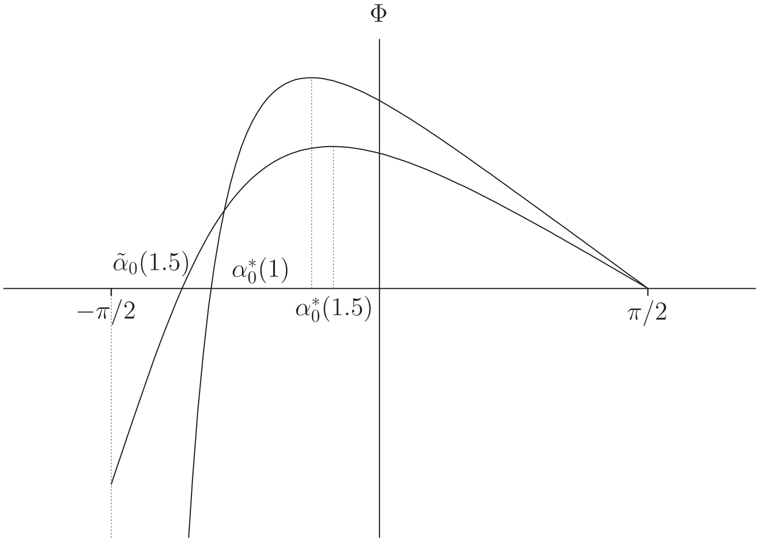

Figure 2.

Next we consider $I(\alpha_0, \zeta)$ and $\Psi(\alpha_0, \zeta)$.

Lemma 5.6. *$I(\alpha_0, \zeta)$ is an increasing function of α_0 in $[-\pi/2, 0]$, reaches a maximum at $\alpha_0 = 0$, then decreases in $(0, \pi/2]$, $I(-\pi/2, \zeta) > 0$, $I(\pi/2, \zeta) = 0$. In particular, $I(\alpha_0, \zeta) > 0$ when $\alpha_0 \in [-\pi/2, \pi/2]$.*

Proof. (5.7) yields

$$\frac{\partial I}{\partial \alpha_0} = -2 \sin \alpha_0 (\zeta + \sin \alpha_0)^{1/2}.$$

Also,

$$I(-\pi/2, \zeta) = 2 \int_0^{\pi/2} \sin \alpha \big[(\zeta + \sin \alpha)^{1/2} - (\zeta - \sin \alpha)^{1/2}\big] \mathrm{d}\alpha > 0,$$

$I(\pi/2, \zeta) = 0$, hence $I(\alpha_0, \zeta) > 0$ when $\alpha_0 \in [-\pi/2, \pi/2)$, and we have derived Lemma 5.6. □

The bracket above (5.18) is positive when $\alpha_0 \in \big(-\pi/2, \alpha_0^*(\zeta)\big)$, so the bracket in (5.15) is an increasing function of $\alpha_0 \in \big(-\pi/2, \alpha_0^*(\zeta)\big)$ and may have at most one zero there. Thus $\partial\Phi/\partial\zeta$ has at most one zero in $\big(-\pi/2, \alpha_0^*(\zeta)\big)$.

A simple calculation yields

$$\frac{\partial \Psi}{\partial \alpha_0} = -\frac{1}{2}\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}(3\Psi + 4\tan\alpha_0). \tag{5.20}$$

Lemma 5.7. *$\partial\Psi/\partial\alpha_0$ has a unique zero $\alpha_0^+(\zeta) < 0$, $\zeta > 1$. Also*

(i) $\Psi\big(\alpha_0^+(\zeta),\zeta\big) = -\frac{4}{3}\tan\alpha_0^+(\zeta)$,

(ii) $\frac{\partial\Psi}{\partial\alpha_0} > 0$, $\alpha_0 \in \Big(-\frac{\pi}{2}, \alpha_0^+(\zeta)\Big)$,
$\frac{\partial\Psi}{\partial\alpha_0} < 0$, $\alpha_0 \in \Big(\alpha_0^+(\zeta), \frac{\pi}{2}\Big)$,

(iii) $\Psi(\alpha_0,\zeta) > 0$, $\alpha_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big)$,

(iv) $\Psi(\alpha_0, 1)$ *decreases from* $\Psi\Big(-\frac{\pi}{2}, 1\Big) = \infty$ *to* $\Psi\Big(\frac{\pi}{2}, 1\Big) = 0$.

Proof. We rewrite (5.5),

$$\Psi(\alpha_0,\zeta) = \frac{2}{(\zeta+\sin\alpha_0)^{3/2}}\left[\int_0^{\pi/2}\sin\alpha(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha + \frac{2}{3}\int_{\alpha_0}^{0}\tan\alpha\,\mathrm{d}(\zeta+\sin\alpha)^{3/2}\right].$$

Integrating by parts one obtains

$$\frac{3}{2}\Psi(\alpha_0,\zeta) + 2\tan\alpha_0 = \frac{1}{(\zeta+\sin\alpha_0)^{3/2}}\left[3\int_0^{\pi/2}\sin\alpha(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha - 2\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{3/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha}\right]. \tag{5.21}$$

We note that

$$\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{3/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha} = \begin{cases}\infty, & \alpha_0 = -\frac{\pi}{2},\\ 0, & \alpha_0 = 0,\\ -\infty, & \alpha_0 = \frac{\pi}{2},\end{cases}$$

and is a decreasing function of $\alpha_0 \in [-\pi/2, \pi/2]$. Consequently the square bracket in (5.21) is an increasing function of $\alpha_0 \in [-\pi/2, \pi/2)$. In particular, when $\alpha_0 \in [-\pi/2, 0]$, it increases from $-\infty$ to

$$3\int_0^{\pi/2} \sin\alpha(\zeta + \sin\alpha)^{1/2} \mathrm{d}\alpha > 0,$$

and therefore vanishes at a unique point $\alpha_0^+ \in (-\pi/2, 0)$. Therefore (5.20) and (5.21) imply (i) and (ii). Furthermore

$$\Psi\Big(-\frac{\pi}{2}, \zeta\Big) = \frac{1}{(\zeta-1)^{3/2}} I\Big(-\frac{\pi}{2}, \zeta\Big) > 0, \tag{5.22}$$

$$\Psi\Big(\frac{\pi}{2}, \zeta\Big) = 0, \tag{5.23}$$

so (ii) implies (iii), and (iv) is obvious from the definition of Ψ. □

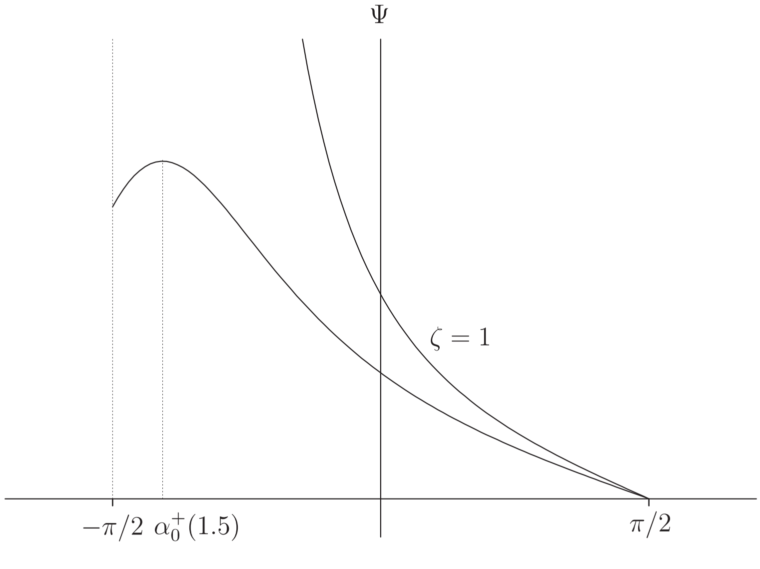

Figure 3.

Lemma 5.8. *We have*

$$\Phi(\alpha_0, \zeta) \le \Psi(\alpha_0, \zeta), \qquad \alpha_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big], \qquad \zeta \ge 1. \tag{5.24}$$

Proof. When $\alpha_0 \geq 0$,

$$\Phi - \Psi = \frac{2}{(\zeta+\sin\alpha_0)^{3/2}} \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha}{(\zeta+\sin\alpha)^{1/2}} (\sin\alpha_0 - \sin\alpha) \mathrm{d}\alpha < 0. \quad (5.25)$$

Also, Lemma 5.2(i) and (5.22), (5.23) give

$$\Phi\Big(-\frac{\pi}{2},\zeta\Big) - \Psi\Big(-\frac{\pi}{2},\zeta\Big) < 0, \quad (5.26)$$

$$\Phi\Big(\frac{\pi}{2},\zeta\Big) - \Psi\Big(\frac{\pi}{2},\zeta\Big) = 0. \quad (5.27)$$

Next,

$$\begin{aligned}\frac{\partial}{\partial\alpha_0} \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha}{(\zeta+\sin\alpha)^{1/2}} (\sin\alpha_0 - \sin\alpha)\mathrm{d}\alpha &= \cos\alpha_0 \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} \\ &= \frac{1}{2}\cos\alpha_0 J(\alpha_0),\end{aligned}$$

so the integral in (5.25) decreases when $\alpha_0 < \tilde{\alpha}_0(\zeta)$, and increases when $\alpha_0 > \tilde{\alpha}_0(\zeta)$. This proves (5.24) in view of (5.26), (5.27). □

Lemma 5.9. *We have*

$$\frac{\partial\Psi}{\partial\zeta} = \frac{\Phi - 3\Psi}{2(\zeta+\sin\alpha_0)} < 0, \quad (5.28)$$

$$\alpha_0^+(\zeta) < \alpha_0^*(\zeta). \quad (5.29)$$

Proof. (5.28) follows from Lemma 5.7(iii) and (5.24). Φ has its maximum at α_0^* and Ψ has its maximum at α_0^+, so

$$\Phi\big(\alpha_0^+(\zeta),\zeta\big) \leq \Phi\big(\alpha_0^*(\zeta),\zeta\big) < \Psi\big(\alpha_0^*(\zeta),\zeta\big) \leq \Psi\big(\alpha_0^+(\zeta),\zeta\big),$$

and therefore

$$(\Psi-\Phi)\big(\alpha_0^+(\zeta),\zeta\big) \geq (\Psi-\Phi)\big(\alpha_0^*(\zeta),\zeta\big). \quad (5.30)$$

But $\Psi - \Phi$ is a decreasing function of α_0; indeed, (5.20) and (5.11) yield

$$\frac{\partial}{\partial\alpha_0}(\Psi-\Phi) = -\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}\Big(\frac{3}{2}\Psi - \frac{1}{2}\Phi\Big) = \cos\alpha_0 \frac{\partial\Psi}{\partial\zeta} < 0, \quad (5.31)$$

and then (5.30) implies (5.29); indeed, since $\frac{\partial}{\partial\alpha_0}(\Psi - \Phi) < 0$, one has $\alpha_0^* \neq \alpha_0^+$. □

Lemma 5.10. $\Psi(\alpha_0, \zeta)$ *has the following convergent power series expansion in* ζ^{-1} *for large* ζ:

$$\begin{aligned}\Psi(\alpha_0,\zeta) =& \frac{2\cos\alpha_0}{\zeta} + \frac{1}{2\zeta^2}\left(\frac{\pi}{2} - \alpha_0 - \frac{5}{2}\sin 2\alpha_0\right) \\ &+ \frac{1}{4\zeta^3}\left(3\Big(\frac{\pi}{2} - \alpha_0\Big)\sin\alpha_0 - \frac{2}{3}\cos\alpha_0(2 - 5\sin^2\alpha_0)\right) \\ &+ O\Big(\frac{1}{\zeta^4}\Big).\end{aligned} \tag{5.32}$$

On the Jacobian $\boldsymbol{\Delta_2}$.

We set

$$\Delta_2 = \frac{\partial(\Psi,\Phi)}{\partial(\alpha_0,\zeta)}, \tag{5.33}$$

and collect the formulas for its evaluation:

$$\frac{\partial\Phi}{\partial\alpha_0} = -\frac{1}{2(\zeta+\sin\alpha_0)}\big((\cos\alpha_0)\Phi + 4\sin\alpha_0\big), \tag{5.34}$$

$$\frac{\partial\Psi}{\partial\alpha_0} = -\frac{1}{2(\zeta+\sin\alpha_0)}\big(3(\cos\alpha_0)\Psi + 4\sin\alpha_0\big), \tag{5.35}$$

$$\frac{\partial\Phi}{\partial\zeta} = -\frac{1}{2(\zeta+\sin\alpha_0)}\left(\Phi + 2(\zeta+\sin\alpha_0)^{1/2}\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}\right), \tag{5.36}$$

$$\frac{\partial\Psi}{\partial\zeta} = \frac{1}{2(\zeta+\sin\alpha_0)}(-3\Psi+\Phi). \tag{5.37}$$

We shall be using

$$K(\alpha_0,\zeta) = 2\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}, \tag{5.38}$$

$$P(Z,\alpha_0,\zeta) = -\frac{1}{2}KZ^2 - JZ + \frac{3}{2}I, \tag{5.39}$$

and the discriminant of the polynomial P,

$$4D(\alpha_0,\zeta) = J^2 + 3IK. \tag{5.40}$$

Lemma 5.11. *We have*

$$(\zeta+\sin\alpha_0)^3\Delta_2 = (\cos\alpha_0)D - \frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}P(\zeta+\sin\alpha_0,\alpha_0,\zeta), \quad (5.41)$$

$$\begin{aligned} &P(\zeta+\sin\alpha_0,\alpha_0,\zeta)\\ &= \int_{\alpha_0}^{\pi/2}(\sin\alpha-\sin\alpha_0)(\sin\alpha_0+3\sin\alpha+4\zeta)\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}, \quad (5.42)\\ &\frac{\partial}{\partial\alpha_0}\big((\zeta+\sin\alpha_0)^3\Delta_2\big)\\ &= -(\sin\alpha_0)D - \frac{2\cos\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}P(\zeta+\sin\alpha_0,\alpha_0,\zeta). \quad (5.43)\end{aligned}$$

Proof. In

$$\begin{aligned}(\zeta+\sin\alpha_0)^3\Delta_2 &= (\zeta+\sin\alpha_0)^3\left(\frac{\partial\Psi}{\partial\alpha_0}\frac{\partial\Phi}{\partial\zeta}-\frac{\partial\Psi}{\partial\zeta}\frac{\partial\Phi}{\partial\alpha_0}\right)\\ &= \Big\{\frac{1}{4}\big(3(\cos\alpha_0)\Psi+4\sin\alpha_0\big)\big(\Phi+(\zeta+\sin\alpha_0)^{1/2}K\big)\\ &\quad+\frac{1}{4}\big((\cos\alpha_0)\Phi+4\sin\alpha_0\big)(\Phi-3\Psi)\Big\}(\zeta+\sin\alpha_0),\end{aligned}$$

we replace Φ and Ψ by their definition in terms of J and I and obtain (5.41). As for (5.42),

$$\begin{aligned}&-P(\zeta+\sin\alpha_0,\alpha_0,\zeta)\\ &= \frac{1}{2}(\zeta+\sin\alpha_0)^2K+(\zeta+\sin\alpha_0)J-\frac{3}{2}I\\ &= \int_{\alpha_0}^{\pi/2}\frac{\big[2(\zeta+\sin\alpha_0)(\zeta+\sin\alpha)-3(\zeta+\sin\alpha)^2+(\zeta+\sin\alpha_0)^2\big]\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}\end{aligned}$$

which yields (5.42). Next we differentiate (5.41) and (5.42):

$$\begin{aligned}&\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_2 = -(\sin\alpha_0)D+\frac{1}{2}\cos\alpha_0\sin\alpha_0\\ &\cdot\left[-3(\zeta+\sin\alpha_0)^{1/2}K-3I\frac{1}{(\zeta+\sin\alpha_0)^{3/2}}-2J\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\right]\\ &-\left(\frac{2\cos\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}-\frac{\sin\alpha_0\cos\alpha_0}{(\zeta+\sin\alpha_0)^{3/2}}\right)P-\frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}\frac{\partial P}{\partial\alpha_0}, \quad (5.44)\end{aligned}$$

$$\frac{\partial}{\partial\alpha_0}P(\zeta+\sin\alpha_0,\alpha_0,\zeta) = -\cos\alpha_0\big[(\zeta+\sin\alpha_0)K+J\big], \quad (5.45)$$

and replacing $\partial P/\partial\alpha_0$ by (5.45) and P by $P(\zeta+\sin\alpha_0,\alpha_0,\zeta)$ in (5.44), the terms with $\cos\alpha_0\sin\alpha_0$ cancel and we obtain (5.43). □

Corollary 5.12. *One has*

$$(\zeta+\sin\alpha_0)^3\Delta_2 \begin{cases} >0, & \alpha_0\in\left[0,\frac{\pi}{2}\right), \\ =0, & \alpha_0=\frac{\pi}{2}, \end{cases} \tag{5.46}$$

$$\frac{\partial}{\partial\alpha_0}\big((\zeta+\sin\alpha_0)^3\Delta_2\big) \begin{cases} <0, & \alpha_0\in\left[0,\frac{\pi}{2}\right), \\ =0, & \alpha_0=\frac{\pi}{2}. \end{cases} \tag{5.47}$$

Proof. When $\alpha_0\in[0,\pi/2)$, $P(\zeta+\sin\alpha_0,\alpha_0,\zeta)>0$, see (5.42). Also $K>0$, $I>0$, so $D>0$, and (5.43) implies (5.47). Consequently $(\zeta+\sin\alpha_0)^3\Delta_2$ is a decreasing function in $[0,\pi/2)$, it vanishes at $\alpha_0=\pi/2$ and therefore it is positive in $[0,\pi/2)$. □

Lemma 5.13.

(i) *$K(\alpha_0,\zeta)$ has a unique zero $\hat\alpha_0(\zeta)<0$,*

(ii) $D(\alpha_0,\zeta)>0$ when $\alpha_0\geq\hat\alpha_0(\zeta)$,

(iii) $D\Big(-\frac{\pi}{2},\zeta\Big)<0$, $P\Big(X,-\frac{\pi}{2},\zeta\Big)>0$ when $\zeta\geq 1$, $D\Big(-\frac{\pi}{2},1\Big)=-\infty$.

(iv) *At $\alpha_0=-\pi/2$,*

$$(\zeta+\sin\alpha_0)^3\Delta_2>0, \qquad \frac{\partial}{\partial\alpha_0}\big((\zeta+\sin\alpha_0)^3\Delta_2\big)<0, \qquad \zeta\geq 1.$$

Proof. $K(\alpha_0,\zeta)>0$ when $\alpha_0\geq 0$, $K(-\pi/2,\zeta)<0$, and K is an increasing function of $\alpha_0\in[-\pi/2,0]$, since

$$\frac{\partial}{\partial\alpha_0}K(\alpha_0,\zeta)=-\frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{3/2}}>0, \quad \alpha_0\in\Big[-\frac{\pi}{2},0\Big).$$

Consequently K has a unique zero $\hat\alpha_0(\zeta)\in(-\pi/2,0)$; in particular $K>0$ on $(\hat\alpha_0,\pi/2)$, and this implies (ii). As for (iii), we have

$$D\Big(-\frac{\pi}{2},\zeta\Big)=\frac{1}{4}J^2\Big(-\frac{\pi}{2},\zeta\Big)+\frac{3}{2}I\Big(-\frac{\pi}{2},\zeta\Big)\int_{-\pi/2}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}.$$

The integral is $-\infty$ at $\zeta = 1$, so $D(-\pi/2, 1) = -\infty$. For large ζ one has

$$J\Big(-\frac{\pi}{2},\zeta\Big) \sim -\frac{\pi}{2\zeta^{3/2}}, \qquad I\Big(-\frac{\pi}{2},\zeta\Big) \sim \frac{\pi}{2\zeta^{1/2}}, \qquad K\Big(-\frac{\pi}{2},\zeta\Big) \sim -\frac{3\pi}{2\zeta^{5/2}},$$

so

$$D\Big(-\frac{\pi}{2},\zeta\Big) \sim -\frac{\pi^2}{2\zeta^3} < 0, \qquad \zeta \sim \infty.$$

Also,

$$\begin{aligned} 8D_\zeta\Big(-\frac{\pi}{2},\zeta\Big) &= J\Big(-\frac{\pi}{2},\zeta\Big)K\Big(-\frac{\pi}{2},\zeta\Big) - I\Big(-\frac{\pi}{2},\zeta\Big)\int_{-\pi/2}^{\pi/2}\frac{18\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{5/2}} \\ &> 0, \end{aligned}$$

consequently $D(-\pi/2, \zeta) < 0$, $\zeta \in [1, \infty)$. Since the discriminant is negative and $-K(-\pi/2, \zeta) > 0$, we have $P(X, -\pi/2, \zeta) > 0$, and this impies (iii). In view of these results (5.41), (5.43) imply (iv). □

Lemma 5.14. *One has*

$$P(\zeta + \sin\alpha_0, \alpha_0, \zeta) > 0, \qquad \alpha_0 \le \hat{\alpha}_0(\zeta). \tag{5.48}$$

Proof. Indeed,

$$\begin{aligned} &2P(\zeta+\sin\alpha_0,\alpha_0,\zeta) \\ &= -(\zeta+\sin\alpha_0)^2K(\alpha_0,\zeta) - 2(\zeta+\sin\alpha_0)J(\alpha_0,\zeta) + 3I(\alpha_0,\zeta) \\ &= -(\zeta+\sin\alpha_0)^2K(\alpha_0,\zeta) + \frac{1}{2}(\zeta+\sin\alpha_0)^{3/2}(3\Psi-\Phi). \end{aligned}$$

The first term on the right hand side is positive when $\alpha_0 \le \hat{\alpha}_0(\zeta)$, and the second term is always positive in view of (5.24). Thus we have established Lemma 5.14. □

Proposition 5.15. $\Delta_2 > 0$ *for* $\zeta \ge 1$ *and* $\alpha_0 \in [-\pi/2, \pi/2)$.

Proof. According to Corollary 5.12 and Lemma 5.13(iv),

$$\left.\begin{aligned} \zeta+\sin\alpha_0)^3\Delta_2 > 0, \\ \frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_2 < 0, \end{aligned}\right\} \text{ when } \alpha_0 \in \Big[0,\frac{\pi}{2}\Big), \quad \alpha_0 = -\frac{\pi}{2}.$$

In particular, if $(\zeta+\sin\alpha_0)^3\Delta_2$ vanishes, it can only vanish in $(-\pi/2,0)$, and it must vanish at least twice there. The derivative $(\partial/\partial\alpha_0)(\zeta+\sin\alpha_0)^3\Delta_2$ must have opposite signs at these consecutive zeros, or, possibly vanish at a double zero.

(i) $\boldsymbol{\Delta_2(\alpha_0,\zeta)\neq 0}$, $\boldsymbol{\alpha_0\leq\hat\alpha_0(\zeta)}$. At the endpoints we have

$$\Delta_2(-\pi/2,\zeta)>0,$$

and, in view of Lemma 5.13(ii) and Lemma 5.14,

$$D\big(\hat\alpha_0(\zeta),\zeta\big)>0,\qquad P\big(\zeta+\sin\hat\alpha_0(\zeta),\hat\alpha_0(\zeta),\zeta\big)>0,$$

hence (5.41) yields

$$\big(\zeta+\sin\hat\alpha_0(\zeta)\big)^3\Delta_2\big(\hat\alpha_0(\zeta),\zeta\big)>0.$$

If $\Delta_2(\bar\alpha_0,\zeta)=0$, $\bar\alpha_0\in\big(-\pi/2,\hat\alpha_0(\zeta)\big)$, then

$$\cos\bar\alpha_0 D(\bar\alpha_0,\zeta)=\frac{2\sin\bar\alpha_0}{(\zeta+\sin\bar\alpha_0)^{1/2}}P(\zeta+\sin\bar\alpha_0,\bar\alpha_0,\zeta)<0 \tag{5.49}$$

according to (5.41) and (5.48), and then (5.48), (5.43) and (5.40) imply that

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_2<0\quad\text{at }\alpha_0=\bar\alpha_0.$$

This contradicts the necessity of $(\partial/\partial\alpha_0)(\zeta+\sin\alpha_0)^3\Delta_2$ having opposite signs at consecutive zeros or vanishing at a double zero of $(\zeta+\sin\alpha_0)^3\Delta_2$, consequently Δ_2 cannot vanish in $\big(-\pi/2,\hat\alpha_0(\zeta)\big)$.

(ii) **$\boldsymbol{\Delta_2}$ cannot vanish in $\boldsymbol{\big(\hat\alpha_0(\zeta),0\big)}$**. For $\alpha_0>\hat\alpha_0(\zeta)$, $K(\alpha_0,\zeta)>0$, so $D(\alpha_0,\zeta)>0$. At a zero of Δ_2 in $\big(\hat\alpha_0(\zeta),0\big)$, (5.43) and (5.41) yield

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_2=-\frac{D(\alpha_0,\zeta)}{\sin\alpha_0}>0,$$

in view of Lemma 5.13(ii). This cannot happen and we have completed the proof of Proposition 5.15. □

The rest of chapter 5 is devoted to a careful description of

The curves $\Gamma_2(\zeta)$.

For fixed $\zeta \geq 1$ we define the curves $\Gamma_2(\zeta)$:

$$(\hat{x}, \hat{t}) = \big(\Phi(\alpha_0, \zeta), \Psi(\alpha_0, \zeta)\big), \qquad \alpha_0 \in [-\pi/2, \pi/2]. \tag{5.50}$$

Lemma 5.16. *We let α_0 start at $\pi/2$ and decrease to $-\pi/2$.*

(i) $\zeta > 1$. *$\Gamma_2(\zeta)$ starts at $(0,0)$ tangent to $\hat{x} - \hat{t} = 0$. $\hat{x}$ increases from 0 to $\Phi\big(\alpha_0^*(\zeta), \zeta\big)$ then decreases to $\Phi(-\pi/2, \zeta) < 0$. $\hat{t}$ increases from 0 to $\Psi\big(\alpha_0^+(\zeta), \zeta\big)$ and then decreases to $\Psi(-\pi/2, \zeta) > 0$.*

(ii) $\zeta = 1$. *$\Gamma_2(1)$ starts at $(0,0)$ tangent to $\hat{x} - \hat{t} = 0$. $\hat{x}$ increases from 0 to $\Phi\big(\alpha_0^*(1), 1\big)$ then decreases to $-\infty$. $\hat{t}$ increases from 0 to ∞, with*

$$\lim_{\alpha_0 \to -\frac{\pi}{2}} \frac{\hat{t}(\alpha_0, 1)}{\hat{x}(\alpha_0, 1)} = -\infty.$$

(iii) $\Gamma_2\Big(-\frac{\pi}{2}, \zeta\Big) = C_1(-\pi/2)$, *when considered as sets.*

Proof. $\zeta \geq 1$:

$$\frac{\partial \hat{t}}{\partial \hat{x}} = \frac{\partial \Psi/\partial \alpha_0}{\partial \Phi/\partial \alpha_0} = \frac{3\Psi + 4\tan\alpha_0}{\Phi + 4\tan\alpha_0} = 1 + \frac{3\Psi - \Phi}{\Phi + 4\tan\alpha_0} \to 1 \quad \text{as } \alpha \to \frac{\pi}{2}.$$

This shows that $\Gamma_2(\alpha_0, \zeta)$ starts at the origin tangent to $\hat{x} - \hat{t} = 0$. The rest of the statements in Lemma 5.16 is a reformulation of our results on Φ and Ψ. □

Next we translate the results of Lemmas 5.1 and 5.2 into statements on the solvability of

$$\hat{x} = \Phi(\alpha_0, \zeta). \tag{5.51}$$

Lemma 5.17. *Given $\hat{x} \in \big[\Phi(-\pi/2, \zeta), \Phi\big(\alpha_0^*(\zeta), \zeta\big)\big]$, $\hat{x} = \Phi(\alpha_0, \zeta)$ has*

(i) *one solution $\alpha_0^{(1)}(\hat{x}, \zeta) \in \big(-\pi/2, \tilde{\alpha}_0(\zeta)\big)$, provided*

$$\hat{x} \in \big[\Phi(-\pi/2, \zeta), 0\big), \tag{5.52}$$

(ii) *two solutions* $\alpha_0^{(j)}(\hat{x}, \zeta)$, $j = 1, 2$,

$$\tilde{\alpha}_0(\zeta) \le \alpha_0^{(1)}(\hat{x}, \zeta) \le \alpha_0^*(\zeta) \le \alpha_0^{(2)}(\hat{x}, \zeta), \tag{5.53}$$

provided

$$\hat{x} \in \big[0, \Phi\big(\alpha_0^*(\zeta), \zeta\big)\big]. \tag{5.54}$$

Lemma 5.18. *One has*

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Phi\big(\alpha_0^*(\zeta), \zeta\big) < 0, \quad \text{and} \tag{5.55}$$

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Phi\Big(-\frac{\pi}{2}, \zeta\Big) > 0. \tag{5.56}$$

Proof. Setting $\alpha_0 = -\pi/2$ in (5.15) we obtain (5.56). Next,

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Phi\big(\alpha_0^*(\zeta), \zeta\big) = \Phi_{\alpha_0}\big(\alpha_0^*(\zeta), \zeta\big)\frac{\mathrm{d}\alpha_0^*}{\mathrm{d}\zeta} + \Phi_\zeta\big(\alpha_0^*(\zeta), \zeta\big) = \Phi_\zeta\big(\alpha_0^*(\zeta), \zeta\big) < 0$$

in view of Lemma 5.4(i). □

A simple consequence is

Corollary 5.19. *When* $\hat{x} \in \big[0, \Phi\big(\alpha_0^*(1), 1\big)\big]$,

$$\hat{x} = \Phi\big(\alpha_0^*(\zeta), \zeta\big) \tag{5.57}$$

has a unique solution, and when $\hat{x} < 0$,

$$\hat{x} = \Phi\Big(-\frac{\pi}{2}, \zeta\Big) \tag{5.58}$$

has a unique solution. We shall denote this unique solution by $\zeta_M(\hat{x})$.

We note that $\zeta_M(\hat{x}) = \max \zeta$ among all those ζ for which either

$$\hat{x} = \Phi(\alpha_0, \zeta), \qquad \hat{x} \in \big[0, \Phi\big(\alpha_0^*(1), 1\big)\big]$$

has a solution, or

$$\hat{x} = \Phi\Big(-\frac{\pi}{2}, \zeta\Big), \qquad \hat{x} < 0$$

has a solution.

If $\hat{x} \in \big(0, \Phi\big(\alpha_0^*(1), 1\big)\big)$, then for every $\zeta \in \big(1, \zeta_M(\hat{x})\big)$, $\hat{x} = \Phi(\alpha_0, \zeta)$ has 2 solutions $\alpha_0^{(1)}(\hat{x}, \zeta)$, $\alpha_0^{(2)}(\hat{x}, \zeta)$; this is a consequence of Lemma 5.17(ii).

When $\hat{x} < 0$, then Lemma 5.17(i) implies that $\hat{x} = \Phi(\alpha_0, \zeta)$ has a unique solution $\alpha_0^{(1)}(\hat{x}, \zeta)$ for each ζ such that $\Phi(-\pi/2, \zeta) < \hat{x}$; according to (5.56) this is equivalent to $\zeta \le \zeta_M(\hat{x})$. We note that in this case, that is $\hat{x} < 0$,

$$\alpha_0^{(1)}(\hat{x}, \zeta_M) = -\frac{\pi}{2}.$$

Lemma 5.20.

(i) $\hat{x}>0$: $\zeta_M(\hat{x})$ *decreases from* ∞ *to* 1 *as* $\hat{x}$ *increases from* 0 *to* $\Phi\big(\alpha_0^*(1), 1\big)$.

(ii) $\hat{x}<0$: $\zeta_M(\hat{x})$ *increases from* 1 *to* ∞ *as* $\hat{x}$ *increases from* $-\infty$ *to* 0.

(iii) $\lim\limits_{\hat{x}\to 0} \alpha_0^{(1)}(\hat{x}, \zeta) = \tilde{\alpha}_0(\zeta)$, $\lim\limits_{\hat{x}\to 0^+} \alpha_0^{(2)}(\hat{x}, \zeta) = \frac{\pi}{2}$, $\frac{\partial \alpha_0^{(1)}}{\partial \hat{x}} > 0$, $\frac{\partial \alpha_0^{(2)}}{\partial \hat{x}} < 0$.

Proof. The results are implied by the picture of the graph of Φ, properties of $\zeta_M(\hat{x})$ and the following simple fact:

$$\lim_{\zeta\to\infty} \Phi(\alpha_0, \zeta) = 0.$$

□

Before we embark on solving

$$\hat{t} = \Psi\big(\alpha_0^{(j)}(\hat{x}, \zeta), \zeta\big) = \Psi^{(j)}(\hat{x}, \zeta), \qquad \zeta \in \big[1, \zeta_M(\hat{x})\big], \tag{5.59}$$

for ζ, it may help to summarize our understanding of the structure of the solutions $\alpha_0^{(j)}(\hat{x}, \zeta)$ of (5.51):

(i) Necessarily

$$\hat{x} \in \big(-\infty, \Phi\big(\alpha_0^*(1), 1\big)\big].$$

(ii) For such an $\hat{x}$,

$$\zeta \in \big[1, \zeta_M(\hat{x})\big].$$

(iii) There are 2 possible roots of (5.51):

$$\alpha_0^{(1)}(\hat{x},\zeta) \in \Big[-\frac{\pi}{2}, \alpha_0^*(\zeta)\Big],$$
$$\alpha_0^{(2)}(\hat{x},\zeta) \in \Big[\alpha_0^*(\zeta), \frac{\pi}{2}\Big).$$

For $\hat{x} < 0$, only $\alpha_0^{(1)}(\hat{x},\zeta)$ is a root. When $0 < \hat{x} \in \big(0, \Phi\big(\alpha_0^*(1), 1\big)\big]$, there are 2 roots, $\alpha_0^{(j)}(\hat{x},\zeta)$, $j = 1, 2$.

Lemma 5.21. *We have*

(i) $\frac{\partial\Psi^{(j)}}{\partial\zeta}(\hat{x},\zeta) = -\frac{\Delta_2}{\partial\Phi/\partial\alpha_0}\Big|_{\alpha_0=\alpha_0^{(j)}(\hat{x},\zeta)}$,

(ii) $\frac{\partial\Psi^{(1)}}{\partial\zeta} < 0$, $\frac{\partial\Psi^{(2)}}{\partial\zeta} > 0$, $\zeta \in \big(1, \zeta_M(\hat{x})\big)$,

(iii) *each equation*

$$\hat{t} = \Psi^{(1)}(\hat{x},\zeta), \qquad \hat{t} = \Psi^{(2)}(\hat{x},\zeta)$$

has at most one solution in ζ.

Proof. From

$$\hat{x} = \Phi\big(\alpha_0^{(j)}(\hat{x},\zeta), \zeta\big)$$

one has

$$\frac{\partial\alpha_0^{(j)}(\hat{x},\zeta)}{\partial\zeta} = -\frac{\partial\Phi/\partial\zeta}{\partial\Phi/\partial\alpha_0}\Big|_{\alpha_0=\alpha_0^{(j)}(\hat{x},\zeta)},$$

and therefore

$$\begin{aligned}\frac{\partial\Psi^{(j)}}{\partial\zeta} &= \Big(\frac{\partial\Psi}{\partial\alpha_0}\Big|_{\alpha_0=\alpha_0^{(j)}(\hat{x},\zeta)}\Big)\frac{\partial\alpha_0^{(j)}(\hat{x},\zeta)}{\partial\zeta} + \frac{\partial\Psi}{\partial\zeta}\Big|_{\alpha_0=\alpha_0^{(j)}(\hat{x},\zeta)}\\ &= -\frac{\Delta_2}{\partial\Phi/\partial\alpha_0}\Big|_{\alpha_0=\alpha_0^{(j)}(\hat{x},\zeta)}\end{aligned}$$

which is (i). $\Delta_2 > 0$, except at $\alpha_0 = \pi/2$. Since $\alpha_0^{(1)} < \alpha_0^*(\zeta) < \alpha_0^{(2)}$,

$$\frac{\partial\Phi}{\partial\alpha_0}(\alpha_0^{(1)},\zeta) > 0, \qquad \frac{\partial\Phi}{\partial\alpha_0}(\alpha_0^{(2)},\zeta) < 0,$$

and we have derived (ii) which implies (iii). □

Remark 5.22. We note that

$$\hat{x} = \Phi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big), \zeta_M(\hat{x})\big),$$

so $\alpha_0^{(1)} = \alpha_0^{(2)} = \alpha_0^*$ is excluded from the above discussion. On the other hand, $\Phi_{\alpha_0}\big(\alpha_0^*(\zeta), \zeta\big) = 0$ implies that

$$\frac{\partial \Psi^{(j)}(\hat{x}, \zeta)}{\partial \zeta} \longrightarrow \begin{cases} -\infty, & \text{j=1,} \\ \infty, & \text{j=2,} \end{cases}$$

as $\alpha_0^{(j)}(\hat{x}, \zeta) \to \alpha_0^*(\zeta)$, $j = 1, 2$.

Corollary 5.23. *The system of equations*

$$\hat{x} = \Phi(\alpha_0, \zeta), \qquad \hat{t} = \Psi(\alpha_0, \zeta) \tag{5.60}$$

has at most 2 *solutions in* $\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta \in [1, \infty)$. *When* $\hat{x} < 0$ *it has at most one solution.*

We shall improve on this result.

Definition 5.24. Let R_2 denote the image of the mapping

$$(\alpha_0, \zeta) \;\to\; \Big(\Phi(\alpha_0, \zeta), \Psi(\alpha_0, \zeta)\Big), \tag{5.61}$$

$$(\alpha_0, \zeta) \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big] \times [1, \infty). \tag{5.62}$$

Evidently $R_2 = \bigcup_{\zeta} \Gamma_2(\zeta)$, $\zeta \in [1, \infty)$.

Lemma 5.25. $\hat{x} < 0$.

(i) *The intersection of* R_2 *with the vertical abscissa at* $\hat{x}$ *is the following segment of* $\hat{t}$*:*

$$\Psi\Big(-\frac{\pi}{2}, \zeta_M(\hat{x})\Big) \le \hat{t} \le \Psi\big(\alpha_0^{(1)}(\hat{x}, 1), 1\big), \tag{5.63}$$

with $\zeta_M(\hat{x})$ *being the unique root of*

$$\hat{x} = \Phi\Big(-\frac{\pi}{2}, \zeta\Big). \tag{5.64}$$

(ii) *The lower end* $\big(\hat{x}, \Psi\big(-\pi/2, \zeta_M(\hat{x})\big)\big)$ *of this segment is on the curve* $C_1(-\pi/2)$.

(iii) *The upper end* $\big(\hat{x}, \Psi\big(\alpha_0^{(1)}(\hat{x},1),1\big)\big)$ *of this segment is on the curve* $\Gamma_2(1)$. *In particular,* $\Gamma_2(1)$ *is above* $C_1(-\pi/2)$.

(iv) *The mapping* (5.61), (5.62) *is* 1-1 *into* $R_2 \cap \{\hat{x} < 0\}$.

(v) $\Gamma_2(\zeta) \cap C_1(-\pi/2) = \big(\Phi(-\pi/2, \zeta_M), \Psi(-\pi/2, \zeta_M)\big)$.

Proof. When $\hat{x} < 0$, (5.51) has a unique root $\alpha_0^{(1)}(\hat{x},\zeta)$ for each $\zeta \in \big[1, \zeta_M(\hat{x})\big]$, and $\Psi\big(\alpha_0^{(1)}(\hat{x},\zeta),\zeta\big) = \Psi^{(1)}(\hat{x},\zeta)$ is a decreasing function of ζ. Consequently,

$$\Psi\Big(-\frac{\pi}{2}, \zeta_M(\hat{x})\Big) = \Psi\big(\alpha_0^{(1)}\big(\hat{x}, \zeta_M(\hat{x})\big), \zeta_M(\hat{x})\big) < \Psi\big(\alpha_0^{(1)}(\hat{x},\zeta),\zeta\big) < \Psi\big(\alpha_0^{(1)}(\hat{x},1),1\big),$$

if $1 < \zeta < \zeta_M(\hat{x})$. The lower end of this interval is the set of points

$$(\hat{x}, \hat{t}) = \Big(\Phi\Big(-\frac{\pi}{2}, \zeta_M(\hat{x})\Big), \Psi\Big(-\frac{\pi}{2}, \zeta_M(\hat{x})\Big)\Big),$$

$\zeta_M(\hat{x}) \in (1,\infty)$, which is the curve $C_1(-\pi/2)$ parametrized by $\zeta_M \in (1,\infty)$. The upper end is the following set of points:

$$(\hat{x}, \hat{t}) = \big(\Phi\big(\alpha_0^{(1)}(\hat{x},1),1\big), \Psi\big(\alpha_0^{(1)}(\hat{x},1),1\big)\big),$$

where $\alpha_0^{(1)}(\hat{x},1)$ runs in

$$-\frac{\pi}{2} = \alpha_0^{(1)}(-\infty,1) < \alpha_0^{(1)}(\hat{x},1) < \alpha_0^{(1)}(0,1) = \tilde{\alpha}_0(1).$$

This is the arc of $\Gamma_2(1)$ in the half-space $\hat{x} < 0$. Thus for a fixed $\hat{x} < 0$ the vertical segment of abscissa $\hat{x}$ in R_2 is the interval (5.63) stretching between $C_1(-\pi/2)$ and $\Gamma_2(1)$, and $\Gamma_2(1)$ is above $C_1(-\pi/2)$.

R_2 is the union of the vertical intervals $\big[\Psi^{(1)}\big(\hat{x}, \zeta_M(\hat{x})\big), \Psi^{(1)}(\hat{x},1)\big]$, $\hat{x} < 0$. $\zeta_M(\hat{x})$ occurs only at $\alpha_0 = -\pi/2$. If $\Gamma_2(\zeta)$ strikes $C_1(-\pi/2)$ at $\alpha_0 > -\pi/2$, then that $\zeta < \zeta_M(\hat{x})$, so for that particular $\hat{x}$, in view of the decreasing behaviour of $\Psi^{(1)}(\hat{x},\zeta)$, the lower end $\Psi^{(1)}\big(\hat{x}, \zeta_M(\hat{x})\big)$ of the segment $\big[\Psi^{(1)}\big(\hat{x}, \zeta_M(\hat{x})\big), \Psi^{(1)}(\hat{x},1)\big]$ would be below $C_1(-\pi/2)$, outside of R_2, and this is a contradiction. Hence $\Gamma_2(\zeta)$ meets $C_1(-\pi/2)$ only at $\alpha_0 = -\pi/2$, and we have derived Lemma 5.25. $\square$

When $\hat{x} > 0$, we separate R_2 into 2 subdomains, namely $\alpha_0 < \alpha_0^*(\zeta)$ and $\alpha_0 > \alpha_0^*(\zeta)$.

Lemma 5.26.

(i) *Given $\hat{x} > 0$, the intersection of the region R_2 with the vertical line of abscissa $\hat{x}$ which corresponds to the root $\alpha_0^{(1)}(\hat{x},\zeta)$ is the segment*

$$\Psi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big),\zeta_M(\hat{x})\big) \le \hat{t} \le \Psi\big(\alpha_0^{(1)}(\hat{x},1),1\big), \tag{5.65}$$

where $\zeta_M(\hat{x})$ is the unique root of $\hat{x} = \Phi\big(\alpha_0^*(\zeta),\zeta\big)$.

(ii) *The upper end of these segments is the upper arc of $\Gamma_2(1)$ between $\hat{x} = 0$ and $\hat{x} = \Phi\big(\alpha_0^*(1),1\big)$.*

(iii) *The lower end of these segments is the curve*

$$(\hat{x},\hat{t}) = \big(\Phi\big(\alpha_0^*(\zeta),\zeta\big),\Psi\big(\alpha_0^*(\zeta),\zeta\big)\big), \tag{5.66}$$

$\zeta \in [1,\infty)$. This curve is the locus of those points of $\Gamma_2(\zeta)$ where $\hat{x}$ is maximum.

(iv) *The points in this part of R_2 are in 1-1 correspondence with (α_0,ζ), $\alpha_0 = \alpha_0^{(1)}(\hat{x},\zeta)$.*

Proof. We note that

$$\alpha_0^{(1)}\big(\hat{x},\zeta_M(\hat{x})\big) = \alpha_0^*\big(\zeta_M(\hat{x})\big) = \alpha_0^{(2)}\big(\hat{x},\zeta_M(\hat{x})\big).$$

Also,

$$\Psi^{(1)}(\hat{x},\zeta) = \Psi\big(\alpha_0^{(1)}(\hat{x},\zeta),\zeta\big) \tag{5.67}$$

is a decreasing function of ζ, so $\hat{t} = \Psi\big(\alpha_0^{(1)}(\hat{x},\zeta),\zeta\big)$ varies in the interval (5.65). The upper end of this interval corresponds to the set of points

$$(\hat{x},\hat{t}) = \big(\Phi\big(\alpha_0^{(1)}(\hat{x},1),1\big),\Psi\big(\alpha_0^{(1)}(\hat{x},1),1\big)\big),$$

with

$$\hat{x} \in \big[0,\Phi\big(\alpha_0^*(1),1\big)\big].$$

This is an arc of $\Gamma_2(1)$ which extends from $\alpha_0 = \tilde{\alpha}_0(1)$, where $\hat{x} = 0$ on $\Gamma_2(1)$, to $\alpha_0 = \alpha_0^*(1)$, where $\hat{x}$ is maximum on $\Gamma_2(1)$. The lower end of the interval (5.65) is the arc

$$(\hat{x},\hat{t}) = \Big(\Phi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big),\zeta_M(\hat{x})\big),\Psi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big),\zeta_M(\hat{x})\big)\Big), \tag{5.68}$$

where $\hat{x}$ acts as the parameter,

$$\hat{x}\in\big[0,\Phi\big(\alpha_0^*(1),1\big)\big].$$

We recall that for $\zeta\in[1,\infty)$, $\Phi\big(\alpha_0^*(\zeta),\zeta\big)$ is the maximum value of $\hat{x}$ along $\Gamma_2(\zeta)$. Consequently the lower end of the interval (5.65) is the curve joining the points of $\Gamma_2(\zeta)$ where $\hat{x}$ is maximal.

Finally (iv) follows from the strictly decreasing behaviour of the function $\Psi^{(1)}(\hat{x},\zeta)$ in ζ, and we have completed the derivation of Lemma 5.26. □

Lemma 5.27.

(i) *When $\hat{x}>0$, the intersection of the region R_2 with the vertical line of abscissa $\hat{x}$ which corresponds to the root $\alpha_0^{(2)}(\hat{x},\zeta)$ is the segment*

$$\Psi\big(\alpha_0^{(2)}(\hat{x},1),1\big)\le\hat{t}\le\Psi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big),\zeta_M(\hat{x})\big). \tag{5.69}$$

(ii) *The lower end of this interval is the lower arc of $\Gamma_2(1)$ between $\hat{x}=0$ and $\hat{x}=\Phi\big(\alpha_0^*(1),1\big)$.*

(iii) *The upper end of this interval is the curve* (5.68).

(iv) *The points $(\hat{x},\hat{t})$ in this part of R_2 are in* 1-1 *correspondence with (α_0,ζ), $\alpha_0=\alpha_0^{(2)}(\hat{x},\zeta)$.*

Proof. Given $\hat{x}>0$,

$$\hat{t}=\Psi^{(2)}(\hat{x},\zeta)=\Psi\big(\alpha_0^{(2)}(\hat{x},\zeta),\zeta\big)$$

is an increasing function of $\zeta\in(1,\infty)$, and the corresponding $\hat{t}$ varies in the interval

$$\Psi\big(\alpha_0^{(2)}(\hat{x},1),1\big)\le\hat{t}\le\Psi\big(\alpha_0^*\big(\zeta_M(\hat{x})\big),\zeta_M(\hat{x})\big),$$

since

$$\alpha_0^{(2)}\big(\hat{x},\zeta_M(\hat{x})\big)=\alpha_0^*\big(\zeta_M(\hat{x})\big).$$

The rest of the argument is similar to the derivation of Lemma 5.26. □

We summarize these results in

Proposition 5.28.

(i) *For* $(\alpha_0, \zeta) \in [-\pi/2, \pi/2) \times [1, \infty)$,

$$(\alpha_0, \zeta) \longrightarrow \big(\Phi(\alpha_0, \zeta), \Psi(\alpha_0, \zeta)\big) \tag{5.70}$$

is a 1-1 *mapping onto the region* R_2, *where* R_2 *is the domain bounded by* $C_1(-\pi/2)$ *and* $\Gamma_2(1)$.

(ii) $R_2 \cap \{\hat{x} < 0\}$ *corresponds to the root* $\alpha_0^{(1)}(\hat{x}, \zeta)$ *of* (5.51).

(iii) *When* $\hat{x} \geq 0$, *the part of* R_2 *between the upper part of* $\Gamma_2(1)$ *and the curve* (5.68) *corresponds to* $\alpha_0^{(1)}(\hat{x}, \zeta)$.

(iv) *When* $\hat{x} \geq 0$, *the part of* R_2 *between the lower part of* $\Gamma_2(1)$ *and the curve* (5.68) *corresponds to* $\alpha_0^{(2)}(\hat{x}, \zeta)$.

(v) *The mapping* (5.70) *is not* C^1 *on the curve* (5.68)*; see Remark* 5.22.

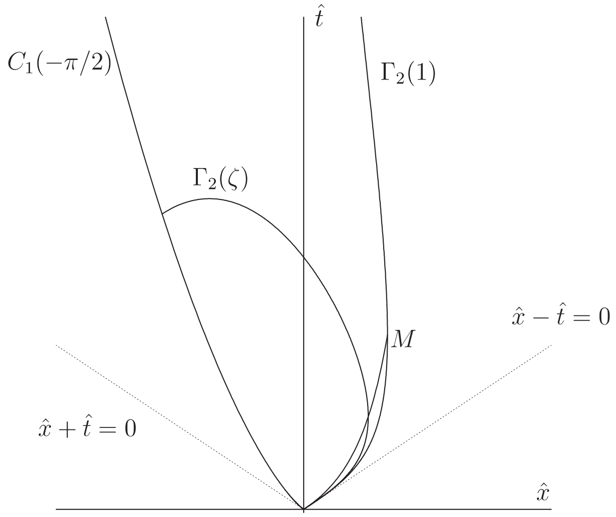

Figure 4.

6. $\alpha_{\text{end}} = \pi - \alpha_0 + 2n\pi$, $n = 1, 2, \ldots$, $\dot{y}(0) > 0$

We still need to consider geodesics which return to the plane after $n = 1, 2, \ldots$ full periods at $\alpha_{\text{end}} = \pi - \alpha_0 + 2n\pi$. The final points are given by formulas (5.1) and (5.2) which may be rewritten in the following form:

$$(\hat{x}, \hat{t}) = \big(\Phi(\alpha_0, \zeta), \Psi(\alpha_0, \zeta)\big) + n\left(\frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}}, \frac{I(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}}\right), \quad (6.1)$$

$\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta \in [1, \infty)$. According to Lemma 4.1, $I(\zeta) > 0$, $I(1) < \infty$, $J(\zeta) < 0$, $J(1) = -\infty$.

Proposition 6.1. (x, y_0, t) *cannot be connected to* $(0, y_0, 0)$ *by an infinite number of geodesics of the form* (6.1).

Remark 6.2. Suppose there are an infinite number of geodesics of the form (6.1) which connect (x, y_0, t) to $(0, y_0, 0)$ with parameters α_{0,n_p}, $\zeta_{n_p} \geq 1$, $p = 1, 2, \ldots$. Clearly $\hat{t} > 0$. Then we have two possibilities:

(i) if $\{n_p\}$ is unbounded, we may choose a strictly increasing subsequence, which we again denote by $n_1, n_2, \ldots$.

(ii) if $\{n_p\}$ is bounded, there is a fixed positive integer q, such that (x, y_0, t) is connected to $(0, y_0, 0)$ by an infinite number of geodesics of the form (6.1) with $n = q$.

In the following 2 Lemmas we shall show that neither case can occur; this will prove Proposition 6.1.

Lemma 6.3. *It is not possible to find a subsequence* n_p*,* $p = 1, 2, \ldots$ *of the positive integers n such that there are an infinite number of geodesics, one for each* n_p*,* $p = 1, 2, \ldots$*, of the form (6.1) which join* (x, y_0, t) *to* $(0, y_0, 0)$*.*

Proof of Lemma 6.3. Assume the opposite. Then there are parameters α_{0,n_p} and ζ_{n_p} with $\alpha_{0,n_p} \in [-\pi/2, \pi/2]$, $\zeta_{n_p} \in [1, \infty)$ such that for the given (x, y_0, t) one has

$$\hat{x} = \Phi(\alpha_{0,n_p}, \zeta_{n_p}) + \frac{n_p J(\zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{1/2}}, \quad (6.2)$$

$$\hat{t} = \Psi(\alpha_{0,n_p}, \zeta_{n_p}) + \frac{n_p I(\zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{3/2}}, \quad (6.3)$$

$p = 1, 2, 3, \ldots$. We may assume that

$$\lim_{p\to\infty} \alpha_{0,n_p} = \bar{\alpha}_0, \qquad \lim_{p\to\infty} \zeta_{n_p} = \bar{\zeta}, \qquad \bar{\zeta} \in [1, \infty]. \quad (6.4)$$

a) $1 < \bar{\zeta} < \infty$. In this case $\Phi(\alpha_{0,n_p}, \zeta_{n_p})$, $\Psi(\alpha_{0,n_p}, \zeta_n)$, $J(\zeta_{n_p})$ and $I(\zeta_{n_p})$ all have finite limits and $I(\bar{\zeta}) > 0$. Consequently the second term on the right hand side of (6.3) has an infinite limit as $n_p \to \infty$ which contradicts the finiteness of $\hat{t}$.

b) $\bar{\zeta} = 1$. Here $\lim\limits_{p\to\infty} J(\zeta_{n_p}) = -\infty$, and (6.2) gives

$$\frac{J(\alpha_{0,n_p}, \zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{1/2}} \equiv \Phi(\alpha_{0,n_p}, \zeta_{n_p}) = -\frac{n_p J(\zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{1/2}} + \hat{x},$$

so that for large p one has

$$J(\alpha_{0,n_p}, \zeta_{n_p}) \sim -n_p J(\zeta_{n_p}) \to \infty.$$

This contradicts the fact that $J(\alpha_0, 1)$ is bounded from above by $J(0, 1)$.

c) $\bar{\zeta} = \infty$. (4.23) and (4.24) yield

$$\frac{n_p J(\zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{1/2}} = -\frac{n_p}{\zeta_{n_p}^2}\left(\frac{\pi}{2} - \frac{\pi}{4\zeta_{n_p}}\sin\alpha_{0,n_p} + \cdots\right),$$

and

$$\frac{n_p I(\zeta_{n_p})}{(\zeta_{n_p} + \sin\alpha_{0,n_p})^{3/2}} = \frac{n_p}{\zeta_{n_p}^2}\left(\frac{\pi}{2} - \frac{3\pi}{4\zeta_{n_p}}\sin\alpha_{0,n_p} + \cdots\right).$$

Also, in view of (5.19) and (5.32) we have

$$\Phi(\alpha_{0,n_p}, \zeta_{n_p}) = \frac{2}{\zeta_{n_p}}(\cos\alpha_{0,n_p} + \cdots), \tag{6.5}$$

$$\Psi(\alpha_{0,n_p}, \zeta_{n_p}) = \frac{2}{\zeta_{n_p}}(\cos\alpha_{0,n_p} + \cdots). \tag{6.6}$$

Each of these right hand sides represents the beginning of a convergent power series in $\zeta_{n_p}^{-1}$. Consequently,

$$\hat{x} = -\frac{n_p}{\zeta_{n_p}^2}\left(\frac{\pi}{2} - \frac{\pi}{4\zeta_{n_p}}\sin\alpha_{0,n_p} + \cdots\right) + O\left(\frac{1}{\zeta_{n_p}}\right), \tag{6.7}$$

$$\hat{t} = \frac{n_p}{\zeta_{n_p}^2}\left(\frac{\pi}{2} - \frac{3\pi}{4\zeta_{n_p}}\sin\alpha_{0,n_p} + \cdots\right) + O\left(\frac{1}{\zeta_{n_p}}\right). \tag{6.8}$$

$\hat{x}$ and $\hat{t}$ are finite and fixed, therefore

$$\lim_{p\to\infty}\left(-\frac{n_p}{\zeta_{n_p}^2}\right) = \frac{2\hat{x}}{\pi} = -E, \qquad \lim_{p\to\infty}\frac{n_p}{\zeta_{n_p}^2} = \frac{2\hat{t}}{\pi} = E \geq 0, \tag{6.9}$$

with some constant E, so

$$\hat{x} = -\frac{\pi E}{2}, \qquad \hat{t} = \frac{\pi E}{2}, \tag{6.10}$$

and $(\hat{x}, \hat{t})$ must be on the critical line $\hat{x} + \hat{t} = 0$, $\hat{t} > 0$. Since x and t are not both zero, on the critical line neither are zero, and therefore $E > 0$. We still need to show that points on the critical line cannot be connected to $(0, y_0, 0)$ by an infinite number of geodesics of the form (6.2), (6.3). To this end we need more information on $\bar{\alpha}_0$. As $n_p/\zeta_{n_p}^2$ has a finite nonzero limit we may write

$$\begin{aligned}
\hat{x} + \frac{\pi}{2}\frac{n}{\zeta_{n_p}^2} &= \frac{n_p\pi}{4\zeta_{n_p}^3}\sin\alpha_{0,n_p} + \frac{2}{\zeta_{n_p}}\cos\alpha_{0,n_p} + O\Big(\frac{1}{\zeta_{n_p}^2}\Big), \\
\hat{t} - \frac{\pi}{2}\frac{n_p}{\zeta_{n_p}^2} &= -\frac{3n_p\pi}{4\zeta_{n_p}^3}\sin\alpha_{0,n_p} + \frac{2}{\zeta_{n_p}}\cos\alpha_{0,n_p} + O\Big(\frac{1}{\zeta_{n_p}^2}\Big).
\end{aligned}$$

We set

$$\frac{n_p}{\zeta_{n_p}^2} = E + \varepsilon(n_p), \qquad \varepsilon(n_p) = o(1), \qquad p \to \infty. \tag{6.11}$$

Then

$$\begin{aligned}
\frac{\pi}{2}\varepsilon(n_p)\zeta_{n_p} &= \frac{\pi}{4}\big(E + \varepsilon(n_p)\big)\sin\alpha_{0,n_p} + 2\cos\alpha_{0,n_p} + O\Big(\frac{1}{\zeta_{n_p}}\Big), \\
-\frac{\pi}{2}\varepsilon(n_p)\zeta_{n_p} &= -\frac{3\pi}{4}\big(E + \varepsilon(n_p)\big)\sin\alpha_{0,n_p} + 2\cos\alpha_{0,n_p} + O\Big(\frac{1}{\zeta_{n_p}}\Big).
\end{aligned}$$

Adding, one has

$$0 = -\frac{\pi}{2}\big(E + \varepsilon(n_p)\big)\sin\alpha_{0,n_p} + 4\cos\alpha_{0,n_p} + O\Big(\frac{1}{\zeta_{n_p}}\Big).$$

Letting $p \to \infty$,

$$0 = -\frac{E\pi}{2}\sin\bar{\alpha}_0 + 4\cos\bar{\alpha}_0,$$

and

$$\tan \bar{\alpha}_0 = \frac{8}{E\pi} > 0 \;\Rightarrow\; \bar{\alpha}_0 \in \Big(0, \frac{\pi}{2}\Big). \tag{6.12}$$

Consequently, as $p \to \infty$,

$$\Psi(\alpha_{0,n_p}, \zeta_{n_p}) \sim \Psi(\bar{\alpha}_0, \zeta_{n_p}) = O\Big(\frac{1}{\zeta_{n_p}}\Big) \to 0,$$

$$\frac{n_p I(\zeta_{n_p})}{(\zeta_{n_p} + \sin \alpha_{0,n_p})^{3/2}} \sim \frac{n_p \sqrt{\zeta_{n_p}} \int_{\bar{\alpha}_0}^{\pi/2} \sin \alpha d\alpha}{\zeta_{n_p}^{3/2}} \sim \sqrt{n_p E} \cos \bar{\alpha}_0 \to \infty,$$

which yields

$$\lim_{p \to \infty} \left(\Psi(\alpha_{0,n_p}, \zeta_{n_p}) + \frac{n_p I(\zeta_{n_p})}{(\zeta_{n_p} + \sin \alpha_{0,n_p})^{3/2}} \right) = \infty.$$

$\hat{t}$ is finite, and we have contradicted (6.3). This shows that we cannot have infinitely many geodesics, represented by (6.2), (6.3) with $\lim_{p\to\infty} n_p = \infty$, connecting points on the critical line with the point $(0, y_0, 0)$. Thus we have completed the proof of Lemma 6.3. □

Lemma 6.4. *Given a fixed positive integer q, one cannot join (x, y_0, t) to $(0, y_0, 0)$ by an infinite number of distinct geodesics of the form*

$$\hat{x} = \Phi_q(\alpha_0, \zeta) = \Phi(\alpha_0, \zeta) + \frac{qJ(\zeta)}{(\zeta + \sin \alpha_0)^{1/2}}, \tag{6.13}$$

$$\hat{t} = \Psi_q(\alpha_0, \zeta) = \Psi(\alpha_0, \zeta) + \frac{qI(\zeta)}{(\zeta + \sin \alpha_0)^{3/2}}, \tag{6.14}$$

$\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta \in [1, \infty)$.

Proof of Lemma 6.4. We start with (6.14) and note that $\Psi(\alpha_0, \zeta)$ and $I(\zeta)$ are both decreasing functions of ζ, hence so is $\Psi_q(\alpha_0, \zeta)$, $\zeta \in [1, \infty)$. Consequently, if (6.14) has a solution $\zeta(\alpha_0)$ for a given α_0, then the solution is unique. For a fixed α_0, $\Psi_q(\alpha_0, \zeta)$ decreases from $\Psi_q(\alpha_0, 1) > 0$ to $\Psi_q(\alpha_0, \infty) = 0$. As for $\Psi_q(\alpha_0, 1)$, it decreases from $\Psi_q(-\pi/2, 1) = \infty$ to $\Psi_q(\pi/2, 1) = qI(1)/2 > 0$. Thus for a fixed α_0, (6.14) has a solution if and only if $\hat{t} \le \Psi_q(\alpha_0, 1)$.

(i) If $\hat{t} \le \Psi_q(\pi/2, 1) = qI(1)/2$, then (6.14) has a unique solution $\zeta(\alpha_0)$

for all

$$\alpha_0 \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]. \tag{6.15}$$

(ii) If $\hat{t} > \Psi_q(\pi/2, 1) = qI(1)/2$, then (6.14) has a unique solution whenever

$$\alpha_0 \in \Big[-\frac{\pi}{2}, \alpha_{0,\hat{t}}\Big], \tag{6.16}$$

where we let $\alpha_{0,\hat{t}}$ denote the unique solution of

$$\hat{t} = \Psi_q(\alpha_{0,\hat{t}}, 1). \tag{6.17}$$

We note that $\zeta(-\pi/2) > 1$, $\zeta(\alpha_{0,\hat{t}}) = 1$ and $\zeta(\pi/2) \geq 1$. We need $\zeta'(\alpha_0)$ at the end points of the intervals (6.15) and (6.16). To this end we differentiate (6.14):

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\alpha_0}\left[\Psi\big(\alpha_0, \zeta(\alpha_0)\big) + \frac{qI\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{3/2}}\right] \\
&= \frac{\partial\Psi}{\partial\alpha_0}\big(\alpha_0, \zeta(\alpha_0)\big) - \frac{3}{2}\frac{qI\big(\zeta(\alpha_0)\big)\cos\alpha_0}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{5/2}} \\
&\quad + \frac{\partial}{\partial\zeta}\left[\Psi(\alpha_0,\zeta) + \frac{qI(\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}\right]_{\zeta=\zeta(\alpha_0)} \cdot \zeta'(\alpha_0) \\
&= -\frac{1}{2}\frac{\cos\alpha_0}{\zeta(\alpha_0)+\sin\alpha_0}\Big(3\Psi\big(\alpha_0,\zeta(\alpha_0)\big) + 4\tan\alpha_0\Big) - \frac{3}{2}\frac{qI\big(\zeta(\alpha_0)\big)\cos\alpha_0}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{5/2}} \\
&\quad + \left[\frac{\partial\Psi}{\partial\zeta}\big(\alpha_0,\zeta(\alpha_0)\big) + \frac{qI'\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{3/2}} - \frac{3}{2}\frac{qI\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{5/2}}\right]\zeta'(\alpha_0).
\end{aligned}$$

Multiplying both sides by $2\big(\zeta(\alpha_0)+\sin\alpha_0\big)$ we obtain

$$\begin{aligned}
0 = &-3\cos\alpha_0\left[\Psi\big(\alpha_0,\zeta(\alpha_0)\big) + \frac{qI\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{3/2}}\right] - 4\sin\alpha_0 \\
&+ \Bigg[\Big\{\Phi\big(\alpha_0,\zeta(\alpha_0)\big) - 3\Psi\big(\alpha_0,\zeta(\alpha_0)\big)\Big\} \\
&\qquad + q\Big\{\frac{J\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{1/2}} - \frac{3I\big(\zeta(\alpha_0)\big)}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{3/2}}\Big\}\Bigg]\zeta'(\alpha_0).
\end{aligned}$$

Both curly brackets are negative, the first by (5.24) and the second because $J(\zeta) < 0$ and $I(\zeta) > 0$. Consequently so is their sum,

$$\Phi_q(\alpha_0, \zeta) - 3\Psi_q(\alpha_0, \zeta) < 0. \tag{6.18}$$

Thus we have

$$\zeta'(\alpha_0) = \frac{3\hat{t}\cos\alpha_0 + 4\sin\alpha_0}{\Phi_q\big(\alpha_0, \zeta(\alpha_0)\big) - 3\Psi_q\big(\alpha_0, \zeta(\alpha_0)\big)}. \tag{6.19}$$

In particular

$$\zeta'\Big(-\frac{\pi}{2}\Big) = \frac{-4}{\Phi_q\Big(-\frac{\pi}{2}, \zeta\Big(-\frac{\pi}{2}\Big)\Big) - 3\Psi_q\Big(-\frac{\pi}{2}, \zeta\Big(-\frac{\pi}{2}\Big)\Big)} > 0, \tag{6.20}$$

and

$$\zeta'\Big(\frac{\pi}{2}\Big) < 0. \tag{6.21}$$

Consequently $\zeta(\alpha_0)$ starts out increasing at $\alpha_0 = -\pi/2$, $\zeta(-\pi/2) > 1$, attains its maximum at $\alpha_{0,q}^+$, where we set

$$\tan\alpha_{0,q}^+ = \frac{3\hat{t}}{4}, \qquad \alpha_{0,0}^+ = \alpha_0^+, \tag{6.22}$$

then decreases to

$$\zeta(\alpha_{0,\hat{t}}) = 1, \text{ or to } \zeta\Big(\frac{\pi}{2}\Big) \geq 1.$$

To complete the proof of Lemma 6.4, which will complete the proof of Proposition 6.1, we still need to show that

$$\hat{x} = \Phi_q\big(\alpha_0, \zeta(\alpha_0)\big) \tag{6.23}$$

has at most a finite number of solutions in $\alpha_0 \in [-\pi/2, \alpha_{0,\hat{t}}]$, or in $[-\pi/2, \pi/2]$. $\Phi_q\big(\alpha_0, \zeta(\alpha_0)\big)$ is an analytic function of α_0. Therefore an infinite number of solutions α_0 of (6.23) implies that $\hat{x}$ is a limit point of the values of $\Phi_q\big(\alpha_0, \zeta(\alpha_0)\big)$ at either one or both end points of the α_0-interval. This cannot happen at $\alpha_{0,\hat{t}}$, because $\Phi(\alpha_{0,\hat{t}}, 1)$ is finite and $J(1) = -\infty$, so we have $\Phi_q(\alpha_{0,\hat{t}}, 1) = -\infty$. It cannot happen at $\alpha_0 = \pm\pi/2$ either, because it has a nonvanishing derivative with respect to α_0 at $\alpha_0 = \pm\pi/2$. Indeed,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\Phi_q\big(\alpha_0,\zeta(\alpha_0)\big)\\ &=\frac{\partial\Phi}{\partial\alpha}\big(\alpha_0,\zeta(\alpha_0)\big)-\frac{1}{2}\frac{qJ\big(\zeta(\alpha_0)\big)\cos\alpha_0}{\big(\zeta(\alpha_0)+\sin\alpha_0\big)^{3/2}}+\frac{\partial\Phi_q}{\partial\zeta}\big(\alpha_0,\zeta(\alpha_0)\big)\zeta'(\alpha_0),\end{aligned}$$

and at $\alpha_0 = -\pi/2$ we have

$$\begin{aligned}&\frac{\mathrm{d}\Phi_q}{\mathrm{d}\alpha_0}\Big(-\frac{\pi}{2},\zeta\Big(-\frac{\pi}{2}\Big)\Big)\\ &=\frac{2}{\zeta(-\frac{\pi}{2})-1}+\frac{\partial}{\partial\zeta}\frac{(q+1)J(\zeta)}{(\zeta-1)^{1/2}}\Big|_{\zeta=\zeta(-\frac{\pi}{2})}\cdot\zeta'\Big(-\frac{\pi}{2}\Big)\\ &=\frac{2}{\zeta(-\frac{\pi}{2})-1}+(q+1)\left[\frac{J'\Big(\zeta(-\frac{\pi}{2})\Big)}{\Big(\zeta(-\frac{\pi}{2})-1\Big)^{1/2}}-\frac{J\Big(\zeta(-\frac{\pi}{2})\Big)}{2\Big(\zeta(-\frac{\pi}{2})-1\Big)^{3/2}}\right]\zeta'\Big(-\frac{\pi}{2}\Big).\end{aligned}$$

The square bracket is positive and so is $\zeta'(-\pi/2)$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\Phi_q\big(\alpha_0,\zeta(\alpha_0)\big)\Big|_{\alpha_0=-\frac{\pi}{2}}>0.$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\Phi_q\big(\alpha_0,\zeta(\alpha_0)\big)\Big|_{\alpha_0=\frac{\pi}{2}}<0.$$

This implies that (6.23) can have at most a finite number of solutions which proves Lemma 6.4, and we have completed the proof of Proposition 6.1. □

7. $\alpha_{\text{end}} = \pi - \alpha_0 + 2n\pi$, $n = 1, 2, \ldots$, $\dot{y}(0) < 0$

We are still interested in formulas (5.1) and (5.2):

$$\hat{x}(\pi-\alpha_0+2n\pi)=\Phi(\alpha_0,\zeta)+\frac{nJ(\zeta)}{(\zeta+\sin\alpha_0)^{1/2}}, \tag{7.1}$$

$$\hat{t}(\pi-\alpha_0+2n\pi)=\Psi(\alpha_0,\zeta)+\frac{nI(\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}, \tag{7.2}$$

$n = 1, 2, \ldots$, although here $\dot{y}(0) < 0$ which is equivalent to

$$\frac{\pi}{2} < \alpha_0 < \frac{3\pi}{2}. \tag{7.3}$$

Note that $n \neq 0$ if $\dot{y}(0) < 0$, see Chapter 3 (ii). We introduce

$$\alpha_0' = \pi - \alpha_0, \qquad \alpha' = \pi - \alpha, \tag{7.4}$$

and obtain

$$\Phi(\alpha_0, \zeta) = \frac{-2}{(\zeta + \sin\alpha_0')^{1/2}} \int_{\alpha_0'}^{\pi/2} \frac{\sin\alpha' \mathrm{d}\alpha'}{(\zeta + \sin\alpha')^{1/2}} = -\Phi(\alpha_0', \zeta),$$
$$\Psi(\alpha_0, \zeta) = \frac{-2}{(\zeta + \sin\alpha_0')^{3/2}} \int_{\alpha_0'}^{\pi/2} \sin\alpha'(\zeta + \sin\alpha')\mathrm{d}\alpha' = -\Psi(\alpha_0', \zeta).$$

Thus

$$\hat{x} = -\,\Phi(\alpha_0', \zeta) + \frac{nJ(\zeta)}{(\zeta + \sin\alpha_0')^{1/2}}, \tag{7.5}$$
$$\hat{t} = -\,\Psi(\alpha_0', \zeta) + \frac{nI(\zeta)}{(\zeta + \sin\alpha_0')^{3/2}}, \tag{7.6}$$

with

$$\alpha_0' = \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]. \tag{7.7}$$

$\boldsymbol{n = 1}$. Here we use

$$\int_0^{\pi} f(\sin\alpha)\mathrm{d}\alpha = 2\int_0^{\pi/2} f(\sin\alpha)\mathrm{d}\alpha,$$

$$\int_{\pi}^{2\pi} f(\sin\alpha)\mathrm{d}\alpha = 2\int_{-\pi/2}^{0} f(\sin\alpha)\mathrm{d}\alpha,$$

and write

$$\hat{x} = \frac{2}{(\zeta + \sin\alpha_0)^{1/2}} \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}} = \varphi(\alpha_0, \zeta), \tag{7.8}$$
$$\hat{t} = \frac{2}{(\zeta + \sin\alpha_0)^{3/2}} \int_{-\pi/2}^{\alpha_0} \sin\alpha(\zeta + \sin\alpha)^{1/2}\mathrm{d}\alpha = \psi(\alpha_0, \zeta), \tag{7.9}$$

with

$$-\frac{\pi}{2} \leq \alpha_0 \leq \frac{\pi}{2}, \qquad \zeta > 1, \tag{7.10}$$

where we returned to using α_0 for α_0'; we note that α_0 of (7.10) is the $\pi - \alpha_0$ of the chapter heading.

$\mathbf{n = 2, 3, \ldots}$. With $m = n - 2$ we have

$$\hat{x} = \varphi(\alpha_0, \zeta) + \frac{mJ(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}}, \tag{7.11}$$

$$\hat{t} = \psi(\alpha_0, \zeta) + \frac{mI(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}}, \tag{7.12}$$

$m = 0, 1, 2, \ldots$, $\alpha_0 \in [-\pi/2, \pi/2]$. We set

$$\varphi(\alpha_0, \zeta) = \frac{j(\alpha_0, \zeta)}{(\zeta + \sin\alpha_0)^{1/2}}, \tag{7.13}$$

$$\psi(\alpha_0, \zeta) = \frac{i(\alpha_0, \zeta)}{(\zeta + \sin\alpha_0)^{3/2}}, \tag{7.14}$$

with

$$j(\alpha_0, \zeta) = 2\int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}}, \tag{7.15}$$

$$i(\alpha_0, \zeta) = 2\int_{-\pi/2}^{\alpha_0} \sin\alpha(\zeta + \sin\alpha)^{1/2}\mathrm{d}\alpha, \tag{7.16}$$

and work with φ, ψ, j and i in a manner similar to the use of Φ, Ψ, J and I in chapter 5.

On the functions φ, ψ, j and i.

Lemma 7.1. (i) *For* $\alpha_0 \in (-\pi/2, \pi/2]$,

$$j(\alpha_0, \zeta) < 0, \quad \text{and } j\Big(-\frac{\pi}{2}, \zeta\Big) = 0, \tag{7.17}$$

(ii) *j has a minimum at* $\alpha_0 = 0$.

Proof. Clearly $j(\pi/2, \zeta) = J(\zeta) < 0$, and

$$\frac{\partial j}{\partial \alpha_0} = \frac{2\sin\alpha_0}{(\zeta + \sin\alpha_0)^{1/2}} = 0 \;\Rightarrow\; \alpha_0 = 0. \tag{7.18}$$

□

Lemma 7.2. (i) *We have*

$$\frac{\partial\varphi}{\partial\alpha_0} = -\frac{1}{2}\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}(\varphi - 4\tan\alpha_0). \tag{7.19}$$

(ii) *Given* $\zeta > 1$, $\partial\varphi/\partial\alpha_0$ *has a unique zero at* $\alpha^*_{0,\varphi}(\zeta) < 0$, *where*

$$\varphi\big(\alpha^*_{0,\varphi}(\zeta),\zeta\big) = 4\tan\alpha^*_{0,\varphi}(\zeta), \tag{7.20}$$

and φ *achieves its minimum at* $\alpha^*_{0,\varphi}(\zeta)$. *In particular,*

$$\varphi\Big(-\frac{\pi}{2},\zeta\Big) = 0, \qquad \varphi\Big(\frac{\pi}{2},\zeta\Big) < 0 \tag{7.21}$$

imply that

$$\varphi(\alpha_0,\zeta) < 0, \qquad \alpha_0 \in \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big]. \tag{7.22}$$

(iii) $\lim\limits_{\zeta\to 1^+}\varphi(\alpha_0,\zeta) = -\infty$, *uniformly when* α_0 *is bounded away from* $-\pi/2$.

(iv)

$$\lim_{\zeta\to 1^+}\alpha^*_{0,\varphi}(\zeta) = -\frac{\pi}{2}. \tag{7.23}$$

Proof. (i) is immediate. (ii) Note that

$$\varphi - 4\tan\alpha_0 = \frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\big(j - 4\sqrt{\zeta+\sin\alpha_0}\tan\alpha_0\big).$$

j is strictly decreasing from 0 and $4\sqrt{\zeta+\sin\alpha_0}\tan\alpha_0$ is strictly increasing to 0 when $\alpha_0 \in (-\pi/2, 0)$, see the proof of Lemma 5.2(ii), so they intersect once at $\alpha^*_{0,\varphi}(\zeta) < 0$, and we have (7.20). Finally one has

$$\lim_{\zeta\to 1^+}\int_{-\pi/2}^{\alpha_0}\frac{\sin\alpha\, d\alpha}{(\zeta+\sin\alpha)^{1/2}} = -\infty,$$

which proves (iii), and then (7.20) and (iii) imply that $\lim\limits_{\zeta\to 1^+}\tan\alpha^*_{0,\varphi}(\zeta) = -\infty$ which yields (iv). □

Lemma 7.3. *For large ζ one has*

$$\begin{aligned}\varphi(\alpha_0,\zeta) =& \frac{2}{\zeta}\Big\{ -\cos\alpha_0 + \frac{1}{2\zeta}\Big(\sin\alpha_0\cos\alpha_0 - \int_{-\pi/2}^{\alpha_0}\sin^2\alpha d\alpha\Big) \\ & - \frac{3}{8\zeta^2}\Big(\sin^2\alpha_0\cos\alpha_0 - \frac{2}{3}\sin\alpha_0\int_{-\pi/2}^{\alpha_0}\sin^2\alpha d\alpha - \int_{-\pi/2}^{\alpha_0}\sin^3\alpha d\alpha\Big) \\ & + \cdots\Big\}. \end{aligned} \tag{7.24}$$

A simple calculation gives

$$\frac{\partial\varphi}{\partial\zeta} = -\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\left[\frac{\varphi}{2(\zeta+\sin\alpha_0)^{1/2}} + \int_{-\pi/2}^{\alpha_0}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}\right]. \tag{7.25}$$

Since $\varphi < 0$ when $\alpha_0 \in (-\pi/2, \pi/2]$, and

$$\int_{-\alpha_0}^{\alpha_0}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}} < 0, \qquad \alpha_0 > 0,$$

we have

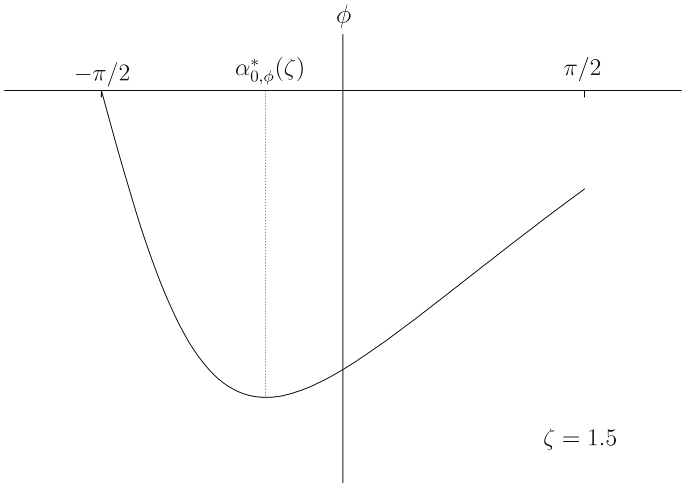

Figure 5.

Lemma 7.4. *For all $\alpha_0 \in (-\pi/2, \pi/2]$,*

$$\frac{\partial\varphi}{\partial\zeta} > 0. \tag{7.26}$$

Lemma 7.5. *For a fixed* $\zeta \geq 1$, (i) $i(\alpha_0, \zeta)$ *has a unique zero* $\tilde{\alpha}_{0,i}(\zeta) > 0$, (ii) i *is minimum at* $\alpha_0 = 0$.

Proof. $i(\pi/2, \zeta) > 0$, $i(\alpha_0, \zeta) < 0$ for $\alpha_0 \leq 0$, and

$$\frac{\partial i}{\partial \alpha_0} = 2 \sin \alpha_0 (\zeta + \sin \alpha_0)^{1/2}.$$

Consequently i is decreasing when $\alpha_0 < 0$, and increasing when $\alpha_0 > 0$. These imply (i) and (ii) and Lemma 7.5. □

We note that

$$\frac{\partial \psi}{\partial \alpha_0} = -\frac{1}{2} \frac{\cos \alpha_0}{\zeta + \sin \alpha_0} (3\psi - 4 \tan \alpha_0). \tag{7.27}$$

Lemma 7.6. (i) *For fixed* $\zeta > 1$, $\partial \psi / \partial \alpha_0$ *has a unique zero* $\alpha^+_{0,\psi}(\zeta) < 0$. *In particular,*

$$3\psi\big(\alpha^+_{0,\psi}(\zeta), \zeta\big) = 4 \tan \alpha^+_{0,\psi}(\zeta). \tag{7.28}$$

ψ *decreases from* $\psi(-\pi/2, \zeta) = 0$ *to* $\psi\big(\alpha^+_{0,\psi}(\zeta), \zeta\big)$ *and then increases to* $\psi(\pi/2, \zeta) > 0$.

(ii) $\psi(\alpha_0, 1)$ *increases from* $\psi(-\pi/2, 1) = -\infty$ *to* $\psi(\pi/2, 1) > 0$.

Proof. We integrate by parts in ψ,

$$\psi = \frac{2}{(\zeta + \sin \alpha_0)^{3/2}} \Bigg(\int_{-\pi/2}^{0} \sin \alpha (\zeta + \sin \alpha)^{1/2} \mathrm{d}\alpha - \frac{2}{3} \int_0^{\alpha_0} (\zeta + \sin \alpha)^{3/2} \frac{\mathrm{d}\alpha}{\cos^2 \alpha} + \frac{2}{3} (\zeta + \sin \alpha_0)^{3/2} \tan \alpha_0 \Bigg),$$

and obtain

$$3\psi - 4 \tan \alpha_0 = \frac{2}{(\zeta + \sin \alpha_0)^{3/2}} \Big(3 \int_{-\pi/2}^{0} \sin \alpha (\zeta + \sin \alpha)^{1/2} \mathrm{d}\alpha - 2 \int_0^{\alpha_0} (\zeta + \sin \alpha)^{3/2} \frac{\mathrm{d}\alpha}{\cos^2 \alpha} \Big). \tag{7.29}$$

As a function of α_0

$$-\int_0^{\alpha_0} (\zeta + \sin\alpha)^{3/2} \frac{\mathrm{d}\alpha}{\cos^2\alpha}$$

decreases in the interval $(-\pi/2, \pi/2)$ from $+\infty$ to $-\infty$, and has a unique zero at $\alpha_0 = 0$. Therefore the bracket on the right side of (7.29) has a unique zero at $\alpha^+_{0,\psi}(\zeta) < 0$. Consequently, (7.27) implies that

$$\frac{\partial\psi}{\partial\alpha_0} < 0, \qquad \alpha_0 < \alpha^+_{0,\psi}(\zeta), \tag{7.30}$$

$$\frac{\partial\psi}{\partial\alpha_0} > 0, \qquad \alpha_0 > \alpha^+_{0,\psi}(\zeta), \tag{7.31}$$

$$\psi(0,\zeta) < 0. \tag{7.32}$$

When $\zeta = 1$, the second integral in the bracket in (7.29) exists even at $\alpha_0 = -\pi/2$. The bracket is a decreasing function of α_0. Integrating by parts the first integral in (7.29) we find that the bracket vanishes at $\alpha_0 = -\pi/2$, therefore it is negative for all $\alpha_0 \in (-\pi/2, \pi/2]$, and $\partial\psi/\partial\alpha_0 > 0$ by (7.27). As the right hand side of (7.29) is ≤ 0, we see that $\psi(-\pi/2, 1) = -\infty$ which implies (ii) and we have established Lemma 7.6. □

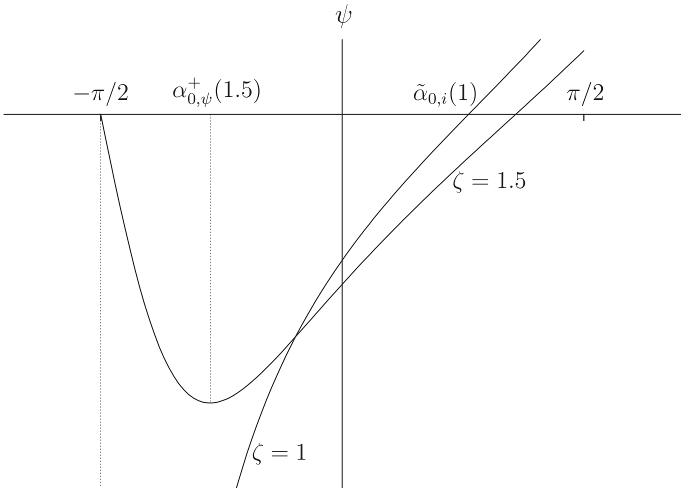

Figure 6.

Lemma 7.7. *For large* ζ,

$$\psi(\alpha_0,\zeta)=\frac{1}{\zeta}\Big\{-2\cos\alpha_0+\frac{1}{\zeta}\Big(3\sin\alpha_0\cos\alpha_0+\int_{-\pi/2}^{\alpha_0}\sin^2\alpha\mathrm{d}\alpha\Big)$$
$$-\frac{1}{\zeta^2}\Big(\frac{15}{4}\sin^2\alpha_0\cos\alpha_0+\frac{3}{2}\sin\alpha_0\int_{-\pi/2}^{\alpha_0}\sin^2\alpha\mathrm{d}\alpha$$
$$+\frac{1}{4}\int_{-\pi/2}^{\alpha_0}\sin^3\alpha\mathrm{d}\alpha\Big)+\cdots\Big\}. \tag{7.33}$$

One has

$$\frac{\partial\psi}{\partial\zeta}=\frac{-3\psi+\varphi}{2(\zeta+\sin\alpha_0)}. \tag{7.34}$$

Lemma 7.8. $\hat{\alpha}_{0,\psi}(\zeta)$ *is the unique zero of* $\partial\psi/\partial\zeta$ *in* $(-\pi/2,\pi/2]$. *In particular,*

$$3\psi\big(\hat{\alpha}_{0,\psi}(\zeta),\zeta\big)=\varphi\big(\hat{\alpha}_{0,\psi}(\zeta),\zeta\big), \tag{7.35}$$

$$\frac{\partial\psi}{\partial\zeta}>0,\qquad \alpha_0<\hat{\alpha}_{0,\psi}(\zeta), \tag{7.36}$$

$$\frac{\partial\psi}{\partial\zeta}<0,\qquad \alpha_0>\hat{\alpha}_{0,\psi}(\zeta). \tag{7.37}$$

Proof. We shall derive (7.36) and (7.37) for $-3\psi+\varphi$.

$$-3\psi+\varphi$$
$$=\frac{1}{(\zeta+\sin\alpha_0)^{3/2}}\int_{-\pi/2}^{\alpha_0}\frac{\sin\alpha}{(\zeta+\sin\alpha)^{1/2}}(\sin\alpha_0-3\sin\alpha-2\zeta)\mathrm{d}\alpha. \tag{7.38}$$

The integral vanishes at $\alpha_0=-\pi/2$. Also, $\psi(\pi/2,\zeta)>0$, $\varphi(\pi/2,\zeta)<0$, and the integral is negative at $\alpha_0=\pi/2$. Furthermore,

$$\frac{\partial}{\partial\alpha_0}\int_{-\pi/2}^{\alpha_0}\frac{\sin\alpha}{(\zeta+\sin\alpha)^{1/2}}(\sin\alpha_0-3\sin\alpha-2\zeta)\mathrm{d}\alpha$$
$$=\frac{1}{2}\cos\alpha_0(\zeta+\sin\alpha_0)^{1/2}(\varphi-4\tan\alpha_0),$$

so the integral increases when $\alpha_0<\alpha^*_{0,\varphi}(\zeta)$, hence positive at $\alpha^*_{0,\varphi}(\zeta)$, and then decreases for $\alpha_0>\alpha^*_{0,\varphi}(\zeta)$. Since the integral is negative at $\alpha_0=\pi/2$, it attains a unique zero at some point $\hat{\alpha}_{0,\psi}(\zeta)$,

$$\alpha^*_{0,\varphi}(\zeta) < \hat{\alpha}_{0,\psi}(\zeta) < \frac{\pi}{2}. \tag{7.39}$$

□

Lemma 7.9. *We have*

$$\varphi < \psi. \tag{7.40}$$

Proof.

$$\varphi - \psi = \frac{2}{(\zeta + \sin\alpha_0)^{3/2}} \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha}{(\zeta + \sin\alpha)^{1/2}} (\sin\alpha_0 - \sin\alpha)\mathrm{d}\alpha.$$

When $\alpha_0 \leq 0$, the integrand is negative and therefore so is $\varphi - \psi$. Also,

$$\frac{\partial}{\partial\alpha_0} \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha}{(\zeta + \sin\alpha)^{1/2}} (\sin\alpha_0 - \sin\alpha)\mathrm{d}\alpha = \frac{1}{2} j \, \cos\alpha_0 < 0$$

by (7.17). Consequently the integral, and therefore $\varphi - \psi$ stay negative for $\alpha_0 > 0$. □

On the Jacobian $\boldsymbol{\Delta_3}$

We are interested in the mapping

$$(\alpha_0, \zeta) \longrightarrow (\hat{x}, \hat{t}) = \big(\varphi(\alpha_0, \zeta), \psi(\alpha_0, \zeta)\big), \tag{7.41}$$

and to this end we let

$$\Delta_3 = \frac{\partial(\psi, \varphi)}{\partial(\alpha_0, \zeta)} \tag{7.42}$$

denote its Jacobian. Then

$$\begin{aligned} \Delta_3 =& \frac{1}{2} \frac{\cos\alpha_0}{(\zeta + \sin\alpha_0)^{3/2}} \left(\frac{\varphi}{2(\zeta + \sin\alpha_0)^{1/2}} + \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{3/2}} \right) \\ &+ \frac{1}{4} \frac{\cos\alpha_0}{(\zeta + \sin\alpha_0)^2} (\varphi - 4\tan\alpha_0)(-3\psi + \varphi). \end{aligned} \tag{7.43}$$

We set

$$k(\alpha_0, \zeta) = 2 \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{3/2}}, \tag{7.44}$$

$$P(X, \alpha_0, \zeta) = -\frac{1}{2}kX^2 - jX + \frac{3}{2}i, \tag{7.45}$$

$$D = \frac{3}{4}ik + \frac{1}{4}j^2, \tag{7.46}$$

so $4D$ is the discriminant of P.

Lemma 7.10. *We have*

$$(\zeta + \sin\alpha_0)^3\Delta_3 = (\cos\alpha_0)D + \frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}P(\zeta+\sin\alpha_0, \alpha_0, \zeta), \tag{7.47}$$

$$\begin{aligned} &P(\zeta+\sin\alpha_0, \alpha_0, \zeta) \\ &= \int_{-\pi/2}^{\alpha_0} (\sin\alpha - \sin\alpha_0)(\sin\alpha_0 + 3\sin\alpha + 4\zeta)\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}. \end{aligned} \tag{7.48}$$

Proof. (7.45) yields

$$\begin{aligned} P(\zeta+\sin\alpha_0, \alpha_0, \zeta) = \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha}{(\zeta+\sin\alpha)^{3/2}}\big[&-(\zeta+\sin\alpha_0)^2 \\ &- 2(\zeta+\sin\alpha_0)(\zeta+\sin\alpha) + 3(\zeta+\sin\alpha)^2\big]\mathrm{d}\alpha \end{aligned}$$

which yields (7.48). As for (7.47), we replace ψ and φ in (7.43) by their integrals, and then a direct calculation gives

$$\begin{aligned} (\zeta+\sin\alpha_0)^3\Delta_3 = \cos\alpha_0 &\left[\frac{3}{4}ki + \frac{1}{4}j^2\right] \\ &+ \sin\alpha_0\left[-2(\zeta+\sin\alpha_0)j + \frac{3i}{(\zeta+\sin\alpha_0)^{1/2}} - (\zeta+\sin\alpha_0)^{3/2}k\right] \end{aligned}$$

which is (7.47). □

Lemma 7.11. *The α_0 derivatives of P and Δ_3 are given by*

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_0}P(\zeta+\sin\alpha_0, \alpha_0, \zeta) = -(\cos\alpha_0)\big[j + (\zeta+\sin\alpha_0)k\big], \tag{7.49}$$

$$\begin{aligned} &\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3 \\ &= -(\sin\alpha_0)D + \frac{2\cos\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}P(\zeta+\sin\alpha_0, \alpha_0, \zeta). \end{aligned} \tag{7.50}$$

Proof. (7.49) follows easily from (7.48). Then (7.47) yields

$$\frac{\partial}{\partial \alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3 = -(\sin\alpha_0)D$$
$$+\cos\alpha_0\left[\frac{3}{2}\sin\alpha_0(\zeta+\sin\alpha_0)^{1/2}k+\frac{3}{2}\frac{\sin\alpha_0}{(\zeta+\sin\alpha_0)^{3/2}}i+\frac{\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}j\right]$$
$$+2\cos\alpha_0\left[\frac{1}{(\zeta+\sin\alpha_0)^{1/2}}-\frac{1}{2}\frac{\sin\alpha_0}{(\zeta+\sin\alpha_0)^{3/2}}\right]P$$
$$+\frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\alpha_0}P(\zeta+\sin\alpha_0,\alpha_0,\zeta).$$

We replace $\mathrm{d}P/\mathrm{d}\alpha_0$ by the right side of (7.49), note that the terms with $\sin\alpha_0\cos\alpha_0$ cancel and obtain (7.50). □

Proposition 7.12. $\Delta_3>0$, $\alpha\in(-\pi/2,\pi/2]$, $\zeta>1$.

Proof. We shall argue as we did in the proof of Proposition 5.15.

1)

$$\Delta_3\begin{cases}=0, & \alpha_0=-\frac{\pi}{2},\\ >0, & \alpha_0\in\left(-\frac{\pi}{2},0\right].\end{cases}\tag{7.51}$$

Indeed i and k are negative when $\alpha\le 0$, so $D>0$ there. Furthermore, the integrand in (7.48) is positive when $\alpha_0\le 0$, consequently

$$P(\zeta+\sin\alpha_0,\alpha_0,\zeta)>0,\qquad \alpha_0\le 0,\tag{7.52}$$

and, in view of (7.50),

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3>0,\qquad \alpha_0\le 0.\tag{7.53}$$

(7.47) and (7.48) imply that $(\zeta+\sin\alpha_0)^3\Delta_3=0$ at $\alpha_0=-\pi/2$, therefore (7.53) implies that

$$(\zeta+\sin\alpha_0)^3\Delta_3>0,\qquad -\frac{\pi}{2}<\alpha_0\le 0,$$

and we have derived (7.51).

2) $\Delta_3>0$ near $\alpha_0=\pi/2$. We note that $D(\pi/2,\zeta)$ agrees with $D(-\pi/2,\zeta)$

of Lemma 5.13(iii) which is negative, so

$$D\Big(\frac{\pi}{2},\zeta\Big)<0,\qquad \zeta\ge 1, \tag{7.54}$$

and $P(X,\pi/2,\zeta)$ has no real roots. Therefore $P(0,\pi/2,\zeta)=(3/2)i(\pi/2,\zeta)>0$ implies that

$$P\Big(X,\frac{\pi}{2},\zeta\Big)>0. \tag{7.55}$$

In view of (7.47),

$$(\zeta+\sin\alpha_0)^3\Delta_3\Big|_{\alpha_0=\pi/2}=\frac{2}{\sqrt{\zeta+1}}P\Big(\zeta+1,\frac{\pi}{2},\zeta\Big)>0, \tag{7.56}$$

which proves 2). Clearly, (7.50) and (7.54) yield

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3\Big|_{\alpha_0=\frac{\pi}{2}}=-D>0. \tag{7.57}$$

3) Thus, if Δ_3 has a zero, it must have a least 2 zeros in $\alpha_0>0$ where $\partial(\zeta+\sin\alpha_0)^3\Delta_3/\partial\alpha_0$ has opposite signs.

4) Δ_3 does not vanish for $\alpha_0\ge\tilde\alpha_{0,i}$; we recall that $\tilde\alpha_{0,i}$ is the unique zero of i, and $\tilde\alpha_{0,i}>0$. Indeed, $i>0$ when $\alpha_0>\tilde\alpha_{0,i}$, and since j and k are always negative, we have

$$P(\zeta+\sin\alpha_0,\alpha_0,\zeta)=-\frac{1}{2}(\zeta+\sin\alpha_0)^2k-(\zeta+\sin\alpha_0)j+\frac{3}{2}i>0, \tag{7.58}$$

$\alpha_0\ge\tilde\alpha_{0,i}$. Furthermore, as $i(\tilde\alpha_{0,i},\zeta)=0$, we have

$$D(\tilde\alpha_{0,i},\zeta)=\frac{1}{4}j^2>0, \tag{7.59}$$

and (7.47) yields

$$(\zeta+\sin\alpha_0)^3\Delta_3\Big|_{\alpha_0=\tilde\alpha_{0,i}}>0. \tag{7.60}$$

Consequently, in view of (7.56), if $(\zeta+\sin\alpha_0)^3\Delta_3$ vanishes in $(\tilde\alpha_{0,i},\pi/2)$, it must have at least 2 zeros there with $\partial(\zeta+\sin\alpha_0)^3\Delta_3/\partial\alpha_0$ having opposite signs at the 2 zeros, or a double zero of Δ_3. This cannot happen. At a zero of Δ_3 we have

$$D = -\frac{2\tan\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}} P(\zeta+\sin\alpha_0, \alpha_0, \zeta), \tag{7.61}$$

see (7.47). Substituting (7.61) into (7.50) we obtain

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3 = \frac{2P(\zeta+\sin\alpha_0,\alpha_0,\zeta)}{\cos\alpha_0(\zeta+\sin\alpha_0)^{1/2}} > 0 \tag{7.62}$$

at a zero of Δ_3 in $(\tilde{\alpha}_{0,i}, \pi/2)$, in view of (7.58). This proves 4). We still need

5) Δ_3 has no zero in $(0, \tilde{\alpha}_{0,i})$. In this interval $i < 0$. Since k is always negative, one has $D > 0$ when $\alpha_0 \le \tilde{\alpha}_{0,i}$, and at a zero of Δ_3 in $(0, \tilde{\alpha}_{0,i})$ we have $P(\zeta+\sin\alpha_0, \alpha_0, \zeta) < 0$ see (7.61). According to (7.62), at a zero of Δ_3 in $(0, \tilde{\alpha}_{0,i})$

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_3 = \frac{2P(\zeta+\sin\alpha_0,\alpha_0,\zeta)}{\cos\alpha_0(\zeta+\sin\alpha_0)^{1/2}} < 0,$$

and at consecutive zeros the slope of $(\zeta+\sin\alpha_0)^3\Delta_3$ cannot change sign, or vanish at a double zero, which again contradicts 3). This completes the proof of Proposition 7.12. □

The curves $C_3(\alpha_0)$.

Let $C_3(\alpha_0)$, $\alpha_0 \in [-\pi/2, \pi/2]$, denote the following curve

$$C_3(\alpha_0) = \left\{\left(\hat{x} = \varphi(\alpha_0,\zeta), \hat{t} = \psi(\alpha_0,\zeta)\right), \qquad 1 < \zeta < \infty\right\}, \tag{7.63}$$

parametrized by ζ, in the $(\hat{x}, \hat{t})$-plane,

$C_3(-\pi/2)$ collapses to $(0,0)$,

$C_3(\pi/2)$ coincides with $C_1(\pi/2)$, since

$$\varphi\left(\frac{\pi}{2},\zeta\right) = \frac{J(\zeta)}{(\zeta+1)^{1/2}}, \qquad \psi\left(\frac{\pi}{2},\zeta\right) = \frac{I(\zeta)}{(\zeta+1)^{3/2}}.$$

Lemma 7.13. (i) *For $\alpha_0 \in (-\pi/2, \pi/2)$, the curves $C_3(\alpha_0)$ start, when $\zeta = \infty$, at the origin $(0,0)$ tangent to the line $\hat{x} = \hat{t}$.*

(ii) *All curves $C_3(\alpha_0)$ stay in the half plane $\hat{t} \ge \hat{x}$.*

(iii) *As* $\zeta \to 1^+$, $\hat{x}(\alpha_0, \zeta) \to -\infty$, $\hat{t}(\alpha_0, \zeta) \to \psi(\alpha_0, 1)$, *so that each curve* $C_3(\alpha_0)$ *has a horizontal asymptote*

$$\hat{t} = \psi(\alpha_0, 1).$$

(iv) *As* ζ *decreases from* ∞ *to* 1, $\hat{x}$ *decreases from* 0 *to* $-\infty$.

(v) *When* ζ *decreases from* ∞ *to* 1, $\hat{t}$ *starts decreasing, reaches a unique minimum, then increases to its asymptotic value* $\psi(\alpha_0, 1)$.

Proof. (i) is a consequence of the large ζ expansions of φ and ψ given in Lemmas 7.3 and 7.7.

(ii) follows from $\varphi < \psi$ as proved in Lemma 7.9.

(iii) is a consequence of the definition of φ and ψ and (iv) follows from Lemma 7.4, which says that the ζ-derivative of φ is positive.

(v) is a consequence of the following □

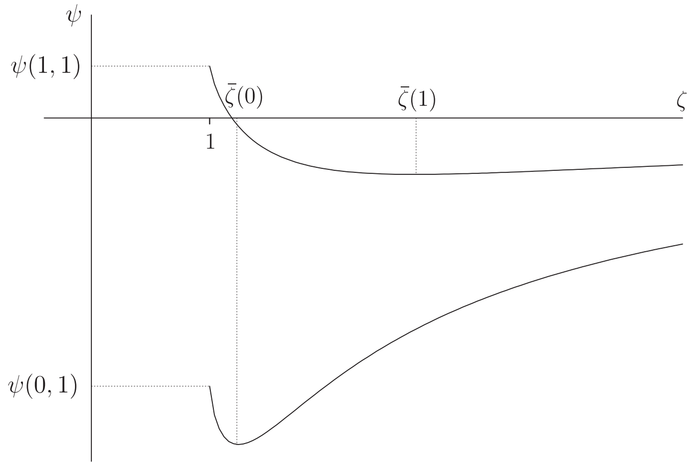

Figure 7.

Lemma 7.14. $\partial\psi/\partial\zeta$ *has a unique zero* $\bar{\zeta}(\alpha_0)$; $\partial\psi/\partial\zeta > 0$ *for* $\zeta > \bar{\zeta}$ *and* $\partial\psi/\partial\zeta < 0$ *for* $\zeta < \bar{\zeta}$.

Proof. We start with (7.34):

$$\frac{\partial\psi}{\partial\zeta} = \frac{-3\psi + \varphi}{2(\zeta + \sin\alpha_0)}.$$

As $\zeta \to 1^+$, ψ stays finite but $\varphi \to -\infty$, so

$$\lim_{\zeta \to 1^+} \frac{\partial \psi}{\partial \zeta} = -\infty.$$

The sign of $\partial\psi/\partial\zeta$ is the sign of $u = -3\psi + \varphi$,

$$\begin{aligned} u(\alpha_0,\zeta) &= \int_{-\pi/2}^{\alpha_0} (\sin\alpha_0 - 3\sin\alpha - 2\zeta)\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} \\ &= \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha(\sin\alpha_0 - \sin\alpha)}{(\zeta+\sin\alpha)^{1/2}}\mathrm{d}\alpha - i(\alpha_0,\zeta). \end{aligned} \tag{7.64}$$

The large ζ expansions of φ and ψ given in Lemmas 7.3 and 7.7 yield

$$-3\psi + \varphi \sim \frac{4}{\zeta}\cos\alpha_0 > 0, \qquad \zeta \to \infty,$$

and $u(\alpha_0,\zeta) > 0$ when $\zeta \sim \infty$. Also, $u(\alpha_0,\zeta) \to -\infty$ as $\zeta \to 1^+$, $\alpha_0 \in (-\pi/2,\pi/2)$. From (7.64),

$$\begin{aligned} \frac{\partial u}{\partial \zeta} &= -\frac{1}{2}\int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha(\sin\alpha_0 - \sin\alpha)}{(\zeta+\sin\alpha)^{3/2}}\mathrm{d}\alpha - \int_{-\pi/2}^{\alpha_0}\frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} \\ &= -\frac{1}{2}v(\alpha_0,\zeta) - \frac{1}{2}j(\alpha_0,\zeta). \end{aligned} \tag{7.65}$$

We note that

(i) $v\Big(-\frac{\pi}{2},\zeta\Big) = 0,$

(ii) $\dfrac{\partial v}{\partial \alpha_0} = \cos\alpha_0 \displaystyle\int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} < 0.$

Consequently $v < 0$ and therefore (7.65) implies that $\partial u/\partial\zeta > 0$. Thus u increases from $-\infty$ to $u(\alpha_0,\infty) > 0$ as ζ increases from 1 to ∞, and $u(\alpha_0,\zeta)$ has a unique zero $\bar{\zeta}(\alpha_0)$, and so does $\partial\psi/\partial\zeta$. □

The region R_3.

We let R_3 denote the image of the mapping (7.41):

$$(\alpha_0,\zeta) \longrightarrow \big(\varphi(\alpha_0,\zeta),\psi(\alpha_0,\zeta)\big) = (\hat{x},\hat{t}),$$

$(\alpha_0,\zeta) \in (-\pi/2,\pi/2] \times (1,\infty)$. In particular R_3 is the union of the curves $C_3(\alpha_0)$, $-\pi/2 < \alpha_0 \le \pi/2$.

Lemma 7.15. (i) *For* $(\hat{x}, \hat{t}) \in R_3$ *the equations*

$$\hat{x} = \varphi(\alpha_0, \zeta), \qquad \hat{t} = \psi(\alpha_0, \zeta)$$

have a unique solution. Consequently the mapping (7.41) *is* 1-1 *and onto its image* R^3.

(ii) *The curves* $C_3(\alpha_0)$ *do not intersect; their* $\hat{t}$ *component increases as* α_0 *increases from* $-\pi/2$ *to* $\pi/2$.

Proof. (i) According to Lemma 7.13(iv) and Lemma 7.4, φ varies between $-\infty$ and 0, and $\partial\varphi/\partial\zeta > 0$. Thus φ is an increasing function of ζ, and for any $\hat{x} < 0$ the equation

$$\hat{x} = \varphi(\alpha_0, \zeta)$$

has a unique solution in ζ, $\zeta = \zeta(\alpha_0, \hat{x})$. Substituting this value of ζ into $\hat{t} = \psi(\alpha_0, \zeta)$, we obtain

$$\hat{t}(\alpha_0, \hat{x}) = \psi\big(\alpha_0, \zeta(\alpha_0, \hat{x})\big). \tag{7.66}$$

Now

$$\hat{x} = \varphi\big(\alpha_0, \zeta(\alpha_0, \hat{x})\big) \tag{7.67}$$

yields

$$0 = \frac{\partial\varphi}{\partial\alpha_0} + \frac{\partial\varphi}{\partial\zeta}\frac{\partial\zeta}{\partial\alpha_0},$$

and

$$\frac{\partial\hat{t}(\alpha_0, \hat{x})}{\partial\alpha_0} = \frac{\Delta_3}{\partial\varphi/\partial\zeta}\bigg|_{\zeta=\zeta(\alpha_0,\hat{x})} > 0, \tag{7.68}$$

in view of Proposition 7.12 and Lemma 7.4. Thus $\hat{t}(\alpha_0, \hat{x})$ is an increasing function of α_0, and (7.66) has at most one solution.

(ii) is a consequence of (i) and of (7.68). □

Lemma 7.16. *The region* R_3 *is bounded by* $C_1(\pi/2) = C_3(\pi/2)$ *and the line* $\hat{x} = \hat{t}$, $\hat{x} < 0$.

Proof. $C_3(-\pi/2)$ degenerates to a point. Nevertheless, $\hat{t}(\alpha_0, \hat{x})$ decreases as α_0 decreases to $-\pi/2$, it is bounded from below by $\hat{x} = \hat{t}$, $\hat{x} < 0$, so it converges to a limit position. We shall show that this limit is the line $\hat{x} = \hat{t}$, $\hat{x} < 0$, i.e.

$$\lim_{\alpha_0 \to -\frac{\pi}{2}} \hat{t}(\alpha_0, \hat{x}) = \hat{x}. \tag{7.69}$$

This will prove that the mapping $\alpha_0 \to \hat{t}(\alpha_0, \hat{x})$ sends $(-\pi/2, \pi/2)$ onto the interval $\big(\hat{x}, \hat{t}(\pi/2, \hat{x})\big]$, and $\hat{t}(\pi/2, \hat{x}) \in C_3(\pi/2)$. The key is the behaviour of $\zeta(\alpha_0, \hat{x})$ as $\alpha_0 \to -\pi/2$. From (7.67) we have

$$\hat{x} = \frac{2}{(\zeta(\alpha_0, \hat{x}) + \sin\alpha_0)^{1/2}} \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta(\alpha_0, \hat{x}) + \sin\alpha)^{1/2}}. \tag{7.70}$$

When $\alpha_0 \to -\pi/2$, the integral has limit zero if $\zeta(\alpha_0, \hat{x})$ is bounded away from 1, so for a fixed $\hat{x} < 0$,

$$\lim_{\alpha_0 \to -\frac{\pi}{2}} \zeta(\alpha_0, \hat{x}) = 1. \tag{7.71}$$

We set

$$\alpha_0 = -\frac{\pi}{2} + \varepsilon_0, \tag{7.72}$$

$$\zeta(\alpha_0, \hat{x}) = 1 + \delta, \qquad \lim_{\varepsilon_0 \to 0} \delta(\varepsilon_0) = 0, \tag{7.73}$$

$$\alpha = -\frac{\pi}{2} + \varepsilon. \tag{7.74}$$

Then

$$\begin{aligned} \int_{-\pi/2}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta(\alpha_0, \hat{x}) + \sin\alpha)^{1/2}} &= -\int_0^{\varepsilon_0} \frac{\cos\varepsilon \mathrm{d}\varepsilon}{(1 + \delta - \cos\varepsilon)^{1/2}} \\ &= -\sqrt{2}\mathrm{Argsinh}\left(\sqrt{\frac{2}{\delta}} \sin\frac{\varepsilon_0}{2}\right) + O(\varepsilon_0^3); \end{aligned}$$

this is easily seen if we introduce $x = \sin(\varepsilon/2)$ as the variable of integration. As $\hat{x} = -|\hat{x}| \neq 0$, (7.70) yields

$$|\hat{x}| = \frac{2\sqrt{2}\mathrm{Argsinh}\Big(\sqrt{\frac{2}{\delta}} \sin\frac{\varepsilon_0}{2}\Big) + O(\varepsilon_0^3)}{\sqrt{\delta + 2\sin^2\frac{\varepsilon_0}{2}}}.$$

The denominator vanishes as $\varepsilon_0 \to 0$, hence so does the numerator. In particular

$$\lim_{\varepsilon_0\to 0} \frac{\varepsilon_0}{\sqrt{2\delta}} = 0.$$

$$|\hat{x}| + O(\varepsilon_0) = \frac{2\sqrt{2}\mathrm{Argsinh}\Big(\frac{\varepsilon_0}{\sqrt{2\delta}}\Big)}{\sqrt{\delta + \frac{\varepsilon_0^2}{2}}},$$

or,

$$\sinh\Big(\frac{1}{4}\big(|\hat{x}| + O(\varepsilon_0)\big)\sqrt{2\delta + \varepsilon_0^2}\Big) = \frac{\varepsilon_0}{\sqrt{2\delta}},$$

and for small ε_0 we have

$$\frac{1}{4}\big(|\hat{x}| + O(\varepsilon_0)\big)\sqrt{2\delta + \varepsilon_0^2} = \frac{\varepsilon_0}{\sqrt{2\delta}}.$$

This leads to a quadratic equation in δ which we solve when ε_0 is small,

$$\delta(\varepsilon_0) = \frac{2\varepsilon_0}{|\hat{x}|} + o(\varepsilon_0). \tag{7.75}$$

Rewriting (7.66),

$$\begin{aligned} &\hat{t}\big(\alpha_0, \zeta(\alpha_0, \hat{x})\big) \\ &= \frac{2}{(\zeta(\alpha_0, \hat{x}) + \sin\alpha_0)^{3/2}} \int_{-\pi/2}^{\alpha_0} \sin\alpha\big(\zeta(\alpha_0, \hat{x}) + \sin\alpha\big)^{1/2} \mathrm{d}\alpha, \end{aligned} \tag{7.76}$$

and we note that using (7.73) and (7.75) we can calculate the integral,

$$\begin{aligned} \int_{-\pi/2}^{\alpha_0} \sin\alpha\big(\zeta(\alpha_0, \hat{x}) + \sin\alpha\big)^{1/2} \mathrm{d}\alpha &\sim -\int_0^{\varepsilon_0} \cos\varepsilon\sqrt{\delta + \frac{1}{2}\varepsilon^2}\mathrm{d}\varepsilon \\ &\sim -\frac{\sqrt{2}}{\sqrt{|\hat{x}|}}\varepsilon_0^{3/2}, \qquad \varepsilon_0 \sim 0. \end{aligned}$$

Therefore

$$\lim_{\alpha_0\to -\pi/2} \hat{t}\big(\alpha_0, \zeta(\alpha_0, \hat{x})\big) = -|\hat{x}| = \hat{x}$$

which is (7.69) and we have completed the proof of Lemma 7.16. □

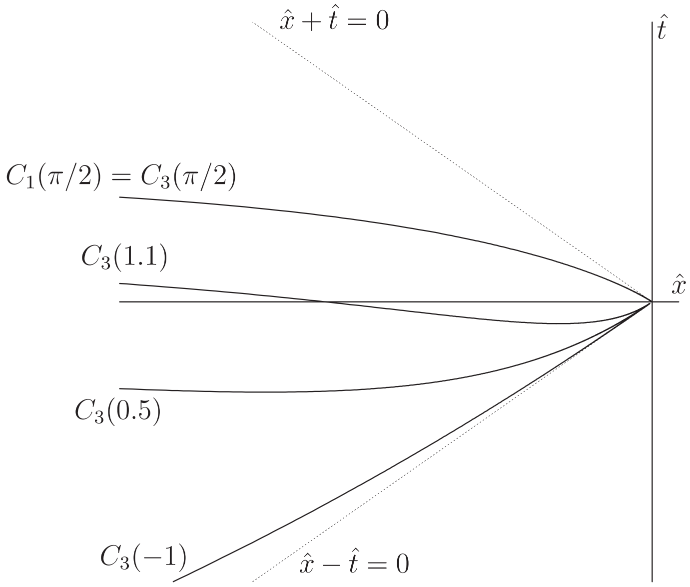

Figure 8. R_3.

In summary,

Proposition 7.17. *The mapping*

$$\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big]\times(1,\infty)\ni(\alpha_0,\zeta)\ \longrightarrow\ (\hat{x},\hat{t})=\big(\varphi(\alpha_0,\zeta),\psi(\alpha_0,\zeta)\big),$$

which corresponds to the first return point on the plane $y=y_0$ *of the geodesics issued from* $(0,y_0,0)$ *with* $\dot{y}(0)<0$, *is* 1-1 *and onto a domain* R_3. R_3 *is bounded by* $C_1(\pi/2)=C_3(\pi/2)$ *from above and by the half-line* $\hat{x}=\hat{t}$, $\hat{x}<0$ *from below. This half-line cannot be reached by the mapping.*

The next statement concerning the effect of the periods (7.11)−(7.12) is immediate.

Lemma 7.18. *The mapping* (7.11)−(7.12) *has its image in the space bounded by* $\hat{t}>\hat{x}$, $\hat{x}<0$.

On the returns after $\mathbf{m=1,2,\ldots}$ periods.

The relevant formulas are given by (7.11) and (7.12). To simplify our notation we replace m by n and look for geodesics with endpoints represented by

$$(\hat{x},\hat{t}) = \big(\varphi(\alpha_0,\zeta),\psi(\alpha_0,\zeta)\big) + n\left(\frac{J(\zeta)}{(\zeta+\sin\alpha_0)^{1/2}},\frac{I(\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}\right), \quad (7.77)$$

$\alpha_0 \in (-\pi/2,\pi/2]$, $\zeta\in(1,\infty)$, $n=1,2,\ldots$.

Proposition 7.19. *(x,y_0,t) cannot be joined to $(0,y_0,0)$ by an infinite number of geodesics of the form* (7.77).

To prove this result we shall imitate the proof of Proposition 6.1. In particular, Remark 6.2 holds as long as we replace formulas (6.1) with formulas (7.77).

Lemma 7.20. *There is no subsequence $\{n_p,\ p=1,2,\ldots\}$ of the positive integers N, such that we have an infinite number of geodesics, one for each n_p, $p=1,2,\ldots$ represented by* (7.77) *which join (x,y_0,t) to $(0,y_0,0)$.*

Proof. Assume the opposite. Then there are parameters $\alpha_{0,n_p}\in(-\pi/2,\pi/2]$ and $\zeta_{n_p}\in(1,\infty)$ such that for the given (x,y_0,t) one has

$$\hat{x} = \varphi(\alpha_{0,n_p},\zeta_{n_p}) + \frac{n_pJ(\zeta_{n_p})}{(\zeta_{n_p}+\sin\alpha_{0,n_p})^{1/2}}, \quad (7.78)$$

$$\hat{t} = \psi(\alpha_{0,n_p},\zeta_{n_p}) + \frac{n_pI(\zeta_{n_p})}{(\zeta_{n_p}+\sin\alpha_{0,n_p})^{3/2}}, \quad (7.79)$$

$p=1,2,\ldots$. We may assume that

$$\lim_{p\to\infty}\alpha_{0,n_p}=\bar{\alpha}_0, \qquad \lim_{p\to\infty}\zeta_{n_p}=\bar{\zeta}, \qquad \bar{\zeta}\in[1,\infty]. \quad (7.80)$$

a) $1<\bar{\zeta}<\infty$. In this case $\varphi(\bar{\alpha}_0,\bar{\zeta})$, $\psi(\bar{\alpha}_0,\bar{\zeta})$, $J(\bar{\zeta})$ and $I(\bar{\zeta})$ are all finite with $I(\bar{\zeta})>0$. Consequently the right hand side of (7.79) grows without limit as $p\to\infty$, which contradicts the finiteness of $\hat{t}$, and a) cannot occur.

b) If $\bar{\zeta}=1$, then (7.78) yields

$$\frac{j(\alpha_{0,n_p},\zeta_{n_p})}{(\zeta_{n_p}+\sin\alpha_{0,n_p})^{1/2}} = \frac{-n_pJ(\zeta_{n_p})}{(\zeta_{n_p}+\sin\alpha_{0,n_p})^{1/2}}+\hat{x},$$

so for large p we have

$$j(\alpha_{0,n_p}, \zeta_{n_p}) \sim -n_p J(\zeta_{n_p}) \;\longrightarrow\; \infty, \tag{7.81}$$

since $J(1) = -\infty$. Since j is bounded by 0, (7.81) leads to a contradiction and $\bar{\zeta} = 1$ is not possible.

c) $\bar{\zeta} = \infty$. The proof follows the argument of the proof of Lemma 6.3(c). Expansions (6.5) and (6.6) are replaced by Lemmas 7.3 and 7.7. (6.7)–(6.10) still hold, and we conclude that if (x, y_0, t) is connected to $(0, y_0, 0)$ by an infinite number of geodesics represented by (7.78), (7.79) with $\lim_{p\to\infty} \zeta_{n_p} = \infty$, then (x, y_0, t) must be on the critical line $\hat{x} + \hat{t} = 0$, $\hat{x} < 0$. That this is impossible is a consequence of the argument following (6.10) in the proof of Lemma 6.3(c); here we get $\tan \bar{\alpha}_0 = -8/(E\pi) < 0$ which implies that $\bar{\alpha}_0 \in (-\pi/2, 0)$. □

Lemma 7.21. *Given a fixed integer q, it is not possible to connect (x, y_0, t) and $(0, y_0, 0)$ by an infinite number of distinct geodesics which can be represented in the form*

$$\hat{x} = \varphi_q(\alpha_0, \zeta) = \varphi(\alpha_0, \zeta) + \frac{qJ(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}}, \tag{7.82}$$

$$\hat{t} = \psi_q(\alpha_0, \zeta) = \psi(\alpha_0, \zeta) + \frac{qI(\zeta)}{(\zeta + \sin\alpha_0)^{3/2}}, \tag{7.83}$$

$\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta \in (1, \infty)$.

Proof of Lemma 7.21. We shall follow the argument of the proof of Lemma 6.4, but exchange the roles of $\hat{x}$ and $\hat{t}$. Lemma 7.4 implies that φ is an increasing function of ζ. So is the second term on the right hand side of (7.82). Indeed,

$$\begin{aligned}
&\frac{\partial}{\partial\zeta}\frac{J(\zeta)}{(\zeta + \sin\alpha_0)^{1/2}} \\
&= \frac{-1}{2(\zeta + \sin\alpha_0)^{3/2}}\left[(\zeta + \sin\alpha_0)\int_0^{2\pi}\frac{\sin\alpha d\alpha}{(\zeta + \sin\alpha)^{3/2}} + J(\zeta)\right] \\
&> 0.
\end{aligned} \tag{7.84}$$

Therefore $\varphi_q(\alpha_0, \zeta)$ is an increasing function of ζ, and if (7.82) has a solution $\zeta(\alpha_0) = \zeta(\alpha_0, \hat{x})$ then $\zeta(\alpha_0)$ is unique. By Lemma 7.13(iv) φ increases from $-\infty$ to 0 as ζ goes from 1 to ∞. This holds for $J(\zeta)$ too, and thus holds for $\varphi_q(\alpha_0, \zeta)$. Consequently for any $\alpha_0 \in (-\pi/2, \pi/2]$ and $\hat{x} \in (-\infty, 0)$, there is a unique $\zeta(\alpha_0)$ such that

$$\hat{x} = \varphi_q\big(\alpha_0, \zeta(\alpha_0)\big). \tag{7.85}$$

We shall show that

$$\hat{t} = \psi_q\big(\alpha_0, \zeta(\alpha_0)\big) \tag{7.86}$$

has at most a finite number of solutions α_0. This will prove Lemma 7.21 and completes the proof of Proposition 7.19. To prove that (7.86) cannot have infinitely many solutions we argue by contradiction. Suppose (7.86) does have infinitely many solutions α_0. Since $\psi_q\big(\alpha_0, \zeta(\alpha_0)\big)$ is analytic in α_0, the infinite set of solutions of (7.86) must have as a limit point at least one of the endpoints of the interval $[-\pi/2, \pi/2]$. We shall show that this cannot happen by proving that the α_0-derivative of $\psi_q\big(\alpha_0, \zeta(\alpha_0)\big)$ does not vanish at $\alpha_0 = \pm\pi/2$. To this end we note that $\zeta(-\pi/2)$ is the unique solution of

$$\frac{qJ\big(\zeta(-\pi/2)\big)}{\sqrt{\zeta(-\pi/2) - 1}} = \hat{x}, \qquad \hat{x} \in (-\infty, 0), \tag{7.87}$$

and $\zeta(\pi/2)$ is the unique solution of

$$\varphi\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big) + \frac{qJ\big(\zeta(\pi/2)\big)}{\sqrt{\zeta(\pi/2) + 1}} = \hat{x}, \tag{7.88}$$

so

$$\zeta_\pm = \zeta\Big(\pm\frac{\pi}{2}\Big) \in (1, \infty). \tag{7.89}$$

(i) $\alpha_0 = -\pi/2$. To find $\zeta'(-\pi/2)$ we differentiate (7.85).

$$\begin{aligned} 0 &= \frac{\partial\varphi}{\partial\alpha_0}\Big|_{\alpha_0=-\pi/2} + \frac{\partial\varphi}{\partial\zeta}\Big|_{\alpha_0=-\pi/2} \cdot \zeta'\Big(-\frac{\pi}{2}\Big) \\ &\quad + \frac{\partial}{\partial\zeta}\Big(\frac{qJ(\zeta)}{(\zeta-1)^{1/2}}\Big)\Big|_{\zeta=\zeta_-} \cdot \zeta'\Big(-\frac{\pi}{2}\Big) \\ &= \frac{-2}{\zeta_- - 1} - \frac{q\zeta'(-\pi/2)}{2(\zeta_- - 1)^{3/2}}\left[(\zeta_- - 1)\int_0^{2\pi} \frac{\sin\alpha \, d\alpha}{(\zeta_- + \sin\alpha)^{3/2}} + J(\zeta_-)\right]. \end{aligned}$$

The content of the square bracket is negative, so

$$\zeta'\Big(-\frac{\pi}{2}\Big) > 0. \tag{7.90}$$

Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\psi_q\big(\alpha_0,\zeta(\alpha_0)\big)\Big|_{\alpha_0=-\pi/2}\\
&= \frac{\partial\psi}{\partial\alpha_0}\Big|_{\alpha_0=-\pi/2} + \frac{\partial\psi}{\partial\zeta}\Big|_{\alpha_0=-\pi/2}\cdot\zeta'\Big(-\frac{\pi}{2}\Big) + \frac{\partial}{\partial\zeta}\Big(\frac{qI(\zeta)}{(\zeta-1)^{3/2}}\Big)\Big|_{\zeta=\zeta_-}\cdot\zeta'\Big(-\frac{\pi}{2}\Big)\\
&= \frac{-2}{\zeta_- - 1} + \Big[-\frac{3qI(\zeta_-)}{2(\zeta_- - 1)^{5/2}} + \frac{J(\zeta_-)}{2(\zeta_- - 1)^{3/2}}\Big]\zeta'\Big(-\frac{\pi}{2}\Big)\\
&< 0,
\end{aligned}$$

since the content of the square bracket is negative.

(ii) $\alpha_0 = \pi/2$. Again,

$$0 = \frac{\partial\varphi}{\partial\alpha_0}\Big|_{\alpha_0=\pi/2} + \Big[\frac{\partial\varphi}{\partial\zeta}\Big|_{\alpha_0=\pi/2} + \frac{\partial}{\partial\zeta}\Big(\frac{qJ(\zeta)}{(\zeta+1)^{1/2}}\Big)\Big|_{\zeta=\zeta_+}\Big]\zeta'\Big(\frac{\pi}{2}\Big).$$

$\partial\varphi/\partial\zeta > 0$ at $\alpha_0 = \pi/2$, see Lemma 7.7, and so is the second term in the square bracket according to (7.84). Also,

$$\frac{\partial\varphi}{\partial\alpha_0}\Big|_{\alpha_0=\pi/2} = \frac{2}{\zeta_+ + 1} > 0,$$

so we have

$$\zeta'\Big(\frac{\pi}{2}\Big) < 0. \tag{7.91}$$

Consequently,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\alpha_0}\psi_q\big(\alpha_0,\zeta(\alpha_0)\big)\Big|_{\alpha_0=\pi/2}\\
&= \frac{\partial\psi}{\partial\alpha_0}\Big|_{\alpha_0=\pi/2,\zeta=\zeta_+} + \frac{\partial\psi}{\partial\zeta}\Big|_{\alpha_0=\pi/2,\zeta=\zeta_+}\cdot\zeta'\Big(\frac{\pi}{2}\Big) + \frac{\partial}{\partial\zeta}\Big(\frac{qI(\zeta)}{(\zeta+1)^{3/2}}\Big)\Big|_{\zeta=\zeta_+}\cdot\zeta'\Big(\frac{\pi}{2}\Big)\\
&= \frac{2}{\zeta_+ + 1} + \frac{1}{2(\zeta_+ + 1)}\Big[\frac{-3I(\zeta_+)}{(\zeta_+ + 1)^{3/2}} + \frac{J(\zeta_+)}{(\zeta_+ + 1)^{1/2}}\Big]\zeta'\Big(\frac{\pi}{2}\Big)\\
&\quad + q\Big[\frac{J(\zeta_+)}{2(\zeta_+ + 1)^{3/2}} - \frac{3I(\zeta_+)}{2(\zeta_+ + 1)^{5/2}}\Big]\zeta'\Big(\frac{\pi}{2}\Big) > 0,
\end{aligned}$$

since both square brackets are negative. This proves Lemma 7.21, and we have completed the proof of Proposition 7.19. □

8. $\zeta \in (-1,1)$, $\dot{y}(0) > 0$

We recall the behaviour of a geodesic when $\zeta = (-1,1)$; for details the reader may consult chapter 2. For fixed ζ and $\tau > 0$, the y-component of the motion is given by formula (2.62),

$$y(\alpha) = \operatorname{sgn}\big(y(\alpha)\big)\frac{(\zeta+\sin\alpha)^{1/2}}{\tau^{1/2}}.$$

α starts at $\alpha_0 \in [-\pi/2, 3\pi/2]$, which is uniquely determined by the following requirements:

$$\begin{aligned} y_0 = y(0) &= \frac{(\zeta+\sin\alpha_0)^{1/2}}{\tau^{1/2}}, &&\text{see (2.61)},\\ \dot{y}(0) = \eta(0) &= \cos\alpha_0, &&\text{see (2.21)}. \end{aligned} \tag{8.1}$$

The x and t components of the geodesic curve are given by formulas (2.63) and (2.64),

$$\begin{aligned} x(\alpha) &= \frac{1}{2\tau^{1/2}}\int_{\alpha_0}^{\alpha}\operatorname{sgn}\big(y(\alpha')\big)\frac{\sin\alpha' d\alpha'}{(\zeta+\sin\alpha')^{1/2}},\\ t(\alpha) &= \frac{1}{2\tau^{3/2}}\int_{\alpha_0}^{\alpha}\operatorname{sgn}\big(y(\alpha')\big)\sin\alpha'(\zeta+\sin\alpha')^{1/2}d\alpha'. \end{aligned}$$

The behaviour of the curve represented by (2.62)−(2.64) is described after formula (2.48). We recall some of the salient points. To begin with, (2.62) restricts α to the interval

$$A(\zeta) \le \alpha \le \pi - A(\zeta), \tag{8.2}$$

with

$$A(\zeta) = \operatorname{Arcsin}(-\zeta) = \sin^{-1}(-\zeta) \in \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big]. \tag{8.3}$$

Then (2.38) implies that α increases when $y(\alpha) > 0$, and decreases when $y(\alpha) < 0$. In particular, the geodesic crosses the $y = 0$ plane everytime $\alpha = A(\zeta)$ or $\alpha = \pi - A(\zeta)$, at which time α changes direction. The maxima and minima of the y-component of the motion occur at $\alpha = \pi/2$, with

$$y\Big(\frac{\pi}{2}\Big) = \pm\frac{\sqrt{1+\zeta}}{\sqrt{\tau}}. \tag{8.4}$$

We shall classify the geodesics which leave $(0, y_0, 0)$ in the increasing y-direction, $\dot{y}(0) > 0$, and return to the $y = y_0$ plane according to the direction in which they pierce the $y = y_0$ plane. The ones that arrive from the left always arrive at $\alpha = \alpha_0$ after a finite number $n = 1, 2, \ldots$ of periods. The ones that arrive from the right always arrive at $\alpha = \pi - \alpha_0$ after a finite number $n = 0, 1, 2, \ldots$ of periods. In short, we have

$$\alpha = \alpha_0, \qquad n = 1, 2, \ldots, \tag{8.5}$$

$$\alpha = \pi - \alpha_0, \qquad n = 0, 1, 2, \ldots \tag{8.6}$$

All the geodesics (8.5) and (8.6) with $n = 1, 2, \ldots$ are nonlocal, that is, they cross the $y = 0$ plane. So are the geodesics whose y-component starts as a decreasing function, $\dot{y}(0) < 0$, or, equivalently $\alpha_0 > \pi/2$. Thus the last batch of local geodesics are given by (8.6) when $n = 0$ and $\alpha_0 \le \pi/2$. We shall show that this family of geodesics fill the missing domain in the half plane $\hat{t} > \hat{x}$, which we shall denote by R_4.

The case $\alpha = \pi - \alpha_0$, $\alpha_0 \le \pi/2$.

Start with $n = 0$, and

$$A(\zeta) \le \alpha_0 \le \frac{\pi}{2}, \tag{8.7}$$

so $\dot{y}(0) > 0$, at least when $\alpha_0 < \pi/2$. $y(\alpha)$ starts at $y_0 = y(\alpha_0)$, increases, reaches a maximum at $\alpha = \pi/2$, turns back and reaches the $y = y_0$ plane at $\alpha = \pi - \alpha_0$ with negative velocity $\dot{y}(\pi - \alpha_0) = -\dot{y}(0)$, see (8.1). We rewrite (2.63) and (2.64),

$$\hat{x} = \frac{1}{(\zeta + \sin\alpha_0)^{1/2}} \int_{\alpha_0}^{\pi-\alpha_0} \frac{\sin\alpha d\alpha}{(\zeta + \sin\alpha)^{1/2}} = \Phi(\alpha_0, \zeta), \tag{8.8}$$

$$\hat{t} = \frac{1}{(\zeta + \sin\alpha_0)^{3/2}} \int_{\alpha_0}^{\pi-\alpha_0} (\zeta + \sin\alpha)^{1/2} \sin\alpha d\alpha = \Psi(\alpha_0, \zeta), \tag{8.9}$$

with

$$\Phi(\alpha_0,\zeta) =\frac{J(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}}, \tag{8.10}$$

$$\Psi(\alpha_0,\zeta) =\frac{I(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{3/2}}. \tag{8.11}$$

Adding a period means that α continues to increase to $\pi - A(\zeta)$, then it decreases all the way to $A(\zeta)$, where it turns around and increases to $\pi-\alpha_0$. This adds

$$\frac{2}{(\zeta+\sin\alpha_0)^{1/2}}\int_{A(\zeta)}^{\pi-A(\zeta)}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha_0)^{1/2}}, \tag{8.12}$$

to $\hat{x}$ and an analogous quantity to $\hat{t}$. After n periods one has

$$\hat{x} = \Phi(\alpha_0,\zeta) + \frac{2n}{(\zeta+\sin\alpha_0)^{1/2}}\int_{A(\zeta)}^{\pi-A(\zeta)}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}, \tag{8.13}$$

$$\hat{t} = \Psi(\alpha_0,\zeta) + \frac{2n}{(\zeta+\sin\alpha_0)^{3/2}}\int_{A(\zeta)}^{\pi-A(\zeta)}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha. \tag{8.14}$$

We note that these geodesics always strike the $y = y_0$ plane from the right, that is with negative velocity. The ones coming from the left have the form (8.5). The functions Φ, Ψ, J and I agree with the same named functions of chapter 5, at least formally. Their domain is different here, $\zeta \in (-1,1)$ and $\alpha_0 \in \big[A(\zeta),\pi/2\big]$, so we need to study their behaviour again.

Whatever other conditions we may set on α_0 in chapter 8, they are always in addition to the sometime unstated condition

$$A(\zeta) \le \alpha_0 \le \frac{\pi}{2}. \tag{8.15}$$

The functions J and Φ, and I and Ψ.

Lemma 8.1. *$J\big(A(\zeta),\zeta\big)$ is a decreasing function of ζ, $\zeta \in (-1,1)$, with a unique zero $\tilde{\zeta}$ which is in the interval* $(0,1)$,

$$J\big(A(\tilde{\zeta}),\tilde{\zeta}\big) = 0. \tag{8.16}$$

Also,

$$\lim_{\zeta\to 1} J\big(A(\zeta),\zeta\big) = -\infty, \tag{8.17}$$

$$\lim_{\zeta\to -1} J\big(A(\zeta),\zeta\big) = \sqrt{2}\pi. \tag{8.18}$$

Proof. We start with

$$J\big(A(\zeta),\zeta)\big) = 2\int_{A(\zeta)}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}, \tag{8.19}$$

and note that $\zeta > 0 \Rightarrow A(\zeta) < 0$ and $\zeta < 0 \Rightarrow A(\zeta) > 0$. Consequently (8.19) shows that

$$\zeta \le 0 \Rightarrow J\big(A(\zeta),\zeta\big) > 0. \tag{8.20}$$

When $\zeta > 0$ we integrate the integral in (8.19) by parts and obtain

$$J\big(A(\zeta),\zeta\big) = 2\int_0^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} - 4\int_{A(\zeta)}^0 (\zeta+\sin\alpha)^{1/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha}. \tag{8.21}$$

Differentiating (8.21) with respect to ζ, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\zeta}J\big(A(\zeta),\zeta\big) &= -\int_0^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} - 2\int_{A(\zeta)}^0 \frac{1}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}\\ &< 0,\end{aligned} \tag{8.22}$$

where we used $A(\zeta) = \operatorname{Arcsin}(-\zeta)$ to drop the nonintegrated term. Furthermore, $\zeta \to 1 \Rightarrow A(\zeta) \to -\pi/2$, and therefore (8.17) is equivalent to $J(-\pi/2, 1) = -\infty$ which can be found in Lemma 5.1. In view of (8.17), (8.20) and (8.22), we have established the existence of a unique zero $\tilde{\zeta}$ of $J\big(A(\zeta),\zeta\big)$ in $(0,1)$ which is the only zero of $J\big(A(\zeta),\zeta\big)$ in $(-1,1)$ courtesy of (8.20). When $\zeta < 0$, we cannot use (8.21) to study $J(\alpha_0,\zeta)$. Instead we return to (8.19) and change the variable of integration to $x = \sin\alpha$:

$$J\big(A(\zeta),\zeta\big) = 2\int_{-\zeta}^1 \frac{1}{\sqrt{\zeta+x}}\frac{x\mathrm{d}x}{\sqrt{1-x^2}} = 2\int_\gamma^1 \frac{1}{\sqrt{x-\gamma}}\frac{x\mathrm{d}x}{\sqrt{1-x^2}}, \tag{8.23}$$

$-\zeta = \gamma > 0$. When $\zeta \sim -1$, we have $\gamma \sim 1$, and

$$\begin{aligned}
\lim_{\zeta\to -1} J\big(A(\zeta),\zeta\big) &= \lim_{\gamma\to 1} 2\int_\gamma^1 \frac{1}{\sqrt{(x-\gamma)(1-x)}}\frac{x\mathrm{d}x}{\sqrt{1+x}} \\
&= \lim_{\gamma\to 1}\sqrt{2}\int_\gamma^1 \frac{\mathrm{d}x}{\sqrt{(x-\gamma)(1-x)}} \\
&= \lim_{\gamma\to 1}\sqrt{2}\int_\gamma^1 \frac{\mathrm{d}x}{\sqrt{\left(\frac{1-\gamma}{2}\right)^2-\left(x-\frac{1+\gamma}{2}\right)^2}} \\
&= \lim_{\gamma\to 1}\sqrt{2}\int_{-\frac{1-\gamma}{2}}^{\frac{1-\gamma}{2}} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1-\gamma}{2}\right)^2-t^2}}, \qquad t = x-\frac{1+\gamma}{2} \\
&= \sqrt{2}\int_{-1}^1 \frac{\mathrm{d}s}{\sqrt{1-s^2}}, \qquad t=\frac{1-\gamma}{2}s \\
&= \sqrt{2}\pi,
\end{aligned} \tag{8.24}$$

and this proves (8.18). This calculation yields more. Namely, if we do not simplify (8.23) for the sake of finding the limit we easily obtain

$$\begin{aligned}
J\big(A(\zeta),\zeta\big) &= \int_{-1}^1 \frac{2\mathrm{d}s}{\sqrt{1-s^2}}\frac{\frac{1-\gamma}{2}s+\frac{1+\gamma}{2}}{\sqrt{1+\frac{1-\gamma}{2}s+\frac{1+\gamma}{2}}} \\
&= \int_{-1}^1 \frac{2\mathrm{d}s}{\sqrt{1-s^2}}\left[\sqrt{2-(1-s)\frac{1-\gamma}{2}}-\frac{1}{\sqrt{2-(1-s)\frac{1-\gamma}{2}}}\right].
\end{aligned} \tag{8.25}$$

The integrand is an increasing function of γ, hence a decreasing function of $\zeta \in (-1,0)$; as an extra bonus, we note that this argument works for $\zeta \in (-1,1)$. Thus we have shown that $J\big(A(\zeta),\zeta\big)$ is a decreasing function of ζ, $\zeta \in (-1,1)$, and thus completed the proof of Lemma 8.1. □

Lemma 8.2. (i) $\zeta \in (-1,0)$. *We have*

$$J(\alpha_0,\zeta) > 0, \qquad \alpha_0 \in \Big[A(\zeta),\frac{\pi}{2}\Big),$$

and $J(\alpha_0,\zeta)$ *is a decreasing function of* α_0.

(ii) $\zeta \in [0, \tilde{\zeta})$. *Here*

$$J(\alpha_0, \zeta) > 0, \qquad \alpha_0 \in \Big[A(\zeta), \frac{\pi}{2}\Big),$$

and $J(\alpha_0, \zeta)$ *has a maximum at* $\alpha_0 = 0$.

(iii) *When* $\zeta \in [\tilde{\zeta}, 1]$, $J(\alpha_0, \zeta)$ *increases from* $J\big(A(\zeta), \zeta\big) \le 0$ *to its maximum* $J(0, \zeta) > 0$, *then decreases to* $J(\pi/2, \zeta) = 0$, *has a unique zero* $\tilde{\alpha}_0(\zeta) \in \big[A(\zeta), 0\big)$ *with* $\tilde{\alpha}_0(\tilde{\zeta}) = A(\tilde{\zeta})$.

(iv) $J(\pi/2, \zeta) = 0$, $\zeta \in (-1, 1]$.

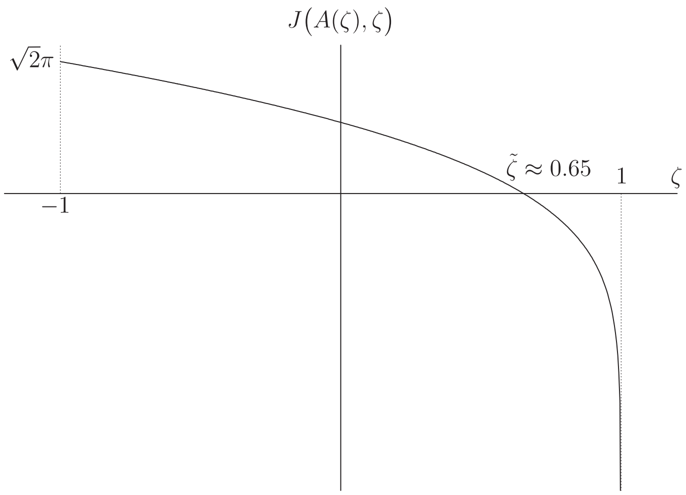

Figure 9.

Proof. We note that

$$\frac{\partial J}{\partial \alpha_0} = -\frac{2 \sin \alpha_0}{(\zeta + \sin \alpha_0)^{1/2}}. \tag{8.26}$$

(i) $\zeta \in (-1, 0)$. Here $0 < A(\zeta) \le \alpha_0$, so $\partial J/\partial \alpha_0 < 0$, and J is a decreasing function of α_0 with $J(\pi/2, \zeta) = 0$. This proves (i).

(ii) $\zeta \in (0, \tilde{\zeta})$. For each fixed ζ, $J(\alpha_0, \zeta)$ increases from $J\big(A(\zeta), \zeta\big) > 0$ to $J(0, \zeta)$, and then decreases to $J(\pi/2, \zeta) = 0$ which implies (ii).

(iii) is a consequence of Lemma 8.1 and (iv) is obvious, and this proves Lemma 8.2. □

We note that

$$\frac{\partial\Phi}{\partial\alpha_0} = -\frac{\cos\alpha_0}{2(\zeta+\sin\alpha_0)}\big(\Phi(\alpha_0,\zeta)+4\tan\alpha_0\big). \tag{8.27}$$

Lemma 8.3. (i) $\zeta\in(-1,\tilde{\zeta})$. *$\Phi(\alpha_0,\zeta)$ decreases from $\Phi\big(A(\zeta),\zeta\big)=\infty$ to $\Phi(\pi/2,\zeta)=0$. In particular,*

$$\Phi(\alpha_0,\zeta)>0,\quad \zeta\in(-1,\tilde{\zeta}),\quad \alpha_0\in\big(A(\zeta),\pi/2\big). \tag{8.28}$$

(ii) $\zeta\in(\tilde{\zeta},1)$. *Here $\partial\Phi/\partial\alpha_0$ has a unique zero $\alpha_0^*(\zeta)$ with*

$$\Phi\big(\alpha_0^*(\zeta),\zeta\big) = -4\tan\alpha_0^*(\zeta), \tag{8.29}$$

$$A(\zeta)<\tilde{\alpha}_0(\zeta)<\alpha_0^*(\zeta)<0. \tag{8.30}$$

$\Phi(\alpha_0,\zeta)$ increases from $\Phi\big(A(\zeta),\zeta\big)=-\infty$ to $\Phi\big(\alpha_0^*(\zeta),\zeta\big)>0$, then decreases to $\Phi(\pi/2,\zeta)=0$.

(iii) $\zeta=\tilde{\zeta}$. *$\Phi(\alpha_0,\tilde{\zeta})$ decreases from $\Phi\big(A(\tilde{\zeta}),\tilde{\zeta}\big)>0$ to $\Phi(\pi/2,\tilde{\zeta})=0$.*

Proof. (i) We start with $\zeta\in(-1,0]$. Here $A(\zeta)\geq 0$, so (8.27) gives

$$\frac{\partial\Phi(\alpha_0,\zeta)}{\partial\alpha_0}<0,\qquad \alpha_0\in\Big[A(\zeta),\frac{\pi}{2}\Big].$$

Since $J\big(A(\zeta),\zeta\big)>0$ and finite, and $\big(\zeta+\sin A(\zeta)\big)^{-1/2}=\infty$, we have (i) when $\zeta=(-1,0]$.

When $\zeta>0$, so that $A(\zeta)<0$, we may integrate by parts. Starting with

$$J(\alpha_0,\zeta)=2\int_0^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}+4\int_{\alpha_0}^{0}\tan\alpha d(\zeta+\sin\alpha)^{1/2}, \tag{8.31}$$

an integration by parts yields

$$\begin{aligned}&J(\alpha_0,\zeta)+4\tan(\alpha_0)(\zeta+\sin\alpha_0)^{1/2}\\&=2\int_0^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}-4\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{1/2}\frac{d\alpha}{\cos^2\alpha}.\end{aligned} \tag{8.32}$$

The function

$$\alpha_0 \longrightarrow -4\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{1/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha} \tag{8.33}$$

is an increasing function on the interval $\big[A(\zeta),\pi/2\big)$, so its minimum is attained at $\alpha_0 = A(\zeta)$. Consequently the right hand side of (8.32) is an increasing function of α_0 in the same interval with its minimum at $\alpha_0 = A(\zeta)$. We note that

$$2\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} - 4\int_{A(\zeta)}^{0}(\zeta+\sin\alpha)^{1/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha} = J\big(A(\zeta),\zeta\big). \tag{8.34}$$

When $\zeta < \tilde{\zeta}$, $J\big(A(\zeta),\zeta\big) > 0$, so we have

$$J(\alpha_0,\zeta) + 4(\tan\alpha_0)(\zeta+\sin\alpha_0)^{1/2} > 0,$$

and from (8.27)

$$\frac{\partial\Phi}{\partial\alpha_0} < 0, \qquad \alpha_0 \in \Big[A(\zeta),\frac{\pi}{2}\Big), \qquad \zeta\in(0,\tilde{\zeta}).$$

This completes the proof of (i). When $\zeta > \tilde{\zeta}$, $J\big(A(\zeta),\zeta\big) < 0$, and the increasing function $J(\alpha_0,\zeta) + 4(\tan\alpha_0)(\zeta+\sin\alpha_0)^{1/2} < 0$ when $\alpha_0 < \tilde{\alpha}_0(\zeta)$, and (8.32) implies that $J(\alpha_0,\zeta) + 4(\tan\alpha_0)(\zeta+\sin\alpha_0)^{1/2} > 0$ when $\alpha_0 \geq 0$. Consequently, $J(\alpha_0,\zeta) + 4(\tan\alpha_0)(\zeta+\sin\alpha_0)^{1/2}$ has a unique zero $\alpha_0^*(\zeta) \in \big(\tilde{\alpha}_0(\zeta),0\big)$, and

$$\frac{\partial\Phi}{\partial\alpha_0} > 0, \qquad \alpha_0 \in \big[A(\zeta),\alpha_0^*(\zeta)\big),$$
$$\frac{\partial\Phi}{\partial\alpha_0} < 0, \qquad \alpha_0 \in \Big(\alpha_0^*(\zeta),\frac{\pi}{2}\Big].$$

Therefore, when $\zeta\in(\tilde{\zeta},1)$, $\Phi(\alpha_0,\zeta)$ increases from $\Phi\big(A(\zeta),\zeta\big) = -\infty$ to its maximum value $\Phi\big(\alpha_0^*(\zeta),\zeta\big) > 0$, and then decreases to $\Phi(\pi/2,\zeta) = 0$. This gives (ii).

(iii) $\zeta = \tilde{\zeta}$. According to Lemma 8.2(iii), $J(\alpha_0,\tilde{\zeta}) > 0$ for $\alpha_0 \in \big(A(\tilde{\zeta}),\pi/2\big)$. Therefore so is $\Phi(\alpha_0,\zeta)$. Also,

$$\lim_{\alpha_0\to A(\tilde{\zeta})} J(\alpha_0,\tilde{\zeta}) = 0.$$

We need more accurate information about this vanishing. From (8.26)

$$\frac{\partial J(\alpha_0,\tilde{\zeta})}{\partial \alpha_0} = -\frac{2\sin\alpha_0}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}}. \tag{8.35}$$

When $\alpha_0 \to A(\zeta)$,

$$\begin{aligned}\sin\alpha_0 &= \sin\big(\alpha_0 - A(\zeta) + A(\zeta)\big)\\ &= \sin\big(\alpha_0 - A(\zeta)\big)\cos A(\zeta) + \cos\big(\alpha_0 - A(\zeta)\big)\sin A(\zeta)\\ &= \big(\alpha_0 - A(\zeta)\big)\cos A(\zeta) + \sin A(\zeta) + O\big([\alpha_0 - A(\zeta)]^2\big),\end{aligned}$$

and therefore one has

$$\begin{aligned}\zeta + \sin\alpha_0 &= \sin\alpha_0 - \sin A(\zeta)\\ &= \big(\alpha_0 - A(\zeta)\big)\cos A(\zeta) + O\big([\alpha_0 - A(\zeta)]^2\big).\end{aligned} \tag{8.36}$$

We may rewrite (8.35) in the following form,

$$\frac{\partial J(\alpha_0,\tilde{\zeta})}{\partial \alpha_0} = \frac{-2\sin A(\tilde{\zeta}) + O\big(\alpha_0 - A(\tilde{\zeta})\big)}{\sqrt{\cos A(\tilde{\zeta})}\big(\alpha_0 - A(\tilde{\zeta})\big)^{1/2}},$$

and integrating one obtains

$$\begin{aligned}J(\alpha_0,\tilde{\zeta}) &= \frac{-4\sin A(\tilde{\zeta})}{\sqrt{\cos A(\tilde{\zeta})}}\big(\alpha_0 - A(\tilde{\zeta})\big)^{1/2} + O\big((\alpha_0 - A(\tilde{\zeta}))^{3/2}\big)\\ &= -4\big(\tan A(\tilde{\zeta})\big)(\tilde{\zeta}+\sin\alpha_0)^{1/2} + O\big((\tilde{\zeta}+\sin\alpha_0)^{3/2}\big).\end{aligned}$$

This yields

$$\lim_{\alpha_0\to A(\tilde{\zeta})} \Phi(\alpha_0,\tilde{\zeta}) = -4\tan A(\tilde{\zeta}) > 0.$$

Finally, (8.32) and (8.34) show that

$$J(\alpha_0,\tilde{\zeta}) + 4(\tan\alpha_0)(\tilde{\zeta}+\sin\alpha_0)^{1/2} \begin{cases} = 0, & \alpha_0 = A(\tilde{\zeta}),\\ > 0, & \alpha_0 > A(\tilde{\zeta}),\end{cases}$$

and therefore

$$\frac{\partial \Phi(\alpha_0,\tilde{\zeta})}{\partial \alpha_0} < 0, \qquad \alpha_0 \in \Big(A(\tilde{\zeta}), \frac{\pi}{2}\Big).$$

This proves (iii), and we have completed the proof of Lemma 8.3. □

One has

$$\frac{\partial\Phi}{\partial\zeta} = \frac{-1}{(\zeta+\sin\alpha_0)^{1/2}}\left[\frac{\Phi}{2(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}\right]. \quad (8.37)$$

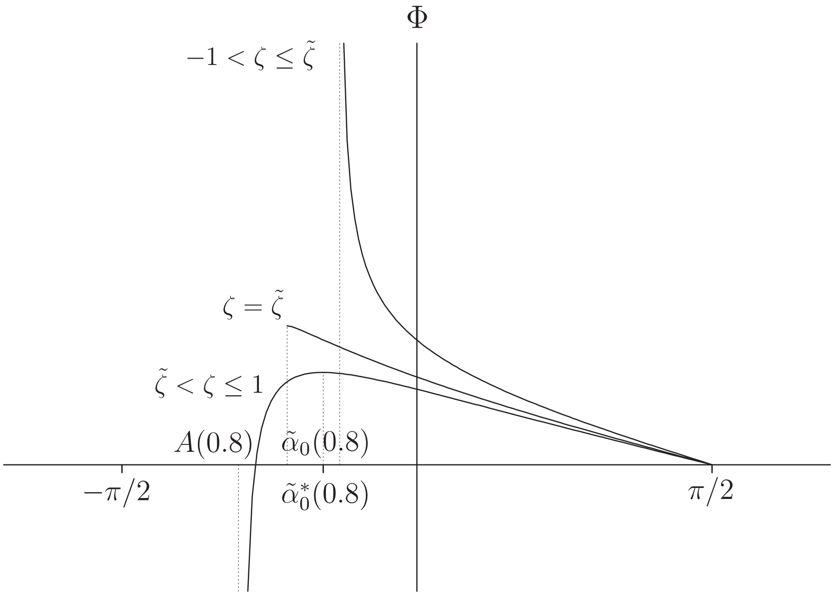

Figure 10.

Lemma 8.4. (i) $\zeta \in (-1, \tilde{\zeta}]$. *Then*

$$\frac{\partial\Phi(\alpha_0,\zeta)}{\partial\zeta} < 0, \qquad \alpha_0 \in \Big(A(\zeta), \frac{\pi}{2}\Big), \quad (8.38)$$

and

$$\lim_{\alpha_0\to A(\zeta)} \frac{\partial\Phi(\alpha_0,\zeta)}{\partial\zeta} = -\infty. \quad (8.39)$$

(ii) $\zeta \in (\tilde{\zeta}, 1)$. *In this case $\partial\Phi/\partial\zeta$, as a function of α_0, has a unique zero which is in $\big(A(\zeta), \alpha_0^*(\zeta)\big)$. Also*

$$\lim_{\alpha_0\to A(\zeta)} \frac{\partial\Phi(\alpha_0,\zeta)}{\partial\zeta} = \infty. \quad (8.40)$$

Proof. First we shall derive the limits (8.39) and (8.40). To this end we need to consider the individual terms in the square bracket of formula (8.37). We start with

$$\begin{aligned} J(\alpha_0,\zeta) &= 2\int_{A(\zeta)}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} + 2\int_{\alpha_0}^{A(\zeta)} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} \\ &= J\big(A(\zeta),\zeta\big) + O\big(\sqrt{\alpha_0 - A(\zeta)}\big), \qquad \alpha_0 \sim A(\zeta). \end{aligned}$$

For the second term in the square bracket we choose a small ε_0, fix it, and let $\alpha_0 \in \big(A(\zeta), A(\zeta)+\varepsilon_0\big)$. Then

$$\begin{aligned} &\int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} \\ &= \int_{\alpha_0}^{\pi/2} \frac{\sin A(\zeta) \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} + \int_{\alpha_0}^{\pi/2} \frac{\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}} \\ &= \frac{\sin A(\zeta)}{\cos^{3/2} A(\zeta)} \int_{\alpha_0}^{A(\zeta)+\varepsilon_0} \left[\frac{1}{\big(\alpha - A(\zeta)\big)^{3/2}} + O\Big(\frac{1}{\big(\alpha - A(\zeta)\big)^{1/2}}\Big)\right] \mathrm{d}\alpha + O(1) \\ &= \frac{\sin A(\zeta)}{\cos^{3/2} A(\zeta)} (-2) \frac{1}{\big(\alpha - A(\zeta)\big)^{1/2}}\Bigg|_{\alpha=\alpha_0}^{\alpha=A(\zeta)+\varepsilon_0} + O(1) \\ &= \frac{2\sin A(\zeta)}{\cos^{3/2} A(\zeta)} \frac{1}{(\alpha_0 - A(\zeta)^{1/2}} + O(1), \end{aligned}$$

and the square bracket in (8.37) has the following behaviour when $\alpha_0 \sim A(\zeta)$, $\zeta \neq \tilde{\zeta}$:

$$\begin{aligned} &\frac{1}{2}\frac{\Phi(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}} \\ &= \frac{J\big(A(\zeta),\zeta\big)}{2\cos\big(A(\zeta)\big)\big(\alpha_0 - A(\zeta)\big)} + O\Big(\frac{1}{\sqrt{\alpha_0 - A(\zeta)}}\Big), \end{aligned} \tag{8.41}$$

hence

$$\lim_{\alpha_0 \to A(\zeta)} \left[\frac{1}{2}\frac{\Phi(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\right] = \begin{cases} \infty, & \zeta < \tilde{\zeta}, \\ -\infty, & \tilde{\zeta} < \zeta. \end{cases} \tag{8.42}$$

When $\zeta = \tilde{\zeta}$ we must make use of the vanishing of $J\big(A(\tilde{\zeta}),\tilde{\zeta}\big)$. To this end we note that $A(\tilde{\zeta}) < 0$, and we can integrate by parts.

$$\begin{aligned}
&\int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} \\
&= \Big(\int_0^{\pi/2} + \int_{\alpha_0}^{0}\Big) \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} \\
&= \int_0^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} - 2\int_{\alpha_0}^{0} \tan\alpha \mathrm{d}(\tilde{\zeta}+\sin\alpha)^{-1/2} \\
&= \int_0^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} + \frac{2\tan\alpha_0}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}} + 2\int_{\alpha_0}^{0} \frac{1}{(\tilde{\zeta}+\sin\alpha)^{1/2}} \frac{\mathrm{d}\alpha}{\cos^2\alpha}.
\end{aligned}$$

For the first term in the square bracket we use $J\big(A(\tilde{\zeta}),\tilde{\zeta}\big) = 0$ and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\Phi(\alpha_0,\tilde{\zeta})}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}} \\
&= \frac{1}{\tilde{\zeta}+\sin\alpha_0} \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}} \\
&= -\frac{1}{\tilde{\zeta}+\sin\alpha_0} \int_{A(\tilde{\zeta})}^{\alpha_0} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}} \\
&= -\frac{1}{\tilde{\zeta}+\sin\alpha_0} \int_{A(\tilde{\zeta})}^{\alpha_0} \left[\frac{\sin A(\tilde{\zeta})}{(\tilde{\zeta}+\sin\alpha)^{1/2}} + (\tilde{\zeta}+\sin\alpha)^{1/2}\right] \mathrm{d}\alpha \\
&= -\frac{\sin A(\tilde{\zeta})}{\tilde{\zeta}+\sin\alpha_0} \int_{A(\tilde{\zeta})}^{\alpha_0} \frac{\mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{1/2}} + O\big(\sqrt{\alpha_0 - A(\tilde{\zeta})}\big) \\
&= -\frac{\sin A(\tilde{\zeta})}{\cos^{3/2}\big(A(\tilde{\zeta})\big)\big(\alpha_0 - A(\tilde{\zeta})\big)} \int_{A(\tilde{\zeta})}^{\alpha_0} \frac{\mathrm{d}\alpha}{\big(\alpha - A(\tilde{\zeta})\big)^{1/2}} + O\big(\sqrt{\alpha_0 - A(\tilde{\zeta})}\big) \\
&= \frac{-2\tan A(\tilde{\zeta})}{\sqrt{\cos A(\tilde{\zeta})}\big(\alpha_0 - A(\tilde{\zeta})\big)^{1/2}} + O\big(\sqrt{\alpha_0 - A(\tilde{\zeta})}\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{\Phi(\alpha_0,\tilde{\zeta})}{(\tilde{\zeta}+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} \\
&= \int_0^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\tilde{\zeta}+\sin\alpha)^{3/2}} + \int_{\alpha_0}^{0} \frac{2}{(\tilde{\zeta}+\sin\alpha)^{1/2}} \frac{\mathrm{d}\alpha}{\cos^2\alpha} + O\big(\sqrt{\alpha_0 - A(\tilde{\zeta})}\big)
\end{aligned}$$

$$\xrightarrow{\alpha_0\to A(\tilde\zeta)} \int_0^{\pi/2} \frac{\sin\alpha d\alpha}{(\tilde\zeta+\sin\alpha)^{3/2}} + 2\int_{A(\tilde\zeta)}^{0} \frac{1}{(\tilde\zeta+\sin\alpha)^{1/2}}\frac{d\alpha}{\cos^2\alpha}$$
$$> 0, \tag{8.43}$$

and

$$\lim_{\alpha_0\to A(\tilde\zeta)} \frac{\partial\Phi}{\partial\zeta}(\alpha_0,\tilde\zeta) = -\infty.$$

As for (8.38) we note that

$$\frac{\partial}{\partial\alpha_0}\left[\frac{1}{2}\frac{\Phi(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}\right]$$
$$= \frac{1}{(\zeta+\sin\alpha_0)^{1/2}}\frac{\partial\Phi}{\partial\alpha_0}. \tag{8.44}$$

According to Lemma 8.3(i) and (iii), $\partial\Phi/\partial\alpha_0 < 0$ for $\zeta \in (-1,\tilde\zeta]$, $\alpha_0 \in \big(A(\zeta),\pi/2\big)$. Consequently, (8.44) implies that

$$\frac{1}{2}\frac{\Phi(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}} \tag{8.45}$$

is a decreasing function of α_0. It vanishes at $\alpha_0 = \pi/2$, so it is positive for $\alpha_0 \in \big(A(\zeta),\pi/2\big)$, and this, together with (8.37) imply (8.38).

(ii) $\zeta \in (\tilde\zeta, 1)$. According to (8.42), (8.45) is $-\infty$ at $\alpha_0 = A(\zeta)$, and it is clearly 0 at $\alpha_0 = \pi/2$. (8.44) and Lemma 8.3(ii) show that

$$\frac{1}{2}\frac{\Phi(\alpha_0,\zeta)}{(\zeta+\sin\alpha_0)^{1/2}} + \int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}$$

increases from $-\infty$ at $\alpha_0 = A(\zeta)$ until α_0 reaches $\alpha_0^*(\zeta)$ where it is positive because after $\alpha_0^*(\zeta)$ it decreases to 0 at $\alpha_0 = \pi/2$. In particular,

$$\frac{\partial\Phi}{\partial\zeta}\big(A(\zeta),\zeta\big) = \infty, \quad \text{and} \quad \frac{\partial\Phi}{\partial\zeta}\big(\alpha_0^*(\zeta),\zeta\big) < 0,$$

see (8.37). Therefore $\partial\Phi/\partial\zeta$, as a function of α_0, has a unique zero in $(A(\zeta), \alpha_0^*(\zeta))$ which yields (ii), and this concludes the proof of Lemma 8.4. □

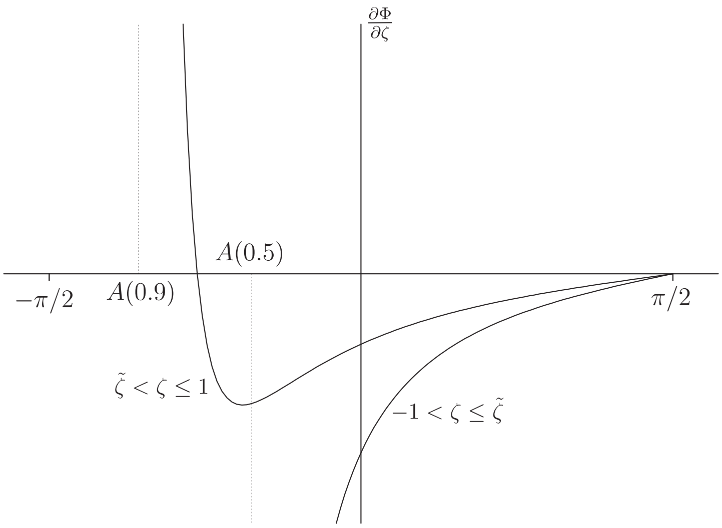

Figure 11.

Lemma 8.5. *For all $\zeta \in (-1, 1)$ and $\alpha_0 \in [A(\zeta), \pi/2]$ one has*

$$I(\alpha_0, \zeta) > 0. \tag{8.46}$$

In particular, (i) *when $\zeta \in (-1, 0]$, $I(\alpha_0, \zeta)$ is a decreasing function of $\alpha_0 \in [A(\zeta), \pi/2]$.*

(ii) *When $\zeta \in (0, 1)$, $I(\alpha_0, \zeta)$ increases from $I\big(A(\zeta), \zeta\big)$ to $I(0, \zeta)$, then decreases to $I(\pi/2, \zeta) = 0$.*

At $\alpha_0 = A(\zeta)$ we have

$$\frac{\partial I}{\partial \alpha_0}\big(A(\zeta), \zeta\big) = 0. \tag{8.47}$$

Proof. (8.47) is an immediate consequence of

$$\frac{\partial I(\alpha_0, \zeta)}{\partial \alpha_0} = -2\sin(\alpha_0)(\zeta + \sin\alpha_0)^{1/2}.$$

When $\zeta \leq 0$, and $A(\zeta) \geq 0$, one has

$$I\big(A(\zeta), \zeta\big) = 2\int_{A(\zeta)}^{\pi/2} (\zeta + \sin\alpha)^{1/2} \sin\alpha \mathrm{d}\alpha > 0.$$

When $\zeta > 0$, and $A(\zeta) < 0$, we write

$$\begin{aligned} I\big(A(\zeta),\zeta\big) =& 2\int_{-A(\zeta)}^{\pi/2} (\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha \\ &+ 2\int_0^{-A(\zeta)} \big[(\zeta+\sin\alpha)^{1/2}-(\zeta-\sin\alpha)^{1/2}\big]\sin\alpha\mathrm{d}\alpha. \end{aligned}$$

Both integrals are positive, so we have derived (8.46). With $A(\zeta) \geq 0$, $I(\alpha_0,\zeta)$ is decreasing which is (i). When $A(\zeta) < 0$, (ii) is immediate, and we have Lemma 8.5. □

(8.46) implies that

$$\Psi\big(A(\zeta),\zeta\big) = \infty. \tag{8.48}$$

An elementary calculation yields

$$\frac{\partial\Psi}{\partial\alpha_0} = -\frac{\cos\alpha_0}{\zeta+\sin\alpha_0}\Big(\frac{3}{2}\Psi + 2\tan\alpha_0\Big). \tag{8.49}$$

Lemma 8.6.

$$\frac{\partial\Psi}{\partial\alpha_0} < 0. \tag{8.50}$$

Proof. When $\alpha_0 \geq 0$, (8.49) implies (8.50). When $\alpha_0 < 0$, we may assume that $A(\zeta) < 0$, and therefore $\zeta > 0$. In this case

$$\begin{aligned} \frac{3}{2}\Psi + 2\tan\alpha_0 =& \frac{3}{(\zeta+\sin\alpha_0)^{3/2}}\int_0^{\pi/2}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha \\ &+ \frac{3}{(\zeta+\sin\alpha_0)^{3/2}}\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha + 2\tan\alpha_0, \end{aligned}$$

and after integrating the second integral by parts we obtain

$$\begin{aligned} \frac{3}{2}\Psi + 2\tan\alpha_0 = \frac{1}{(\zeta+\sin\alpha_0)^{3/2}}\Bigg[&3\int_0^{\pi/2}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha \\ &- 2\int_{\alpha_0}^{0}(\zeta+\sin\alpha)^{3/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha}\Bigg]. \end{aligned} \tag{8.51}$$

The second integral in the square bracket is a decreasing function of α_0, therefore the square bracket is an increasing function of α_0, and, if it has a zero, it has at most one. According to (8.48), $\Psi\big(A(\zeta),\zeta\big) = \infty$. Also, $\Psi(0,\zeta) > 0$, so $3\Psi/2 + \tan\alpha_0 > 0$ at $\alpha_0 = A(\zeta)$ and at $\alpha_0 = 0$. Thus, if it has a zero when $\alpha_0 \in \big(A(\zeta), 0\big)$, it must have at least 2 zeros which contradicts the statement that the square bracket in (8.51) may have at most one zero. Consequently, $3\Psi/2 + \tan\alpha_0$ has no zero in $\big[A(\zeta), 0\big)$, it is positive there, and this proves (8.50). $\square$

Lemma 8.7. *For all* $\zeta \in (-1, 1)$ *and* $\alpha_0 \in \big[A(\zeta), \pi/2\big]$ *we have*

$$\Psi - \Phi \geq 0, \tag{8.52}$$

and

$$\frac{\partial \Psi}{\partial \zeta} < 0. \tag{8.53}$$

Proof. We note that

$$\Psi - \Phi = \frac{2}{(\zeta + \sin\alpha_0)^{3/2}} \int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0) \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}}, \tag{8.54}$$

which is positive when $\alpha_0 \geq 0$. When $\alpha_0 < 0$, we may assume that $A(\zeta) < 0$, so $\zeta > 0$, and consider the derivative

$$\frac{\partial}{\partial \alpha_0} \int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0) \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}} = -\frac{1}{2}\cos(\alpha_0) J(\alpha_0, \zeta). \tag{8.55}$$

(i) $\zeta \in (0, \tilde{\zeta})$. In this case $J(\alpha_0, \zeta) > 0$ for $\alpha_0 \in \big[A(\zeta), 0\big]$, see Lemma 8.2(ii), therefore

$$\frac{\partial}{\partial \alpha_0} \int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0) \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}} < 0.$$

The integral is positive for $\alpha_0 = 0$, therefore it is positive for $\alpha_0 \in \big[A(\zeta), 0\big]$, and then (8.54) implies (8.52).

(ii) $\zeta \in [\tilde{\zeta}, 1)$. According to Lemma 8.2(iii),

$$J(\alpha_0, \zeta) < 0, \qquad \alpha_0 \in \big[A(\zeta), \tilde{\alpha}_0(\zeta)\big),$$
$$J(\alpha_0, \zeta) > 0, \qquad \alpha_0 \in \big(\tilde{\alpha}_0(\zeta), 0\big],$$

and therefore (8.55) yields

$$\frac{\partial}{\partial \alpha_0} \int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0) \frac{\sin\alpha \, d\alpha}{(\zeta + \sin\alpha)^{1/2}} \begin{cases} > 0, & \alpha_0 \in \big[A(\zeta), \tilde{\alpha}_0(\zeta)\big), \\ < 0, & \alpha_0 \in \big(\tilde{\alpha}_0(\zeta), 0\big]. \end{cases} \tag{8.56}$$

The integral is positive at $\alpha_0 = 0$, so it is positive when $\alpha_0 \in \big(\tilde{\alpha}_0(\zeta), 0\big]$, in particular it reaches its maximum at $\alpha_0 = \tilde{\alpha}_0(\zeta)$, and this proves (8.52) for $\alpha_0 \in \big(\tilde{\alpha}_0(\zeta), 0\big]$. Next we note that at $\alpha_0 = A(\zeta) = \text{Arcsin}(-\zeta)$,

$$\begin{aligned} \int_{A(\zeta)}^{\pi/2} \big(\sin\alpha - \sin A(\zeta)\big) \frac{\sin\alpha \, d\alpha}{(\zeta + \sin\alpha)^{1/2}} &= \int_{A(\zeta)}^{\pi/2} (\zeta + \sin\alpha)^{1/2} \sin\alpha \, d\alpha \\ &= \frac{1}{2} I\big(A(\zeta), \zeta\big) \\ &> 0, \end{aligned}$$

see (8.46). Since the integral is increasing in $\big[A(\zeta), \tilde{\alpha}_0(\zeta)\big)$, it remains positive there, and this concludes the derivation of (8.52). As for (8.53), we note that

$$\frac{\partial \Psi}{\partial \zeta} = \frac{\Phi - 3\Psi}{2(\zeta + \sin\alpha_0)}, \tag{8.57}$$

and $\Psi > 0$ and so is $\Psi - \Phi$, and we have (8.53). This completes the proof of Lemma 8.7. □

The Jacobian $\boldsymbol{\Delta_4}$

We need to understand the behaviour of the mapping

$$(\alpha_0, \zeta) \longrightarrow (\hat{x}, \hat{t}) = \big(\Phi(\alpha_0, \zeta), \Psi(\alpha_0, \zeta)\big), \tag{8.58}$$

$$(\alpha_0, \zeta) \in \bigcup_{\zeta \in (-1,1)} \Big(\Big[A(\zeta), \frac{\pi}{2}\Big] \times \{\zeta\} \Big). \tag{8.59}$$

To that end we shall calculate its Jacobian

$$\Delta_4 = \frac{\partial(\Psi, \Phi)}{\partial(\alpha_0, \zeta)}. \tag{8.60}$$

K, P and its discriminant D are still given by (5.38), (5.39), and (5.40). The formal calculations of Lemma 5.11 go through unchanged and we restate the formulas:

$$(\zeta + \sin\alpha_0)^3 \Delta_4 = (\cos\alpha_0) D - \frac{2\sin\alpha_0}{(\zeta + \sin\alpha_0)^{1/2}} P(\zeta + \sin\alpha_0, \alpha_0, \zeta), \tag{8.61}$$

$$P(\zeta + \sin\alpha_0, \alpha_0, \zeta) = \int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0)(\sin\alpha_0 + 3\sin\alpha + 4\zeta) \frac{\sin\alpha \, d\alpha}{(\zeta + \sin\alpha)^{3/2}}, \tag{8.62}$$

$$\frac{\partial}{\partial\alpha_0}\big((\zeta + \sin\alpha_0)^3 \Delta_4\big) = -(\sin\alpha_0) D - \frac{2\cos\alpha_0}{(\zeta + \sin\alpha_0)^{1/2}} P(\zeta + \sin\alpha_0, \alpha_0, \zeta). \tag{8.63}$$

Proposition 8.8. *$\Delta_4 > 0$ on (8.59) except at $\pi/2$ where it vanishes.*

We shall follow the argument of the proof of Proposition 5.15.

Lemma 8.9. *For all $\zeta \in (-1, 1)$, we have*

$$\Delta_4 > 0, \quad 0 \le \alpha_0 < \frac{\pi}{2}, \qquad \Delta_4(\pi/2, \zeta) = 0. \tag{8.64}$$

In particular,

$$\Delta_4 > 0 \quad \text{when } \zeta \in (-1, 0], \quad \alpha_0 \in \Big[A(\zeta), \frac{\pi}{2}\Big). \tag{8.65}$$

Proof. In view of Lemma 8.5, $I(\alpha_0, \zeta) > 0$ when $\alpha_0 \in \big[A(\zeta), \pi/2\big)$, and $K > 0$ when $0 \le \alpha_0 < \pi/2$, obviously. Consequently $D > 0$ when $\alpha_0 \ge 0$ and $D(\pi/2) = 0$. Also, (8.62) implies that

$$P(\zeta + \sin\alpha_0, \alpha_0, \zeta) \begin{cases} > 0, & \alpha_0 \in \big[0, \frac{\pi}{2}\big), \\ = 0, & \alpha_0 = \frac{\pi}{2}, \end{cases} \tag{8.66}$$

and then (8.63) yields

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_4 \begin{cases} <0, & \alpha_0\in[0,\frac{\pi}{2}),\\ =0, & \alpha_0=\frac{\pi}{2}. \end{cases} \tag{8.67}$$

$(\zeta+1)^3\Delta_4(\pi/2,\zeta)=0$, therefore (8.67) implies that $(\zeta+\sin\alpha_0)^3\Delta_4>0$ for $\alpha_0\in[0,\pi/2)$, and we have derived (8.64) and Lemma 8.9. □

This still leaves the question of $\alpha_0<0$ which occurs with $\zeta>0$.

Lemma 8.10. *Suppose $\zeta\in(0,1)$. Then*

$$\lim_{\alpha_0\to A(\zeta)} D(\alpha_0,\zeta)=-\infty, \tag{8.68}$$

and

$$\lim_{\alpha_0\to A(\zeta)} P(\zeta+\sin\alpha_0,\alpha_0,\zeta)=\frac{3}{2}I\big(A(\zeta),\zeta\big)>0. \tag{8.69}$$

Therefore,

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_4<0, \tag{8.70}$$

$$(\zeta+\sin\alpha_0)^3\Delta_4>0 \tag{8.71}$$

when $A(\zeta)<\alpha_0=A(\zeta)+\varepsilon_0<0$, $\varepsilon_0>0$ sufficiently small.

Proof. (8.46) yields

$$\lim_{\alpha_0\to A(\zeta)} I(\alpha_0,\zeta)=I\big(A(\zeta),\zeta\big)>0,$$

and Lemma 8.1 implies that

$$J\big(A(\zeta),\zeta\big)=\lim_{\alpha_0\to A(\zeta)} J(\alpha_0,\zeta)$$

is finite. We choose a small ε_0 and set $\alpha_0=A(\zeta)+\varepsilon_0$. The proof of Lemma 8.4 yields

$$\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{3/2}}=\frac{2\sin A(\zeta)}{\cos^{3/2}A(\zeta)}\frac{1}{\sqrt{\varepsilon_0}}+O(1).$$

Therefore we have

$$D\big(A(\zeta)+\varepsilon_0,\zeta\big) = 3I\big(A(\zeta),\zeta\big)\frac{\sin A(\zeta)}{\cos^{3/2}A(\zeta)}\frac{1}{\sqrt{\varepsilon_0}} + O(1)$$

which implies (8.68), since $A(\zeta) < 0$. Next we note that

$$\begin{aligned}P(\zeta+\sin\alpha_0,\alpha_0,\zeta) = &-(\zeta+\sin\alpha_0)^2\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}\\ &-(\zeta+\sin\alpha_0)J(\alpha_0,\zeta)+\frac{3}{2}I(\alpha_0,\zeta),\end{aligned}$$

see (5.39). With $\alpha_0 = A(\zeta)+\varepsilon_0$, the first term is $O(\varepsilon_0^{3/2})$ and the second term is $O(\varepsilon_0)$. Therefore

$$P(\zeta+\sin\alpha_0,\alpha_0,\zeta) = \frac{3}{2}I\big(A(\zeta),\zeta\big)+O(\varepsilon_0), \tag{8.72}$$

and this yields (8.69). In view of (8.68) and (8.69), both terms on the right hand side of (8.63) are negative, assuming that $A(\zeta) < \alpha_0 = A(\zeta)+\varepsilon_0 < 0$, $\varepsilon_0 > 0$ sufficiently small, and this implies (8.70). Finally we come to $(\zeta+\sin\alpha_0)^3\Delta_4$, represented by (8.61). Substituting the definition (5.40) of D and formula (8.72) into (8.61) we obtain

$$\begin{aligned}(\zeta+\sin\alpha_0)^3\Delta_4 = &\frac{3}{2}I(\alpha_0,\zeta)\left[(\cos\alpha_0)\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}-\frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}\right]\\ &+\frac{1}{4}J(\alpha_0,\zeta)^2+O(\sqrt{\varepsilon_0}).\end{aligned}$$

An integration by parts yields

$$\begin{aligned}&(\cos\alpha_0)\int_{\alpha_0}^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}-\frac{2\sin\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}\\ &=(\cos\alpha_0)\bigg\{\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}-2\int_{\alpha_0}^{0}\tan\alpha\mathrm{d}\Big(\frac{1}{(\zeta+\sin\alpha)^{1/2}}\Big)\\ &\qquad\qquad-\frac{2\tan\alpha_0}{(\zeta+\sin\alpha_0)^{1/2}}\bigg\}\\ &=(\cos\alpha_0)\left\{\int_0^{\pi/2}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{3/2}}+2\int_{\alpha_0}^{0}\frac{1}{(\zeta+\sin\alpha)^{1/2}}\frac{\mathrm{d}\alpha}{\cos^2\alpha}\right\}>0,\end{aligned}$$

and the curly bracket stays finite, positive and nonzero as $\alpha_0 \to A(\zeta)$. This proves (8.71). $\square$

Proof of Proposition 8.8. In view of Lemma 8.9 we need to prove that $\Delta_4 > 0$ only when $\alpha_0 \in \big[A(\zeta), 0\big)$, $A(\zeta) < 0$. In particular $\zeta \in (0,1)$. According to (8.61),

$$(\zeta + \sin\alpha_0)^3 \Delta_4\Big|_{\alpha_0=0} > 0, \tag{8.73}$$

and (8.71) yields

$$(\zeta + \sin\alpha_0)^3 \Delta_4\Big|_{\alpha_0=A(\zeta)} > 0. \tag{8.74}$$

$K(\alpha_0, \zeta)$ is an increasing function of $\alpha_0 \in \big[A(\zeta), 0\big]$, and the proof of Lemma 8.4 and the definition of $K(\alpha_0, \zeta)$ show that

$$K(\alpha_0, \zeta) \begin{cases} = -\infty, & \alpha_0 = A(\zeta), \\ > 0, & \alpha_0 = 0. \end{cases} \tag{8.75}$$

Consequently $K(\alpha_0, \zeta)$ has a zero $\hat{\alpha}_0(\zeta) \in \big(A(\zeta), 0\big)$.

(i) $\boldsymbol{A(\zeta) \le \alpha_0 \le \hat{\alpha}_0(\zeta)}$. For such α_0,

$$\begin{aligned} &P(\zeta + \sin\alpha_0, \alpha_0, \zeta) \\ &= -\frac{1}{2}K(\alpha_0, \zeta)(\zeta + \sin\alpha_0)^2 - J(\alpha_0, \zeta)(\zeta + \sin\alpha_0) + \frac{3}{2}I(\alpha_0, \zeta) \\ &= -\frac{1}{2}K(\alpha_0, \zeta)(\zeta + \sin\alpha_0)^2 + \frac{1}{4}(\zeta + \sin\alpha_0)^{3/2}(3\Psi - \Phi) \\ &> 0. \end{aligned} \tag{8.76}$$

If Δ_4 vanishes in $\big[A(\zeta), \hat{\alpha}_0(\zeta)\big]$, then at the zero we have

$$D = \frac{2\sin\alpha_0}{(\zeta + \sin\alpha_0)^{1/2}\cos\alpha_0} P(\zeta + \sin\alpha_0, \alpha_0, \zeta) < 0, \tag{8.77}$$

see (8.61). Therefore (8.63) and (8.77) give

$$\frac{\partial}{\partial\alpha_0}(\zeta + \sin\alpha_0)^3 \Delta_4 = -\frac{2P(\zeta + \sin\alpha_0, \alpha_0, \zeta)}{(\zeta + \sin\alpha_0)^{1/2}\cos\alpha_0} < 0$$

at the zero of Δ_4, which shows that Δ_4 can have at most one zero when $\alpha_0 \in \big[A(\zeta), \hat{\alpha}_0(\zeta)\big]$. On the other hand $(\zeta + \sin\alpha_0)^3\Delta_4 > 0$ at $\alpha_0 = A(\zeta)$, see (8.71), and at $\alpha_0 = \hat\alpha_0(\zeta)$ (8.61) yields

$$\begin{aligned}&\big(\zeta + \sin\hat\alpha_0(\zeta)\big)^3\Delta_4\big(\hat\alpha_0(\zeta),\zeta\big)\\&= \frac{1}{4}J^2\big(\hat\alpha_0(\zeta),\zeta\big)\cos\hat\alpha_0(\zeta) - \frac{2\sin\hat\alpha_0(\zeta)}{\big(\zeta+\sin\hat\alpha_0(\zeta)\big)^{1/2}}P\big(\zeta+\sin\hat\alpha_0(\zeta),\hat\alpha_0(\zeta),\zeta\big) > 0,\end{aligned}$$

in view of (8.76). Therefore $(\zeta+\sin\alpha_0)^3\Delta_4$ cannot have only one zero when $\alpha_0 \in \big[A(\zeta),\hat\alpha_0(\zeta)\big]$, so it has none.

(ii) $\boldsymbol{\hat\alpha_0(\zeta) < \alpha_0 < 0}$. Here $K > 0$ and therefore $D > 0$. At a zero of $(\zeta+\sin\alpha_0)^3\Delta_4$, in this interval, one has

$$\frac{\partial}{\partial\alpha_0}(\zeta+\sin\alpha_0)^3\Delta_4 = -\frac{D}{\sin\alpha_0} > 0,$$

in view of (8.61) and (8.63), and $(\zeta+\sin\alpha_0)^3\Delta_4$ has at most one zero when $\alpha_0 \in \big(\hat\alpha_0(\zeta),0\big)$. Again, $(\zeta+\sin\alpha_0)^3\Delta_4 > 0$ at $\alpha_0 = \hat\alpha_0(\zeta)$ and also at $\alpha_0 = 0$, where its value is $D(0,\zeta) > 0$, thus $(\zeta+\sin\alpha_0)^3\Delta_4$ cannot vanish when $\alpha_0 \in \big(\hat\alpha_0(\zeta),0\big)$. This concludes the proof of Proposition 8.8. □

The curves $\boldsymbol{\Gamma_4(\zeta)}$.

For each fixed $\zeta \in (-1,1]$ we let $\Gamma_4(\zeta)$ denote the curve

$$\hat x = \Phi(\alpha_0,\zeta), \quad \hat t = \Psi(\alpha_0,\zeta), \quad \alpha_0 \in \Big[A(\zeta),\frac{\pi}{2}\Big] \tag{8.78}$$

in the $(\hat x,\hat t)$ plane. When $\zeta = 1$, the curve $\Gamma_4(1)$ agrees with the curve $\Gamma_2(1)$ of (5.50), one of the boundary curves of R_2. We rewrite some of the results on Φ and Ψ as results on $\hat x$ and $\hat t$.

Lemma 8.11. (i) $\tilde\zeta < \zeta < 1$. *As α_0 decreases from $\pi/2$ to $A(\zeta)$, $\hat t$ increases from 0 to ∞ along $\Gamma(\zeta)$. $\hat x$ increases from $\Phi(\pi/2,\zeta) = 0$ to $\Phi\big(\alpha_0^*(\zeta),\zeta\big)$ and then decreases to $\Phi\big(A(\zeta),\zeta\big) = -\infty$.*

(ii) $\zeta = \tilde\zeta$. *As α_0 decreases from $\pi/2$ to $A(\tilde\zeta)$, $\hat t$ increases from 0 to ∞ and $\hat x$ increases from 0 to $\Phi\big(A(\tilde\zeta),\tilde\zeta\big) = -4\tan A(\tilde\zeta) > 0$.*

(iii) $\zeta \in (-1,\tilde\zeta)$. *Both $\hat t$ and $\hat x$ increase from 0 to ∞ as α_0 decreases from $\pi/2$ to $A(\zeta)$.*

(iv) *All curves $\Gamma_4(\zeta)$, $\zeta \in (-1,1]$ start tangent to the line $\hat t = \hat x$ at $(0,0)$ when $\alpha_0 = \pi/2$ and stay in the half space $\hat t > \hat x$.*

Proof. (i), (ii) and (iii) are reformulations of Lemma 8.3 for Φ, and formula (8.48) and Lemma 8.6 for Ψ.

(iv) is a consequence of (8.52) and (8.27), (8.49); the last two imply that $\mathrm{d}\hat{t}/\mathrm{d}\hat{x} = 1$ at $\alpha_0 = \pi/2$. □

Lemma 8.12. *The mapping* (8.58) *is* 1-1 *onto its image, therefore the curves* $\Gamma_4(\zeta)$ *do not cross each other. As* ζ *decreases in* $(-1, 1]$*, the curves* $\Gamma_4(\zeta)$ *move to the right.*

Proof. For a given $\zeta \in (-1, 1]$, $\Psi(\alpha_0, \zeta)$ decreases from ∞ to 0 as α_0 increases from $A(\zeta)$ to $\pi/2$, see (8.48) and Lemma 8.6. Therefore for any $\zeta \in (-1, 1]$ and $\hat{t} \in (0, \infty)$,

$$\hat{t} = \Psi(\alpha_0, \zeta) \tag{8.79}$$

has a unique solution $\alpha_0(\hat{t}, \zeta)$; we note that $\alpha_0(0, \zeta) = \pi/2$ for all ζ. So, if $(\hat{x}, \hat{t})$ is in the image of the mapping (8.58), we must have

$$\hat{x} = \Phi\big(\alpha_0(\hat{t}, \zeta), \zeta\big). \tag{8.80}$$

But

$$\frac{\partial}{\partial \zeta}\Phi\big(\alpha_0(\hat{t}, \zeta), \zeta\big) = \frac{\Delta_4}{\frac{\partial \Psi}{\partial \alpha_0}}\bigg|_{\alpha_0 = \alpha_0(\hat{t}, \zeta)} < 0, \tag{8.81}$$

since by Proposition 8.8, $\Delta_4 > 0$, except at $\alpha_0 = \pi/2$ where it vanishes, and $\partial\Psi/\partial\alpha_0 < 0$ by Lemma 8.6. Thus (8.80) has a unique solution $\zeta \in (-1, 1]$, and the system of equations

$$\hat{x} = \Phi(\alpha_0, \zeta), \qquad \hat{t} = \Psi(\alpha_0, \zeta)$$

is uniquely solvable if $(\hat{x}, \hat{t})$ is in the image of the mapping (8.58). For a given $\hat{t}_0$, the point of intersection of the line $\hat{t} = \hat{t}_0$ with $\Gamma(\zeta)$ moves to the right when ζ decreases from 1 toward -1; this is a consequence of (8.81). □

The region $\mathbf{R_4}$

Definition 8.13. We shall denote by R_4 the image set of the mapping (8.58).

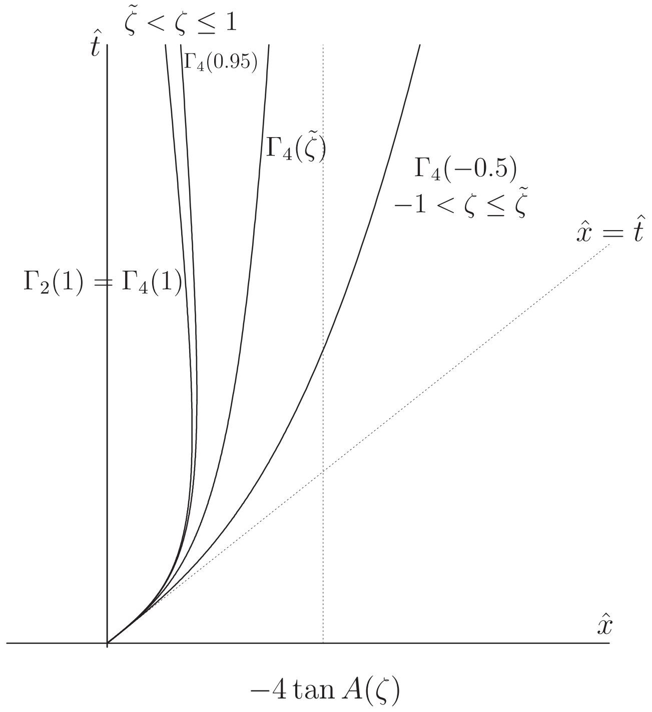

Figure 12.

Proposition 8.14. *The region R_4 is the set in the $(\hat{x}, \hat{t})$-plane bounded on the left by the curve $\Gamma_4(1)$, included in R_4, and on the right by the half line $\hat{t} = \hat{x}$, $\hat{x} > 0$, not included in R_4. R_4 is in* 1-1 *correspondence with the domain*

$$\bigcup_{-1<\zeta\le 1} \Big(\Big[A(\zeta), \frac{\pi}{2} \Big) \times \{\zeta\} \Big)$$

via the mapping (8.58).

Proof. Choose $\hat{x} > 0$ and let $\alpha_0(\hat{x}, \zeta)$ denote the unique solution of

$$\hat{x} = \Phi(\alpha_0, \zeta)$$

for $\zeta \in (-1, \tilde{\zeta})$, see Lemma 8.3(i). To prove Proposition 8.14 it suffices to show that

$$\lim_{\zeta \to -1} \Psi\big(\alpha_0(\hat{x}, \zeta), \zeta\big) = \hat{x}. \tag{8.82}$$

In other words, the vertical line with abscissa $\hat{x}$ intersects R_4 in the interval $(\hat{x}, \infty)$; note that the line $\hat{x} = \hat{t}$, $\hat{x} > 0$, corresponds to $\zeta = -1$ where the mapping (8.58) is not defined, but see Remark 8.15. The limit (8.82) is equivalent to letting $A = A(\zeta) \to \pi/2$, and we shall look at the behaviour of $\alpha_0(\hat{x}, \zeta)$ as $A \to \pi/2$ first. Recall that

$$\hat{x} = \frac{2}{(\sin\alpha_0 - \sin A)^{1/2}} \int_{\alpha_0}^{\pi/2} \frac{\sin\alpha d\alpha}{(\sin\alpha - \sin A)^{1/2}}, \tag{8.83}$$

and we are interested in

$$A \le \alpha_0 \le \alpha \le \frac{\pi}{2}, \qquad A \to \frac{\pi}{2}.$$

Then

$$\begin{aligned} \sin\alpha - \sin A &= \sin(\alpha - A)\cos A + \cos(\alpha - A)\sin A - \sin A \\ &= \left[(\alpha - A) - \frac{1}{3!}(\alpha - A)^3 + \cdots\right] \sin\left(\frac{\pi}{2} - A\right) \\ &\quad + \left[1 - \frac{1}{2!}(\alpha - A)^2 + \cdots\right] \cos\left(\frac{\pi}{2} - A\right) - \cos\left(\frac{\pi}{2} - A\right) \\ &= (\alpha - A)\left[\frac{\pi}{2} - A - \frac{1}{3!}\left(\frac{\pi}{2} - A\right)^3\right] \\ &\quad - \frac{1}{2!}(\alpha - A)^2\left[1 - \frac{1}{2!}\left(\frac{\pi}{2} - A\right)^2\right] + O\left((\alpha - A)^2\left(\frac{\pi}{2} - A\right)^3\right), \end{aligned}$$

and with

$$\varepsilon = \frac{\pi}{2} - A$$

we have

$$\sin\alpha - \sin A = (\alpha - A)\varepsilon - \frac{1}{2}(\alpha - A)^2 + O\big((\alpha - A)\varepsilon^3\big). \tag{8.84}$$

Similarly,

$$\sin\alpha_0 - \sin A = (\alpha_0 - A)\varepsilon - \frac{1}{2}(\alpha_0 - A)^2 + O\big((\alpha_0 - A)\varepsilon^3\big). \tag{8.85}$$

We note that

$$0 \le \alpha - A \le \alpha_0 - A \le \varepsilon,$$

and rewrite the integrand in (8.83),

$$\begin{aligned}
\frac{\sin\alpha}{(\sin\alpha-\sin A)^{1/2}} &= \frac{\sin A+(\sin\alpha-\sin A)}{(\sin\alpha-\sin A)^{1/2}}\\
&= \frac{\cos\varepsilon+O\big((\alpha-A)\varepsilon\big)}{(\sin\alpha-\sin A)^{1/2}}\\
&= \frac{1+O(\varepsilon^2)}{(\alpha-A)^{1/2}\Big(\varepsilon-\frac{1}{2}(\alpha-A)+O(\varepsilon^3)\Big)^{1/2}}\\
&= \frac{1+O(\varepsilon^2)}{(\alpha-A)^{1/2}\Big(\varepsilon-\frac{1}{2}(\alpha-A)\Big)^{1/2}}\big(1+O(\varepsilon^2)\big)^{-1/2}\\
&= \frac{1+O(\varepsilon^2)}{\Big((\alpha-A)\varepsilon-\frac{1}{2}(\alpha-A)^2\Big)^{1/2}},
\end{aligned}$$

since $\varepsilon-(\alpha-A)/2>\varepsilon/2$. A similar calculation yields

$$\frac{1}{(\sin\alpha_0-\sin A)^{1/2}}=\frac{1+O(\varepsilon^2)}{\Big((\alpha_0-A)\varepsilon-\frac{1}{2}(\alpha_0-A)^2\Big)^{1/2}}.$$

With this approximation (8.83) takes the following form:

$$\hat{x}=\frac{2\big(1+O(\varepsilon^2)\big)}{\Big((\alpha_0-A)\varepsilon-\frac{1}{2}(\alpha_0-A)^2\Big)^{1/2}}\int_{\alpha_0}^{\pi/2}\frac{\mathrm{d}\alpha}{\Big((\alpha-A)\varepsilon-\frac{1}{2}(\alpha-A)^2\Big)^{1/2}}.$$

In the integral we set $u=\alpha-A$,

$$\begin{aligned}
\int_{\alpha_0}^{\pi/2}\frac{\mathrm{d}\alpha}{\Big((\alpha-A)\varepsilon-\frac{1}{2}(\alpha-A)^2\Big)^{1/2}} &= \sqrt{2}\int_{\alpha_0-A}^{\varepsilon}\frac{\mathrm{d}u}{\big(\varepsilon^2-(\varepsilon-u)^2\big)^{1/2}}\\
&= \sqrt{2}\,\mathrm{Arcsin}\Big(1-\frac{\alpha_0-A}{\varepsilon}\Big),
\end{aligned}$$

so we have

$$\hat{x}=\frac{2\sqrt{2}\,\mathrm{Arcsin}\Big(1-\frac{\alpha_0-A}{\varepsilon}\Big)}{(\alpha_0-A)^{1/2}\varepsilon^{1/2}\Big(1-\frac{\alpha_0-A}{2\varepsilon}\Big)^{1/2}}\big(1+O(\varepsilon^2)\big). \tag{8.86}$$

$\hat{x}$ is fixed and the denominator vanishes as $\varepsilon \to 0$, so the numerator must also vanish when $\varepsilon \to 0$, and this implies that $\alpha_0 - A \sim \varepsilon$, as $\varepsilon \to 0$. To find out how near $\alpha_0 - A$ and ε are, we set

$$1 - \frac{\alpha_0 - A}{\varepsilon} = \gamma = o(1), \qquad \varepsilon \to 0, \tag{8.87}$$

and substitute this into (8.86),

$$\sin\Big(\big(1 + O(\varepsilon^2)\big)\frac{1}{4}\hat{x}\varepsilon(1-\gamma^2)^{1/2}\Big) = \gamma,$$

or,

$$\big(1 + O(\varepsilon^2)\big)\frac{\hat{x}\varepsilon}{4}\Big(1-\gamma^2\Big)^{1/2} + O(\varepsilon^3) = \gamma \;\Rightarrow\; \gamma = O(\varepsilon),$$

and therefore

$$\big(1 + O(\varepsilon^2)\big)\frac{\hat{x}\varepsilon}{4}\Big(1-\frac{1}{2}\gamma^2\Big) + O(\varepsilon^3) = \gamma,$$

$$\frac{\hat{x}\varepsilon}{4}\big(1 + O(\varepsilon^2)\big) + O(\varepsilon^3) = \gamma,$$

so

$$\gamma = \frac{1}{4}\hat{x}\varepsilon + O(\varepsilon^2).$$

Therefore (8.87) yields

$$\alpha_0 - A = \varepsilon - \frac{1}{4}\hat{x}\varepsilon^2 + O(\varepsilon^3). \tag{8.88}$$

We are ready to derive (8.82). Recall the integral in Ψ:

$$\begin{aligned}
&\int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin A)^{1/2}\sin\alpha \mathrm{d}\alpha \\
&= \int_{\alpha_0}^{\pi/2} \Big((\alpha - A)\varepsilon - \frac{1}{2}(\alpha - A)^2\Big)^{1/2}\big(1 + O(\varepsilon^2)\big)\big(\sin A + O(\varepsilon^2)\big)\mathrm{d}\alpha \\
&= \big(1 + O(\varepsilon^2)\big)\int_{\alpha_0}^{\pi/2} \Big((\alpha - A)\varepsilon - \frac{1}{2}(\alpha - A)^2\Big)^{1/2}\mathrm{d}\alpha.
\end{aligned}$$

But,

$$\int_{\alpha_0}^{\pi/2}\Big((\alpha-A)\varepsilon-\frac{1}{2}(\alpha-A)^2\Big)^{1/2}\mathrm{d}\alpha$$
$$=\frac{1}{\sqrt{2}}\int_{\alpha_0-A}^{\varepsilon}\big(\varepsilon^2-(u-\varepsilon)^2\big)^{1/2}\mathrm{d}u=\frac{\varepsilon^2}{\sqrt{2}}\int_0^{\mathrm{Arcsin}(1-\frac{\alpha_0-A}{\varepsilon})}\cos^2\theta\mathrm{d}\theta$$
$$=\frac{\varepsilon^2}{2\sqrt{2}}\Big[\mathrm{Arcsin}\Big(1-\frac{\alpha_0-A}{\varepsilon}\Big)+\frac{1}{2}\sin\Big(2\mathrm{Arcsin}\Big(1-\frac{\alpha_0-A}{\varepsilon}\Big)\Big)\Big],$$

and then

$$1-\frac{\alpha_0-A}{\varepsilon}=\frac{\hat{x}\varepsilon}{4}\big(1+O(\varepsilon)\big)$$

implies that

$$\int_{\alpha_0}^{\pi/2}(\sin\alpha-\sin A)^{1/2}\sin\alpha\mathrm{d}\alpha=\frac{\hat{x}\varepsilon^3}{4\sqrt{2}}\big(1+O(\varepsilon)\big).$$

Consequently,

$$\Psi\big(\alpha_0(\hat{x},\zeta),\zeta\big)=\frac{2\frac{\hat{x}\varepsilon^3}{4\sqrt{2}}\big(1+O(\varepsilon)\big)}{(\alpha_0-A)^{3/2}\varepsilon^{3/2}\Big(1-\frac{\alpha_0-A}{2\varepsilon}\Big)^{3/2}}\xrightarrow{\varepsilon\to 0}\hat{x},$$

since $1-(\alpha_0-A)/2\varepsilon\to 1/2$ as $\varepsilon\to 0$. Thus we have derived (8.82) and this concludes the proof of Proposition 8.14. □

Remark 8.15. We recall that when $\hat{t}=\hat{x}$, or $t=y_0^2x$, the geodesics joining (x,y_0,t) to $(0,y_0,0)$ are given by (2.33). For these geodesics $\tau=0$, $\eta=0$, $\xi=\pm 1$ and $y(s)$ stays in the $y=y_0$-plane; $\xi=1\Rightarrow x>0$, and $\xi=-1\Rightarrow x<0$.

In view of the symmetry that led to (2.10) we have completed the proof of Theorem 1.2(ii) and of Theorem 1.3. For the sake of completeness we add chapter 9 with a discussion of the behaviour of nonlocal geodesics on the $y=y_0$-plane.

9. On nonlocal Geodesics

In Chapters 2–8 we established Theorem 1.2(ii) and Theorem 1.3, based on the study of local geodesics, that is geodesics which do not leave the step 2 domain $y > 0$; the point of issue of these geodesics is $(0, y_0, 0)$, $y_0 > 0$. In our last chapter we shall discuss the behaviour of geodesics which cross the $y = 0$ plane, or, at least contain points of the $y = 0$ plane, before returning to the y_0-plane, $y_0 > 0$. There are 3 such classes of geodesics and they all have $\xi = -\zeta \in [-1, 1]$, so $\alpha_0 \in \big[A(\zeta), \pi - A(\zeta)\big]$.

(i) Geodesics that end on the y_0-plane at α_0, $\alpha_0 \in \big[A(\zeta), \pi - A(\zeta)\big]$, after n periods. The y-component may start in either the positive or negative direction; in particular (8.5) is included.

(ii) Geodesics that start in the positive y-direction, return to the y_0-plane at $\pi - \alpha_0$, and then do $n = 1, 2, \ldots$ periods. These include (8.6) with $n = 1, 2, \ldots$.

(iii) Geodesics which start in the negative y-direction, return to the y_0-plane at $\pi - \alpha_0$, then do $n = 1, 2, \ldots$ periods.

Finally we shall include a short discussion of the case $y_0 = 0$; this was already dealt with in much more detail in [9].

Geodesics with $\alpha_{\text{end}} = \alpha_0$ after n periods

To obtain the formulas for these curves we use (2.63) and (2.64) while carefully following the motion of $\alpha \in \big[A(\zeta), \pi - A(\zeta)\big]$ as described in the paragraph after (2.48). Assuming $\alpha_0 \in \big[A(\zeta), \pi/2\big]$, after one period

$$x = \frac{1}{2\tau^{1/2}} \int_{\alpha_0}^{\alpha_{\text{end}}} \operatorname{sgn}\big(y(\alpha)\big) \frac{\sin \alpha \mathrm{d}\alpha}{(\zeta + \sin \alpha)^{1/2}} \tag{9.1}$$

yields

$$\begin{aligned} x &= \frac{1}{2\tau^{1/2}} \left\{ \int_{\alpha_0}^{\pi - A(\zeta)} - \int_{\pi - A(\zeta)}^{A(\zeta)} + \int_{A(\zeta)}^{\alpha_0} \right\} \frac{\sin \alpha \mathrm{d}\alpha}{(\zeta + \sin \alpha)^{1/2}} \\ &= \frac{1}{\tau^{1/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} \frac{\sin \alpha \mathrm{d}\alpha}{(\zeta + \sin \alpha)^{1/2}}, \end{aligned} \tag{9.2}$$

and after n periods we have

$$x = \frac{n}{\tau^{1/2}} J\big(A(\zeta), \zeta\big), \qquad \zeta \in [-1, 1), \tag{9.3}$$

see (8.19). Similarly,

$$t = \frac{n}{\tau^{3/2}} I\big(A(\zeta), \zeta\big). \tag{9.4}$$

These formulas are also valid for $\alpha_0 \in \big(\pi/2, \pi - A(\zeta)\big]$. We set $\gamma = -\zeta$, and note that

$$\frac{\mathrm{d}A(\zeta)}{\mathrm{d}\gamma} = \frac{1}{\sqrt{1-\gamma^2}} > 0, \tag{9.5}$$

so we shall use γ, and $\alpha_0 = A(\zeta)$, interchangeably as new variables. In particular it is helpful to use

$$J(\alpha_0) = 2\int_{\alpha_0}^{\pi/2} \frac{\sin\alpha \mathrm{d}\alpha}{(\sin\alpha - \sin\alpha_0)^{1/2}}, \tag{9.6}$$

$$I(\alpha_0) = 2\int_{\alpha_0}^{\pi/2} (\sin\alpha - \sin\alpha_0)^{1/2} \sin\alpha \mathrm{d}\alpha; \tag{9.7}$$

as the arguments are angles we don't envisage any confusion about the notation. Clearly

$$J\big(A(\zeta), \zeta\big) = J\big(A(\zeta)\big), \tag{9.8}$$

$$I\big(A(\zeta), \zeta\big) = I\big(A(\zeta)\big), \tag{9.9}$$

and after n full periods the return mapping (9.3), (9.4) takes the form

$$(x, t) = n\left(\frac{J(\alpha_0)}{\tau^{1/2}}, \frac{I(\alpha_0)}{\tau^{3/2}}\right), \tag{9.10}$$

with $(\alpha_0, \tau) \in (-\pi/2, \pi/2] \times (0, \infty)$; we note that $\alpha_0 = A(\zeta)$. A simple calculation along the lines of (8.24) and (8.25) yields the derivatives of I and J. In particular,

$$\frac{\partial J\big(A(\zeta)\big)}{\partial\gamma} = \frac{1}{2}\int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} \left(\frac{1}{\Big(2 - \frac{1-\gamma}{2}(1-s)\Big)^{1/2}} + \frac{1}{\Big(2 - \frac{1-\gamma}{2}(1-s)\Big)^{3/2}} \right) \mathrm{d}s, \tag{9.11}$$

and therefore

$$\frac{\partial J\big(A(\zeta)\big)}{\partial\gamma} > 0. \tag{9.12}$$

It is worth mentioning that the integral in (9.11) is well defined as long as $2-\frac{1-\gamma}{2}(1-s)\neq 0$, $s\in[-1,1]$, which is certainly true if $-1<\gamma=-\zeta\leq 1$. Similarly,

$$\begin{aligned}
I\big(A(\zeta)\big) &= 2\int_{A(\zeta)}^{\pi/2}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha\\
&= 2\int_{-\zeta}^{1}\sqrt{\zeta+x}\mathrm{d}\big(-\sqrt{1-x^2}\big)\\
&= -2\int_{\gamma}^{1}\sqrt{x-\gamma}\mathrm{d}\sqrt{1-x^2}=\int_{\gamma}^{1}\frac{\sqrt{1-x^2}}{\sqrt{x-\gamma}}\mathrm{d}x\\
&= \int_{\gamma}^{1}\frac{(1-x)\sqrt{1+x}}{\sqrt{(1-x)(x-\gamma)}}\mathrm{d}x\\
&= \int_{-\frac{1-\gamma}{2}}^{\frac{1-\gamma}{2}}\frac{\Big(1-t-\frac{1+\gamma}{2}\Big)\Big(1+t+\frac{1+\gamma}{2}\Big)^{1/2}}{\sqrt{\Big(\frac{1-\gamma}{2}\Big)^2-t^2}}\mathrm{d}t\\
&= \int_{-1}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}}\Big(1-\frac{1-\gamma}{2}s-\frac{1+\gamma}{2}\Big)\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{1/2},
\end{aligned}\tag{9.13}$$

and simplifying one has

$$I\big(A(\zeta)\big)=\frac{1-\gamma}{2}\int_{-1}^{1}\sqrt{\frac{1-s}{1+s}}\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{1/2}\mathrm{d}s>0. \tag{9.14}$$

For future reference we note that

$$\frac{\partial}{\partial\gamma}\frac{I}{1-\gamma}=\frac{1}{4}\int_{-1}^{1}\sqrt{\frac{1-s}{1+s}}\frac{\frac{1}{2}(1-s)\mathrm{d}s}{\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{1/2}}>0, \tag{9.15}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big(J\big(A(\zeta)\big)-\frac{I\big(A(\zeta)\big)}{1+\zeta}\Big)\\
&=\frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1-s}{1+s}}\left(\frac{1-\frac{1}{4}(1-s)}{\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{1/2}}+\frac{1}{\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{3/2}}\right)\mathrm{d}s,
\end{aligned}\tag{9.16}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big(J\big(A(\zeta)\big) - \frac{I\big(A(\zeta)\big)}{1+\zeta}\Big) > 0. \tag{9.17}$$

Lemma 9.1. (i) $J(\alpha_0)$ *is an increasing function of* $\alpha_0 = A(\zeta) \in (-\pi/2, \pi/2]$, *where* J *increases from* $J(-\pi/2) = -\infty$ *to* $J(\pi/2) = \sqrt{2}\pi$. $J(\alpha_0)$ *has a unique zero at* $\tilde{\alpha}_0 = A(\tilde{\zeta}) \in (-\pi/2, 0)$.

(ii) *We have*

$$\lim_{\zeta\to 1} I\big(A(\zeta)\big) = \frac{4\sqrt{2}}{3}, \tag{9.18}$$

$$\lim_{\zeta\to -1} \frac{I\big(A(\zeta)\big)}{\zeta+1} = \frac{\sqrt{2}\pi}{2}, \tag{9.19}$$

and

$$\frac{\mathrm{d}I(\alpha_0)}{\mathrm{d}\alpha_0} = -\frac{1}{2}\cos\alpha_0 J(\alpha_0). \tag{9.20}$$

Since $A(1) = -\pi/2$ *and* $A(-1) = \pi/2$, (9.19), (9.20) *imply that* $I(\alpha_0)$ *increases from* $I(-\pi/2) = \frac{4\sqrt{2}}{3}$ *to* $I(\tilde{\alpha}_0) > 0$, *then decreases to* $I(\pi/2) = 0$. *In particular,*

$$I(\alpha_0) > 0, \qquad \alpha_0 \in [-\pi/2, 0). \tag{9.21}$$

(iii) $J^3(\alpha_0)/I(\alpha_0)$ *increases from* $(J^3/I)(-\pi/2) = -\infty$ *to* $(J^3/I)(\pi/2) = \infty$.

Proof. (i) is just a restatement of Lemma 8.1. As for (ii), (9.14) yields

$$\begin{aligned}\lim_{\zeta\to -1} \frac{I\big(A(\zeta)\big)}{\zeta+1} &= \frac{\sqrt{2}}{2}\int_{-1}^{1}\sqrt{\frac{1-s}{1+s}}\mathrm{d}s \\ &= \frac{\sqrt{2}}{4}\int_{-1}^{1}\left(\sqrt{\frac{1-s}{1+s}} + \sqrt{\frac{1+s}{1-s}}\right)\mathrm{d}s \\ &= \frac{\sqrt{2}}{2}\int_{-1}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}} \\ &= \sqrt{2}\frac{\pi}{2},\end{aligned} \tag{9.22}$$

which is (9.19). Also

$$I\Big(-\frac{\pi}{2}\Big) = 2\int_{-1}^{1} \sqrt{1+x}\frac{x\mathrm{d}x}{\sqrt{1-x^2}} = 2\int_{-1}^{1} \frac{x\mathrm{d}x}{\sqrt{1-x}} = \frac{4\sqrt{2}}{3},$$

which gives (9.18), and (9.20) is clear.

(iii) $J(\pi/2) = \sqrt{2}\pi$ and $I(\pi/2) = 0^+$, hence $(J^3/I)(\pi/2) = \infty$. Also, $J(-\pi/2) = -\infty$ and $I(-\pi/2) > 0$, so $(J^3/I)(-\pi/2) = -\infty$. Next we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha_0}\frac{J^3(\alpha_0)}{I(\alpha_0)} &= \frac{3J^2(\alpha_0)}{I(\alpha_0)}\frac{\mathrm{d}J(\alpha_0)}{\mathrm{d}\alpha_0} - \frac{J^3(\alpha_0)}{I^2(\alpha_0)}\frac{\mathrm{d}I(\alpha_0)}{\mathrm{d}\alpha_0} \\ &= \frac{1}{I^2(\alpha_0)}\left(\frac{1}{2}\cos\alpha_0 J^4(\alpha_0) + 3J^2(\alpha_0)I(\alpha_0)\frac{\mathrm{d}J(\alpha_0)}{\mathrm{d}\alpha_0}\right) \\ &> 0, \end{aligned}$$

since $I(\alpha_0) > 0$ and so is $\mathrm{d}J(\alpha_0)/\mathrm{d}\alpha_0$. This completes the proof of Lemma 9.1. □

Proposition 9.2. *The mapping*

$$(\alpha_0, \tau) \to (x, t) = \left(\frac{J(\alpha_0)}{\tau^{1/2}}, \frac{I(\alpha_0)}{\tau^{3/2}}\right) \tag{9.23}$$

sends the domain $(-\pi/2, \pi/2) \times (0, \infty)$ *onto the half plane* (x, t), $t > 0$ *in a* 1-1 *manner.*

Proof. For any $t > 0$ and $x \in (-\infty, \infty)$,

$$\frac{x^3}{t} = \frac{J^3(\alpha_0)}{I(\alpha_0)} \tag{9.24}$$

has a unique solution $\alpha_0 \in (-\pi/2, \pi/2)$, and then τ is obtained from

$$\tau^{3/2} = \frac{I(\alpha_0)}{t}. \tag{9.25}$$

□

Remark 9.3. We note that $t = 0$ implies that $\alpha_0 = \pi/2$. This is equivalent to $\zeta = -1$, and then (2.51) gives the x-axis itself as the trajectory of the geodesic. Thus $A(\zeta) = \pi/2$ is not allowed when $y_0 \neq 0$.

Corollary 9.4. *After completing n full periods* $n=1,2,\ldots$, *the geodesics of* (9.3), (9.4) *send the domain* $(-\pi/2,\pi/2)\times(0,\infty)$ *onto the half plane* (x,y_0,t), $t>0$ *in a* 1-1 *manner.*

Remark 9.5. This suggests that nonlocal geodesics, in our case this refers to those which cross the $y=0$ plane, are unreliable when giving information about the local behaviour of the manifold around the point $(0,y_0,0)$. We recall that the $y=0$ plane is the boundary of the uniformly step 2 domain $y>0$ which contains the point $(0,y_0,0)$, $y_0>0$.

Geodesics with $\alpha_{\mathrm{end}}=\pi-\alpha_0$ after $n+1/2$ periods, $\dot y(\alpha_0)>0$.

These are the geodesics of (8.6) explicitly given by formulas (8.13), (8.14) which we write in the following form:

$$\hat x=\Phi^{(n)}(\alpha_0,\zeta)=\Phi(\alpha_0,\zeta)+2n\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+\sin\alpha_0)^{1/2}},\tag{9.26}$$

$$\hat t=\Psi^{(n)}(\alpha_0,\zeta)=\Psi(\alpha_0,\zeta)+2n\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+\sin\alpha_0)^{3/2}},\tag{9.27}$$

$n=0,1,2,\ldots$. Chapter 8 contains the discussion of the $n=0$ case, and here we shall consider the cases when $n=1,2,\ldots$.

Proposition 9.6. (x,y_0,t) *cannot be joined to* $(0,y_0,0)$ *by an infinite number of distinct geodesics of the form* (9.26), (9.27).

We shall follow the argument developed in the proof of Proposition 6.1. In particular we shall refute the two possibilities of Remark 6.2.

Lemma 9.7. *The function*

$$\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+\sin\alpha_0)^{3/2}}\tag{9.28}$$

is a strictly decreasing function of ζ. *It is also a strictly decreasing function of* α_0 *with its minimum at* $\alpha_0=\pi/2$, *where we have*

$$\lim_{\zeta\to-1}\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}=\infty,\tag{9.29}$$

$$\lim_{\zeta\to 1}\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}=\frac{2}{3}. \tag{9.30}$$

Proof. Clearly

$$\left(\frac{\zeta+1}{\zeta+\sin\alpha_0)}\right)^{3/2}=\left(1+\frac{1-\sin\alpha_0}{\zeta+\sin\alpha_0}\right)^{3/2} \tag{9.31}$$

is a decreasing function of ζ, and by (9.15) so is

$$\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}. \tag{9.32}$$

Therefore so is their product which is (9.28). Also (9.29) and (9.30) are immediate consequences of (9.19) and (9.18), respectively. □

Lemma 9.8. *Let*

$$f(\zeta)=J\big(A(\zeta),\zeta\big)-\frac{I\big(A(\zeta),\zeta\big)}{\zeta+1}. \tag{9.33}$$

Then $f(\zeta)$ and $f'(\zeta)$ are both decreasing functions of ζ. $f(\zeta)$ decreases from $f(-1)=\pi/\sqrt{2}$ to $-\infty$ as $\zeta\to 1$. $f(0)>0$, and therefore $f(\zeta)$ has a unique zero $\zeta_0\in(0,1)$. Furthermore, one has

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{f(\zeta)}{(\zeta+1)^{1/2}}<0. \tag{9.34}$$

Proof. We recall that $\gamma=-\zeta$. Then (9.17) shows that $f(\zeta)$ is decreasing, and (9.16) yields

$$f'(\zeta)=\int_{-1}^{1}\sqrt{\frac{1-s}{1+s}}\left(\frac{-\frac{1}{2}\big(1-\frac{1}{4}(1-s)\big)}{\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{1/2}}-\frac{\frac{1}{2}}{\Big(2-\frac{1-\gamma}{2}(1-s)\Big)^{3/2}}\right)\mathrm{d}s. \tag{9.35}$$

The integrand is a strictly decreasing function of ζ, hence $f'(\zeta)$ is a decreasing function. $f(-1)$ and $f(1)$ can be obtained from Lemmas 8.1 and 9.1. Next we have

$$
\begin{aligned}
f(0) = 2\int_{-1}^{1} \frac{\mathrm{d}s}{\sqrt{1-s^2}} \left(\sqrt{2-\frac{1}{2}(1-s)} - \frac{1}{\sqrt{2-\frac{1}{2}(1-s)}} \right) \\
-\frac{1}{2}\int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} \sqrt{2-\frac{1}{2}(1-s)}\mathrm{d}s \\
= \frac{\sqrt{2}}{4}\int_{-1}^{1} \frac{\mathrm{d}s}{\sqrt{1-s^2}} \frac{(3+s)^2-8}{\sqrt{3+s}} \\
= \frac{\sqrt{2}}{4}\left(\int_{-1}^{-(3-2\sqrt{2})} + \int_{-(3-2\sqrt{2})}^{3-2\sqrt{2}} + \int_{3-2\sqrt{2}}^{1} \right) \frac{\mathrm{d}s}{\sqrt{1-s^2}} \frac{(3+s)^2-8}{\sqrt{3+s}}. \quad (9.36)
\end{aligned}
$$

The integrand is a strictly increasing function of $s \in (-1, 1)$, it is negative if $s < -(3-2\sqrt{2})$ and positive when $s > -(3-2\sqrt{2})$. To prove that $f(0) > 0$, it suffices to show that

$$
\begin{aligned}
&\Big(\int_{-1}^{-(3-2\sqrt{2})} + \int_{3-2\sqrt{2}}^{1} \Big) \frac{\mathrm{d}s}{\sqrt{1-s^2}} \frac{(3+s)^2-8}{\sqrt{3+s}} \\
&= \int_{3-2\sqrt{2}}^{1} \frac{\mathrm{d}s}{\sqrt{1-s^2}} \left(\frac{(3+s)^2-8}{\sqrt{3+s}} - \frac{(3-s)^2-8}{\sqrt{3-s}} \right) \\
&> 0. \qquad (9.37)
\end{aligned}
$$

This is true, because the integrand in the third integral of (9.37) is positive. Therefore $f(0) > 0$, and there is a unique $\zeta_0 \in (0, 1)$ with $f(\zeta_0) = 0$. As for (9.34), we note that

$$
\frac{\mathrm{d}}{\mathrm{d}\zeta} \frac{f(\zeta)}{\sqrt{\zeta+1}} = \frac{2f'(\zeta)(\zeta+1) - f(\zeta)}{2(\zeta+1)^{3/2}},
$$

and this is negative if

$$
\frac{f'(\zeta)}{f(\zeta)} > \frac{1}{2(\zeta+1)}, \quad \zeta > \zeta_0. \qquad (9.38)
$$

Set $F(\zeta) = -f(\zeta)$. Then $F(\zeta_0) = 0$, and F and F' both are strictly increasing functions. Consequently

$$
F'(\zeta) > \frac{F(\zeta)}{\zeta - \zeta_0}, \qquad (9.39)
$$

which implies

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{F'(\zeta)}{F(\zeta)} > \frac{1}{\zeta - \zeta_0} > \frac{1}{2(\zeta+1)}, \tag{9.40}$$

and we have derived (9.38), and therefore (9.34). □

Remark 9.9. From (9.35) one has

$$\begin{aligned} f'(-1) &= \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(J\big(A(\zeta),\zeta\big) - \frac{I\big(A(\zeta),\zeta\big)}{\zeta+1} \right)\Bigg|_{\zeta=-1} \\ &= -\frac{1}{2}\int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} \left(\frac{1-\frac{1}{4}(1-s)}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \right) \mathrm{d}s \\ &= -\frac{1}{8\sqrt{2}} \int_{-1}^{1} \sqrt{\frac{1-s}{1+s}} (5+s)\mathrm{d}s \\ &= -\frac{1}{8\sqrt{2}} \int_{-1}^{1} \left(4\sqrt{\frac{1-s}{1+s}} + \sqrt{1-s^2} \right) \mathrm{d}s \\ &= -\frac{9\pi}{16\sqrt{2}}, \end{aligned} \tag{9.41}$$

so

$$f'(-1) < -1, \tag{9.42}$$

and $f(\zeta)$ starts decreasing at $\zeta = -1$ quite rapidly.

Lemma 9.7 implies

$$\min_{\alpha_0,\zeta} \frac{I\big(A(\zeta),\zeta\big)}{(\zeta + \sin\alpha_0)^{3/2}} = \min_{\zeta} \frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}} = \frac{2}{3}, \tag{9.43}$$

and since $\Psi(\alpha_0, \zeta) > 0$, we have

Corollary 9.10. *For sufficiently large n there are no geodesics of the form* (9.26), (9.27) *joining* (x, y_0, t) *and* $(0, y_0, 0)$.

This refutes possibility (i) in Remark 6.2 and we are left with eliminating the second possibility which will prove Proposition 9.6.

Lemma 9.11. *Given a fixed positive integer q, it is not possible to join* (x, y_0, t) *and* $(0, y_0, 0)$ *by an infinite number of distinct geodesics which can* *be represented by*

$$(\hat{x}, \hat{t}) = \big(\Phi^{(q)}(\alpha_0, \zeta), \Psi^{(q)}(\alpha_0, \zeta)\big), \tag{9.44}$$

$\zeta \in (-1, 1)$, $\alpha_0 \in \big[A(\zeta), \pi/2\big]$.

Proof. For a given $\zeta \in (-1, 1)$, $\Psi^{(q)}(\alpha_0, \zeta)$ is a decreasing function of $\alpha_0 \in \big[A(\zeta), \pi/2\big]$, $\Psi^{(q)}(\alpha_0, \zeta)$ decreases from $\Psi^{(q)}\big(A(\zeta), \zeta\big) = \infty$ to

$$\min_{\alpha_0} \Psi^{(q)}(\alpha_0, \zeta) = \Psi^{(q)}\Big(\frac{\pi}{2}, \zeta\Big) = 2q \frac{I\big(A(\zeta), \zeta\big)}{(\zeta + 1)^{3/2}}. \tag{9.45}$$

Consequently,

$$\hat{t} = \Psi^{(q)}(\alpha_0, \zeta) \tag{9.46}$$

has a solution $\alpha_0(\hat{t}, \zeta)$ if and only if

$$\hat{t} \geq 2q \frac{I\big(A(\zeta), \zeta\big)}{(\zeta + 1)^{3/2}}. \tag{9.47}$$

If the solution $\alpha_0(\hat{t}, \zeta)$ exists, it is unique. According to the proof of the first statement of Lemma 9.7

$$\frac{I\big(A(\zeta), \zeta\big)}{(\zeta + 1)^{3/2}}$$

is a strictly decreasing function of ζ. So to have any $\zeta \in (-1, 1)$ for which (9.46) has a solution $\alpha_0(\hat{t}, \zeta)$, we must have

$$\hat{t} \geq 2q \frac{I\Big(-\frac{\pi}{2}, 1\Big)}{2^{3/2}} = \frac{4q}{3}. \tag{9.48}$$

In this case for each $\zeta \in [\zeta_{\hat{t}}, 1]$ (9.46) has a solution $\alpha_0(\hat{t}, \zeta)$, where $\zeta_{\hat{t}}$ is uniquely defined by

$$\hat{t} = 2q \frac{I\big(A(\zeta_{\hat{t}}), \zeta_{\hat{t}}\big)}{(\zeta_{\hat{t}} + 1)^{3/2}}. \tag{9.49}$$

Given $\alpha_0(\hat{t}, \zeta)$, $\zeta \in [\zeta_{\hat{t}}, 1]$, we shall show that

$$\hat{x} = \Phi^{(q)}\big(\alpha_0(\hat{t}, \zeta), \zeta\big) \tag{9.50}$$

can have, at most, a finite number of solutions ζ, $\zeta \in [\zeta_{\hat{t}}, 1]$. As in Lemma 6.4, in view of the analyticity of $\Phi^{(q)}\big(\alpha_0(\hat{t},\zeta),\zeta\big)$ in the ζ variable, it suffices to show that the end points of the interval $[\zeta_{\hat{t}}, 1]$ cannot be limit points of the set of solutions of (9.50). We note that

$$\alpha_0(\hat{t},\zeta_{\hat{t}}) = \frac{\pi}{2}. \tag{9.51}$$

Also, $\alpha_0(\hat{t},1)$ is the solution of

$$\hat{t} = \Psi^{(q)}\big(\alpha_0(\hat{t},1),1\big), \tag{9.52}$$

so, $-\pi/2 = A(1) < \alpha_0(\hat{t},1) \le \pi/2$. Since

$$\Phi^{(q)}\big(\alpha_0(\hat{t},1),1\big) = -\infty, \tag{9.53}$$

the upper end point of $[\zeta_{\hat{t}}, 1]$ cannot be a limit point of solutions of (9.50). As for the lower end point $\zeta_{\hat{t}}$, we shall show that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Phi^{(q)}\big(\alpha_0(\hat{t},\zeta),\zeta\big)\Big|_{\zeta=\zeta_{\hat{t}}} \neq 0, \tag{9.54}$$

and therefore $\zeta_{\hat{t}}$ cannot be a limit point of solutions of (9.50). This will prove Lemma 9.11. First we need the ζ derivative of $\Psi^{(q)}\big(\alpha_0(\hat{t},\zeta),\zeta\big)$ at $\zeta = \zeta_{\hat{t}}$ to obtain $\alpha_0'(\zeta_{\hat{t}})$, where we set $\alpha_0(\zeta) = \alpha_0(\hat{t},\zeta)$. Recall that $\alpha_0(\hat{t},\zeta_{\hat{t}}) = \pi/2$, so

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\Psi\big(\alpha_0(\zeta),\zeta\big) + 2q\frac{I\big(A(\zeta),\zeta\big)}{\big(\zeta+\sin\alpha_0(\zeta)\big)^{3/2}}\right)\Bigg|_{\alpha_0(\zeta)=\frac{\pi}{2}} \\ &= \left(\frac{\partial\Psi}{\partial\alpha_0}\big(\alpha_0(\zeta),\zeta\big)\Big|_{\alpha_0(\zeta)=\frac{\pi}{2}}\right)\alpha_0'(\zeta) + 2q\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}, \end{aligned}$$

or,

$$0 = -\frac{2}{\zeta_{\hat{t}}+1}\alpha_0'(\zeta_{\hat{t}}) + 2q\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}\Big|_{\zeta=\zeta_{\hat{t}}}.$$

Next we find (9.54):

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\Phi\big(\alpha_0(\zeta),\zeta\big)+2q\frac{J\big(A(\zeta),\zeta\big)}{\big(\zeta+\sin\alpha_0(\zeta)\big)^{1/2}}\right)\Bigg|_{\alpha_0(\zeta)=\frac{\pi}{2}}\\
&=\left(\frac{\partial\Phi}{\partial\alpha_0}\big(\alpha_0(\zeta),\zeta\big)\Bigg|_{\alpha_0(\zeta)=\frac{\pi}{2}}\right)\alpha_0'(\zeta)+2q\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+1)^{1/2}}\\
&=-\frac{2}{\zeta+1}\alpha_0'(\zeta)+2q\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+1)^{1/2}},
\end{aligned}$$

and therefore one has

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Phi^{(q)}\big(\alpha_0(\hat{t},\zeta),\zeta\big)\Big|_{\zeta=\zeta_{\hat{t}}}=2q\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+1)^{1/2}}-\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}\right]_{\zeta=\zeta_{\hat{t}}}<0,$$

see (9.34). This proves Lemma 9.11 and we have completed the proof of Proposition 9.6. □

Geodesics with $\alpha_{\text{end}}=\pi-\alpha_0$ after $n+\frac{1}{2}$ periods, $\dot{y}(0)<0$.

These are the type (iii) curves of the introduction to chapter 9. To obtain their formulas we use (2.63) and (2.64) again while carefully following the motion of $\alpha\in\big[A(\zeta),\pi-A(\zeta)\big]$ as described in the paragraph after (2.48). Since $\alpha_0\in\big[\pi/2,\pi-A(\zeta)\big]$, after one-half period (9.1) yields

$$\begin{aligned}
x(\pi-\alpha_0)=&\frac{1}{2\tau^{1/2}}\left(\int_{\alpha_0}^{\pi-A(\zeta)}-\int_{\pi-A(\zeta)}^{A(\zeta)}+\int_{A(\zeta)}^{\pi-\alpha_0}\right)\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}\\
=&\frac{1}{2\tau^{1/2}}\left(2\int_{A(\zeta)}^{\pi-\alpha_0}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}+\int_{A(\zeta)}^{\pi-A(\zeta)}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}\right).
\end{aligned}\tag{9.55}$$

One full period is

$$\frac{2}{2\tau^{1/2}}\int_{A(\zeta)}^{\pi-A(\zeta)}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}=\frac{J\big(A(\zeta),\zeta\big)}{\tau^{1/2}},$$

so adding n full periods one finds

$$x(\pi-\alpha_0)=\frac{1}{\tau^{1/2}}\Big(\int_{A(\zeta)}^{\pi-\alpha_0}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}+\Big(n+\frac{1}{2}\Big)J\big(A(\zeta),\zeta\big)\Big).$$

We change α_0 to $\alpha_0'=\pi-\alpha_0$, so $\alpha_0'\in\big[A(\zeta),\pi/2\big)$, and

$$x(\alpha_0')=\frac{1}{\tau^{1/2}}\Big(\int_{A(\zeta)}^{\alpha_0'}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}+\Big(n+\frac{1}{2}\Big)J\big(A(\zeta),\zeta\big)\Big),$$

which leads to

$$\hat{x}=\frac{2}{(\zeta+\sin\alpha_0)^{1/2}}\Big(\int_{A(\zeta)}^{\alpha_0}\frac{\sin\alpha d\alpha}{(\zeta+\sin\alpha)^{1/2}}+\Big(n+\frac{1}{2}\Big)J\big(A(\zeta),\zeta\big)\Big), \tag{9.56}$$

and, similarly,

$$\hat{t}=\frac{2}{(\zeta+\sin\alpha_0)^{3/2}}\Big(\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{1/2}\sin\alpha d\alpha+\Big(n+\frac{1}{2}\Big)I\big(A(\zeta),\zeta\big)\Big), \tag{9.57}$$

where we dropped the $'$ from α_0'. Note that for a fixed $\alpha_0\in[-\pi/2,\pi/2]$ we have

$$-\sin\alpha_0\le\zeta<1. \tag{9.58}$$

$\hat{x}$ and $\hat{t}$ represent fixed numbers, but we shall use $\hat{x}^{(n)}(\alpha_0,\zeta)$ and $\hat{t}^{(n)}(\alpha_0,\zeta)$ to represent the right hand sides of (9.56) and (9.57), respectively.

Proposition 9.12. *(x,y_0,t) cannot be joined to $(0,y_0,0)$ by an infinite number of distinct geodesics represented by* (9.56), (9.57).

The proof again consists of refuting the 2 possibilities of Remark 6.2.

Lemma 9.13. *$\hat{t}^{(n)}(\alpha_0,\zeta)$ is a decreasing function of ζ. It decreases from $\hat{t}^{(n)}(\alpha_0,-\sin\alpha_0)=\infty$ to $\hat{t}^{(n)}(\alpha_0,1)>0$ in the interval* (9.58), *$\zeta\in[-\sin\alpha_0,1)$. More precisely one has*

$$\hat{t}^{(n)}(\alpha_0,\zeta)\ge\hat{t}^{(n)}(\alpha_0,1)>\frac{2(2-\sqrt{2})}{3}+\frac{4n}{3}. \tag{9.59}$$

Proof. We start with

$$\frac{\partial}{\partial\zeta}\frac{\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha_0)^{3/2}}$$
$$=\frac{\frac{1}{2}\int_{A(\zeta)}^{\alpha_0}\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}}{(\zeta+\sin\alpha_0)^{3/2}}-\frac{3}{2}\frac{\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha_0)^{5/2}}$$
$$=\frac{1}{2(\zeta+\sin\alpha_0)^{5/2}}\int_{A(\zeta)}^{\alpha_0}\big(\zeta+\sin\alpha_0-3(\zeta+\sin\alpha)\big)\frac{\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha)^{1/2}}$$
$$=\frac{1}{2(\zeta+\sin\alpha_0)^{5/2}}\int_{A(\zeta)}^{\alpha_0}\left(\frac{\sin\alpha_0-\sin\alpha}{(\zeta+\sin\alpha)^{1/2}}-2(\zeta+\sin\alpha)^{1/2}\right)\sin\alpha\mathrm{d}\alpha.$$

We integrate by parts:

$$\int_{A(\zeta)}^{\alpha_0}\frac{\sin\alpha_0-\sin\alpha}{(\zeta+\sin\alpha)^{1/2}}\sin\alpha\mathrm{d}\alpha$$
$$=2\int_{A(\zeta)}^{\alpha_0}(\sin\alpha_0-\sin\alpha)\tan\alpha\mathrm{d}(\zeta+\sin\alpha)^{1/2}$$
$$=-2\int_{A(\zeta)}^{\alpha_0}\left(-\sin\alpha+(\sin\alpha_0-\sin\alpha)\frac{1}{\cos^2\alpha}\right)(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha$$
$$=2\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha-2\int_{A(\zeta)}^{\alpha_0}(\sin\alpha_0-\sin\alpha)\frac{(\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha}{\cos^2\alpha},$$

so one has

$$\frac{\partial}{\partial\zeta}\frac{\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)\sin\alpha\mathrm{d}\alpha}{(\zeta+\sin\alpha_0)^{3/2}}=-\frac{\int_{A(\zeta)}^{\alpha_0}(\sin\alpha_0-\sin\alpha)(\zeta+\sin\alpha)^{1/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha}}{(\zeta+\sin\alpha_0)^{5/2}}$$
$$<0. \tag{9.60}$$

According to Lemma 9.7

$$\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+\sin\alpha_0)^{3/2}}$$

is a decreasing function of ζ, so we have the first statement of Lemma 9.13. In view of (9.60)

$$\min_{\alpha_0,\zeta}\hat{t}^{(0)}(\alpha_0,\zeta)=\min_{\alpha_0}\hat{t}^{(0)}(\alpha_0,1)=\min_{\alpha_0}\frac{2\int_{-\pi/2}^{\alpha_0}(1+\sin\alpha)^{1/2}\sin\alpha\mathrm{d}\alpha+\frac{4\sqrt{2}}{3}}{(1+\sin\alpha_0)^{3/2}},$$

where we used (9.18). The numerator is always positive. Indeed, it is smallest at $\alpha_0 = 0$, where

$$\int_{-\pi/2}^{0} (1+\sin\alpha)^{1/2} \sin\alpha \mathrm{d}\alpha = \int_{-1}^{0} \sqrt{1+x}\frac{x\mathrm{d}x}{\sqrt{1-x^2}} = \frac{2}{3}(\sqrt{2}-2),$$

and therefore

$$\int_{-\pi/2}^{0} (1+\sin\alpha)^{1/2} \sin\alpha \mathrm{d}\alpha + \frac{2\sqrt{2}}{3} = \frac{4}{3}(\sqrt{2}-1) > 0. \tag{9.61}$$

Consequently,

$$\min_{\alpha_0,\zeta} \hat{t}^{(0)}(\alpha_0,\zeta) > \frac{\frac{8}{3}(\sqrt{2}-1)}{2^{3/2}} = \frac{2(2-\sqrt{2})}{3} > 0,$$

and,

$$\min_{\alpha_0,\zeta} \hat{t}^{(n)}(\alpha_0,\zeta) > \frac{2(2-\sqrt{2})}{3} + \frac{n\frac{8\sqrt{2}}{3}}{2\sqrt{2}} = \frac{2(2-\sqrt{2})}{3} + \frac{4n}{3}$$

which is (9.59). Finally, $(\zeta+\sin\alpha_0)^{3/2}\hat{t}^{(n)}(\alpha_0,\zeta)$ is positive and bounded away from zero, so $\hat{t}^{(n)}(\alpha_0,-\sin\alpha_0) = \infty$, and we have completed the proof of Lemma 9.13. □

Clearly, (9.59) yields

Corollary 9.14. *For sufficiently large n, there are no geodesics of the form* (9.56), (9.57) *which join a fixed (x, y_0, t) to $(0, y_0, 0)$.*

We still need to prove that the second possibility of Remark 6.2 cannot occur; that will prove Proposition 9.12.

Lemma 9.15. *$\partial\hat{t}^{(n)}(\alpha_0,\zeta)/\partial\alpha_0$ has a unique zero $\alpha_{0,\zeta}^{(n)}$ in the interval $\big[A(\zeta),\pi/2\big]$. Indeed,*

$$0 < \alpha_{0,\zeta}^{(n)} < \frac{\pi}{2}. \tag{9.62}$$

For each fixed ζ, $\hat{t}^{(n)}(\alpha_0,\zeta)$ decreases from $\hat{t}^{(n)}\big(A(\zeta),\zeta\big) = \infty$ to $\hat{t}^{(n)}(\alpha_{0,\zeta}^{(n)},\zeta)$ > 0, then increases to

$$\hat{t}^{(n)}\Big(\frac{\pi}{2},\zeta\Big) = 2(n+1)\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}. \tag{9.63}$$

Proof. We have

$$\frac{\partial \hat{t}^{(n)}(\alpha_0,\zeta)}{\partial \alpha_0} = \frac{\cos\alpha_0}{\zeta+\sin\alpha_0}\big(2\tan\alpha_0 - 3\hat{t}^{(n)}(\alpha_0,\zeta)\big). \tag{9.64}$$

We note that

$$\frac{\partial \hat{t}^{(n)}(\alpha_0,\zeta)}{\partial \alpha_0} < 0, \qquad \alpha_0 \le 0. \tag{9.65}$$

Also, (9.64) yields

$$\frac{\partial \hat{t}^{(n)}}{\partial \alpha_0}\big(A(\zeta),\zeta\big) = -\infty, \quad \frac{\partial \hat{t}^{(n)}}{\partial \alpha_0}\Big(\frac{\pi}{2},\zeta\Big) = \frac{2}{\zeta+1} > 0. \tag{9.66}$$

Consequently, $\partial\hat{t}^{(n)}/\partial\alpha_0$ has at least one zero. To see that it has no more than one zero, we note that

$$\frac{\partial}{\partial \alpha_0}\frac{(\zeta+\sin\alpha_0)^{1/2}}{\cos\alpha_0}\frac{\partial \hat{t}^{(n)}}{\partial \alpha_0} = \frac{2}{\cos^2\alpha_0}(\zeta+\sin\alpha_0)^{3/2} > 0. \tag{9.67}$$

Therefore $\partial\hat{t}^{(n)}/\partial\alpha_0$ has exactly one zero, and it is in $(0,\pi/2)$ according to (9.65). This completes the proof of Lemma 9.15. □

We need Lemma 9.15 only when $\zeta = 1$. The next result is not needed in the argument but it helps to clarify the structure.

Lemma 9.16. *$\alpha_{0,\zeta}^{(n)}$ is a decreasing function of ζ,*

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\alpha_{0,\zeta}^{(n)} < 0. \tag{9.68}$$

Proof. We drop the "n" in $\alpha_{0,\zeta}^{(n)}$, and use

$$\alpha_{0,\zeta}' = \frac{\mathrm{d}\alpha_{0,\zeta}}{\mathrm{d}\zeta}, \tag{9.69}$$

and note that $\alpha_0 = \alpha_{0,\zeta}$ is the unique solution of

$$2\tan\alpha_0 - 3\hat{t}^{(n)}(\alpha_0,\zeta) = 0. \tag{9.70}$$

Therefore,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\zeta}\big(2\tan\alpha_{0,\zeta} - 3\hat{t}^{(n)}(\alpha_{0,\zeta},\zeta)\big)$$
$$= \frac{\partial}{\partial\alpha_0}\big(2\tan\alpha_0 - 3\hat{t}^{(n)}(\alpha_0,\zeta)\big)\Big|_{\alpha_0=\alpha_{0,\zeta}} \cdot \alpha_{0,\zeta}' + \frac{\partial}{\partial\zeta}\big(2\tan\alpha_{0,\zeta} - 3\hat{t}^{(n)}(\alpha_{0,\zeta},\zeta)\big),$$

thus

$$\frac{\partial}{\partial\alpha_0}\big(2\tan\alpha_0 - 3\hat{t}^{(n)}(\alpha_0,\zeta)\big)\Big|_{\alpha_0=\alpha_{0,\zeta}} \cdot \alpha_{0,\zeta}' = 3\frac{\partial\hat{t}^{(n)}}{\partial\zeta}(\alpha_{0,\zeta},\zeta) < 0,$$

or,

$$\Big(\frac{2}{\cos^2\alpha_{0,\zeta}} - 3\frac{\partial\hat{t}^{(n)}}{\partial\alpha_0}(\alpha_{0,\zeta},\zeta)\Big)\alpha_{0,\zeta}' < 0. \tag{9.71}$$

Integration by parts yields

$$\int_{A(\zeta)}^{\alpha_0} \sin\alpha\big((\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha\big)$$
$$= \frac{2}{3}\tan\alpha_0(\zeta+\sin\alpha_0)^{3/2} - \frac{2}{3}\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{3/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha},$$

and therefore,

$$\frac{\partial\hat{t}^{(n)}}{\partial\alpha_0} = \frac{2\cos\alpha_0}{(\zeta+\sin\alpha_0)^{5/2}}\left(\int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{3/2}\frac{\mathrm{d}\alpha}{\cos^2\alpha} - \frac{3}{2}\Big(n+\frac{1}{2}\Big)I\big(A(\zeta),\zeta\big)\right).$$

We integrate by parts again,

$$\int_{A(\zeta)}^{\alpha_0}\frac{\zeta+\sin\alpha}{\cos^2\alpha}\big((\zeta+\sin\alpha)^{1/2}\mathrm{d}\alpha\big) = \frac{2}{3}\int_{A(\zeta)}^{\alpha_0}\frac{\zeta+\sin\alpha}{\cos^3\alpha}\mathrm{d}(\zeta+\sin\alpha)^{3/2}$$
$$= \frac{2}{3}\frac{(\zeta+\sin\alpha_0)^{5/2}}{\cos^3\alpha_0} - \int_{A(\zeta)}^{\alpha_0}(\zeta+\sin\alpha)^{3/2}\left(\frac{1}{\cos^2\alpha} + \frac{3\sin\alpha(\zeta+\sin\alpha)}{\cos^4\alpha}\right)\mathrm{d}\alpha.$$

Therefore,

$$\frac{\partial\hat{t}^{(n)}}{\partial\alpha_0} = \frac{2\cos\alpha_0}{(\zeta+\sin\alpha_0)^{5/2}}\left(\frac{2}{3}\frac{(\zeta+\sin\alpha_0)^{5/2}}{\cos^3\alpha_0}\right.$$
$$\left. - \int_{A(\zeta)}^{\alpha_0}(1+3\zeta\sin\alpha+2\sin^2\alpha)\frac{(\zeta+\sin\alpha)^{3/2}}{\cos^4\alpha}\mathrm{d}\alpha - \frac{3}{2}\Big(n+\frac{1}{2}\Big)I\big(A(\zeta),\zeta\big)\right),$$

and

$$\begin{aligned}&\frac{2}{\cos^2\alpha_0}-3\frac{\partial\hat{t}^{(n)}}{\partial\alpha_0}(\alpha_0,\zeta)\\&=\frac{3\cos\alpha_0}{(\zeta+\sin\alpha_0)^{5/2}}\left(\int_{A(\zeta)}^{\alpha_0}(1+3\zeta\sin\alpha+2\sin^2\alpha)\frac{(\zeta+\sin\alpha)^{3/2}}{\cos^2\alpha}\mathrm{d}\alpha\right.\\&\qquad\left.+\frac{3}{2}\Big(n+\frac{1}{2}\Big)I\big(A(\zeta),\zeta\big)\right)\\&>0.\end{aligned}$$

Indeed, with $-\zeta<\sin\alpha$,

$$1+3\zeta\sin\alpha+2\sin^2\alpha>1-3\sin^2\alpha+2\sin^2\alpha>0.$$

In view of (9.71) we have (9.68) and Lemma 9.16. □

Definition 9.17. Fix an $\alpha_0\in[-\pi/2,\pi/2]$ and a $\hat{t}>\hat{t}^{(n)}(\alpha_0,1)$. We shall denote by $\zeta^{(n)}(\alpha_0,\hat{t})$ the unique solution of

$$\hat{t}=\hat{t}^{(n)}(\alpha_0,\zeta). \tag{9.72}$$

Our results on $\hat{t}^{(n)}$ imply the following results on $\zeta^{(n)}$.

Lemma 9.18. (i) *Let*

$$\hat{t}^{(n)}(\alpha_{0,1}^{(n)},1)\le\hat{t}\le\hat{t}^{(n)}\Big(\frac{\pi}{2},1\Big). \tag{9.73}$$

Then $\zeta^{(n)}$ exists when α_0 is in the interval

$$\alpha_0^{(1)}\le\alpha_0\le\alpha_0^{(2)}, \tag{9.74}$$

where

$$\hat{t}=\hat{t}^{(n)}(\alpha_0^{(j)},1),\qquad j=1,2. \tag{9.75}$$

We note that

$$\alpha_0^{(1)}\le\alpha_{0,1}^{(n)}\le\alpha_0^{(2)}. \tag{9.76}$$

(ii) *Let*

$$\hat{t} > \hat{t}^{(n)}\Big(\frac{\pi}{2}, 1\Big). \tag{9.77}$$

Then $\zeta^{(n)}(\alpha_0, \hat{t})$ *exists for all* α_0 *in the interval*

$$\alpha_0^{(1)} \le \alpha_0 \le \frac{\pi}{2}, \tag{9.78}$$

where $\alpha_0^{(1)}$ *is the solution of (9.75).*

(iii) *No solution* $\zeta^{(n)}(\alpha_0, \hat{t})$ *exists when*

$$\hat{t} < \hat{t}^{(n)}(\alpha_{0,1}^{(n)}, 1). \tag{9.79}$$

Lemma 9.19. *Given a fixed positive integer* q, (x, y_0, t) *cannot be joined to* $(0, y_0, 0)$ *by an infinite number of distinct geodesics which have representations in the following form:*

$$(\hat{x}, \hat{t}) = \big(\hat{x}^{(q)}(\alpha_0, \zeta), \hat{t}^{(q)}(\alpha_0, \zeta)\big), \tag{9.80}$$

$\alpha_0 \in [-\pi/2, \pi/2]$, $\zeta \in (-\sin\alpha_0, 1)$.

Proof. As $\hat{x}^{(q)}\big(\alpha_0, \zeta(\alpha_0)\big)$ is an analytic function of α_0 in its interval of existence, it suffices to show that the end points of that interval are not limit points of solutions α_0 of

$$\hat{x} = \hat{x}^{(q)}\big(\alpha_0, \zeta(\alpha_0)\big); \tag{9.81}$$

here we used $\zeta(\alpha_0) = \zeta^{(n)}(\alpha_0, \hat{t})$. Note that

$$\lim_{\alpha_0 \to \alpha_0^{(j)}} \zeta(\alpha_0) = 1, \qquad j = 1, 2, \tag{9.82}$$

and therefore

$$\lim_{\alpha_0 \to \alpha_0^{(j)}} \hat{x}^{(q)}\big(\alpha_0, \zeta(\alpha_0)\big) = -\infty, \qquad j = 1, 2. \tag{9.83}$$

So $\alpha_0^{(j)}$, $j = 1, 2$ cannot be limit points of solutions of (9.81). This leaves us with the end point $\alpha_0 = \pi/2$. Here (9.77) implies that

$$-1 < \zeta\Big(\frac{\pi}{2}\Big) < 1. \tag{9.84}$$

We shall show that

$$\frac{\mathrm{d}\hat{x}^{(q)}}{\mathrm{d}\alpha_0}\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big) \neq 0, \tag{9.85}$$

which proves that the end point $\alpha_0 = \pi/2$ cannot be a limit point of solutions of (9.81). Differentiating $\hat{t} = \hat{t}^{(q)}\big(\alpha_0, \zeta(\alpha_0)\big)$ we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}\hat{t}^{(q)}}{\mathrm{d}\alpha_0}\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big)\\
&= \frac{\partial\hat{t}^{(q)}}{\partial\alpha_0}\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big) + \frac{\partial\hat{t}^{(q)}}{\partial\zeta}\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big)\cdot\zeta'\Big(\frac{\pi}{2}\Big),\\
&= \frac{2}{\zeta\big(\frac{\pi}{2}\big)+1} + 2(q+1)\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{I\big(A(\zeta),\zeta\big)}{(\zeta+1)^{3/2}}\bigg|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}\cdot\zeta'\Big(\frac{\pi}{2}\Big),
\end{aligned}$$

which gives us $\zeta'(\pi/2)$; note that $\zeta'(\pi/2) > 0$. Next, we differentiate $\hat{x}^{(q)}\big(\alpha_0, \zeta(\alpha_0)\big)$ at $\alpha_0 = \pi/2$:

$$\begin{aligned}
&\frac{\mathrm{d}\hat{x}^{(q)}}{\mathrm{d}\alpha_0}\Big(\frac{\pi}{2}, \zeta\Big(\frac{\pi}{2}\Big)\Big)\\
&= \frac{2}{\zeta\big(\frac{\pi}{2}\big)+1} + 2(q+1)\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+1)^{1/2}}\bigg|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}\cdot\zeta'\Big(\frac{\pi}{2}\Big)\\
&= \frac{2}{\zeta\big(\frac{\pi}{2}\big)+1} + 2(q+1)\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{J\big(A(\zeta),\zeta\big)}{(\zeta+1)^{1/2}}\bigg|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}\frac{-\frac{1}{\zeta\left(\frac{\pi}{2}\right)+1}}{(q+1)\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{I\left(A(\zeta),\zeta\right)}{(\zeta+1)^{3/2}}\Big|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}}\\
&= \frac{2\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{I\left(A(\zeta),\zeta\right)}{(\zeta+1)^{3/2}} - \frac{J\left(A(\zeta),\zeta\right)}{(\zeta+1)^{1/2}}\right)\Big|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}}{\left(\zeta\left(\frac{\pi}{2}\right)+1\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{I\left(A(\zeta),\zeta\right)}{(\zeta+1)^{3/2}}\Big|_{\zeta=\zeta\left(\frac{\pi}{2}\right)}}\\
&> 0
\end{aligned}$$

in view of (9.34) and the proof of Lemma 9.7. Note that $\zeta(\pi/2) = \zeta^{(q)}(\pi/2, \hat{t})$, so the derivative does depend on q. This gives (9.85), and we have completed the proof of Lemma 9.19, and established Proposition 9.12. □

$y_0 = 0$.

For geodesics that leave the origin one always has

$$-1 \leq \zeta \leq 1, \tag{9.86}$$

and (2.61) implies that

$$\alpha_0 = A(\zeta) = \mathrm{Arcsin}(-\zeta). \tag{9.87}$$

Formulas (2.62)–(2.64) hold with

$$A(\zeta) \leq \alpha \leq \pi - A(\zeta). \tag{9.88}$$

A given geodesic with parameters (ζ, τ) returns to the plane $y = 0$ at

$$\alpha = \pi - A(\zeta) \text{ after } n \text{ periods}, \qquad n = 0, 1, 2, \dots \tag{9.89}$$

$$\alpha = A(\zeta) \text{ after } n \text{ periods}, \qquad n = 1, 2, \dots. \tag{9.90}$$

We may assume that $\dot{y}(\alpha_0) > 0$.

(i) $\alpha = \pi - A(\zeta)$ after n periods, $n = 0, 1, 2, \dots$. Here

$$x = \frac{1}{2\tau^{1/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}} + \frac{2n}{2\tau^{1/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} \frac{\sin\alpha \mathrm{d}\alpha}{(\zeta + \sin\alpha)^{1/2}},$$

$$t = \frac{1}{2\tau^{3/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} (\zeta + \sin\alpha)^{1/2} \sin\alpha \mathrm{d}\alpha + \frac{2n}{2\tau^{3/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} (\zeta + \sin\alpha)^{1/2} \sin\alpha \mathrm{d}\alpha,$$

or,

$$x = (2n+1) \frac{J\big(A(\zeta), \zeta\big)}{2\tau^{1/2}}, \tag{9.91}$$

$$t = (2n+1) \frac{I\big(A(\zeta), \zeta\big)}{2\tau^{3/2}}. \tag{9.92}$$

(ii) $\alpha = A(\zeta)$ after n periods, $n = 1, 2, \ldots$. Here

$$x = \frac{2n}{2\tau^{1/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} \frac{\sin\alpha d\alpha}{(\zeta + \sin\alpha)^{1/2}} = 2n \frac{J\big(A(\zeta), \zeta\big)}{2\tau^{1/2}}, \tag{9.93}$$

$$t = \frac{2n}{2\tau^{3/2}} \int_{A(\zeta)}^{\pi - A(\zeta)} (\zeta + \sin\alpha)^{1/2} \sin\alpha d\alpha = 2n \frac{I\big(A(\zeta), \zeta\big)}{2\tau^{3/2}}. \tag{9.94}$$

Combining (9.91)–(9.94) we have

$$x = p \frac{J\big(A(\zeta), \zeta\big)}{2\tau^{1/2}}, \tag{9.95}$$

$$t = p \frac{I\big(A(\zeta), \zeta\big)}{2\tau^{3/2}}, \tag{9.96}$$

$p = 1, 2, \ldots$. Formally this mapping agrees with (9.3), (9.4), and Proposition 9.2 yields

Proposition 9.20. (i) *For each $p = 1, 2, \ldots$, the mapping*

$$(\alpha_0, \tau) \longrightarrow (x, t) = p \left(\frac{J\big(A(\zeta), \zeta\big)}{2\tau^{1/2}}, \frac{I\big(A(\zeta), \zeta\big)}{2\tau^{3/2}} \right) \tag{9.97}$$

sends the domain $(-\pi/2, \pi/2) \times (0, \infty)$ onto the half plane $(x, 0, t)$, $t > 0$ in a 1–1 manner; $\alpha_0 = A(\zeta)$ replaces the parameter ζ.

(ii) *When $A(\zeta) = \pi/2$, i.e. $\zeta = -1$, we have the 1–1 mapping*

$$\tau \longrightarrow (x, 0, 0) = p \left(\frac{J\left(\frac{\pi}{2}, -1\right)}{2\tau^{1/2}}, 0, 0 \right) = p \Big(\frac{\sqrt{2}\pi}{2\tau^{1/2}}, 0, 0 \Big) \tag{9.98}$$

of $(0, \infty)$ onto the half line $x \in (0, \infty)$ for each $p = 1, 2, \ldots$.

(iii) *When $\xi = 1$ ($\zeta = -1$) and $\tau > 0$, Hamilton's equations give us another set of geodesics*

$$x(s) = s, \quad y = 0, \quad t = 0, \tag{9.99}$$

see (2.51), with $x(x) = x > 0$; recall that s is arclength. For each fixed $x \in (0, \infty)$ this geodesic may be looked upon as a continuous set of geodesics joining $(x, 0, 0)$ to $(0, 0, 0)$, one for each $\tau \in (0, \infty)$.

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Department of Mathematics, University of Paris VI, France.
E-mail: gaveau@ccr.jussieu.fr

Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada.
E-mail: greiner@math.toronto.edu