

GREEN'S FUNCTION OF BOLTZMANN EQUATION, 3-D WAVES

BY

TAI-PING LIU AND SHIH-HSIEN YU

Abstract

We study the Green's function for the linearized Boltzmann equation. For the short-time period, the Green's function is dominated by the particle-like waves; and for large-time, by the fluid-like waves exhibiting the weak Huygens principle. The fluid-like waves are constructed by the spectral analysis and complex analytic techniques, making uses of the rotational symmetry of the equation in the space variables. The particle-like waves are constructed by a Picard iteration, making uses of the exchange of regularity in the microscopic velocity with the regularity in the space variables through a Mixture Lemma. We obtain the pointwise estimates in the space and time variables of the Green's function through a long-short waves and particle-wave decompositions.

1. Introduction

Consider the Boltzmann equation for the hard sphere model

$$f_t + \xi \cdot \nabla_x f = B(f, f), \quad (x, t, \xi) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3, \quad (1.1)$$

Received May 30, 2005 and in revised form December 11, 2005.

Communicated by Jih-Hsin Cheng.

AMS Subject Classification: 76P05, 35L65, 82C40.

Key words and phrases: Boltzmann equation, Green's function, Mixture Lemma, particle-fluid waves, weak Huygens principle.

The research of the first author is supported in part by Institute of Mathematics, Academia Sinica, Taipei and NSF Grant DMS-9803323. The research of the second author is supported by CityU Strategic Research Grant 7001565.

$$\begin{aligned}
B(f, g) \equiv & \frac{1}{2} \int_{\substack{\mathbb{R}^3 \times \mathbb{S}^2 \\ (\xi - \xi_*) \cdot \Omega > 0}} [-f(\xi)g(\xi_*) - f(\xi_*)g(\xi) + f(\xi')g(\xi'_*) + f(\xi'_*)g(\xi')] \\
& \times |(\xi - \xi_*) \cdot \Omega| d\xi_* d\Omega, \\
& \begin{cases} \xi' = \frac{\xi + \xi_*}{2} - \{\Omega \cdot (\xi - \xi_*)\}\Omega, \\ \xi'_* = \frac{\xi + \xi_*}{2} + \{\Omega \cdot (\xi - \xi_*)\}\Omega. \end{cases}
\end{aligned}$$

We consider the perturbation of the Boltzmann solution around a Maxwellian M :

$$\begin{cases} \mathbf{f} = M + \sqrt{M}\mathbf{u}, \\ \partial_t \mathbf{u} + \xi \cdot \nabla_x \mathbf{u} = L\mathbf{u} + \Gamma(\mathbf{u}). \end{cases} \quad (1.2)$$

The Maxwellian is global and, for simplicity, its mean velocity is assumed to be zero, and the Boltzmann constant, its density and temperature are assumed to be one:

$$M \equiv \frac{e^{-\frac{|\xi|^2}{2}}}{(2\pi)^{3/2}}. \quad (1.3)$$

L is the linearized collision operator:

$$L\mathbf{h} \equiv \frac{2B\left(M, \sqrt{M}\mathbf{h}\right)}{\sqrt{M}}, \quad (1.4)$$

and Γ is the nonlinear term

$$\Gamma(\mathbf{h}) \equiv \frac{B(\sqrt{M}\mathbf{h}, \sqrt{M}\mathbf{h})}{\sqrt{M}}. \quad (1.5)$$

The main goal of the present paper is to study the wave structure of the solutions of the Boltzmann equation, (1.1), or, equivalently, (1.2). We are interested in the precise, pointwise structure of the solutions around a Maxwellian state. There are particle-like, entropy, rotational, and Huygens waves. For the study of the basic structure of these waves, it suffices to consider the linearized Boltzmann equation

$$\begin{cases} \partial_t \mathbf{g} + \xi \cdot \nabla_x \mathbf{g} = L\mathbf{g}, \\ \mathbf{g}(x, 0, \xi) = \mathbf{g}_0(x, \xi), \end{cases}$$

and study its Green's function:

$$\begin{cases} \partial_t \mathbb{G} + \xi \cdot \nabla_x \mathbb{G} = L\mathbb{G}, \\ \mathbb{G}(x, 0, \xi, \xi_0) = \delta(x)\delta(\xi - \xi_0). \end{cases} \quad (1.6)$$

There are two main reasons for studying the Green's function. The first is that the general solution can be expressed as the convolution of the initial data and the source with the Green's function. In fact, one may solve (1.2) by this approach. The second reason is particular to Boltzmann equation, namely to study its particle-fluid dual property. In fact, Boltzmann equation occupies the middle ground between the interacting particles system and the continuum equations of fluid dynamics. It is well-known that the fluid dynamics equations can be derived from the Boltzmann equation through various limits of zero Knudsen number, see [15, 39] and references therein. On the other hand, the particle-like behavior of the Boltzmann equation allows for its description of phenomena such as the ghost effects and thermal creep that the fluid dynamics equations cannot, [31]. The Green's function describes the dispersion and dissipation around a thermo-equilibrium state M of the particles that initially concentrate at the origin $x = 0$ with uniform microscopic velocity ξ_0 . The short-time behavior is dominated by the particle-like waves. For large-time, the dissipative fluid-like behavior emerges. It is the transition from particle-like behavior to fluid-like behavior that the Green's function describes. Thus we aim at the pointwise description of the Green's function. With sufficient understanding of the Green's function, it is possible to study the wave structure of solutions to the full Boltzmann equation.

The linearized collision operator (1.4) consists of a multiplicative operator $\nu(\xi)$ and an integral operator \mathbb{K} , (2.1),

$$Lg(\xi) = -\nu(\xi)g(\xi) + \mathbb{K}g(\xi).$$

For the hard sphere model,

$$\nu(\xi) \sim 1 + |\xi|.$$

The integral operator \mathbb{K} is compact and its kernel $\mathbb{K}(\xi, \xi_*)$ has singularity at $\xi = \xi_*$, (2.1). the hyperbolic nature of the Boltzmann equation, the kernel

is written as the sum of $K(\xi, \xi_*) = K_0(\xi, \xi_*) + K_1(\xi, \xi_*)$, with K_1 the regular part without the singularity at $\xi = \xi_*$, and K_0 the hyperbolic part, (4.1). The particle-like waves are constructed based on the solution operators \mathbb{S}^t for the damped transport equation

$$\mathbf{h}_t + \xi \cdot \nabla_x \mathbf{h} + \nu(\xi) \mathbf{h} = 0, \quad \mathbf{h}(x, t, \xi) \equiv \mathbb{S}^t \mathbf{h}(x, 0, \xi)$$

and the operator \mathbb{O}_D^t for the truncated linearized Boltzmann equation:

$$\mathbf{j}_t + \xi \cdot \nabla_x \mathbf{j} + \nu(\xi) \mathbf{j} = K_0 \mathbf{j}, \quad \mathbf{j}(x, t) \equiv \mathbb{O}_D^t \mathbf{j}_0(x).$$

A Picard iteration based on these two operators is designed to extract the particle-like waves. The remaining part of the solution is regular. These are explained in Section 4.

For the construction of fluid-like waves, we use the Fourier transform and concentrate on the low frequency, the long waves. The reason for this is that the fluid-like waves are the result of the equilibrating mechanism of the collisions. Since the Boltzmann equation is dissipative, as evidence of the H-Theorem, long waves survive and represent the large-time behavior of the solutions. By Fourier transform we have, symbolically, the expression of the Green's function

$$\mathbb{G}(x, t, \xi) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \eta + (-i\xi \cdot \eta + L)t} d\eta. \quad (1.7)$$

The analysis of long waves is done by studying the spectrum of $-i\xi \cdot \eta + L$ near origin, Section 5. There is a long tradition of the study of the spectrum and the relation of the Boltzmann equation and the Navier-Stokes equations for gas dynamics, [11, 20]. For our study of the wave structure in the later sections, we need to study the analyticity property of the point spectrum near origin. For this we start with the one space dimensional case and make essential uses of the rotational symmetry of the Boltzmann equation for the three dimensional case. This allows us to use the complex analytic techniques in the study of Huygens waves, Section 6, and contact and rotational waves, Section 7.

By simple scaling, one sees that, for Boltzmann solutions, the large-time behavior corresponds to the small Knudsen number. In the zero Knudsen number limit, the solutions of the Boltzmann equation would approach the

solutions of the Euler equations. Thus we briefly review the waves for the linearized Euler equations in gas dynamics in Section 3. The Huygens waves are the solutions of the linear wave equation for the density. Other waves correspond to momentum and entropy. The explicit expression of inviscid waves gives us the basic information from which we design the wave pairings on the level of the Boltzmann equation in Section 5. Such pairings are physically correct and therefore have the desired analyticity properties that are needed for the study of wave structure in Sections 6 and 7.

The Green's function has been studied for planar waves, $x \in \mathbb{R}^1$, [27], and shown to have the dual particle-fluid property. The present paper is concerned with the three dimensional case, $x \in \mathbb{R}^3$, which has richer geometric features for the waves. On the other hand, it should be pointed out that nonlinearity plays a stronger role in the one-dimensional case. Thus in [27], the asymptotic state of the solutions contains the nonlinear kinetic Burgers waves. For three dimensional case, there is the stronger dispersion of waves and therefore the solutions to the full Boltzmann equation behave closer to the Green's function than in the one dimensional case. For brevity, we will not study the full Boltzmann equation in the present paper.

The particle-like waves are constructed along the same line as [27] with the help of the Mixture Lemma straightforwardly generalized here. However, the particle-like waves for the three dimensional waves contain δ -waves of lower dimensions, Section 9. The analyticity properties that are needed for the study of the fluid-like waves here are analytically and geometrically more intricate and interesting than for the one-dimensional case. We expect our general techniques in Section 5 to be applicable to other systems endowed with the natural rotational symmetry as the Boltzmann equation.

With the particle-like waves and fluid-like waves constructed, the remaining term is regular and small time-asymptotically. This allows us to apply the calculus and use the weighted energy method for the complete study of wave structure, first for rough initial values at Section 8 and then for the Green's function at Section 9.

There is the macro-micro decomposition, [28] that decompose a function $u = u(\xi)$ into macro, fluid part P_0u and micro, non-fluid part P_1u . The projection P_0u maps into the kernel space $\{\sqrt{M}, \xi^1\sqrt{M}, \xi^2\sqrt{M}, \xi^3\sqrt{M}, |\xi|^2\sqrt{M}\}$ of the linearized collision operator L . The fluid part P_0u is further decomposed here into isotropic part $P_0^{iso}u$, projection into $\{\sqrt{M}, |\xi|^2\sqrt{M}\}$

and the momentum part \mathbf{P}_0^m , projection into $\{\xi^1\sqrt{\mathbf{M}}, \xi^2\sqrt{\mathbf{M}}, \xi^3\sqrt{\mathbf{M}}\}$, (2.6). For the underlying Maxwellian state, (1.3), the one-dimensional fluid waves have speed $\lambda_1 = -\mathbf{c}$, $\lambda_2 = 0$, $\lambda_3 = \mathbf{c}$, where $\mathbf{c} = \sqrt{5/3}$ is the sound speed for the monatomic gases that the Boltzmann equation models, Section 3. The Boltzmann equation is dissipative as the result of the H-Theorem. Thus it is related to the Navier-Stokes equations in gas dynamics. The dissipation parameters A_i , $i = 1, \dots, 5$ for the Boltzmann equation are related to the viscosities μ , μ' and heat conductivity κ for the Navier-Stokes equations as follows:

$$\begin{cases} A_2 = \frac{1}{6}(\mathbf{P}_1\xi^1|\xi|^2\sqrt{\mathbf{M}}, (-L)^{-1}\mathbf{P}_1\xi^1|\xi|^2\sqrt{\mathbf{M}}) \equiv \kappa, \\ A_4 = A_5 = (\mathbf{P}_1\xi^1\xi^2\sqrt{\mathbf{M}}, (-L)^{-1}\mathbf{P}_1\xi^1\xi^2\sqrt{\mathbf{M}}) \equiv \mu, \\ \frac{5}{2}(\mathbf{P}_1\xi^1\xi^1\sqrt{\mathbf{M}}, (-L)^{-1}\mathbf{P}_1\xi^1\xi^1\sqrt{\mathbf{M}}) \equiv \mu', \\ A_1 = A_3 = \kappa + \mu'. \end{cases}$$

In our main results below, we see that the macroscopic part and microscopic part have different behavior.

Theorem 1.1.(Main Theorem I) *The Green's function $\mathbb{G}(x, t)$ as an L_ξ^2 operator-valued function satisfies, for some constant $C > 0$,*

$$\begin{aligned} \|\mathbb{G}(x, t)\|_{L_\xi^2} &\leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right) \\ &\quad + C \begin{cases} \frac{1}{(1+t)(\sqrt{1+t} + |x|)} & \text{for } |x| \leq ct + \sqrt{1+t}, \\ 0 & \text{for } |x| \geq ct + \sqrt{t}, \end{cases} \end{aligned} \quad (1.8)$$

$$\|\mathbf{P}_0^{iso}\mathbb{G}(x, t)\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right), \quad (1.9)$$

$$\|\mathbb{G}(x, t)\mathbf{P}_1\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^2} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^{5/2}} + e^{-(|x|+t)/C} \right), \quad (1.10)$$

$$\|\mathbf{P}_1\mathbb{G}(x, t)\mathbf{P}_1\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{5/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^3} + e^{-(|x|+t)/C} \right). \quad (1.11)$$

In fact, it is possible to describe the leading particle-like and fluid waves as in the next theorem. We will use the notation $\alpha(\xi)\sqrt{\mathbf{M}} \otimes \langle \beta(\xi)\sqrt{\mathbf{M}} \rangle$ to denote the projection operator of mapping a given function $g = g(\xi)$ into:

$$\alpha\sqrt{\mathbf{M}} \otimes \langle \beta\sqrt{\mathbf{M}} \rangle g(\xi) = \alpha(\xi)\sqrt{\mathbf{M}}(\xi) \int_{\mathbb{R}^3} \beta(\xi_*)\sqrt{\mathbf{M}}(\xi_*)g(\xi_*)d\xi_*.$$

Theorem 1.2. (Main Theorem II) *The Green's function $\mathbb{G}(x, t; \xi, \xi_0)$ can be written as the sum:*

$$\mathbb{G}(x, t; \xi, \xi_0) = [h_0 + h_1 + h_2 + \mathbb{G}_1^0 + \mathbb{G}_2^0 + \mathbb{G}_3^0 + \mathbb{G}_4^0 + \mathbb{G}_5^0 + \bar{\mathbb{G}}](x, t; \xi, \xi_0). \quad (1.12)$$

Here the particle-like waves are giving as:

$$\left\{ \begin{array}{l} h_0(x, t, \xi) = e^{-\nu(\xi_0)t} \delta(x - \xi_0 t) \delta(\xi - \xi_0), \\ h_1(x, t, \xi) = \int_0^t K(\xi, \xi_0) e^{-\nu(\xi)(t-s) - \nu(\xi_0)s} \delta(x - (t-s)\xi - s\xi_0) ds, \\ h_2(x, t, \xi) = \int_0^t \int_{\mathbb{R}^3} \int_0^{s_1} e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_0)s} K(\xi, \xi_1) K(\xi_1, \xi_0) \\ \quad \times \delta(x - (t-s_1)\xi - (s_1-s)\xi_1 - s\xi_0) ds d\xi_1 ds_1. \end{array} \right. \quad (1.13)$$

The leading fluid waves are:

$$\left\{ \begin{array}{l} \mathbb{G}_1^0 + \mathbb{G}_3^0 = \frac{ct}{4\pi} \iint_{|y|=1} H_1(x + cty) dS_y + \frac{1}{4\pi} \iint_{|y|=1} H_2(x + cty) dS_y \\ \quad + \frac{ct}{4\pi} \iint_{|y|=1} \nabla H_2(x + cty) \cdot y dS_y, \\ \text{Huygens waves,} \\ \mathbb{G}_2^0 = \frac{1}{6} (4\pi A_2 t)^{-3/2} e^{-\frac{|x|^2}{4A_2 t}} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} \rangle, \text{ entropy waves,} \\ \mathbb{G}_4^0 + \mathbb{G}_5^0 = (4\pi A_4 t)^{-3/2} e^{-\frac{|x|^2}{4A_4 t}} \sum_{j=1}^3 \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^j \sqrt{\mathbf{M}} \rangle \\ \quad - \sum_{j,k=1}^3 \left[\int_0^t A_4 (4\pi A_4 \tau)^{-3/2} e^{-\frac{|x|^2}{4A_4 \tau}} d\tau \right]_{x_j x_k} \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^k \sqrt{\mathbf{M}} \rangle, \\ \text{rotational waves.} \end{array} \right. \quad (1.14)$$

Here

$$\left\{ \begin{array}{l} H_1 = \frac{\sqrt{10}}{6} \sum_{j=1}^3 ((4\pi A_1 t)^{-3/2} e^{-\frac{|x|^2}{4A_1 t}})_{x_j} \\ \quad \times \left(\xi^j \sqrt{\mathbf{M}} \otimes \left\langle |\xi|^2 \sqrt{\mathbf{M}} \right\rangle + |\xi|^2 \sqrt{\mathbf{M}} \otimes \left\langle \xi^j \sqrt{\mathbf{M}} \right\rangle \right), \\ H_2 = 5A_1 \sum_{j,k=1}^3 \left(\int_0^t (4\pi A_1 \tau)^{-3/2} e^{-\frac{|x|^2}{4A_1 \tau}} d\tau \right)_{x_j x_k} \xi^j \sqrt{\mathbf{M}} \otimes \left\langle \xi^k \sqrt{\mathbf{M}} \right\rangle \\ \quad + \frac{1}{18} (4\pi A_1 t)^{-3/2} e^{-\frac{|x|^2}{4A_1 t}} |\xi|^2 \sqrt{\mathbf{M}} \otimes \left\langle |\xi|^2 \sqrt{\mathbf{M}} \right\rangle. \end{array} \right.$$

The particle-like waves \mathbf{h}_0 and \mathbf{h}_1 are generalized functions and \mathbf{h}_2 is a regular function. The remaining term is continuous and decays faster than the fluid waves:

$$\left\{ \begin{array}{l} \left[\int_{\mathbb{R}^3} |\bar{\mathbf{G}}(x, t; \xi, \xi_0)|^2 d\xi \right]^{1/2} \\ \leq C(t+1)^{-1/2} \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right) \\ + C \begin{cases} \frac{1}{(1+t)(\sqrt{1+t} + |x|)} & \text{for } |x| \leq ct + \sqrt{1+t}, \\ 0 & \text{for } |x| \geq ct + \sqrt{t}, \end{cases} \end{array} \right. \quad (1.15)$$

Remark 1.3. The particle-like waves, as an operator in L_ξ^2 decay exponentially in $|x|$ and t , Section 9. From Theorem 3.1, Lemma 3.2 and Lemma 3.3, the leading fluid waves have the same decay properties as stated in Main Theorem I. Thus, Main Theorem II is a more precise description of waves than Main Theorem I. We will, however, prove Main Theorem I first in Section 9 and present the leading fluid waves in Section 10.

There are two general types of existence theory for the Boltzmann equation: For existence theory of smooth solution based on energy method, see [16, 24, 30, 33, 34, 36] For existence of weak solution using upper-lower solutions approach, see [1, 18] and renormalized weak solution see [10], and also [38].

The study of pointwise estimates of nonlinear waves for general dissipative system is an interesting subject because it offers explicit expression of the coupling of nonlinear waves. For viscous conservation laws, it is initiated by [21]. The study of Green's function for viscous conservation laws was

done first by [40] and later in [17, 22, 23] For the study of dissipative finite differences, see [25, 26]. Although there are the common features between our present study for the Boltzmann equation and other dissipative partial differential equations, there is also the major differences, chiefly because the Boltzmann equation is, essentially, differential equations of infinite dimensions. This accounts for the rich wave structure and demands the elaborate combinations of analytical and physical considerations of long-short waves, particle-wave, and short-long times decompositions. For the study of Green's function for stationary Boltzmann equation, see [8].

The pointwise study of the Green's function is the basic and necessary starting point for more general quantitative analysis of the Boltzmann equation. Such an analysis is needed for the physical understanding of the crucial initial, shock, and particularly the boundary layer behavior of the solutions of the Boltzmann equation. It would be very interesting and important to study the Green's function for the Boltzmann equation with the boundary effects, c.f. [31, 32]. Such a study would rely on the knowledge of the Green's function for the initial value problem. This and other issues, however, are left to the future.

2. Preliminaries

The contents in this section are mostly standard and can be found in or easily derived from [7].

For a hard sphere model, the collision operator L can expressed explicitly, [14]:

$$\left\{ \begin{array}{l} Lg(\xi) = -\nu(\xi)g(\xi) + Kg(\xi), \\ Kg(\xi) \equiv \int_{\mathbb{R}^3} K(\xi, \xi_*)g(\xi_*) d\xi_*, \\ K(\xi, \xi_*) \equiv \frac{2}{\sqrt{2\pi}|\xi - \xi_*|} \exp\left(-\frac{(|\xi|^2 - |\xi_*|^2)^2}{8|\xi - \xi_*|^2} - \frac{|\xi - \xi_*|^2}{8}\right) \\ \quad - \frac{|\xi - \xi_*|}{2} \exp\left(-\frac{(|\xi|^2 + |\xi_*|^2)}{4}\right), \\ \nu(\xi) \equiv \frac{1}{\sqrt{2\pi}} \left(2e^{-\frac{|\xi|^2}{2}} + 2\left(|\xi| + \frac{1}{|\xi|}\right) \int_0^{|\xi|} e^{-\frac{u^2}{2}} du\right), \\ \nu(\xi) \sim 1 + |\xi|. \end{array} \right. \quad (2.1)$$

Consider the space L_ξ^2 with inner product

$$(g, h) \equiv \int_{\mathbb{R}^3} g(\xi)h(\xi)d\xi.$$

The weighted norm space $L_{\xi,\beta}^\infty$, also in the ξ variables, is

$$\|\mathbf{g}(x, t)\|_{L_{\xi,\beta}^\infty} \equiv \sup_{\xi \in \mathbb{R}^3} (1 + |\xi|)^\beta |\mathbf{g}(x, t, \xi)|. \quad (2.2)$$

We will also need the Sobolev spaces:

$$\|\mathbf{f}\|_{H_x^j(L_\xi^2)} \equiv \left(\sum_{|\alpha| \leq j} \int_{\mathbb{R}^3} (\partial_x^\alpha \mathbf{f}, \partial_x^\alpha \mathbf{f}) dx \right)^{\frac{1}{2}} \quad (2.3)$$

Lemma 2.1. *The operator \mathbf{K} is a compact operator and \mathbf{K}_ξ and \mathbf{K}_{ξ_*} are bounded operators in L_ξ^2 , where*

$$\begin{cases} \mathbf{K}_\xi \mathbf{g}(\xi) \equiv \int_{\mathbb{R}^3} K_\xi(\xi, \xi_*) \mathbf{g}(\xi_*) d\xi_*, \\ \mathbf{K}_{\xi_*} \mathbf{g}(\xi) \equiv \int_{\mathbb{R}^3} K_{\xi_*}(\xi, \xi_*) \mathbf{g}(\xi_*) d\xi_*. \end{cases}$$

The boundedness properties are due to that both $K_\xi(\xi, \xi_*)$ and $K_{\xi_*}(\xi, \xi_*)$ are integrable in ξ_* variable.

Lemma 2.2. *For any $\beta \geq 0$ there exist positive constants $C(\beta)$ and C_1 such that*

$$\begin{cases} \|\mathbf{K} \mathbf{j}\|_{L_{\xi,\beta+1}^\infty} \leq C(\beta) \|\mathbf{j}\|_{L_{\xi,\beta}^\infty}, \\ \|\mathbf{K} \mathbf{j}\|_{L_{\xi,0}^\infty} \leq C_1 \|\mathbf{j}\|_{L_\xi^2}. \end{cases}$$

Lemma 2.3. *For any $\beta \geq 1$ there exists a positive constant $C(\beta)$ such that the nonlinear term $\Gamma(\mathbf{g})$, (1.5), satisfies*

$$\|\Gamma(\mathbf{g})\|_{L_{\xi,\beta-1}^\infty} \leq C(\beta) \|\mathbf{g}\|_{L_{\xi,\beta}^\infty}^2.$$

The kernel of L is spanned by the orthogonal basis:

$$\begin{aligned} \mathfrak{B} &\equiv \{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4\}, \\ \ker(L) &\equiv \text{span}(\mathfrak{B}), \\ \begin{cases} \chi_0 &\equiv \sqrt{M} \\ \chi_i &\equiv \xi^i \sqrt{M} \text{ for } i = 1, 2, 3, \\ \chi_4 &\equiv \frac{1}{\sqrt{6}}(|\xi|^2 - 3)\sqrt{M}. \end{cases} & (2.4) \\ (\chi_i, \chi_j) &= \delta_i^j. \end{aligned}$$

L is negative definite on $\ker(L)^\perp \equiv \{\mathbf{g} \in L_\xi^2 \mid (\mathbf{g}, \chi_j) = 0, j = 0, 1, \dots, 4\}$.

Lemma 2.4.([5]) *The operator L is symmetric and non-positive definite:*

$$(Lg, h) = (g, Lh), \quad (Lg, g) \leq 0$$

Moreover, there exists $\nu_0 > 0$ such that

$$(\mathbf{g}, \mathbf{g}) \leq -\nu_0(\mathbf{g}, L\mathbf{g}) \text{ for any } \mathbf{g} \in \ker(L)^\perp.$$

From the expression (1.7) of the Green's function, there is a need to generalize the above to the consideration of the eigenvalues λ of the operator $-i\eta \cdot \xi + L$ in functional space L_ξ^2 for each given fixed $\eta \in \mathbb{R}^3$:

$$(-i\eta \cdot \xi + L)\psi = \lambda\psi, \quad (\lambda, \psi) \in \mathbb{C} \times L_\xi^2.$$

$\sigma(\eta) \equiv \{\lambda \in \mathbb{C} \mid \text{there exists non-trivial } \mathbf{e} \in L_\xi^2 \text{ such that } (-i\eta \cdot \xi + L)\mathbf{e} = \lambda\mathbf{e}\}$.

From Lemma 2.4, $\sigma(0)$ consists of zero of multiplicity five. The following lemma, [11], says that around the imaginary axis the spectrum consists of only the curves bifurcating from the origin.

Lemma 2.5.([11])

- (1) $\sigma(\eta) \subset \{z \in \mathbb{C} \mid \text{Re}(z) \leq 0\}$.
- (2) *There exist $\kappa_0 > 0$ and $\kappa_1 > 0$ such that for $|\eta| > \kappa_0$*

$$\sigma(\eta) \subset \{z \in \mathbb{C} \mid \text{Re}(z) \leq -\kappa_1\};$$

and for $|\eta| \leq \kappa_0$, in the region $\{z \in \mathbb{C} | 0 \geq \operatorname{Re}(z) \geq -\kappa_1\}$, the set $\sigma(\eta)$ consists of five smooth curves $\sigma_1(|\eta|)$, $\sigma_2(|\eta|)$, $\sigma_3(|\eta|)$, $\sigma_4(|\eta|)$, $\sigma_5(|\eta|)$ through the origin.

We will study the fluid-like waves, which are the long, low frequency waves. For this, we make the following *long wave-short wave decomposition* of the Green's function:

$$\begin{cases} \mathbb{G}_L^t \mathbf{g}(x) \equiv \int_{\mathbb{R}^3} \mathbb{G}_L(x-y, t) \mathbf{g}(y) dy, \\ \mathbb{G}_S^t \mathbf{g}(x) \equiv \int_{\mathbb{R}^3} \mathbb{G}_S(x-y, t) \mathbf{g}(y) dy, \\ \mathbb{G}_L(x, t) \equiv \frac{1}{(2\pi)^3} \int_{|\eta| < \kappa_0/2} e^{ix \cdot \eta + (-i\xi \cdot \eta + L)t} d\eta, \\ \mathbb{G}_S(x, t) \equiv \frac{1}{(2\pi)^3} \int_{|\eta| \geq \kappa_0/2} e^{ix \cdot \eta + (-i\xi \cdot \eta + L)t} d\eta. \end{cases}$$

The solution $\mathbf{h}(x, t)$ of linearized Boltzmann equation can be represented by

$$\begin{aligned} \mathbf{h}(x, t) &= \frac{1}{(2\pi)^3} \left[\int_{|\eta| < \kappa_0/2} e^{ix \cdot \eta + (-i\eta \cdot \xi + L)t} \hat{\mathbf{h}}(\eta, 0) d\eta \right. \\ &\quad \left. + \int_{|\eta| \geq \kappa_0/2} e^{ix \cdot \eta + (-i\eta \cdot \xi + L)t} \hat{\mathbf{h}}(\eta, 0) d\eta \right] \\ &\equiv \mathbb{G}_L^t \mathbf{h}(x, 0) + \mathbb{G}_S^t \mathbf{h}(x, 0). \end{aligned}$$

From the spectral property of Lemma 2.5 one has the following:

Lemma 2.6. *The exists $\nu_1 > 0$, $C > 0$, and C_j such that*

$$\begin{cases} \|(\mathbb{G}_L^t + \mathbb{G}_S^t) \mathbf{h}\|_{L_x^2(L_\xi^2)} \leq C \|\mathbf{h}\|_{L_x^2(L_\xi^2)}, \\ \|\mathbb{G}_L \mathbf{h}\|_{H_x^j(L_\xi^2)} \leq C_j \|\mathbf{h}\|_{L_x^2(L_\xi^2)}, \\ \|\mathbb{G}_S^t \mathbf{h}\|_{L_x^2(L_\xi^2)} \leq C e^{-\nu_1 t} \|\mathbf{h}\|_{L_x^2(L_\xi^2)}, \end{cases} \quad (2.5)$$

where $\|\cdot\|_{H_x^j(L_\xi^2)}$ is given in (2.3).

A macro-micro decomposition (P_0, P_1) on L_ξ^2 was introduced in [28] The

kernel of the operator L is the *fluid, macro* part, which is now further separated into the isotropic pressure and entropy waves, \mathbf{P}_0^{iso} , and the momentum waves, \mathbf{P}_0^m :

$$\left\{ \begin{array}{l} \mathbf{g} \equiv \mathbf{P}_0 \mathbf{g} + \mathbf{P}_1 \mathbf{g}, \text{ (macro-micro decomposition)} \\ \mathbf{P}_1 \mathbf{g} \equiv \mathbf{g} - \mathbf{P}_0 \mathbf{g}. \text{ (micro components),} \\ \mathbf{P}_0 \mathbf{g} \equiv \mathbf{P}_0^{iso} \mathbf{g} + \mathbf{P}_0^m \mathbf{g}, \text{ (isotropic-non isotropic decomposition for} \\ \hspace{15em} \text{macroscopic components)} \\ \mathbf{P}_0^{iso} \mathbf{g} \equiv (\chi_0, \mathbf{g}) \chi_0 + (\chi_4, \mathbf{g}) \chi_4, \text{ (isotropic components)} \\ \mathbf{P}_0^m \mathbf{g} \equiv \sum_{j=1}^3 (\chi_j, \mathbf{g}) \chi_j. \text{ (momentum components)} \end{array} \right. \quad (2.6)$$

The orthogonal complement of the fluid, macro part in L_ξ^2 is called the *non-fluid, micro* part. From Lemma 2.4, the linear collision operator is negative on the non-fluid part.

3. Euler Waves

The linear convection $\xi \cdot \nabla_x$ has a matrix representation in the macroscopic component with respect to the basis \mathfrak{B} of (2.4):

$$[\mathbf{P}_0 \xi \cdot \nabla_x \mathbf{P}_0]_{\mathfrak{B}} = \sum_{j=1}^3 [\mathbf{P}_0 \xi^j \mathbf{P}_0]_{\mathfrak{B}} \partial_{x^j}.$$

Due to the rotational symmetry, in our later analysis we only need the matrix representation of the one-dimensional $[\mathbf{P}_0 \xi^1 \mathbf{P}_0]_{\mathfrak{B}}$:

$$[\mathbf{P}_0 \xi^1 \mathbf{P}_0]_{\mathfrak{B}} \equiv ((\chi_i, \xi^1 \chi_j))_{5 \times 5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \end{pmatrix}.$$

This matrix can be diagonalized:

$$\begin{aligned}
 [\mathbf{P}_0 \xi^1 \mathbf{P}_0]_{\mathfrak{B}} &= \begin{pmatrix} \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & \sqrt{\frac{3}{2}} & 0 & 0 \\ -\sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{5}{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\mathbf{c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\times \begin{pmatrix} \sqrt{\frac{3}{2}} & -\sqrt{\frac{2}{3}} & \sqrt{\frac{3}{2}} & 0 & 0 \\ -\sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{5}{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^{-1},
 \end{aligned}$$

where \mathbf{c} is the speed of the acoustic wave around the state $\rho = \theta = 1$ with velocity zero, (1.2), (1.3),

$$\mathbf{c} = \sqrt{\frac{5}{3}}.$$

The eigenvectors of $\mathbf{P}_0 \xi^1 \mathbf{P}_0$ are:

$$\begin{cases} \mathbf{E}_1 = \sqrt{\frac{3}{2}}\chi_0 - \sqrt{\frac{5}{2}}\chi_1 + \chi_4, \\ \mathbf{E}_2 = -\sqrt{\frac{2}{3}}\chi_0 + \chi_4, \\ \mathbf{E}_3 = \sqrt{\frac{3}{2}}\chi_0 + \sqrt{\frac{5}{2}}\chi_1 + \chi_4, \\ \mathbf{E}_4 = \chi_2, \\ \mathbf{E}_5 = \chi_3; \end{cases} \quad (3.1)$$

$$\begin{cases} \mathbf{P}_0 \xi^1 \mathbf{E}_1 = -\mathbf{c} \mathbf{E}_1, \\ \mathbf{P}_0 \xi^1 \mathbf{E}_2 = 0, \\ \mathbf{P}_0 \xi^1 \mathbf{E}_3 = \mathbf{c} \mathbf{E}_3, \\ \mathbf{P}_0 \xi^1 \mathbf{E}_4 = 0, \\ \mathbf{P}_0 \xi^1 \mathbf{E}_5 = 0. \end{cases}$$

Here, \mathbf{E}_1 and \mathbf{E}_3 are related to the vector components of sound waves propagating forwards and backwards; \mathbf{E}_2 is related to the vector components of local fluid velocity; and both \mathbf{E}_4 and \mathbf{E}_5 are orthogonal to x -direction, which is the direction of wave propagation. These are the eigen modes for the linearized Euler equations, obtained from the linearized Boltzmann equation (1.6) by assuming that the solutions consist of only fluid part. We now derive the linearized Euler equations by assuming that a solution $h(x, t, \xi)$ of the linearized Boltzmann equation is of the form

$$\mathbf{h}(x, t) = \sum_{j=0}^4 \mathbf{h}^j(x, t) \chi_j,$$

then substitute this into the conservation laws

$$\begin{cases} (M^{\frac{1}{2}}, \mathbf{h}_t + \xi \cdot \nabla_x \mathbf{h} - L\mathbf{h}) = 0, \\ (\xi^j M^{\frac{1}{2}}, \mathbf{h}_t + \xi \cdot \nabla_x \mathbf{h} - L\mathbf{h}) = 0 \text{ for } j = 1, 2, 3, \\ (\frac{|\xi|^2}{2} M^{\frac{1}{2}}, \mathbf{h}_t + \xi \cdot \nabla_x \mathbf{h} - L\mathbf{h}) = 0. \end{cases} \quad (3.2)$$

By straightforward computations, the system (3.2) is the linearized Euler equations for the conservative quantities $\rho = (\sqrt{M}, \mathbf{h})$, $m = (\xi \sqrt{M}, \mathbf{h})$, $E = (\frac{|\xi|^2}{2} \sqrt{M}, \mathbf{h})$:

$$\begin{cases} \rho_t + \nabla_x \cdot m = 0, \\ m_t + \frac{2}{3} \nabla_x E = 0, \\ E_t + \frac{5}{2} \nabla_x \cdot m = 0. \end{cases} \quad (3.3)$$

Here, in consistent with the Boltzmann equation, the above Euler equations are polytropic and for Mon-atomic gases. The eigenvalues and eigenvectors for the Euler flux have been given in (3.1). In the above form (3.3), we have

$$\rho_{tt} = -\nabla_x \cdot m_t = \nabla_x \cdot \left(\frac{2}{3} \nabla_x E \right),$$

$$\left(\frac{5}{2} \rho - E \right)_t = 0. \quad (3.4)$$

This yields the wave equation:

$$(\partial_t^2 - \mathbf{c}^2 \Delta_x) \rho_t = 0, \quad \text{where } \mathbf{c} = \sqrt{\frac{5}{3}}. \quad (3.5)$$

The wave equation for $\rho_t(x, t)$ is solved by Kirchhoff's formula. This yields the formulas for $\rho(x, t)$ and $E(x, t)$ in terms of the density $\rho(x, t)$ from (3.4):

$$\begin{cases} \rho(x, t) = \rho(x, 0) + \int_0^t \rho_t(x, s) ds, \\ E(x, t) = E(x, 0) + \frac{5}{2} \int_0^t \rho_t(x, s) ds. \end{cases} \quad (3.6)$$

This in turn yields, from (3.3), the momentum:

$$m(x, t) = m(x, 0) + \frac{2}{3} \int_0^t \nabla_x E(x, s) ds. \quad (3.7)$$

The above procedure yields explicit solution formula for the linearized Euler solutions. For the much more complex Boltzmann equation, we will use the Fourier transformation. For this, we will now recast the above procedure for the Euler equations in the Fourier variables. The Fourier transform of the system (3.3) is

$$\begin{pmatrix} \hat{\rho} \\ \hat{m} \\ \hat{E} \end{pmatrix}_t + i \begin{pmatrix} 0 & \eta^t & 0 \\ 0 & 0 & \frac{2}{3}\eta \\ 0 & \frac{5}{2}\eta^t & 0 \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{m} \\ \hat{E} \end{pmatrix} = 0$$

One can represent the solution $(\hat{\rho}, \hat{m}, \hat{E})$ as follows:

$$\begin{pmatrix} \hat{\rho}(\eta, t) \\ \hat{m}(\eta, t) \\ \hat{E}(\eta, t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{i}{\mathbf{c}} \frac{\sin(\mathbf{c}|\eta|t)}{|\eta|} \eta^t & \frac{2}{3\mathbf{c}^2} (-1 + \cos(\mathbf{c}|\eta|t)) \\ 0 & \frac{\cos(\mathbf{c}|\eta|t)}{|\eta|^2} \eta \otimes \eta^t & \frac{2i}{3\mathbf{c}} \frac{\sin(\mathbf{c}|\eta|t)}{|\eta|} \eta \\ 0 & \frac{i3\mathbf{c}}{2} \frac{\sin(\mathbf{c}|\eta|t)}{|\eta|} \eta^t & \cos(\mathbf{c}|\eta|t) \end{pmatrix} \begin{pmatrix} \hat{\rho}(\eta, 0) \\ \hat{m}(\eta, 0) \\ \hat{E}(\eta, 0) \end{pmatrix}.$$

Here, the tensor product $\eta \otimes \eta^t$ represents a 3×3 sub-matrix $(\eta^j \eta^k)_{jk}$ for $j, k = 1, 2, 3$. Thus the solution operator involves the Riesz transform of a particular form. As the system is invscid, Dirac- δ functions are involved also. The resolution of the wave equation, the exhibition of Huygens principle, by the Kirchhoff method through the Fourier transform method is based on the following:

Theorem 3.1.(Kirchhoff) *Let $w(x, t)$ be a function given by its 3-D*

Fourier transformation:

$$\begin{cases} \hat{w} = \frac{\sin(\mathbf{c}|\eta|t)}{\mathbf{c}|\eta|}, \\ \hat{w}_t = \cos(\mathbf{c}|\eta|t). \end{cases}$$

Then, for any functions $g(x)$ and $h(x)$ one has that

$$w * g(x) = \frac{t}{4\pi} \iint_{|y|=1} g(x + \mathbf{c}ty) d\mathbf{S}_y, \quad (3.8)$$

$$w_t * h(x) = \frac{1}{4\pi} \iint_{|y|=1} h(x + \mathbf{c}ty) d\mathbf{S}_y + \frac{\mathbf{c}t}{4\pi} \iint_{|y|=1} \nabla h(x + \mathbf{c}ty) \cdot y d\mathbf{S}_y. \quad (3.9)$$

To study the dissipation of the Huygens waves in the Boltzmann solutions, we will need the following viscous version.

Lemma 3.2. *Let $w(x, t)$ be the inverse Fourier transformation of $\sin(\mathbf{c}|\eta|t)$ given in Theorem 3.1. For any positive integer $l \geq 0$, one has that*

$$\left| w * \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} \right| \leq O(1) \frac{e^{-\frac{(|x|-\mathbf{c}t)^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}}, \quad (3.10)$$

$$\left| w_t * \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} \right| \leq O(1) \frac{e^{-\frac{(|x|-\mathbf{c}t)^2}{2C(t+1)}}}{(t+1)^{\frac{l+1}{2}}}. \quad (3.11)$$

Proof. By the Kirchhoff formula (3.8), we have

$$J_1 \equiv w * \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} = a_0 t \iint_{|y|=1} \frac{e^{-\frac{|x-\mathbf{c}ty|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} dy.$$

Now, we consider this integration in two cases.

Case 1. $\|x\| - ct \leq O(1)\sqrt{1+t}$.

$$\begin{aligned} J_1 &= O(1)(t+1) \iint_{|y|=1} \frac{e^{-O(1)t|y|^2}}{(t+1)^{\frac{l}{2}}} \\ &= O(1)(t+1) \frac{1}{(t+1)^{\frac{l}{2}}} \left(\frac{1}{\sqrt{1+t}} \right)^2 = O(1) \frac{1}{(1+t)^{\frac{l}{2}}}. \end{aligned} \quad (3.12)$$

Case 2. $\|x\| - ct \geq O(1)\sqrt{1+t}$.

Since

$$\min_{|y|=1} |x - cty| = \|x\| - ct,$$

we have

$$e^{-\frac{|x-cty|^2}{C(t+1)}} \leq e^{-\frac{|x-cty|^2}{2C(t+1)} - \frac{(\|x\|-ct)^2}{2C(t+1)}}; \quad (3.13)$$

and

$$\begin{aligned} J_1 &= (t+1) \iint_{|y|=1} \frac{e^{-\frac{|x-cty|^2}{2C(t+1)} - \frac{|x-cty|^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}} dy \\ &\leq (t+1) e^{-\frac{(\|x\|-ct)^2}{2C(t+1)}} \iint_{|y|=1} \frac{e^{-\frac{|x-cty|^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}} dy \leq \frac{e^{-\frac{(\|x\|-ct)^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}}. \end{aligned} \quad (3.14)$$

From (3.12) and (3.14) we conclude (3.10). (3.11) is shown similarly. \square

In carrying out the viscous version of the procedure (3.6) and (3.7) we will need the following:

Lemma 3.3. For $|x| < ct$

$$\left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \leq \frac{C}{(1+t)(\|x\| + \sqrt{1+t})};$$

and for $|x| > ct$

$$\left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \leq \frac{C e^{-\frac{(\|x\|-ct)^2}{2Ct}}}{(1+t)^2}.$$

Proof. From (3.13), we have that

$$\left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \leq \left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{2Ct}} e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \quad (3.15)$$

The proof of the lemma using (3.15) is separated into three cases.

Case 1. $|x| \leq O(1)\sqrt{1+t}$.

$$\begin{aligned} & \left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\ & \leq \left| \left(\int_0^{\sqrt{1+t}} + \int_{\sqrt{1+t}}^t \right) \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{2Ct}} e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\ & \leq \int_0^{\sqrt{1+t}} \frac{\tau}{(1+t)^{5/2}} d\tau + O(1) \int_{\sqrt{1+t}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{te^{-\frac{(|x|-c\tau)^2}{2Ct}}}{\tau^2} d\tau = \frac{O(1)}{(1+t)^{\frac{3}{2}}}. \end{aligned}$$

Case 2. $\sqrt{1+t} \leq |x| \leq ct + \sqrt{1+t}$.

$$\begin{aligned} & \left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\ & \leq \left| \left(\int_0^{\sqrt{1+t}} + \int_{\sqrt{1+t}}^t \right) \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{2Ct}} e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\ & \leq \int_0^{\sqrt{1+t}} \frac{\tau e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\tau + O(1) \int_{\sqrt{1+t}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{te^{-\frac{(|x|-c\tau)^2}{2Ct}}}{\tau^2} d\tau = \frac{O(1)}{(1+t)|x|}. \end{aligned}$$

Case 3. $|x| \geq ct + \sqrt{1+t}$.

$$\begin{aligned}
& \left| \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\
& \leq \left| \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-c\tau y|^2}{2Ct}} e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^{5/2}} dS_y \right\} d\tau \right| \\
& \leq \int_0^{\frac{t}{2}} \tau e^{-\frac{c^2 t}{4C}} e^{-\frac{(|x|-c\tau)^2}{2Ct}} d\tau + O(1) \int_{\frac{t}{2}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{t e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{\tau^2} d\tau \\
& = \frac{O(1) e^{-\frac{(|x|-c\tau)^2}{2Ct}}}{(1+t)^2}.
\end{aligned}$$

These complete the proof of the lemma. \square

Lemma 3.4. *For any given positive integer $l, k \geq 3$, one has*

$$\iiint_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|^2}{C(t+1)} - \frac{|y|^2}{C(s+1)}}}{(t-s+1)^{\frac{l}{2}} (s+1)^{\frac{k}{2}}} dy \leq \frac{O(1)}{(t-s+1)^{\frac{l-3}{2}} (s+1)^{\frac{k-3}{2}}} \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(t+1)^{\frac{3}{2}}}.$$

This lemma is a simple consequence of semi-group property of heat kernel. We omit its proof.

4. Particle-like Waves

Boltzmann equation for hard spheres is basically semi-linear hyperbolic and thus the roughness of the initial data will propagate into the later time. To apply calculus, we need to extract waves of increasing regularity so that the remaining part is sufficiently smooth. This is done as follows: First, we extract an *essential kinetic wave* from the solution to the initial value problem for the linearized Boltzmann equation. Then we apply Picard's iteration to yield *particle-like waves* with increasing regularities. The procedure is a slight generalization of that introduced in [27] for one space dimension. Rewrite the linearized Boltzmann equation as follows:

$$\partial_t \mathbf{h} + \xi \cdot \nabla_x \mathbf{h} + \nu(\xi) \mathbf{h} = (\mathbf{K}_0 + \mathbf{K}_1) \mathbf{h}.$$

The operator \mathbf{K} is decomposed into $\mathbf{K}_0 \equiv \mathbf{K}_{0,D}$ and $\mathbf{K}_1 \equiv \mathbf{K}_{1,D}$:

$$\left\{ \begin{array}{l} \mathbf{K}_i z(\xi) \equiv \int_{\mathbb{R}} K_i(\xi, \xi_*) z(\xi_*) d\xi_* \text{ for } i = 0, 1, \\ K_0(\xi, \xi_*) = ch_0 \left(\frac{|\xi - \xi_*|}{D\nu_0} \right) K(\xi, \xi_*), \\ K_1(\xi, \xi_*) = \left(1 - ch_0 \left(\frac{|\xi - \xi_*|}{D\nu_0} \right) \right) K(\xi, \xi_*), \\ ch_0(r) \equiv 1 \text{ for } r \in [-1, 1], \\ \text{supp}(ch_0) \subset [-2, 2], \text{ } ch_0 \in C_c^\infty(\mathbb{R}), \text{ } \chi_0 \geq 0. \end{array} \right. \quad (4.1)$$

Here the cutoff parameter D will be chosen to be small. The operator \mathbf{K}_1 is a smoothing operator because it does not inherit the singular nature of the kernel $K(\xi, \xi_*)$ at $\xi = \xi_*$.

Lemma 4.1. *For any $\mathbf{h} \in L_\xi^2$ and any given $i \geq 0$,*

$$\|\nabla_\xi^i \mathbf{K}_1 \mathbf{h}\|_{L_\xi^2} = O(1) \|\mathbf{h}\|_{L_\xi^2}.$$

Proof. From the definition of \mathbf{K}_1 in (4.1),

$$\partial_\xi^\alpha \mathbf{K}_1 \mathbf{h}(\xi) \equiv \int_{\mathbb{R}^3} \mathbf{h}(\xi_*) \partial_\xi^\alpha \left(\left(1 - ch_0 \left(\frac{|\xi - \xi_*|}{D\nu_0} \right) \right) K(\xi, \xi_*) \right) d\xi_*.$$

The function $K(\xi, \xi_*)$ is smooth for $|\xi - \xi_*| > D\nu_0$. Thus, $((1 - ch_0(\frac{|\xi - \xi_*|}{D\nu_0})) K(\xi, \xi_*))$ is a globally smooth function. It is easy to see that for any $i \geq 0$ the function $\partial_\xi^i ((1 - ch_0(\frac{|\xi - \xi_*|}{D\nu_0})) K(\xi, \xi_*)) \in L_\xi^1$ and so defines a bounded operator from L_ξ^2 to L_ξ^2 , and the lemma is proved. \square

Definition 4.2. Denote by \mathbb{S}^t and \mathbb{O}_D^t the solution operators of the following two initial value problems:

$$\left\{ \begin{array}{l} \mathbf{g}_t + \xi \cdot \nabla_x \mathbf{g} + \nu(\xi) \mathbf{g} = 0, \\ \mathbf{g}(x, 0) = \mathbf{g}_0(x) \in L_{\xi, \beta}^\infty \text{ for } x \in \mathbb{R}^3, \\ \mathbf{g}(x, t) \equiv \mathbb{S}^t \mathbf{g}_0(x), \\ \mathbf{j}_t + \xi \cdot \nabla_x \mathbf{j} + \nu(\xi) \mathbf{j} = \mathbf{K}_0 \mathbf{j}, \\ \mathbf{j}(x, 0) = \mathbf{j}_0(x) \in L_{\xi, \beta}^\infty \text{ for } x \in \mathbb{R}^3, \\ \mathbf{j}(x, t) \equiv \mathbb{O}_D^t \mathbf{j}_0(x). \end{array} \right.$$

where $\beta > 5/2$ and $0 < D \ll 1$.

The function $\mathbb{O}_D^t h_0(x)$ is called the “essential kinetic wave” of solution of the linear Boltzmann equation with initial value h_0 .

Lemma 4.3. *For any $\beta \geq 0$, there exists positive constants C and C_β such that the operator \mathbb{S}^t satisfies*

$$\begin{cases} \|\mathbb{S}^t\|_{L_x^\infty(L_{\xi,\beta}^\infty)} \leq C_\beta e^{-\nu_0 t}, \\ \|\mathbb{S}^t\|_{L_x^2(L_\xi^2)} \leq C e^{-\nu_0 t}. \end{cases}$$

Proof. The first estimate is a direct consequence of the following solution formula of hyperbolic equation with damping.

$$\mathbb{S}^t \mathbf{g}_0(x, \xi) = e^{-\nu(\xi)t} \mathbf{g}_0(x - \xi t, \xi). \quad (4.2)$$

The second estimate is proved by the energy method as in the proof of the next lemma. We omit the proof. \square

Lemma 4.4. *There exist positive constants C_0 and C_1 such that for any $D \in (0, C_0)$ the operator \mathbb{O}_D^t satisfies*

$$\|\mathbb{O}_D^t\|_{L_x^2(L_\xi^2)} \leq C_1 e^{-\nu_0 t/2}.$$

Proof. First, we regard \mathbb{O}_D^t as an operator on $L_x^2(L_\xi^2)$, and consider the initial value problem

$$\begin{cases} \mathbf{j}_t + \xi \cdot \nabla_x \mathbf{j} + \nu(\xi) \mathbf{j} - \mathbf{K}_0 \mathbf{j} = 0 \\ \mathbf{j}(x, 0) \equiv \mathbf{g}_0(x). \end{cases}$$

Consider the energy estimate

$$\int_{\mathbb{R}^3} (\mathbf{j}, \mathbf{j}_t + \xi \cdot \nabla_x \mathbf{j} + \nu(\xi) \mathbf{j} - \mathbf{K}_0 \mathbf{j}) dx = 0.$$

Since the L_ξ^2 norm of $|\mathbf{K}_0|$ is $O(1)D$, this results in

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mathbf{j}, \mathbf{j}) dx < (-\nu_0 + O(1)D) \int_{\mathbb{R}^3} (\mathbf{j}, \mathbf{j}) dx.$$

Thus we may choose D sufficiently small so that $-\nu_0 + O(1)D < 0$. This implies that there exist positive constants C_1, C_0 such that, for $D \in (0, C_0)$

$$\frac{d}{dt} \int_{\mathbb{R}^3} (\mathbf{j}, \mathbf{j}) dx \leq -\frac{\nu_0}{2} \int_{\mathbb{R}^3} (\mathbf{j}, \mathbf{j}) dx,$$

whence the lemma follows:

$$\|\mathbf{j}\|_{L_x^2(L_\xi^2)} \leq e^{-\frac{\nu_0 t}{2}} \|\mathbf{g}_0\|_{L_x^2(L_\xi^2)}. \quad \square$$

This lemma results in the existence of the operator \mathbb{O}_D^t in the functional space $L_x^2(L_\xi^2)$ and the global decaying rate in time. We next use the Picard's iteration to analyze the operator \mathbb{O}_D^t in the sup norm.

Lemma 4.5. *The operator \mathbb{O}_D^t is also a bounded operator on $L_x^\infty(L_{\xi, \beta}^\infty)$ for any $\beta \geq 0$, that is, there exist positive constants C_0 and C_1 such that, for any $D \in (0, C_0)$,*

$$\|\mathbb{O}_D^t \mathbf{g}_0\|_{L_x^\infty(L_{\xi, \beta}^\infty)} \leq C_1 e^{-\nu_0 t/2} \|\mathbf{g}_0\|_{L_x^\infty(L_{\xi, \beta}^\infty)}.$$

Proof. From Lemma 2.2,

$$\|\mathbf{K}_0 \mathbf{h}\|_{L_{\xi, \beta+1}^\infty} \leq C_\beta D \|\mathbf{h}\|_{L_{\xi, \beta}^\infty} \quad \text{for } \beta \geq 0. \quad (4.3)$$

From this and Lemma 4.3,

$$\begin{aligned} & \left\| \int_0^t \cdots \int_0^{s_k} \mathbb{S}^{t-s_1} \mathbf{K}_0 \mathbb{S}^{s_1-s_2} \mathbf{K}_0 \mathbb{S}^{s_2-s_3} \mathbf{K}_0 \cdots \mathbb{S}^{s_k-s_{k+1}} \mathbf{K}_0 \mathbb{S}^{s_{k+1}} ds_{k+1} \cdots ds_1 \right\|_{L_x^\infty(L_{\xi, \beta}^\infty)} \\ & \leq (C_\beta)^{k+1} D^{k+1} e^{-\nu_0 t/2}. \end{aligned}$$

Thus, Picard's iteration gives a convergent geometric sequence in $L_x^\infty(L_{\xi, \beta}^\infty)$ for sufficiently small $D > 0$:

$$\begin{aligned} \mathbb{O}_D^t &= \mathbb{S}^t + \int_0^t \mathbb{S}^{t-s_1} \mathbf{K}_0 \mathbb{S}^{s_1} ds_1 + \int_0^t \int_0^{s_1} \mathbb{S}^{t-s_1} \mathbf{K}_0 \mathbb{S}^{s_1-s_2} \mathbf{K}_0 \mathbb{S}^{s_2} ds_2 ds_1 + \cdots \\ & \quad + \int_0^t \cdots \int_0^{s_k} \mathbb{S}^{t-s_1} \mathbf{K}_0 \mathbb{S}^{s_1-s_2} \mathbf{K}_0 \mathbb{S}^{s_2-s_3} \mathbf{K}_0 \cdots \mathbb{S}^{s_k-s_{k+1}} \mathbf{K}_0 \mathbb{S}^{s_{k+1}} ds_{k+1} \cdots ds_1 \\ & \quad + \cdots, \end{aligned} \quad (4.4)$$

and the lemma follows. \square

The following lemma yields the significant hyperbolic property of the operator \mathbb{S}^t and \mathbb{O}_D^t . Note that the property $\nu(\xi) \sim 1 + |\xi|$ for hard sphere model is crucially used in its proof.

Lemma 4.6. *For any given $\beta \geq 0$, there exists sufficiently small $D > 0$ such that*

$$\|\mathbb{S}^t \mathbf{g}_0(x)\|_{L_{\xi, \beta}^\infty} \leq O(1) e^{-\frac{2\nu_0 t}{3}} \left[\max_{|y-x| < t} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} + \max_{|y-x| > t} e^{-\frac{\nu_1 |y-x|}{3}} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} \right]. \quad (4.5)$$

$$\|\mathbb{O}_D^t \mathbf{g}_0(x)\|_{L_{\xi, \beta}^\infty} \leq O(1) e^{-\frac{\nu_0 t}{2}} \left[\max_{|y-x| < t} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} + \max_{|y-x| > t} e^{-\frac{\nu_1 |y-x|}{4}} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} \right], \quad (4.6)$$

where ν_1 are relate to the function $\nu(\xi)$ given in (2.1), see (4.8) below; and ν_0 is given in Lemma 2.4.

Proof. We use the representation (4.2) for \mathbb{S}^t . For $|\xi| \leq 1$,

$$\begin{aligned} |\mathbb{S}^t \mathbf{g}_0(x, \xi)| &\leq e^{-\nu(\xi)t} (1 + |\xi|)^{-\beta} \|\mathbf{g}_0(x - \xi t, \cdot)\|_{L_{\xi, \beta}^\infty} \\ &\leq e^{-2\nu_0 t/3} (1 + |\xi|)^{-\beta} \max_{|y-x| < t} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty}. \end{aligned} \quad (4.7)$$

For $|\xi| > 1$, we use the basic property of hard sphere collision model

$$\frac{1}{2} \nu(\xi) \sim \nu_1 |\xi| \text{ as } |\xi| \rightarrow \infty, \quad (4.8)$$

$$\nu_1 |\xi| t \leq \nu(\xi) t \text{ for } \xi \in \mathbb{R}^3.$$

Thus, for $|\xi| > 1$

$$\begin{aligned} |\mathbb{S}^t \mathbf{g}_0(x, \xi)| &\leq e^{-\nu(\xi)t/3 - 2\nu_0 t/3} (1 + |\xi|)^{-\beta} \|\mathbf{g}_0(x - \xi t, \cdot)\|_{L_{\xi, \beta}^\infty} \\ &\leq e^{-\nu_1 |\xi| t/3 - 2\nu_0 t/3} (1 + |\xi|)^{-\beta} \|\mathbf{g}_0(x - \xi t, \cdot)\|_{L_{\xi, \beta}^\infty} \\ &\leq \max_{|y-x| > t} e^{-\nu_1 |x-y|/3 - 2\nu_0 t/3} (1 + |\xi|)^{-\beta} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty}. \end{aligned} \quad (4.9)$$

The estimate (4.5) follows from (4.7) and (4.9). From the construction of \mathbb{O}_D^t in Lemma 4.5, one can view \mathbb{O}_D^t as a small perturbation of \mathbb{S}^t . From

(4.3) and (4.5), one has that

$$\begin{aligned} & \left\| \int_0^t \cdots \int_0^{s_k} \mathbb{S}^{t-s_1} \mathbb{K}_0 \mathbb{S}^{s_1-s_2} \mathbb{K}_0 \mathbb{S}^{s_2-s_3} \mathbb{K}_0 \cdots \mathbb{S}^{s_k-s_{k+1}} \mathbb{K}_0 \mathbb{S}^{s_{k+1}} \mathbf{g}_0 ds_{k+1} \cdots ds_1 \right\|_{L_{\xi, \beta}^\infty} \\ & \leq (C_\beta D)^{k+1} e^{-\nu_0 t/2} \left[\max_{|y-x|<t} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} + \max_{|y-x|>t} e^{-\nu_1 |y-x|/4} \|\mathbf{g}_0(y)\|_{L_{\xi, \beta}^\infty} \right]. \end{aligned} \quad (4.10)$$

Thus, the Picard's iteration in (4.4) converges for sufficiently small $D > 0$ and (4.6) follows. \square

The iteration (4.4) gives rise to the operator of alternating the operations \mathbb{K} and \mathbb{S} that has increasing regularizing effects that we now study.

Definition 4.7. For any $\mathbf{g}_0 \in L_x^2(L_\xi^2)$, k -th degree *Mixture operator* \mathbb{M}_k^t is given as follows:

$$\mathbb{M}_k^t \mathbf{g}_0 \equiv \int_0^t \int_0^{s_1} \cdots \int_0^{s_{2k-1}} \mathbb{S}^{t-s_1} \mathbb{K} \mathbb{S}^{s_1-s_2} \mathbb{K} \mathbb{S}^{s_2-s_3} \mathbb{K} \cdots \mathbb{S}^{s_k-s_{k+1}} \mathbb{K} \mathbb{S}^{s_{2k}} \mathbf{g}_0 ds_{2k} \cdots ds_1.$$

Lemma 4.8. (Mixture Lemma) *For each given $k \geq 0$, there exists positive constant C_k such that*

$$\|\nabla_x^k \mathbb{M}_k^t \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \leq C_k e^{-\nu_0 t/2} \left(\|\mathbf{g}_0\|_{L_x^2(L_\xi^2)} + \|\nabla_\xi^k \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \right).$$

Proof. This proof is a modification of the 1-D version given in [27] and is proved by induction in k . For the case $k = 1$, it is to show that there exists $C_1 > 0$ such that for any $\mathbf{g}_0 \in L_x^2(L_\xi^2)$ the following inequality holds:

$$\begin{aligned} & \left\| \nabla_x \int_0^t \int_0^{s_1} \mathbb{S}^{t-s_1} \mathbb{K} \mathbb{S}^{s_1-s} \mathbb{K} \mathbb{S}^s \mathbf{g}_0 ds ds_1 \right\|_{L_x^2(L_\xi^2)} \\ & \leq C_1 e^{-\nu_0 t} \left(\|\mathbf{g}_0\|_{L_x^2(L_\xi^2)} + \|\nabla_\xi \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \right). \end{aligned}$$

We consider the derivative with respect to x^1 and decompose it into two parts:

$$\partial_{x^1} \int_0^t \left(\int_0^{s_1/2} + \int_{s_1/2}^{s_1} \right) \mathbb{S}^{t-s_1} \mathbb{K} \mathbb{S}^{s_1-s} \mathbb{K} \mathbb{S}^s \mathbf{g}_0 ds ds_1 \equiv \mathbf{I}_1 + \mathbf{I}_2.$$

Here \mathbf{I}_1 and \mathbf{I}_2 represent “large time scale versus short time scale in a time scale $s_1 - s$ ”. For large time scale, we use Fourier analysis; for short time scale, the direct characteristic method for PDE is used. Consider the Fourier transformation of \mathbf{I}_1 with respect to x^1 :

$$\begin{aligned}
& \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{-ix \cdot \eta} \mathbf{I}_1(x, \xi) dx \\
&= \mathcal{F} \left(\partial_{x^1} \int_0^t \int_0^{s_1/2} \mathbb{S}^{t-s_1} \mathbb{K} \mathbb{S}^{s_1-s} \mathbb{K} \mathbb{S}^s \mathbf{g}_0 ds ds_1 \right) (\eta, \xi) \\
&= i\eta^1 \int_0^t \int_{\mathbb{R}^3} \int_0^{s_1/2} \int_{\mathbb{R}^3} e^{-[i\eta \cdot \xi + \nu(\xi)](t-s_1) - [i\eta \cdot \xi_1 + \nu(\xi_1)](s_1-s) - [i\eta \cdot \xi_2 + \nu(\xi_2)]s} \\
& \quad K(\xi, \xi_1) K(\xi_1, \xi_2) \hat{\mathbf{g}}_0(\eta, \xi_2) d\xi_2 ds d\xi_1 ds_1. \quad (4.11)
\end{aligned}$$

Substitute the identity

$$i\eta^1 e^{-i\xi_1 \cdot \eta(s_1-s)} = -\frac{1}{s_1-s} \partial_{\xi_1^1} e^{-i\xi_1 \cdot \eta(s_1-s)}$$

into (4.11) to result in

$$\begin{aligned}
& \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{-ix \cdot \eta} \mathbf{I}_1(x, \xi) dx \\
&= -\int_0^t \int_{\mathbb{R}^3} \int_0^{s_1/2} \int_{\mathbb{R}^3} e^{-[i\eta \cdot \xi + \nu(\xi)](t-s_1) - [i\eta \cdot \xi_1 + \nu(\xi_1)](s_1-s) - [i\eta \cdot \xi_2 + \nu(\xi_2)]s} \\
& \quad \times \frac{\partial \nu(\xi_1)}{\partial \xi_1^1} K(\xi, \xi_1) K(\xi_1, \xi_2) \hat{\mathbf{g}}_0(\eta, \xi_2) d\xi_2 ds d\xi_1 ds_1 \\
& \quad + \int_0^t \int_{\mathbb{R}^3} \int_0^{s_1/2} \int_{\mathbb{R}^3} e^{-[i\eta \cdot \xi + \nu(\xi)](t-s_1) - [i\eta \cdot \xi_1 + \nu(\xi_1)](s_1-s) - [i\eta \cdot \xi_2 + \nu(\xi_2)]s} \\
& \quad \times \frac{1}{s_1-s} \frac{\partial (K(\xi, \xi_1) K(\xi_1, \xi_2))}{\partial \xi_1^1} \hat{\mathbf{g}}_0(\eta, \xi_2) d\xi_2 ds d\xi_1 ds_1 \\
&\equiv \hat{\mathbf{I}}_1^1(\eta, t, \xi) + \hat{\mathbf{I}}_1^2(\eta, t, \xi). \quad (4.12)
\end{aligned}$$

From Lemma 2.1 and (4.12), there exists C such that

$$\|\hat{\mathbf{I}}_1^2(\eta, t, \cdot)\|_{L_\xi^2} \leq C e^{-\nu_0 t} \|\hat{\mathbf{g}}_0(\eta, \cdot)\|_{L_\xi^2} \text{ for all } \eta \in \mathbb{R}^3. \quad (4.13)$$

For $\hat{\mathbf{I}}_1^1$, we have from the smoothness of $\nu(\xi)$, (2.1), that there exists a

positive constant C such that

$$\|\hat{\mathbf{I}}_1^1(\eta, t, \cdot)\|_{L_\xi^2} \leq C e^{-\nu_0 t} \|\hat{\mathbf{g}}_0(\eta, \cdot)\|_{L_\xi^2} \text{ for all } \eta \in \mathbb{R}^3. \quad (4.14)$$

From (4.13), (4.14), and the Parseval's identity, there exists C such that

$$\|\mathbf{I}_1(\cdot, t, \cdot)\|_{L_x^2(L_\xi^2)} \leq C e^{-\nu_0 t} \|\mathbf{g}_0\|_{L_x^2(L_\xi^2)}.$$

For the term \mathbf{I}_2 , we use characteristic curves to analyze as follows:

$$\begin{aligned} \mathbf{I}_2(x, t, \xi) &= \int_0^t \int_{s_1/2}^{s_1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_3)s} K(\xi, \xi_1) K(\xi_1, \xi_2) \\ &\quad \partial_{x_1} \mathbf{g}_0(x - \xi(t-s_1) - \xi_1(s_1-s_2) - \xi_2 s_2, \xi_2) d\xi_2 d\xi_1 ds_2 ds_1. \end{aligned}$$

This is rewritten in a new set of variables V_1 and V_2 :

$$\begin{cases} V_1 \equiv \xi - \xi_1, \\ V_2 \equiv \xi_1 - \xi_2, \end{cases}$$

$$\begin{aligned} &\mathbf{I}_2(x, t, \xi) \\ &= \int_0^t \int_{s_1/2}^{s_1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_3)s} K(\xi, \xi_1) K(\xi_1, \xi_2) \\ &\quad \times \frac{1}{s_2} \left[\partial_{V_2^1} \mathbf{g}_0(x - \xi t + s_1 V_1 + s_2 V_2, \xi_2) - \mathbf{g}_{0;1}(x - \xi t + s^1 V_1 + s^2 V_2, \xi_2) \right] \\ &\quad dV_2 dV_1 ds_2 ds_1, \end{aligned}$$

where

$$f_{0;j}(x, \xi) \equiv \frac{\partial f_0(x, \xi)}{\partial \xi^j} \text{ where } j = 1, 2, 3.$$

By integration by parts,

$$\begin{aligned} \mathbf{I}_2(x, t, \xi) &= - \int_0^t \int_{s_1/2}^{s_1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{s_2} e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_3)s} \\ &\quad \times \left[\frac{\partial}{\partial V_2^1} [K(\xi, \xi_1) K(\xi_1, \xi_2)] \mathbf{g}_0(x - \xi t + s_1 V_1 + s_2 V_2, \xi_2) \right. \\ &\quad \left. + K(\xi, \xi_1) K(\xi_1, \xi_2) \mathbf{g}_{0;1}(x - \xi t + s^1 V_1 + s^2 V_2, \xi_2) \right] dV_2 dV_1 ds_2 ds_1. \quad (4.15) \end{aligned}$$

From the boundedness of the operator $(K(\xi, \xi_1)K(\xi_1, \xi_2))_{V_2^1}$ we have

$$\|\mathbf{I}_2(\cdot, t, \cdot)\|_{L_x^2(L_\xi^2)} \leq O(1)e^{-\nu_0 t} \left(\|\mathbf{g}_0\|_{L_x^2(L_\xi^2)} + \|\partial_{\xi^1} \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \right).$$

Hence, for $i = 1$,

$$\begin{aligned} & \|\partial_{x^i} \int_0^t \int_0^{s_1} \mathbb{S}^{t-s_1} \mathbf{K} \mathbb{S}^{s_1-s} \mathbf{K} \mathbb{S}^s \mathbf{g}_0 ds ds_1\|_{L_x^2(L_\xi^2)} \\ & \leq O(1)e^{-\nu_0 t} (\|\mathbf{g}_0\|_{L_x^2(L_\xi^2)} + \|\nabla_\xi \mathbf{g}_0\|_{L_x^2(L_\xi^2)}). \end{aligned} \quad (4.16)$$

Similarly, (4.16) is also valid for $i = 2, 3$ and the lemma is proved for $k = 1$.

Next, suppose that the lemma holds for $k = l$. We have

$$\begin{aligned} \partial_{x^1}^{l+1} \mathbb{M}_{t+1}^t &= \partial_{x^1}^{l+1} \int_0^t \int_0^{t_1} \mathbb{S}^{t-t_1} \mathbf{K} \mathbb{S}^{t_1-t_2} \mathbf{K} \mathbb{M}_l^{t_2} dt_2 dt_1 \\ &= \int_0^t \int_0^{t_1} \partial_{x^1} \mathbb{S}^{t-t_1} \mathbf{K} \mathbb{S}^{t_1-t_2} \mathbf{K} \partial_{x^1}^l \mathbb{M}_l^{t_2} dt_2 dt_1. \end{aligned}$$

We may use the same proof for the case $k = 1$ to yield

$$\begin{aligned} & \|\partial_{x^1}^{l+1} \mathbb{M}_{t+1}^t \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \\ &= O(1) \int_0^t e^{-\nu_0(t-s)} \left(\|\partial_{x^1}^l \mathbb{M}_l^s \mathbf{g}_0\|_{L_x^2(L_\xi^2)} + \|\partial_\xi \partial_{x^1}^l \mathbb{M}_l^s \mathbf{g}_0\|_{L_x^2(L_\xi^2)} \right) ds. \end{aligned}$$

This and induction hypothesis will imply the lemma for $k = l + 1$. \square

Remark 4.9. Note that in the above proof, we need two operations of \mathbf{K} to make the change of variables, (4.12) to work. In other words, to gain one degree of regularity we need two operations of \mathbf{K} . This accounts for the definition of the mixture operator. Note also that, as in the overall behavior of the Boltzmann solutions we have been discussing, the large-time part \mathbf{I}_1 describes long waves and is effectively studied by Fourier transform, c.f. Section 5. On the other hand, for the short-time part \mathbf{I}_2 , it is the particle-like behavior and so characteristic method is used.

Consider the initial value problem for the linearized Boltzmann equation

$$\begin{cases} \partial_t \mathbf{g} + \xi \cdot \nabla_x \mathbf{g} = L \mathbf{g}, \\ \mathbf{g}(x, 0) = \mathbf{g}_{in}(x). \end{cases}$$

Here, the initial value \mathbf{g}_{in} has compact support

$$\begin{cases} \mathbf{g}_{in}(x) \in L_x^2(L_\xi^2), \\ \sup_{|x|<1} \|\mathbf{g}_{in}(x)\|_{L_{\xi,3}^\infty} \leq 1, \end{cases}$$

but with no assumption on its regularity.

First, we extract the essential kinetic wave from the solution as follows

$$\bar{\mathbf{g}} \equiv \mathbf{g} - \mathbb{O}^t \mathbf{g}_{in}.$$

The equation for $\bar{\mathbf{g}}$ is

$$\begin{cases} \partial_t \bar{\mathbf{g}} + \xi \cdot \nabla_x \bar{\mathbf{g}} - L \bar{\mathbf{g}} = \mathbf{K}_1 \mathbb{O}^t \mathbf{g}_{in}, \\ \bar{\mathbf{g}}(x, 0) \equiv 0. \end{cases} \quad (4.17)$$

Remark 4.10. That we define the essential kinetic waves with the operator \mathbb{O}^t instead of the damped transport operator \mathbb{S}^t is so that the equation for the remainder $\bar{\mathbf{g}}$, (4.17), has the source $\mathbf{K}_1 \mathbb{O}^t \mathbf{g}_{in}$, which is smooth in the ξ variables by Lemma 4.1. This allows us to carry the next step of constructing increasingly regular, in the x variables, waves using the Mixture Lemma.

Next we define a finite Picard's iteration to produce particle-like waves \mathbf{A}_k :

$$\begin{aligned} \mathbf{R}_k(x, t) &\equiv \bar{\mathbf{g}}(x, t) - \mathbf{A}_k, \\ \mathbf{A}_k &\equiv \int_0^t \mathbb{S}^{t-s} \mathbf{K}_1 \mathbb{O}^s \mathbf{g}_{in} ds + \int_0^t \int_0^s \mathbb{S}^{t-s} \mathbf{K} \mathbb{S}^{s-s_1} \mathbf{K}_1 \mathbb{O}^{s_1} \mathbf{g}_{in} ds_1 ds \\ &\quad + \sum_{j=1}^k \left(\int_0^t \mathbb{M}_j^{t-s} \mathbf{K}_1 \mathbb{O}^s \mathbf{g}_{in} ds + \int_0^t \int_0^s \mathbb{M}_j^{t-s_1} \mathbf{K} \mathbb{S}^{s_1-s} \mathbf{K}_1 \mathbb{O}^s \mathbf{g}_{in} ds ds_1 \right), \\ \mathbf{B}_k &\equiv \int_0^t \int_0^s \mathbb{M}_k^{t-s_1} \mathbf{K} \mathbb{S}^{s_1-s} \mathbf{K}_1 \mathbb{O}^s \mathbf{g}_{in} ds ds_1. \end{aligned} \quad (4.18)$$

From Lemmas 4.4 and 4.5, 4.6 the kinetic waves $A_k(x, t)$ satisfy

$$\begin{cases} \|A_k\|_{L_x^\infty(L_\xi^2)} \leq C_k e^{-\nu_0 t/2}, \\ \|A_k\|_{L_x^2(L_\xi^2)} \leq O(1) e^{-\nu_0 t/2}. \end{cases} \quad (4.19)$$

and, by the Mixture Lemma and Lemma 4.1,

$$\|\nabla_x^k B_k\|_{L_x^2(L_\xi^2)} \leq O(1) e^{-\nu_0 t/2}. \quad (4.20)$$

The remainder R_k solves

$$\begin{cases} \partial_t R_k + \xi \cdot \nabla_x R_k - L R_k = K B_k, \\ R_k(x, 0) \equiv 0. \end{cases} \quad (4.21)$$

Since \mathbb{G}^t is a bounded operator on $L_x^2(L_\xi^2)$, we have from (4.21) and (4.20) that the remainder is smooth, that is, there exists $C > 0$ such that

$$\|\nabla_x^k R_k\|_{L_x^2(L_\xi^2)} \leq C \text{ for } t \geq 0. \quad (4.22)$$

Thus we have the decomposition of the solution

$$\mathbf{g} = \mathbb{O}^t \mathbf{g}_{in} + A_k + R_k,$$

with the first and second terms decay exponentially and the third term smooth.

5. Discrete Spectrum near Origin

The Boltzmann equation is independent of any reference inertial coordinates. Thus, there exists a rotational symmetry for the velocity variables. In this section, we will make uses of this basic property to reduce the spectral properties for three dimensional operator from those for the one-dimensional operator, the planar wave case. In particular, we will study the analytic property of the spectral data. This is the first step in studying the long waves that will be carried out in the next two sections.

5.1. $SO(3)$ transformations

Define a $SO(3)$ -action on L_ξ^2 as follows

$$\begin{aligned} \iota : (\mathfrak{g}, f) \in SO(3) \times L_\xi^2 &\longmapsto L_\xi^2, \\ \iota(\mathfrak{g}, f) &\longmapsto \mathfrak{g}f, \\ \mathfrak{g}f(\xi) &\equiv f(\mathfrak{g}\xi), \end{aligned}$$

where $\mathfrak{g}\xi$ is the image of the linear isometric transformation, $\mathfrak{g} : \xi \in \mathbb{R}^3 \mapsto \mathfrak{g}\xi \in \mathbb{R}^3$. Thus, each $\mathfrak{g} \in SO(3)$ defines an operator on L_ξ^2 :

$$\mathfrak{g} : f \in L_\xi^2 \longmapsto \mathfrak{g}f \in L_\xi^2.$$

Lemma 5.1. *The collision operator Q is invariant under the $SO(3)$ transformation:*

$$Q\mathfrak{g} = \mathfrak{g}Q \text{ for any } \mathfrak{g} \in SO(3), \text{ i.e.}$$

$$\begin{array}{ccc} L_\xi^2 & \xrightarrow{Q} & L_\xi^2 \\ \downarrow \mathfrak{g} & & \downarrow \mathfrak{g} \\ L_\xi^2 & \xrightarrow{Q} & L_\xi^2 \end{array}$$

Proof. For any $\mathfrak{g} \in SO(3)$ and $\Omega \in S^2$,

$$\begin{cases} (\mathfrak{g}\xi)' = \frac{\mathfrak{g}\xi + \mathfrak{g}\xi_*}{2} - (\mathfrak{g}\Omega \cdot (\mathfrak{g}\xi - \mathfrak{g}\xi_*))\mathfrak{g}\Omega = \mathfrak{g}\left(\frac{\xi + \xi_*}{2} - (\Omega \cdot (\xi - \xi_*))\Omega\right) = \mathfrak{g}\xi', \\ (\mathfrak{g}\xi_*)' = \frac{\mathfrak{g}\xi + \mathfrak{g}\xi_*}{2} + (\mathfrak{g}\Omega \cdot (\mathfrak{g}\xi - \mathfrak{g}\xi_*))\mathfrak{g}\Omega = \mathfrak{g}\left(\frac{\xi + \xi_*}{2} + (\Omega \cdot (\xi - \xi_*))\Omega\right) = \mathfrak{g}\xi_*'. \end{cases} \quad (5.1)$$

For any $f \in L_\xi^2$,

$$\begin{aligned} \mathfrak{g}Q(f)(\xi) &= Q(f)(\mathfrak{g}\xi) \\ &= \int_{\substack{(\mathfrak{g}\xi - \bar{\xi}_*) \cdot \bar{\Omega} > 0 \\ \bar{\Omega} \in S^2 \\ \bar{\xi}_* \in \mathbb{R}^3}} (-f(\mathfrak{g}\xi)f(\bar{\xi}_*) + f((\mathfrak{g}\xi)')f(\bar{\xi}_*')) C(\mathfrak{g}\xi - \bar{\xi}_*, \bar{\Omega}) d\bar{\Omega} d\bar{\xi}_*. \end{aligned} \quad (5.2)$$

Let $\bar{\Omega} = \mathfrak{g}\Omega$ and $\bar{\xi}_* = \mathfrak{g}\xi_*$. Then, from (5.1) we have

$$\begin{cases} (\mathfrak{g}\xi)' = \mathfrak{g}\xi', \\ \bar{\xi}'_* = \mathfrak{g}\xi'_*. \end{cases} \quad (5.3)$$

From (5.2) and (5.3),

$$\begin{aligned} \mathfrak{g}Q(f)(\xi) &= Q(f)(\mathfrak{g}\xi) = \int_{\substack{(\xi-\xi_*) \cdot \Omega > 0 \\ \Omega \in S^2 \\ \xi_* \in \mathbb{R}^3}} (-f(\mathfrak{g}\xi)f(\mathfrak{g}\xi_*) + f(\mathfrak{g}\xi')f(\mathfrak{g}\xi'_*)) C(\xi - \xi_*, \Omega) d\Omega d\xi_* \\ &= Q(\mathfrak{g}f)(\xi). \end{aligned} \quad \square$$

Corollary 5.2. *The linearized collision operator L is invariant under $SO(3)$, i.e. for any $\mathfrak{g} \in SO(3)$*

$$\mathfrak{g}L = L\mathfrak{g}.$$

Lemma 5.3. *Let $\mathfrak{g} \in SO(3)$ be any transformation which maps $\eta/|\eta| \equiv (\omega^1, \omega^2, \omega^3)$ to $(1, 0, 0)$. Then,*

$$\begin{cases} \mathfrak{g}\chi_0 = \chi_0, \\ \mathfrak{g}\chi_1 = \sum_{j=1}^3 \omega^j \chi_j, \\ \mathfrak{g}\chi_4 = \chi_4. \end{cases} \quad (5.4)$$

This lemma is a direct consequence of $SO(3)$ action. The proof is omitted. Here, $\mathfrak{g}\chi_2$ and $\mathfrak{g}\chi_3$ are not specified, as the element $\mathfrak{g} \in SO(3)$ mapping $\eta/|\eta|$ to $(1, 0, 0)$ is not unique.

5.2. 1-D problem spectrum

First, we consider a special eigenvalue-eigenvector, (ρ, \mathbf{e}) , problem:

$$(-i\eta^1 \xi^1 + L)\mathbf{e} = \rho \mathbf{e} \text{ for } \eta^1 \in \mathbb{C}. \quad (5.5)$$

Denote by $\rho(\zeta)$ the point spectrum of the operator $-i\zeta\xi^1 + L$:

$$\rho(\zeta) \equiv \{\sigma \in \mathbb{R} \mid \text{the problem } (-i\zeta\xi^1 + L - \sigma)\mathbf{e} = 0 \text{ has a nontrivial solution } \mathbf{e} \in L_\xi^2\}$$

Lemma 5.4. *There exist $\kappa_0 > 0$ such that for any $|\zeta| \leq \kappa_0$ (1)*

$$\rho(\zeta) \subset \{z \in \mathbb{C} \mid \text{Re}(z) \leq 0\}. \quad (5.6)$$

(2) *the functions*

$$\rho(\zeta) = \{\rho_1(\zeta), \rho_2(\zeta), \rho_3(\zeta), \rho_4(\zeta), \rho_5(\zeta)\}$$

are real analytic functions in $\epsilon \equiv i\zeta$ around $\epsilon = 0$:

$$\rho_j(\zeta) = i\zeta \sum_{k=0}^{\infty} (i\zeta)^k \rho_{j,k}, \quad \rho_{j,k} \in \mathbb{R}, \quad (5.7)$$

i.e.

$$\begin{cases} \rho_j(\zeta) = i\zeta A_j^1(\zeta^2) + A_j^2(\zeta^2); \\ A_j^k(x) : \text{ real analytic functions in } x \text{ for } j = 1, \dots, 5 \text{ and } k = 1, 2 \end{cases} \quad (5.8)$$

and there exist normalized eigenfunctions $\mathbf{e}_j(\zeta) \in L_\xi^2$ which are analytic in ζ :

$$\begin{cases} \mathbf{e}_j(0) = \mathbf{E}_j, \quad j = 1, 2, 3, 4, 5, \\ \mathbf{e}_j(\zeta) = \sum_{k=0}^{\infty} (i\zeta)^k \mathbf{e}_{j,k}, \\ \mathbf{e}_{j,k} \in L_\xi^2, \\ (\mathbf{e}_j(\zeta), \mathbf{e}_k(\zeta)) = \delta_j^k, \quad j, k \in \{1, 2, 3, 4, 5\}. \end{cases} \quad (5.9)$$

Furthermore, these normalized eigenfunctions $\mathbf{e}_j(\zeta)$ can be written in the form:

$$\mathbf{e}_j(\zeta) = \mathbf{e}_j^0(\zeta^2) + i\zeta \mathbf{e}_j^1(\zeta^2), \quad \mathbf{e}_j^0(0) = \mathbf{E}_j,$$

where $\mathbf{e}_j^0(x)$ and $\mathbf{e}_j^1(x)$ are L_ξ^2 -valued analytic functions in $x \in \mathbb{R}$.

Proof. (5.6) is already given in Lemma 2.5. We now prove that $\rho_j(\zeta)$ are real analytic function in $\epsilon = i\zeta$ and (5.9).

Replace $\rho(\zeta)$ by $\epsilon\gamma(\zeta) = i\zeta\gamma(\zeta)$. We rewrite the eigenvalue problem (5.5) in the following form

$$(-\epsilon\xi^1 + L)\mathbf{e} = \epsilon\gamma\mathbf{e}, \quad (5.10)$$

and solve $\gamma(\zeta)$.

Apply Macro-Micro decomposition to (5.10) and \mathbf{e} :

$$-\mathbf{P}_0\xi^1(\mathbf{P}_0\mathbf{e} + \mathbf{P}_1\mathbf{e}) = \gamma\mathbf{P}_0\mathbf{e}, \quad (5.11)$$

$$\mathbf{P}_1(-\epsilon\xi^1(\mathbf{P}_0\mathbf{e} + \mathbf{P}_1\mathbf{e}) + L\mathbf{P}_1\mathbf{e}) = \epsilon\gamma\mathbf{P}_1\mathbf{e}. \quad (5.12)$$

From (5.12), one can express $\mathbf{P}_1\mathbf{e}$ as

$$\mathbf{P}_1\mathbf{e} = (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1\mathbf{P}_0\mathbf{e}). \quad (5.13)$$

Then substitute (5.13) into (5.11) to yield

$$[-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1)) - \gamma] \mathbf{P}_0\mathbf{e} = 0. \quad (5.14)$$

Since $\dim(\mathbf{P}_0L_\xi^2) = 5$, one can represent $\mathbf{P}_0\mathbf{e}$ in (5.14) as a vector in \mathbb{R}^5 and $[-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1)) - \gamma]$ can be identified as a 5×5 analytic matrix in both η and ρ . Thus, in order to have a nontrivial solution $\mathbf{P}_0\mathbf{e}$ to (5.14) one needs to impose that

$$\det [-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1)) - \gamma]_{5 \times 5} = 0. \quad (5.15)$$

Since L is a symmetric operator, both $(L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)$ and $\mathbf{P}_1(L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}\mathbf{P}_1$ are symmetric operators. Now, we consider

$$\begin{aligned} & (\chi_j, -\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))\chi_k) \\ &= (\chi_j, -\mathbf{P}_0\xi^1\chi_k) + (\chi_j, -\mathbf{P}_0\xi^1(L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1)\chi_k) \\ &= -(\mathbf{P}_0\xi^1\chi_j, \chi_k) - (\mathbf{P}_1\xi^1\chi_j, (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1)\chi_k) \\ &= -(\mathbf{P}_0\xi^1\chi_j, \chi_k) - ((L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}\mathbf{P}_1\xi^1\chi_j, (\epsilon\mathbf{P}_1\xi^1)\chi_k) \\ &= -(\mathbf{P}_0\xi^1\chi_j, \chi_k) - \epsilon(\mathbf{P}_0\xi^1(L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}\mathbf{P}_1\xi^1\chi_j, \chi_k) \\ &= (\chi_k, -\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))\chi_j) \end{aligned} \quad (5.16)$$

Hence $[-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))]_{5 \times 5}$ is a symmetric matrix.

Next, we denote by $\Lambda_j(\epsilon, \gamma)$, $j = 1, \dots, 5$, the five eigenvalues of the matrix $[-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))]_{5 \times 5}$:

$$\begin{cases} \det \left([-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))]_{5 \times 5} - \Lambda_j(\epsilon, \gamma)id \right) = 0, \\ \left([-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))]_{5 \times 5} - \Lambda_j(\epsilon, \gamma)id \right) \Big|_{\epsilon=0} \mathbf{E}_j = 0, \end{cases} \quad (5.17)$$

where id is a 5×5 identity matrix.

Thus

$$\prod_{j=1}^5 \Lambda_j(\epsilon, \lambda) = \det \left([-\mathbf{P}_0\xi^1(1 + (L - \epsilon\mathbf{P}_1\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1\xi^1))]_{5 \times 5} \right)$$

is analytic in η and ϵ . Note that there is the five multiplicity of zero eigenvalues at $\eta = 0$ for the operator L . Even as we factor out $\epsilon = i\zeta$, there is still the three multiplicity. Thus one cannot conclude directly the analyticity of each of the eigenvalues as claimed in the lemma. Our procedure below is to design a cascading of reducing the problem to a 3×3 matrix with distinct eigenvalues corresponding to a one-dimensional waves.

By symmetry, (5.17), we have

$$\Lambda_4 \equiv \Lambda_5. \quad (5.18)$$

Now, we introduce 1-D and 2-D macro-micro decomposition $(\mathbf{P}_0^{1D}, \mathbf{P}_1^{1D})$ and $(\mathbf{P}_0^{2D}, \mathbf{P}_1^{2D})$ for functions \mathbf{g} satisfying $(\chi_2, \mathbf{g}) = (\chi_3, \mathbf{g}) = 0$:

$$\begin{cases} \mathbf{P}_0^{1D} \mathbf{g} = (\chi_0, \mathbf{g})\chi_0 + (\chi_1, \mathbf{g})\chi_1 + (\chi_4, \mathbf{g})\chi_4 \\ \mathbf{P}_1^{1D} \mathbf{g} = \mathbf{g} - \mathbf{P}_0^{1D} \mathbf{g}; \end{cases}$$

and for $(\chi_3, \mathbf{g}) = 0$,

$$\begin{cases} \mathbf{P}_0^{2D} \mathbf{g} = (\chi_0, \mathbf{g})\chi_0 + (\chi_1, \mathbf{g})\chi_1 + (\chi_2, \mathbf{g})\chi_2 + (\chi_4, \mathbf{g})\chi_4 \\ \mathbf{P}_1^{2D} \mathbf{g} = \mathbf{g} - \mathbf{P}_0^{2D} \mathbf{g}. \end{cases}$$

Through the above two macro-micro decompositions, one sees that $\Lambda_1(\epsilon, \gamma)$, $\Lambda_2(\epsilon, \gamma)$, $\Lambda_3(\epsilon, \epsilon)$ are the eigenvalues of a 3×3 matrix:

$$[-\mathbf{P}_0^{1D}\xi^1(1 + (L - \epsilon\mathbf{P}_1^{1D}\xi^1 - \epsilon\gamma)^{-1}(\epsilon\mathbf{P}_1^{1D}\xi^1))]_{3 \times 3}.$$

From (5.16), we conclude that the above the matrix is real and symmetric in (ϵ, γ) . This 3×3 matrix has 3 distinct eigenvalues for (ϵ, γ) around origin. Thus the three eigenvalues are analytic in (ϵ, γ) around origin. Hence,

$$\Lambda_j(\epsilon, \gamma) \text{ are analytic function in } (\epsilon, \gamma) \text{ around origin for } j = 1, 2, 3. \quad (5.19)$$

Similarly, $\Lambda_1, \dots, \Lambda_4$ are eigenvalues of the 4×4 matrix

$$\left[-\mathbf{P}_0^{2D} \xi^1 (1 + (L - \epsilon \mathbf{P}_1^{2D} \xi^1 - \epsilon \gamma)^{-1} (\epsilon \mathbf{P}_1^{2D} \xi^1)) \right]_{4 \times 4},$$

and so

$$\prod_{j=1}^4 \Lambda_j(\epsilon, \gamma) = \det \left(\left[-\mathbf{P}_0^{2D} \xi^1 (1 + (L - \epsilon \mathbf{P}_1^{2D} \xi^1 - \epsilon \gamma)^{-1} (\epsilon \mathbf{P}_1^{2D} \xi^1)) \right]_{4 \times 4} \right) \quad (5.20)$$

is analytic in (ϵ, γ) . Thus, from (5.20) and (5.19)

$$\Lambda_4(\epsilon, \gamma) = \frac{\det \left(\left[-\mathbf{P}_0^{2D} \xi^1 (1 + (L - \epsilon \mathbf{P}_1^{2D} \xi^1 - \epsilon \gamma)^{-1} (\epsilon \mathbf{P}_1^{2D} \xi^1)) \right]_{4 \times 4} \right)}{\det \left(\left[-\mathbf{P}_0^{1D} \xi^1 (1 + (L - \epsilon \mathbf{P}_1^{1D} \xi^1 - \epsilon \gamma)^{-1} (\epsilon \mathbf{P}_1^{1D} \xi^1)) \right]_{3 \times 3} \right)}$$

is analytic in (ϵ, γ) . From (5.18) we have

$$\Lambda_j(\epsilon, \gamma) \text{ are real analytic in } (\epsilon, \gamma) \text{ around origin for } j = 1, \dots, 5.$$

With the five eigenvalues $\Lambda_j(\epsilon, \gamma)$, $j = 1, \dots, 5$, the problem (5.15) have solutions given by the implicit relations

$$\Lambda_j(\epsilon, \gamma_j) = \gamma_j, \text{ for } j = 1, \dots, 5.$$

Thus the real analyticity of $\Lambda_j(\epsilon, \gamma_j)$ yields that $\gamma_j(\zeta) \equiv \frac{\rho_j(\zeta)}{i\zeta}$ is a real analytic function in $\epsilon \equiv i\zeta$. This proves (5.7) and (5.8).

The proof of (5.9) is a simple consequence of the eigenvector $\mathbf{v}_j(\epsilon, \gamma)$ of the operator $-\mathbf{P}_0 \xi^1 (1 + (L - \epsilon \mathbf{P}_1 \xi^1 - \epsilon \gamma)^{-1} (\epsilon \mathbf{P}_1 \xi^1 \mathbf{P}_0))$ being normalized as $(\mathbf{v}_j, \mathbf{E}_j) = 1$ and its analyticity in the variables (ϵ, γ) , $(\epsilon \equiv i\zeta)$. Then, the eigenvector $\mathbf{e}_j(\zeta)$ is given by

$$\begin{cases} \bar{\mathbf{e}}_j(\zeta) \equiv (1 + (L - i\zeta \mathbf{P}_1 \xi^1 - i\zeta \gamma_j(\zeta))^{-1} (i\zeta \mathbf{P}_1 \xi^1 \mathbf{P}_0)) \mathbf{v}_j(i\zeta, \gamma_j(\zeta)), \\ \mathbf{e}_j(\zeta) \equiv \frac{\bar{\mathbf{e}}_j(\zeta)}{(\bar{\mathbf{e}}_j(\zeta), \bar{\mathbf{e}}_j(\zeta))^{1/2}}. \end{cases}$$

Finally the orthogonal property is a simple consequence of a symmetry of the operator $-i\zeta\xi^1 + L$. \square

5.3. 3-D spectrum

The eigenvalues and the eigenvectors given in Lemma 5.4 can be related by the following lemmas.

Lemma 5.5. *For $|\zeta|$ sufficiently small and $\zeta \in \mathbb{R}$,*

$$\begin{cases} \rho_3(\zeta) = \rho_1(-\zeta), \\ \mathfrak{J}\mathbf{e}_3(-\zeta) = \mathbf{e}_1(\zeta), \end{cases}$$

where $\mathfrak{J} \in SO(3)$,

$$\mathfrak{J}(\xi^1, \xi^2, \xi^3) \equiv (-\xi^1, \xi^2, \xi^3), \quad \mathfrak{J}^2 = \text{Identity}.$$

Proof. Consider the identity

$$\mathfrak{J}(-i\zeta\xi^1 + L)\mathfrak{J}\mathbf{e}_1 = \rho_1(\zeta)\mathfrak{J}\mathbf{e}_1.$$

From Corollary 5.2 and that $\mathfrak{J}\xi^1\mathfrak{J} = -\xi^1$,

$$(i\zeta\xi^1 + L)\mathfrak{J}\mathbf{e}_1 = \rho_1\mathfrak{J}\mathbf{e}_1.$$

From the property $\mathfrak{J}\mathbf{e}_1(0) = \mathbf{E}_3$, this concludes that

$$\begin{cases} \rho_1(\zeta) = \rho_3(-\zeta), \\ \mathfrak{J}\mathbf{e}_1(\zeta) = \mathbf{e}_3(-\zeta). \end{cases}$$

The second identity of the lemma then follows from that $\mathfrak{J}^2 = \text{Identity}$. \square

Lemma 5.6. *For $\zeta \in \mathbb{R}$, the eigenvalues $\rho_2(\zeta)$, $\rho_4(\zeta)$, and $\rho_5(\zeta)$ satisfy*

$$\begin{cases} \rho_2(\zeta) = A_2\zeta^2(1 + \mathcal{J}_2(\zeta^2)) \in \mathbb{R}, \\ \rho_4(\zeta) = \rho_5(\zeta) = A_4\zeta^2(1 + \mathcal{J}_4(\zeta^2)) \in \mathbb{R}, \\ \mathcal{J}_2(0) = \mathcal{J}_4(0) = 0, \end{cases}$$

where $\mathcal{J}_2(z)$ and $\mathcal{J}_4(z)$ are real analytic functions in z .

Proof. The proof for the above two identities are similar, we only prove the first one.

First, we take the complex conjugate of the equation for $\mathbf{e}_2(\zeta)$:

$$[i\bar{\zeta}\xi^1 + L]\bar{\mathbf{e}}_2(\zeta) = \bar{\rho}_2(\zeta)\bar{\mathbf{e}}_2(\zeta).$$

Since $\bar{\mathbf{e}}_2(0) = \mathbf{E}_2$, this yields that $\bar{\mathbf{e}}_2(\zeta)$ is the second eigenvector for the $i\bar{\zeta}\xi^1 + L$. Thus, its eigenvalue is $\bar{\rho}_2(\zeta)$; and it follows that

$$\rho_2(-\bar{\zeta}) = \bar{\rho}_2(\zeta). \quad (5.21)$$

Now, substitute (5.21) into the expansion series (5.7) to result in

$$i(-\bar{\zeta}) \sum_{k=0}^{\infty} (-i\bar{\zeta})^k \rho_{2,k} = \overline{\left(i\zeta \sum_{k=0}^{\infty} (i\zeta)^k \rho_{2,k} \right)}.$$

With the property that $\rho_{2,k} \in \mathbb{R}$ for all k , one concludes that

$$0 = i\zeta \sum_{j=0}^{\infty} (i\zeta)^{2j} \rho_{2,2j} \text{ for } |\zeta| \ll 1.$$

Thus,

$$\rho_{2,2j} = 0 \text{ for all } j,$$

$$\begin{aligned} \rho_2(\zeta) &= i\zeta \sum_{k=0}^{\infty} (i\zeta)^{2k+1} \rho_{2,2k+1} = \zeta^2 \sum_{k=0}^{\infty} \rho_{2,2k+1} (-1)^{k+1} \zeta^{2k} \\ &\equiv A_2 \zeta^2 (1 + \mathcal{J}_2(\zeta^2)) \in \mathbb{R} \text{ for } \zeta \in \mathbb{R}. \end{aligned} \quad \square$$

Corollary 5.7. *For $\zeta \in \mathbb{R}$, the spectrum property of $\rho_j(\eta)$ in (5.8) can be expressed as*

$$\begin{cases} \rho_1(\zeta) = i\zeta A_1^1(\zeta^2) - A_1^2(\zeta^2), \\ \rho_2(\zeta) = -A_2^2(\zeta^2), \\ \rho_3(\zeta) = -i\zeta A_1^1(\zeta^2) - A_1^2(\zeta^2), \\ \rho_4(\zeta) = \rho_5(\zeta) = -A_4^2(\zeta^2), \end{cases} \quad (5.22)$$

where $A_j^k(x)$ are real analytic functions and satisfy

$$\begin{cases} A_1^1(0) = \mathbf{c}, A_1^2(0) = A_4^2(0) = 0, \\ \frac{d}{dx} A_1^2(0) = A_1 > 0, A_1^2(x) = A_1 x(1 + \mathcal{J}_1(x)) \\ \frac{d}{dx} A_2^2(0) = A_2 > 0, A_2^2(x) = A_2 x(1 + \mathcal{J}_2(x)); \\ \frac{d}{dx} A_4^2(0) = A_4 > 0, A_4^2(x) = A_4 x(1 + \mathcal{J}_4(x)), \end{cases} \quad (5.23)$$

where $\mathcal{J}_j(0) = 0$ for $j = 1, 2, 4$. Furthermore,

$$A_j = -(\mathbf{P}_1 \xi^1 \mathbf{E}_j, L^{-1} \mathbf{P}_1 \xi^1 \mathbf{E}_j) \text{ for } j = 1, \dots, 5.$$

Let $\mathbf{e}_j(\zeta)$ be the j -th eigenvector of the operator $-i\zeta \xi^1 + L$, $(-i\zeta \xi^1 + L)\mathbf{e}_j(\zeta) = \rho_j(\zeta)\mathbf{e}_j(\zeta)$. Set

$$\begin{cases} \mathbf{e}_j(\zeta) = \mathbf{a}_j(\zeta) + \mathbf{b}_j(\zeta), \\ \mathbf{a}_j(\zeta) \equiv \mathbf{P}_0 \mathbf{e}_j(\zeta), \quad \mathbf{b}_j(\zeta) \equiv \mathbf{P}_1 \mathbf{e}_j(\zeta), \\ \mathbf{a}_j(\zeta) \equiv \mathbf{a}_j^1(\zeta)\chi_0 + \mathbf{a}_j^2(\zeta)\chi_1 + \mathbf{a}_j^3(\zeta)\chi_4, \\ \mathbf{a}_j^k(\zeta) = \mathbf{a}_{j,0}^k(\zeta^2) + i\zeta \mathbf{a}_{j,1}^k(\zeta^2). \end{cases} \quad (5.24)$$

Here, the functions $\mathbf{a}_{j,l}^k(x)$ are real-valued analytic function in $\zeta \in \mathbb{R}$.

Corollary 5.8. *The macroscopic components $\mathbf{e}_1(\zeta)$ and $\mathbf{e}_3(\zeta)$ satisfy*

$$\begin{cases} \mathbf{a}_1(\zeta) = (\mathbf{a}_{1,0}^1(\zeta^2) + i\zeta \mathbf{a}_{1,1}^1(\zeta^2)) \chi_0 + (\mathbf{a}_{1,0}^2(\zeta^2) + i\zeta \mathbf{a}_{1,1}^2(\zeta^2)) \chi_1 \\ \quad + (\mathbf{a}_{1,0}^3(\zeta^2) + i\zeta \mathbf{a}_{1,1}^3(\zeta^2)) \chi_4, \\ \mathbf{a}_2(\zeta) = \mathbf{a}_{2,0}^1(\zeta^2)\chi_0 + i\zeta \mathbf{a}_{2,1}^2(\zeta^2)\chi_1 + \mathbf{a}_{2,0}^3(\zeta^2)\chi_4, \\ \mathbf{a}_3(\zeta) = (\mathbf{a}_{1,0}^1(\zeta^2) - i\zeta \mathbf{a}_{1,1}^1(\zeta^2)) \chi_0 + (-\mathbf{a}_{1,0}^2(\zeta^2) + i\zeta \mathbf{a}_{1,1}^2(\zeta^2)) \chi_1 \\ \quad + (\mathbf{a}_{1,0}^3(\zeta^2) - i\zeta \mathbf{a}_{1,1}^3(\zeta^2)) \chi_4. \end{cases} \quad (5.25)$$

We finally consider the eigenvalue-eigenvector $(\lambda(\eta), \psi(\eta))$ problem for the general three dimensional waves:

$$(-i\eta \cdot \xi + L)\psi = \lambda\psi \text{ for } \eta \in \mathbb{R}^3. \quad (5.26)$$

For a given $\eta \in \mathbb{R}^3$, let $\mathbf{g} \in SO(3)$ be a transformation which maps $\eta/|\eta|$ to $(1, 0, 0)$. We then have

$$\mathbf{g}^{-1}(-i\eta \cdot \xi + L)\mathbf{g} = -i|\eta|\xi^1 + L. \quad (5.27)$$

Thus, from Lemmas 5.4 and 5.5, and (5.27) we have the following eigenvalues and eigenvectors for (5.26):

$$\begin{cases} (-i\eta \cdot \xi + L)\psi_j = \rho_j(|\eta|)\psi_j, \\ \sigma_j(\eta) = \rho_j(|\eta|) \text{ for } j = 1, \dots, 5, \\ \psi_1(\eta) = \mathbf{g}\mathbf{e}_1(|\eta|), \\ \psi_2(\eta) = \mathbf{g}\mathbf{e}_2(|\eta|), \\ \psi_3(\eta) = \mathbf{g}\mathcal{I}\mathbf{e}_1(-|\eta|), \\ \psi_4(\eta) = \mathbf{g}\mathbf{e}_4(|\eta|), \\ \psi_5(\eta) = \mathbf{g}\mathbf{e}_5(|\eta|). \end{cases} \quad (5.28)$$

Apply the macro-micro decomposition to ψ_j :

$$\begin{cases} \psi_j(\eta) = \alpha_j(\eta) + \beta_j(\eta), \\ \alpha_j \equiv \mathbf{P}_0\psi_j, \quad \beta_j \equiv \mathbf{P}_1\psi_j. \end{cases}$$

The above expressions relate the eigenvalues and eigenfunctions of 3-D problem to those of 1-D problem. We still need to clarify some analytic property of $\psi_j(\eta)$ since the group action \mathbf{g} does give the analytic property explicitly.

In (5.28), we have a basic relation between 1-D eigenfunctions $\mathbf{e}_j(|\zeta|)$ and 3-D eigenfunctions $\psi_j(\eta)$. The following lemma make use further details structure of the 1-D eigenfunctions to obtain a closed form expression of the eigenfunction $\psi_j(\eta)$.

Lemma 5.9. *For $\eta \in \mathbb{R}^3$ with $|\eta| \ll 1$, the eigenvector $\psi_j(\eta)$ of $-i\eta \cdot \xi + L$ satisfies, for $j = 1, 2, 3$,*

$$\begin{aligned} \psi_j(\eta) &= \left(1 + [L - i\mathbf{P}_1\eta \cdot \xi - \rho_j(|\eta|)]^{-1} i\mathbf{P}_1\eta \cdot \xi \right) \\ &\quad \times \left(\mathbf{a}_j^1(|\eta|)\chi_0 + \mathbf{a}_j^2(|\eta|) \sum_{l=1}^3 \frac{\eta^l \chi_l}{|\eta|} + \mathbf{a}_j^3(|\eta|)\chi_4 \right). \end{aligned} \quad (5.29)$$

Proof. First, we apply the macro-micro decomposition to $(-i\eta \cdot \xi + L)\psi_j = \rho_j(|\eta|)\phi_j$:

$$\begin{aligned} -i\mathbf{P}_0\eta \cdot \xi (\alpha_j + \beta_j) &= \rho_j(|\eta|)\alpha_j, \\ -i\mathbf{P}_1\eta \cdot \xi(\alpha_j + \beta_j) + L\beta_j &= \rho_j(|\eta|)\beta_j. \end{aligned} \quad (5.30)$$

From (5.30), we have

$$\beta_j = [L - i\mathbf{P}_1\eta \cdot \xi - \rho(|\eta|)]^{-1}\mathbf{P}_1i(\eta \cdot \xi)\alpha_j. \quad (5.31)$$

Next, from (5.28) we have

$$\psi_j(\eta) = \mathbf{g}e_j(|\eta|) = \mathbf{g}(\mathbf{a}_j(|\eta|) + \mathbf{b}_j(|\eta|) = \mathbf{g}\mathbf{a}_j(|\eta|) + \mathbf{g}\mathbf{b}_j(|\eta|). \quad (5.32)$$

Since the $SO(3)$ group action preserves the macro-micro structure, we have from (5.32),

$$\alpha_j(\eta) = \mathbf{P}_0\psi_j(\eta) = \mathbf{g}\mathbf{a}_j(|\eta|). \quad (5.33)$$

From Corollary 5.4, we have from the expressions in (5.24) and (5.33) that

$$\alpha_j(\eta) = \mathbf{a}_j^1(|\eta|)\chi_0 + \mathbf{a}_j^2(|\eta|) \sum_{j=1}^3 \frac{\eta^j \chi_j}{|\eta|} + \mathbf{a}_j(|\eta|)\chi_4. \quad (5.34)$$

Then, we substitute (5.34) into (5.31) to obtain β_j ; and combine the result with (5.34) to yield $\psi_j(\eta)$. This completes the proof of the lemma. \square

Corollary 5.10.

$$\left\{ \begin{aligned} \alpha_1(\eta) &= (\mathbf{a}_{1,0}^1(|\eta|^2) + i|\eta|\mathbf{a}_{1,0}^1(|\eta|^2)) \chi_0 + (\mathbf{a}_{1,0}^2(|\eta|^2) + i|\eta|\mathbf{a}_{1,0}^2(|\eta|^2)) \sum_{j=1}^3 \frac{\eta^j \chi_j}{|\eta|} \\ &\quad + (\mathbf{a}_{1,0}^3(|\eta|^2) + i|\eta|\mathbf{a}_{1,0}^3(|\eta|^2)) \chi_4, \\ \alpha_3(\eta) &= (\mathbf{a}_{1,0}^1(|\eta|^2) - i|\eta|\mathbf{a}_{1,0}^1(|\eta|^2)) \chi_0 + (-\mathbf{a}_{1,0}^2(|\eta|^2) + i|\eta|\mathbf{a}_{1,0}^2(|\eta|^2)) \sum_{j=1}^3 \frac{\eta^j \chi_j}{|\eta|} \\ &\quad + (\mathbf{a}_{1,0}^3(|\eta|^2) - i|\eta|\mathbf{a}_{1,0}^3(|\eta|^2)) \chi_4. \end{aligned} \right. \quad (5.35)$$

This corollary is a consequence of (5.34), and (5.28), (5.4).

5.4. Pairings

We will encounter different types of integrations such as contour integral, two dimensional double integral, and three dimensional triple integral. We will explicitly specify them by \oint , \iint , and \iiint to avoid confusion. For each given $\eta \in \mathbb{R}^3$, one can define an eigen-projections Π_η to the space spanned by $\{\psi_j(\eta)\}_{j=1}^5$ and its complement Π_η^\perp as follows:

$$\begin{cases} \Pi_\eta \mathbf{h} \equiv \frac{1}{2\pi i} \oint_\Gamma [z + i\xi \cdot \eta - L]^{-1} \mathbf{h} dz, \\ \Pi_\eta^\perp \equiv 1 - \Pi_\eta. \end{cases} \quad (5.36)$$

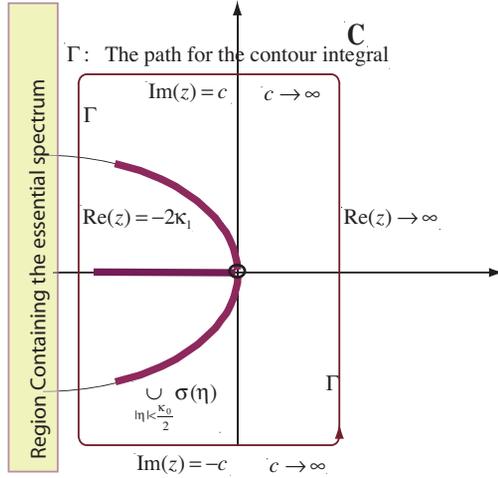


Figure A

Here the path Γ encloses the point spectrum curves $\sigma(\eta)$, $|\eta| \leq \frac{\kappa_0}{2}$, as illustrated in Figure A. Thus, the operator $[z + i\eta \cdot \xi - L]$ is invertible when $z \in \Gamma$ and $|\eta| \leq \frac{\kappa_0}{2}$. With this, the operator $\mathbb{G}_L(x, t)$ is decomposed into:

$$\begin{aligned} \mathbb{G}_L(x, t) &= \frac{1}{(2\pi)^3} \iiint_{|\eta| < \frac{\kappa_0}{2}} e^{ix\eta + (-i\xi \cdot \eta + L)t} (\Pi_\eta + \Pi_\eta^\perp) d\eta \\ &\equiv \mathbb{G}_{L;0}(x, t) + \mathbb{G}_{L;\perp}(x, t), \end{aligned}$$

where the triple integral is over $\{|\eta| \leq \frac{\kappa_0}{2}\} \subset \mathbb{R}^3$. When $|\eta| \leq \frac{\kappa_0}{2}$, the project

operator Π_η can be represented explicitly by the eigenvectors of $-i\xi^1\eta + L$.

$$\Pi_\eta \mathbf{h} = \sum_{j=1}^5 \psi_j(\eta) \otimes \langle \psi_j(\eta) | \mathbf{h}, \tag{5.37}$$

where the notation $\psi_j(\eta) \otimes \langle \psi_j(\eta) |$ is an operator on L_ξ^2 defined as follows:

For any $\mathbf{j}, \mathbf{k} \in L_\xi^2$,

$$\begin{cases} \mathbf{j} \otimes \langle \mathbf{k} | : \mathbf{l} \in L_\xi^2 \mapsto \mathbf{j} \otimes \langle \mathbf{k} | \mathbf{l} \in L_\xi^2, \\ \mathbf{j} \otimes \langle \mathbf{k} | \equiv (\mathbf{k}, \mathbf{l}) \mathbf{j}. \end{cases}$$

From (5.37), one also has that

$$\psi_4 \otimes \langle \psi_4(\eta) | + \psi_5 \otimes \langle \psi_4(\eta) | = \Pi_\eta - \sum_{j=1}^3 \psi_j \otimes \langle \psi_j(\eta) |. \tag{5.38}$$

With this eigen-projection Π_η , one has a rather explicit expression for $\mathbb{G}_{L;0}(x, t)$ and $\mathbb{G}_{L;\perp}(x, t)$:

$$\left\{ \begin{array}{l} \mathbb{G}_{L;0}(x, t) = \frac{1}{(2\pi)^3} \sum_{j=1}^5 \iiint_{|\eta| < \frac{\kappa_0}{2}} e^{ix \cdot \eta + \sigma_j(\eta)t} \psi_j(\eta) \otimes \langle \psi_j(\eta) | d\eta \\ \mathbb{G}_{L;\perp}(x, t) = \frac{1}{(2\pi)^3} \iiint_{|\eta| < \frac{\kappa_0}{2}} e^{ix \cdot \eta + (-i\xi^1 \eta + L)t} \Pi_\eta^\perp d\eta, \\ \mathbb{G}_{L;0}^t \mathbf{h}(x) \equiv \iiint_{\mathbb{R}^3} \mathbb{G}_{L;0}(x - y, t) \mathbf{h}(y) dy, \\ \mathbb{G}_{L;\perp}^t \mathbf{h}(x) \equiv \iiint_{\mathbb{R}^3} \mathbb{G}_{L;\perp}(x - y, t) \mathbf{h}(y) dy. \end{array} \right. \tag{5.39}$$

Applying the spectrum gap, Lemma 2.4, of the operator $(-i\xi \cdot \eta + L)$ when $|\eta| \ll 1$, there exists $\kappa_2 > 0$ and positive constant C such that, for $|\eta| \leq \kappa_2/2$,

$$\|e^{(i\eta^1 \eta + L)t} \Pi_\eta^\perp \mathbf{k}\|_{L_\xi^2} \leq C e^{-\kappa_2 t} \|\mathbf{k}\|_{L_\xi^2} \text{ for any } \mathbf{k} \in L_\xi^2. \tag{5.40}$$

Since the Fourier transform of the operator $\mathbb{G}_{L;\perp}^t$ has compact support inside $|\eta| \leq \kappa_0/2$, we have from (5.40) that there exists $C > 0$ such that for any

$$\mathbf{g} \in L_x^2(L_\xi^2),$$

$$\|\mathbb{G}_{L;\perp}^t \mathbf{g}\|_{L_x^\infty(L_\xi^2)} \leq C e^{-t/C} \|\mathbf{g}\|_{L_x^2(L_\xi^2)}. \quad (5.41)$$

From (5.28), Lemmas 5.5, and 5.6 we have the pairings of the eigenvalues $\sigma_j(\eta)$. These pairings and (5.38) leads to the pairing structure in the representation of $\mathbb{G}_{L;0}(x, t)$ as follows

$$\begin{aligned} \mathbb{G}_{L;0}(x, t) &= \iint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} \left[e^{\sigma_1(\eta)t} \psi_1(\eta) \otimes \langle \psi_1(\eta) | + e^{\sigma_3(-\eta)t} \psi_3(\eta) \otimes \langle \psi_3(\eta) | \right] d\eta \\ &\quad + \iint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} e^{\sigma_2(\eta)t} \psi_2(\eta) \otimes \langle \psi_2(\eta) | d\eta \\ &\quad + \iint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} e^{\sigma_4(\eta)t} \left[\Pi_\eta - \sum_{j=1}^3 \psi_j(\eta) \otimes \langle \psi_j(\eta) | \right] d\eta. \end{aligned} \quad (5.42)$$

We rearrange the pairings in (5.42) to define the following pairings:

Huygens Pairing $\hat{\mathfrak{H}}(\eta, t) \equiv \sum_{j \in \{1,3\}} e^{\sigma_j(\eta)t} \psi_j(\eta) \otimes \langle \psi_j(\eta) - \sum_{j \in \{1,3\}} e^{\sigma_j(\eta)t} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) $
Contact Pairing $\hat{\mathfrak{C}}(\eta, t) \equiv e^{\sigma_2(\eta)t} \psi_2(\eta) \otimes \langle \psi_2(\eta) $
Rotational Pairing $\hat{\mathfrak{R}}_n(\eta, t) \equiv e^{\sigma_4(\eta)t} \left(\Pi_\eta - \sum_{j=1}^3 \psi_j(\eta) \otimes \langle \psi_j(\eta) + \sum_{j \in \{1,3\}} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) \right)$
Riesz Pairings $\mathfrak{P}\hat{\mathfrak{R}}(\eta, t) \equiv \sum_{j \in \{1,3\}} (e^{\sigma_j(\eta)t} - e^{\sigma_4(\eta)t}) \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) ,$ $\mathfrak{P}\hat{\mathfrak{R}}(\eta, t) = \mathfrak{P}\hat{\mathfrak{R}}_1(\eta, t) + \mathfrak{P}\hat{\mathfrak{R}}_2(\eta, t),$ $\mathfrak{P}\hat{\mathfrak{R}}_1(\eta, t) \equiv \sum_{j \in \{1,3\}} e^{\sigma_j(\eta)t} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) $ $\quad - e^{A_1^2(\eta ^2)t} \sum_{j \in \{1,3\}} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) ,$ $\mathfrak{P}\hat{\mathfrak{R}}_2(\eta, t) \equiv (e^{A_1^2(\eta ^2)t} - e^{\sigma_4(\eta)t}) \sum_{j \in \{1,3\}} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \mathbf{P}_0^m \psi_j(\eta) ,$

(5.43)

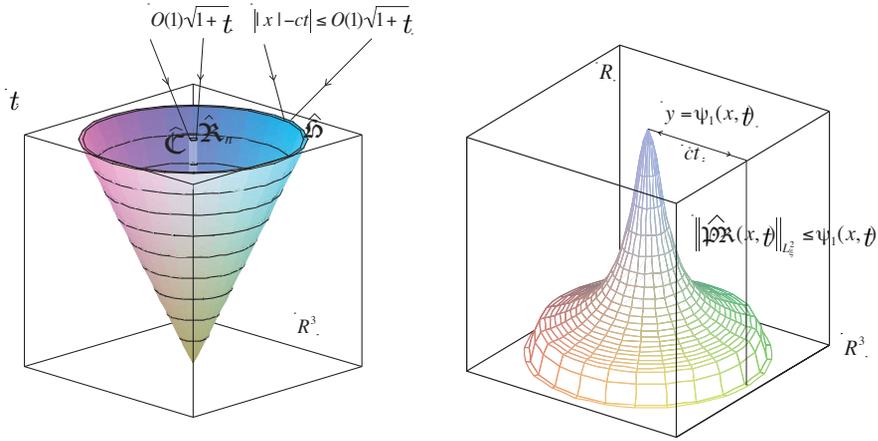
Here, \mathbf{P}_0^m is the momentum component of macro projection, (2.6). Thus we

have the decomposition of the long waves in the Green's function into the Huygens wave \mathfrak{H} , contact wave \mathfrak{C} and others:

$$\mathbb{G}_{L;0} = \mathfrak{H} + \mathfrak{C} + \mathfrak{R} + \mathfrak{P}\mathfrak{R}.$$

Here, as understood before, the Fourier inverse is defined for long waves part, for instance,

$$\mathfrak{H}(x, t) \equiv \frac{1}{(2\pi)^3} \iiint_{|\eta| < \kappa_0/2} e^{i\eta \cdot x} \hat{\mathfrak{H}}(\eta, t) d\eta.$$



Remark 5.11. The above pairings are designed with two things in mind. On the one hand, waves of different physical nature are separated. Thus, in the Huygens pairing the momentum component $\sum_{j \in \{1,3\}} e^{\sigma_j(\eta)t} P_0^m \psi_j(\eta) \otimes \langle P_0^m \psi_j(\eta) |$ is excluded and only the pressure component is kept. This physical consideration turns out to be natural analytically. In fact, the pairings make it possible to study the crucial analyticity properties of each of the pairings. This prepares us for the application of complex analytic techniques and the Kirchhoff formulas as stated in Section 3. The long wave structure of those pairings are illustrated in the figures above and will be justified in the next two sections.

The inverse Fourier transformation \mathfrak{H} of Huygens Pairing $\hat{\mathfrak{H}}$ is an isotropic wave. It is concentrated around the surface of the acoustic cone. Its inviscid version is the Huygens principle in the wave equation. The other two pairings $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{R}}_n$ carry waves along and perpendicular to macroscopic velocity direction, respectively. Thus, they are concentrated around the

center of the acoustic cone. They are realized as contact waves and shear waves in acoustic equation. The Riesz pairing $\mathfrak{P}\mathfrak{R}$ is due to the coupling of isotropic waves and shear waves within a cone. Its disturbance fills the cone with algebraic rates. It should be noted that in planar wave equation there is no such pairing since there is no shear waves. The Riesz pairing is so named because the pairing $\mathfrak{P}\mathfrak{R}(\eta, t)$ contains the factors $\eta^j \eta^k / |\eta|^2$, which are the Fourier transform of Riesz operators. For the corresponding invscid waves see (3.5)–(3.7).

6. Huygens Waves

The Huygens pairing $\hat{\mathfrak{H}}$ gives rise to Huygens waves \mathfrak{H} corresponding to the pressure waves for the wave equation (3.5) in the Euler equations. This will be analyzed by making uses of the Kirchhoff's formula, Theorem 3.1, and the related subsequent Lemmas 3.2, 3.3 and 3.4. The essential element is the analyticity property in the following lemma.

Lemma 6.1. *The Huygens pairing $\hat{\mathfrak{H}}(\eta, t)$ can be written in the following form*

$$\hat{\mathfrak{H}}(\eta, t) = e^{-A_1^2(|\eta|^2)t} \left[\cos(A_1^1(|\eta|^2)|\eta|t) \mathcal{E}_1(\eta) + \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \mathcal{E}_2(\eta) \right], \quad (6.1)$$

where $\mathcal{E}_1(\eta)$ and $\mathcal{E}_2(\eta)$ are analytic functions in η with the properties

$$\mathcal{E}_1(0) \neq 0, \quad \mathcal{E}_2(0) = 0. \quad (6.2)$$

Proof. From (5.22) and (5.28),

$$\begin{cases} \sigma_1(\eta) = & i|\eta|A_1^1(|\eta|^2) - A_1^2(|\eta|^2), \\ \sigma_3(\eta) = - & i|\eta|A_1^1(|\eta|^2) - A_1^2(|\eta|^2). \end{cases} \quad (6.3)$$

Then, from (6.3) and (5.35),

$$\begin{aligned} \mathcal{I}_1(\eta, t) \equiv & e^{\sigma_1(\eta)t} (\alpha_1(\eta) \otimes \langle \alpha_1(\eta) | - \mathbf{P}_0^m \alpha_1(\eta) \otimes \langle \mathbf{P}_0^m \alpha_1(\eta) |) \\ & + e^{\sigma_3(\eta)t} (\alpha_3(\eta) \otimes \langle \alpha_3(\eta) | - \mathbf{P}_0^m \alpha_3(\eta) \otimes \langle \mathbf{P}_0^m \alpha_3(\eta) |) \end{aligned}$$

$$\begin{aligned}
&= e^{-A_1^2(|\eta|^2)t} [(\cos(A_1^1(|\eta|^2)|\eta|t) + i \sin(A_1^1(|\eta|^2)|\eta|t)) \\
&\quad \times (\alpha_1(\eta) \otimes \langle \alpha_1(\eta) | - P_0^m \alpha_1(\eta) \otimes \langle P_0^m \alpha_1(\eta) |) \\
&\quad + (\cos(A_1^1(|\eta|^2)|\eta|t) - i \sin(A_1^1(|\eta|^2)|\eta|t)) \\
&\quad \times (\alpha_3(\eta) \otimes \langle \alpha_3(\eta) | - P_0^m \alpha_3(\eta) \otimes \langle P_0^m \alpha_3(\eta) |)]. \tag{6.4}
\end{aligned}$$

Now, from (5.35) there are analytic functions \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} in $\eta \in \mathbb{R}^3$ so that $\alpha_1(\eta)$ and $\alpha_3(\eta)$ are related by

$$\begin{cases} \alpha_1(\eta) = \mathcal{A}(\eta) + |\eta| \left(\mathcal{B}(\eta) + \frac{\mathcal{C}(\eta)}{|\eta|^2} \right), \\ \alpha_3(\eta) = \mathcal{A}(\eta) - |\eta| \left(\mathcal{B}(\eta) + \frac{\mathcal{C}(\eta)}{|\eta|^2} \right), \\ P_0^m \alpha_1(\eta) = \mathcal{D}(\eta) + \frac{\mathcal{C}(\eta)}{|\eta|}, \\ P_0^m \alpha_3(\eta) = \mathcal{D}(\eta) - \frac{\mathcal{C}(\eta)}{|\eta|}. \end{cases} \tag{6.5}$$

Here, the analytic function $\mathcal{C}(\eta)$ contains a factor $\eta \cdot \xi$. Thus,

$$\mathcal{C}(0) = 0. \tag{6.6}$$

Then, we substitute (6.5) into (6.4) to result in

$$\mathcal{I}_1(\eta, t) = e^{-A_1^2(|\eta|^2)t} \cos(A_1^1(|\eta|^2)|\eta|t) \mathcal{O}_1(\eta) + e^{-A_1^2(|\eta|^2)t} \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \mathcal{O}_2(\eta), \tag{6.7}$$

where both $\mathcal{O}_1(\eta)$ and $\mathcal{O}_2(\eta)$ are analytic functions in $\eta \in \mathbb{R}^3$ determined by the analytic functions \mathcal{A} , \mathcal{B} , \mathcal{C} , and \mathcal{D} :

$$\begin{cases} \mathcal{O}_1(\eta) = 2\mathcal{A}(\eta) \otimes \langle \mathcal{A}(\eta) | + 2\mathcal{B}(\eta) \otimes \langle \mathcal{C}(\eta) | + 2\mathcal{C}(\eta) \otimes \langle \mathcal{B}(\eta) | \\ \quad - 2\mathcal{D}(\eta) \otimes \langle \mathcal{D}(\eta) | + |\eta|^2 2\mathcal{B}(\eta) \otimes \langle \mathcal{B}(\eta) |, \\ \mathcal{O}_2(\eta) = 2\mathcal{A}(\eta) \otimes \langle \mathcal{C}(\eta) | + 2\mathcal{C}(\eta) \otimes \langle \mathcal{A}(\eta) | - 2\mathcal{C}(\eta) \otimes \langle \mathcal{D}(\eta) | \\ \quad - 2\mathcal{D}(\eta) \otimes \langle \mathcal{C}(\eta) | + 2|\eta|^2 (\mathcal{A}(\eta) \otimes \langle \mathcal{B}(\eta) | + 2\mathcal{B}(\eta) \otimes \langle \mathcal{A}(\eta) |). \end{cases}$$

From this and (6.6), we have

$$\mathcal{O}_2(0) = 0. \tag{6.8}$$

Next, we consider

$$\begin{aligned} \mathcal{J}_2(\eta, t) &\equiv e^{\sigma_1(\eta)t} (\alpha_1(\eta) \otimes \langle \beta_1(\eta) | + \beta_1(\eta) \otimes \langle \alpha_1(\eta) |) \\ &\quad + e^{\sigma_3(\eta)t} (\alpha_3(\eta) \otimes \langle \beta_3(\eta) | + \beta_3(\eta) \otimes \langle \alpha_3(\eta) |). \end{aligned} \quad (6.9)$$

From (5.31) and (5.22), $\beta_1(\eta)$ and $\beta_3(\eta)$ are related by

$$\begin{cases} \beta_1(\eta) = (L - iP_1\eta \cdot \xi - i|\eta|A_1^1(|\eta|^2) + A_1^2(|\eta|^2))^{-1}P_1i\eta \cdot \xi \alpha_1, \\ \beta_3(\eta) = (L - iP_1\eta \cdot \xi + i|\eta|A_1^1(|\eta|^2) + A_1^2(|\eta|^2))^{-1}P_1i\eta \cdot \xi \alpha_3. \end{cases} \quad (6.10)$$

Here, $L - iP_1\eta \cdot \xi$, $i|\eta|A_1^1(|\eta|^2)$, and $A_1^2(|\eta|^2)$ are commutative; and the operator L is bounded from below; and $A_1^2(0) = 0$. With these properties, we can expand the operators $(L - iP_1\eta \cdot \xi - i|\eta|A_1^1(|\eta|^2) + A_1^2(|\eta|^2))^{-1}$ and $(L - iP_1\eta \cdot \xi + i|\eta|A_1^1(|\eta|^2) + A_1^2(|\eta|^2))^{-1}$ when $|\eta| \ll 1$ as follows

$$\begin{aligned} &\frac{1}{(L - iP_1\eta \cdot \xi \mp i|\eta|A_1^1(|\eta|^2) + A_1^2(|\eta|^2))} \\ &= \frac{1}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))(1 + \frac{\mp i|\eta|A_1^1(|\eta|^2)}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))})} \\ &= \frac{1}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))} \sum_{j=0}^{\infty} \left(\frac{\pm i|\eta|A_1^1(|\eta|^2)}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))} \right)^j \\ &= \frac{1 \pm \frac{i|\eta|A_1^1(|\eta|^2)}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))}}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))} \sum_{j=0}^{\infty} (-1)^j \left(\frac{|\eta|^2 A_1^1(|\eta|^2)^2}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))^2} \right)^j \\ &= \frac{1 \pm \frac{i|\eta|A_1^1(|\eta|^2)}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))}}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))(1 + \frac{|\eta|^2 A_1^1(|\eta|^2)^2}{(L - iP_1\eta \cdot \xi + A_1^2(|\eta|^2))^2})} \\ &\equiv \mathcal{M}_1(\eta) \pm |\eta| \mathcal{M}_2(\eta). \end{aligned} \quad (6.11)$$

Both $\mathcal{M}_1(\eta)$ and $\mathcal{M}_2(\eta)$ are operators analytic in $\eta \in \mathbb{R}^3$.

Now, combine (6.10), (6.5), and (6.11) to conclude

$$\begin{cases} \beta_1(\eta) = \mathcal{A}_1(\eta) + |\eta| \mathcal{B}_1(\eta), \\ \beta_3(\eta) = \mathcal{A}_1(\eta) - |\eta| \mathcal{B}_1(\eta), \end{cases} \quad (6.12)$$

where both $\mathcal{A}_1(\eta)$ and $\mathcal{B}_1(\eta)$ are analytic functions in $\eta \in \mathbb{R}^3$. Substitute

(6.5) and (6.12) into (6.9) to yield that

$$\mathcal{I}_2(\eta, t) = e^{-A_1^2(|\eta|^2)t} \cos(A_1^1(|\eta|^2)|\eta|t) \mathcal{O}_3(\eta) + e^{-A_1^2(|\eta|^2)t} \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \mathcal{O}_4(\eta),$$

where both $\mathcal{O}_3(\eta)$ and $\mathcal{O}_4(\eta)$ are analytic functions in $\eta \in \mathbb{R}^3$.

Next, for

$$\mathcal{I}_3(\eta, t) \equiv e^{\sigma_1(\eta)t} \beta_1(\eta) \otimes \langle \beta_1(\eta) | + e^{\sigma_3(\eta)t} \beta_3(\eta) \otimes \langle \beta_3(\eta) |,$$

use (6.12) and the argument leading to (6.7) to conclude that

$$\mathcal{I}_3(\eta, t) = e^{-A_1^2(|\eta|^2)t} \cos(A_1^1(|\eta|^2)|\eta|t) \mathcal{O}_5(\eta) + e^{-A_1^2(|\eta|^2)t} \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \mathcal{O}_6(\eta), \quad (6.13)$$

where \mathcal{O}_4 and \mathcal{O}_6 are analytic functions in $\eta \in \mathbb{R}^3$. (6.1) follows from (6.7), (6.9), and (6.13).

Both pairings \mathcal{I}_2 and \mathcal{I}_3 contains microscopic components β_1 and β_3 . From (6.10), both β_1 and β_3 contain $\eta \cdot \xi$ factors. Thus,

$$\mathcal{I}_2(0, t) = \mathcal{I}_3(0, t) = 0.$$

This and (6.8) implies that $\mathcal{E}_2(0) = 0$, and (6.4) is proved. \square

Before applying the inverse Fourier transformation to the pressure Huygens pairing, we need to analyze the phase component $e^{iIm(\rho_1(|\eta|)t)}$ in the pairing. Expand

$$Im(\rho_1(|\eta|)t) = |\eta|A_1^1(|\eta|^2)t = \mathbf{c}|\eta|t + |\eta|\mathcal{H}(|\eta|^2)t,$$

where \mathcal{H} is an analytic function in $|\eta|^2$ with

$$\mathcal{H}(0) = 0.$$

We have the following identity to split wave with velocity \mathbf{c} :

$$\begin{cases} \cos(|\eta|A_1^1(|\eta|^2)t) = \cos(\mathbf{c}|\eta|t) \cos(|\eta|\mathcal{H}(|\eta|^2)t) - \sin(\mathbf{c}|\eta|t) \sin(|\eta|\mathcal{H}(|\eta|^2)t), \\ \sin(|\eta|A_1^1(|\eta|^2)t) = \sin(\mathbf{c}|\eta|t) \cos(|\eta|\mathcal{H}(|\eta|^2)t) + \cos(\mathbf{c}|\eta|t) \sin(|\eta|\mathcal{H}(|\eta|^2)t). \end{cases} \quad (6.14)$$

Lemma 6.2. *For any analytic functions $\mathcal{H}(\zeta)$ in $\zeta \in \mathbb{R}$, both functions $\cos(|\eta|\mathcal{H}(|\eta|^2)t)$ and $\sin(|\eta|\mathcal{H}(|\eta|^2)t)/|\eta|$ are analytic in $\eta \in \mathbb{R}^3$.*

Proof. Consider the power series expansions of both $\cos(|\eta|\mathcal{H}(|\eta|^2)t)$ and $\sin(|\eta|\mathcal{H}(|\eta|^2)t)/|\eta|$:

$$\left\{ \begin{array}{l} \cos(|\eta|\mathcal{H}(|\eta|^2)t) = 1 - \frac{|\eta|^2 \mathcal{H}(|\eta|^2)^2 t^2}{2!} + \frac{|\eta|^4 \mathcal{H}(|\eta|^2)^4 t^4}{4!} \\ \quad - \frac{|\eta|^6 \mathcal{H}(|\eta|^2)^6 t^6}{6!} + \dots, \\ \frac{\sin(|\eta|\mathcal{H}(|\eta|^2)t)}{|\eta|} = 1 - \frac{|\eta|^2 \mathcal{H}(|\eta|^2)^2 t^2}{3!} + \frac{|\eta|^4 \mathcal{H}(|\eta|^2)^4 t^4}{5!} \\ \quad - \frac{|\eta|^6 \mathcal{H}(|\eta|^2)^6 t^6}{7!} + \dots. \end{array} \right. \quad (6.15)$$

The RHS's of (6.15) are analytic function in $|\eta|^2$. Thus, they are analytic in $\eta \in \mathbb{R}^3$. \square

Lemma 6.3. *For any given Mach constant $\mathcal{M} > 1$, there exists $C > 0$ such that the Huygens wave $\mathfrak{H}(x, t)$ satisfies*

$$\begin{aligned} \|\mathfrak{H}(x, t)\|_{L_\xi^2} &= \left\| \frac{1}{(2\pi)^3} \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{H}}(\eta, t) d\eta \right\|_{L_\xi^2} \\ &\leq C \left[\frac{e^{-\frac{(|x| - ct)^2}{Ct}}}{(1+t)^2} + e^{-t/C} \right] \text{ for } |x| \leq \mathcal{M}ct. \end{aligned} \quad (6.16)$$

Proof. With (6.14), one has the following

$$\begin{aligned} & \iiint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{H}}(\eta, t) d\eta \\ &= \iiint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} e^{-A_1^2(|\eta|^2)t} \cos(\mathbf{c}|\eta|t) \\ & \quad \times \left(\cos(|\eta|\mathcal{H}(|\eta|^2)t) \mathcal{E}_1(\eta) + \frac{\sin(|\eta|\mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2(\eta) \right) d\eta \\ &+ \iiint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} e^{-A_1^2(|\eta|^2)t} \frac{\sin(\mathbf{c}|\eta|t)}{|\eta|} \\ & \quad \times \left(-\sin(|\eta|\mathcal{H}(|\eta|^2)t) |\eta| \mathcal{E}_1(\eta) + \cos(|\eta|\mathcal{H}(|\eta|^2)t) \mathcal{E}_2(\eta) \right) d\eta \\ &= w_t * I_1 + w * I_2, \end{aligned}$$

where

$$\left\{ \begin{array}{l} I_1(x, t) \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} e^{-A_1^2(|\eta|^2)t} \\ \quad \times \left(\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1(\eta) + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2(\eta) \right) d\eta, \\ I_2(x, t) \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} e^{-A_1^2(|\eta|^2)t} \\ \quad \times \left(-\sin(|\eta| \mathcal{H}(|\eta|^2)t) |\eta| \mathcal{E}_1(\eta) + \cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_2(\eta) \right) d\eta. \end{array} \right. \quad (6.17)$$

Thus, from the Kirchhoff formula one only needs the data of $I_1(x, t)$, $\nabla_x I_1(x, t)$, $I_2(x, t)$, and $\nabla_x I_2(x, t)$ with $|x| \leq (\mathcal{M} + 1)ct$ in order to determine $\iiint_{|\eta| < \kappa_0/2} e^{ix \cdot \eta} \mathfrak{H}(\eta, t) d\eta$ for $|x| \leq \mathcal{M}ct$.

The analysis for $I_1(x, t)$ and $I_2(x, t)$ with $|x| \leq (\mathcal{M} + 1)ct$ are identical; we estimate $I_1(x, t)$ only. For given $x \in \mathbb{R}^3$, we can find a element $\mathfrak{g}^x \in SO(3)$ so that $\mathfrak{g}^x \frac{x}{|x|} = (1, 0, 0)$. We have $(\mathfrak{g}^x)^{-1}(x \cdot \eta) \mathfrak{g}^x = |x| \eta^1$. With the change of coordinates, the functions \mathcal{E}_1 and \mathcal{E}_2 remain analytic in $\eta \in \mathbb{R}^3$. Without ambiguity, we still use \mathcal{E}_i to denote $(\mathfrak{g}^x)^{-1} \mathcal{E}_i \mathfrak{g}^x$:

$$I_1(x, t) = \iiint_{|\eta| < \kappa_0/2} e^{i|x|\eta^1} e^{-A_1^2(|\eta|^2)t} \left[\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1 + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2 \right] d\eta. \quad (6.18)$$

The integration is divided into two regions:

$$\begin{aligned} I_1(x, t) &= \left(\iiint_{\mathbb{B}} + \iiint_{\{|\eta| \leq \kappa_0/2\} \cap \mathbb{B}^c} \right) e^{i|x|\eta^1 - A_1^2(|\eta|^2)t} \\ &\quad \times \left[\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1 + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2 \right] d\eta \\ &\equiv I_1^1(x, t) + I_1^2(x, t). \end{aligned}$$

where

$$\mathbb{B} \equiv \left[-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}} \right] \times \left[-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}} \right] \times \left[-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}} \right].$$

For $\eta \in \{|\eta| \leq \kappa_0/2\} \cap \mathbb{B}^c$, we have $|\eta| \geq \kappa_0/\sqrt{8}$, and so there exists $C > 0$

such that

$$|I_1^2(x, t)| < C e^{-t/C} \text{ for } x \in \mathbb{R}^3. \quad (6.19)$$

To analyze $I_1^1(x, t)$ we use complex analytic technique for (x, t) in a finite Mach number region. This can be done because, by Lemma 6.2, the function $e^{-A_1^2(|\eta|^2)t} \left[\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1 + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2 \right]$ is analytic in η . Use (5.23) to write $A_1^2(|\eta|^2) = A_1 |\eta|^2 (1 + \mathcal{J}_1(|\eta|^2))$, and so

$$\begin{aligned} I_1^1(x, t) &= \iiint_{\mathbb{B}} e^{-\frac{|x|^2}{4tA_1}} e^{-A_1(\eta^1 - i\frac{|x|}{2A_1 t})^2 t - A_1((\eta^2)^2 + (\eta^3)^2)t + \mathbf{o}(1)|\eta|^2 t} \\ &\quad \times \left[\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1 + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2 \right] d\eta \\ &= e^{-\frac{|x|^2}{4tA_1}} \iint_{\mathbb{B}_2} \int_{\Gamma(-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}}, \frac{|x|}{tM_0})} e^{-A_1(\eta^1 - i\frac{|x|}{2A_1 t})^2 t - A_1((\eta^2)^2 + (\eta^3)^2)t + \mathbf{o}(1)|\eta|^2 t} \\ &\quad \times \left[\cos(|\eta| \mathcal{H}(|\eta|^2)t) \mathcal{E}_1 + \frac{\sin(|\eta| \mathcal{H}(|\eta|^2)t)}{|\eta|} \mathcal{E}_2 \right] d\eta^1 d\eta^2 d\eta^3, \end{aligned}$$

where

$$\mathbb{B}_2 \equiv \left\{ (\eta^2, \eta^3) \in \mathbb{R}^2 : |\eta^2|, |\eta^3| \leq \frac{\kappa_0}{2\sqrt{2}} \right\}$$

and the contour $\Gamma(\frac{-\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}}, c)$ is chosen with a parameter c , Figure B:

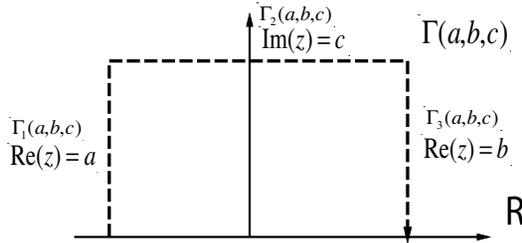


Figure B

$$\begin{aligned} \Gamma(a, b, c) &= \Gamma_1(a, b, c) \cup \Gamma_2(a, b, c) \cup \Gamma_3(a, b, c), \\ \Gamma_1(a, b, c) &\equiv \{ \eta | \operatorname{Re}(\eta) = a, \operatorname{Im}(\eta) \text{ is from } 0 \text{ to } c \}, \\ \Gamma_2(a, b, c) &\equiv \{ \eta | \operatorname{Im}(\eta) = c, \operatorname{Re}(\eta) \text{ is from } a \text{ to } b \}, \\ \Gamma_3(a, b, c) &\equiv \{ \eta | \operatorname{Re}(\eta) = b, \operatorname{Im}(\eta) \text{ is from } c \text{ to } 0 \}. \end{aligned}$$

On the contour $\Gamma(-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}}, \frac{|x|}{tM_0})$ one has the growth rate of

$$|\cos(|\eta|\mathcal{H}(|\eta|^2)t)|, \left| \frac{\sin(|\eta|\mathcal{H}(|\eta|^2)t)}{|\eta|} \right| \leq O(1)e^{o(1)|\eta|^2t}.$$

Thus, for $|x| < (\mathcal{M} + 1)ct$

$$I_1^1(x, t) \leq C \left(\frac{1}{t^{3/2}} e^{-\frac{|x|^2}{Ct}} + e^{-t/C} \right). \quad (6.20)$$

Hence, from (6.19) and (6.20)

$$I_1(x, t) \leq C \left(\frac{1}{t^{3/2}} e^{-\frac{|x|^2}{Ct}} + e^{-t/C} \right) \text{ for } |x| \leq (\mathcal{M} + 1)ct. \quad (6.21)$$

With the Gaussian-like structure of $I_1(x, t)$ in (6.21), each differentiation gives an additional decaying rate of $(1+t)^{-1/2}$. This can be shown by the same procedure for obtaining (6.21):

$$|\nabla_x I_1(x, t)| \leq C \left(\frac{1}{t^2} e^{-\frac{|x|^2}{Ct}} + e^{-t/C} \right) \text{ for } |x| \leq (\mathcal{M} + 1)ct. \quad (6.22)$$

From (6.2), \mathcal{E}_2 contains an η factor. Thus, the integrand for $I_2(x, t)$ in (6.17) has one more η factor compared to that for $I_1(\eta, t)$ in (6.17). This extra factor η results in $I_2(x, t)$ gaining an extra decaying factor $(1+t)^{-1/2}$:

$$I_2(x, t) \leq C \left(\frac{e^{-\frac{|x|^2}{Ct}}}{t^2} + e^{-t/C} \right) \text{ for } |x| \leq (\mathcal{M} + 1)ct. \quad (6.23)$$

Since the estimates for $I_1(x, t)$ and $I_2(x, t)$ in (6.21), (6.22), and (6.23) are valid in $|x| \leq (\mathcal{M} + 1)c$, the Kirchhoff formula in (3.8) and (3.9) is valid for $|x| \leq \mathcal{M}ct$. Thus, Lemma 3.2 yields, for some constant $C > 0$,

$$\begin{aligned} |w_t * I_1(x, t)| &\leq C \left[(1+t)^{-2} e^{-\frac{(|x|-ct)^2}{Ct}} + e^{-t/C} \right], \\ |w * I_2(x, t)| &\leq C \left[(1+t)^{-2} e^{-\frac{(|x|-ct)^2}{Ct}} + e^{-t/C} \right]. \end{aligned}$$

This concludes the long wave structure (6.16) of the Huygens waves. \square

7. Contact and Rotational Waves

The study of contact and rotational waves follows that for the Huygens waves. Thus we will not carry out again explicitly the complex analytic techniques that has been used there. We start with showing the analytic structure of the pairings.

Lemma 7.1. *The contact pairing $\hat{\mathcal{C}}(\eta, t)$ can be written in the form*

$$\hat{\mathcal{C}}(\eta, t) = e^{-A_2^2(|\eta|^2)t} \mathcal{E}_3(\eta), \quad (7.1)$$

where $\mathcal{E}_3(\eta)$ is an analytic function in $\eta \in \mathbb{R}^3$.

Proof. From (5.4), (5.25), and (5.33),

$$\alpha_2(\eta) = P_1 \mathbf{g} a_2(|\eta|) = a_{2,0}^1(|\eta|^2) \chi_0 + i a_{2,1}^2(|\eta|^2) \sum_{j=1}^3 \eta^j \chi_j + a_{2,0}^3(|\eta|^2) \chi_4. \quad (7.2)$$

Thus, $\alpha_2(\eta)$ is an analytic function of $\eta \in \mathbb{R}^3$. From (5.22), $\sigma_2(\eta) = \rho_2(|\eta|) = A_2^2(|\eta|^2)$ is an analytic function in η . Thus, $(L - iP_1 \eta \cdot \xi + A_2^2(|\eta|^2))$ is an analytic function in $\eta \in \mathbb{R}^3$. This, (7.2), and (5.29) yield that $\psi_2(\eta)$ is an analytic function in $\eta \in \mathbb{R}^3$. This concludes the lemma:

$$e^{-A_2^2(|\eta|^2)t} \psi_2(\eta) \otimes \langle \psi_2(\eta) | \text{ analytic in } \eta \in \mathbb{R}^3. \quad (7.3)$$

□

Lemma 7.2. *The rotational pairing $\hat{\mathfrak{R}}_n(\eta, t)$ can be rewritten in the form*

$$\hat{\mathfrak{R}}_n(\eta, t) = e^{-A_4^2(|\eta|^2)t} \mathcal{E}_4(\eta), \quad (7.4)$$

where $\mathcal{E}_4(\eta)$ is an analytic function in $\eta \in \mathbb{R}^3$.

Proof. In this pairing, due to (5.36) and (7.3) the components Π_η and $\psi_2(\eta) \otimes \psi_2(\eta)$ are analytic. We just need to show that $\sum_{j \in \{1,3\}} (\psi_j(\eta) \otimes \langle \psi_j(\eta) | - P_0^m \psi_j(\eta) \otimes \langle P_0^m \psi_j(\eta) |)$ is analytic.

Use (6.7), we have

$$\begin{aligned}
\mathcal{I}_4(\eta, t) &\equiv e^{-A_4^2(|\eta|^2)t}(\alpha_1(\eta) \otimes \langle \alpha_1(\eta) | - P_0^m \alpha_1(\eta) \otimes \langle P_0^m \alpha_1(\eta) | \\
&\quad + \alpha_3(\eta) \otimes \langle \alpha_3(\eta) | - P_0^m \alpha_3(\eta) \otimes \langle P_0^m \alpha_3(\eta) |) \\
&= e^{-A_4^2(|\eta|^2)t}(2\mathcal{A}(\eta) \otimes \langle \mathcal{A}(\eta) | + 2\mathcal{B}(\eta) \otimes \langle \mathcal{C}(\eta) | + 2\mathcal{C}(\eta) \otimes \langle \mathcal{B}(\eta) | \\
&\quad - 2\mathcal{D}(\eta) \otimes \langle \mathcal{D}(\eta) | + 2|\eta|^2 \mathcal{B}(\eta) \otimes \langle \mathcal{B}(\eta) |).
\end{aligned}$$

This shows that $\mathcal{I}_4(\eta, t)$ is analytic in $\eta \in \mathbb{R}^3$.

Similarly, using (6.5) and (6.12), one can show that both $\mathcal{I}_5(\eta, t)$ and $\mathcal{I}_6(\eta, t)$ are analytic, where

$$\begin{cases} \mathcal{I}_5(\eta, t) \equiv e^{-A_4^2(|\eta|^2)t} \left(\sum_{j \in \{1,3\}} \alpha_j(\eta) \otimes \langle \beta_j(\eta) | + \beta_j(\eta) \otimes \langle \alpha_j(\eta) | \right) \\ \mathcal{I}_6(\eta, t) \equiv e^{-A_4^2(|\eta|^2)t} \left(\sum_{j \in \{1,3\}} \beta_j(\eta) \otimes \langle \beta_j(\eta) | \right). \end{cases}$$

This completes the proof of the lemma. \square

Lemma 7.3. *The first Riesz pairing can be written as*

$$\begin{aligned}
\hat{\mathfrak{R}}_1(\eta, t) &= e^{-A_1^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} \eta^j \eta^k \\
&\quad \times \left(\mathcal{E}_{j,k}^1 \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} + \mathcal{E}_{j,k}^2 \int_0^t \frac{\sin(A_1^1(|\eta|^2)|\eta|\tau)}{|\eta|} d\tau \right), \quad (7.5)
\end{aligned}$$

where $\mathcal{E}_{j,k}^l$, $l = 1, 2$, are analytic functions in $\eta \in \mathbb{R}^3$.

Proof. The lemma follows directly from (5.25) and (5.33):

$$\begin{aligned}
&\hat{\mathfrak{R}}_1(\eta, t) \\
&= \sum_{j \in \{1,3\}} e^{\sigma_j(\eta)t} P_0^m \psi_j(\eta) \otimes \langle P_0^m \psi_j(\eta) | \\
&\quad - e^{-A_1^2(|\eta|^2)t} \sum_{j \in \{1,3\}} P_0^m \psi_j(\eta) \otimes \langle P_0^m \psi_j(\eta) |
\end{aligned}$$

$$\begin{aligned}
&= 2e^{-A_1^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} \left(a_{1,0}^2(|\eta|^2)^2 \frac{(1 - \cos(A_1^1(|\eta|^2)|\eta|t))}{|\eta|^2} \right. \\
&\quad \left. - 2a_{1,0}^2(|\eta|^2)a_{1,1}^2(|\eta|^2) \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \right) \eta^j \eta^k \chi_j \otimes \langle \chi_k | \\
&\quad + 2e^{-A_1^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} a_{1,1}^2(|\eta|^2)^2 (1 - \cos(A_1^1(|\eta|^2)|\eta|t)) \eta^j \eta^k \chi_j \otimes \langle \chi_k | \\
&= 2e^{-A_1^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} \left(a_{1,0}^2(|\eta|^2)^2 \frac{A_1^1(|\eta|^2) \int_0^t \sin(A_1^1(|\eta|^2)|\eta|\tau) d\tau}{|\eta|} \right. \\
&\quad \left. - 2a_{1,0}^2(|\eta|^2)a_{1,1}^2(|\eta|^2) \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \right) \eta^j \eta^k \chi_j \otimes \langle \chi_k | \\
&\quad + 2e^{-A_1^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} a_{1,1}^2(|\eta|^2)^2 |\eta|^2 \int_0^t \frac{\sin(A_1^1(|\eta|^2)|\eta|\tau)}{|\eta|} \eta^j \eta^k \chi_j \otimes \langle \chi_k |.
\end{aligned}$$

□

Lemma 7.4. *When $|\eta|$ is sufficient small, then the second Riesz pairing $\hat{\mathfrak{R}}_2(\eta, t)$ can be written as*

$$\begin{aligned}
&\text{if } |A_4^2(|\eta|)| \leq |A_1^2(|\eta|)|, \\
\hat{\mathfrak{R}}_2(\eta, t) &= e^{-\frac{A_4^2(|\eta|^2)t}{2}} \sum_{1 \leq j, k \leq 3} \eta^j \eta^k \\
&\quad \int_0^t \left(e^{-\frac{A_4^2(|\eta|^2)\tau}{2}} \mathcal{O}_{j,k}^1 - e^{-\frac{(2A_1^2(|\eta|^2) - A_4^2(|\eta|^2))\tau}{2}} \mathcal{O}_{j,k}^2 \right) d\tau; \\
&\text{if } |A_1^2(|\eta|)| \leq |A_4^2(|\eta|)|, \\
\hat{\mathfrak{R}}_2(\eta, t) &= e^{-\frac{A_1^2(|\eta|^2)t}{2}} \sum_{1 \leq j, k \leq 3} \eta^j \eta^k \\
&\quad \int_0^t \left(e^{-\frac{A_1^2(|\eta|^2)\tau}{2}} \mathcal{O}_{j,k}^1 - e^{-\frac{(2A_4^2(|\eta|^2) - A_1^2(|\eta|^2))\tau}{2}} \mathcal{O}_{j,k}^2 \right) d\tau, \quad (7.6)
\end{aligned}$$

where $\mathcal{O}_{j,k}^l$ are analytic functions in $\eta \in \mathbb{R}^3$.

Proof. Without loss of generality, we consider the case $|A_4^2(|\eta|)| \leq |A_1^2(|\eta|)|$ only.

This case is a consequence of (5.25) and (5.33):

$$\begin{aligned}
& \mathfrak{P}\mathfrak{R}_2(\eta, t) \\
&= (e^{-A_4^2(|\eta|^2)t} - e^{-A_1^2(|\eta|^2)t}) \sum_{j \in \{1,3\}} \mathbf{P}_0^m \psi_j(\eta) \otimes \langle \psi_j(\eta) | \\
&= e^{-\frac{A_4^2(|\eta|^2)t}{2}} \int_0^t \left(\frac{A_4^2(|\eta|^2)}{2} e^{-\frac{A_4^2(|\eta|^2)\tau}{2}} \right. \\
&\quad \left. - \frac{2A_1^2(|\eta|^2) - A_4^2(|\eta|^2)}{2} e^{-\frac{(2A_1^2(|\eta|^2) - A_4^2(|\eta|^2))\tau}{2}} \right) d\tau \\
&\quad \cdot \sum_{1 \leq j,k \leq 3} \eta^j \eta^k \chi_j \otimes \langle \chi_k | \left(-2a_{1,1}^2(|\eta|^2)^2 + 2 \frac{a_{1,0}^2(|\eta|^2)^2}{|\eta|^2} \right) \\
&= e^{-\frac{A_4^2(|\eta|^2)t}{2}} \sum_{1 \leq j,k \leq 3} \eta^j \eta^k \int_0^t \left(e^{-\frac{A_4^2(|\eta|^2)\tau}{2}} \mathcal{O}_{j,k}^1 - e^{-\frac{(2A_1^2(|\eta|^2) - A_4^2(|\eta|^2))\tau}{2}} \mathcal{O}_{j,k}^2 \right) d\tau, \quad (7.7)
\end{aligned}$$

where $\mathcal{O}_{j,k}^l$ are analytic functions in $\eta \in \mathbb{R}^3$. □

Corollary 7.5. *For the contact and rotational waves $\mathfrak{C}(x, t)$ and $\mathfrak{R}_n(x, t)$, there exists $C > 0$ such that*

$$\begin{aligned}
\|\mathfrak{C}(x, t)\|_{L_\xi^2} &= \left\| \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{C}}(\eta, t) d\eta \right\|_{L_\xi^2}, \\
\|\mathfrak{R}_n(x, t)\|_{L_\xi^2} &= \left\| \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{R}}_n(\eta, t) d\eta \right\|_{L_\xi^2} \\
&\leq C \left[\frac{e^{-\frac{|x|^2}{Ct}}}{(1+t)^{3/2}} + e^{-t/C} \right] \text{ for } |x| \leq (\mathcal{M} + 1)ct. \quad (7.8)
\end{aligned}$$

Proof. Due to (7.1) and (7.4), the procedure of using complex analysis in (6.18)–(6.21) can be applied directly to yield this corollary. □

Lemma 7.6. *For the second Riesz wave $\mathfrak{P}\mathfrak{R}_2(x, t)$, there exists $C > 0$ such that*

$$\begin{aligned}
\|\mathfrak{P}\mathfrak{R}_2(x, t)\|_{L_\xi^2} &= \left\| \iiint_{|\eta| \leq \kappa_0/a2} e^{ix \cdot \eta} \hat{\mathfrak{P}\mathfrak{R}}_2(\eta, t) d\eta \right\|_{L_\xi^2} \\
&\leq C \left[\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{\frac{3}{2}}} + e^{-t/C} \right] \text{ for } |x| \leq \mathcal{M}ct. \quad (7.9)
\end{aligned}$$

Proof. From (7.6),

$$\begin{aligned}
& \iint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{P}}\mathfrak{R}_2(\eta, t) d\eta \\
&= \iint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} e^{-A_4^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} \eta^j \eta^k \left(\int_0^t e^{(A_1^2(|\eta|^2) - A_4^2(|\eta|^2))\tau} d\tau \mathcal{O}_{j,k} \right) d\eta \\
&= \int_0^t \sum_{1 \leq j, k \leq 3} \left(\iint_{|\eta| \leq \kappa_0/2} \left(e^{ix \cdot \eta} e^{-A_4^2(|\eta|^2)t} \eta^j \eta^k \right) e^{(A_1^2(|\eta|^2) - A_4^2(|\eta|^2))\tau} \mathcal{O}_{j,k} d\eta \right) d\tau. \quad (7.10)
\end{aligned}$$

The lemma follows by applying the procedure in (6.18) - (6.21) to (7.10):

$$\begin{aligned}
& \left\| \iint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{P}}\mathfrak{R}_2(\eta, t) d\eta \right\|_{L_\xi^2} \leq O(1) \int_0^t \left(\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{5/2}} + e^{-2t/C} \right) d\tau \\
& \leq O(1) \left(\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{3/2}} + e^{-t/C} \right) \text{ for } |x| \leq (\mathcal{M} + 1)ct. \quad \square
\end{aligned}$$

Lemma 7.7. For the first Riesz wave $\mathfrak{P}\mathfrak{R}_1(x, t)$, there exists $C > 0$ and

$\mathcal{M} > 1$ such that

$$\left\| \mathfrak{P}\mathfrak{R}_1(x, t) \right\|_{L_\xi^2} \leq C \begin{cases} \frac{e^{-\frac{(|x|-ct)^2}{C(t+1)}}}{(1+t)^2} + e^{-t/C} & \text{for } |x| \in [ct, \mathcal{M}ct], \\ \frac{1}{t(|x| + \sqrt{t+1})} & \text{for } |x| \leq ct. \end{cases} \quad (7.11)$$

Proof. From (7.5),

$$\begin{aligned}
& \iint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \hat{\mathfrak{P}}\mathfrak{R}_1(\eta, t) d\eta \\
&= \iint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} e^{-A_4^2(|\eta|^2)t} \sum_{1 \leq j, k \leq 3} \eta^j \eta^k \left(\mathcal{E}_{j,k}^1 \frac{\sin(A_1^1(|\eta|^2)|\eta|t)}{|\eta|} \right. \\
& \quad \left. + \int_0^t \mathcal{E}_{j,k}^2 \frac{\sin(A_1^1(|\eta|^2)|\eta|\tau)}{|\eta|} d\tau \right) d\eta. \quad (7.12)
\end{aligned}$$

By (6.14) and Kirchoff formula (3.8), (7.12) becomes

$$\begin{aligned}
& \iiint_{|\eta| \leq \kappa_0/2} e^{ix \cdot \eta} \mathfrak{P} \mathfrak{R}_1(\eta, t) d\eta \\
= & \sum_{1 \leq j, k \leq 3} w(x, t) * I_{j,k}^1(x, t) + w_t(x, t) * I_{j,k}^2(x, t) \\
& + \int_0^t w(x, \tau) * J_{j,k}^1(x, \tau; t) + w_\tau(x, \tau) * J_{j,k}^2(x, \tau; t) d\tau, \quad (7.13)
\end{aligned}$$

where

$$\begin{cases}
I_{j,k}^1(x, t) & \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{i\eta \cdot x - A_1^2(|\eta|^2)t} \eta^j \eta^k \mathcal{E}_{j,k}^1(\eta) \cos(\mathcal{H}(|\eta|^2)|\eta|t) d\eta, \\
I_{j,k}^2(x, t) & \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{i\eta \cdot x - A_1^2(|\eta|^2)t} \eta^j \eta^k \mathcal{E}_{j,k}^1(\eta) \frac{\sin(\mathcal{H}(|\eta|^2)|\eta|t)}{|\eta|} d\eta, \\
J_{j,k}^1(x, \tau; t) & \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{i\eta \cdot x - A_1^2(|\eta|^2)t} \eta^j \eta^k \mathcal{E}_{j,k}^2(\eta) \cos(\mathcal{H}(|\eta|^2)|\eta|\tau) d\eta, \\
J_{j,k}^2(x, \tau; t) & \equiv \iiint_{|\eta| \leq \kappa_0/2} e^{i\eta \cdot x - A_1^2(|\eta|^2)t} \eta^j \eta^k \mathcal{E}_{j,k}^2(\eta) \frac{\sin(\mathcal{H}(|\eta|^2)|\eta|\tau)}{|\eta|} d\eta.
\end{cases}$$

Again, by the procedure in (6.18)-(6.21) we conclude that there exists $C > 0$ such that for $|x| \leq (\mathcal{M} + 1)ct$

$$\begin{aligned}
\|D_x^\gamma I_{j,k}^l(x, t)\|_{L_\xi^2}, \|D_x^\gamma J_{j,k}^l(x, \tau; t)\|_{L_\xi^2} & \leq C \left(\frac{e^{-\frac{|x|^2}{Ct}}}{(t+1)^{(5+|\gamma|)/2}} + e^{-t/C} \right) \\
& \text{for } 1 \leq j, k \leq 3, l = 1, 2, \tau \in [0, t].
\end{aligned}$$

Substitute this into (7.13) and use the Kirchoff formula, Theorem 3.1, and Lemma 3.4 to conclude this lemma. \square

8. Global Wave Structure

The purpose of this section is study the wave structure of solutions to the linearized Boltzmann equation when the initial value is concentrated around $x = 0$. This gives us a clear picture of the propagation of waves. Thus we consider the following initial value problem:

$$\begin{cases}
\partial_t \mathbf{g} + \xi \cdot \nabla_x \mathbf{g} = L\mathbf{g}, \\
\mathbf{g}(x, 0) = \mathbf{g}_{in}(x).
\end{cases} \quad (8.1)$$

Here, the function \mathbf{g}_{in} has compact support and is in the weighted norm, (2.2),

$$\begin{cases} \mathbf{g}_{in}(x) \equiv 0 \text{ for } |x| \geq 1, \\ \sup_{|x| < 1} \|\mathbf{g}_{in}(x)\|_{L_{\xi,3}^{\infty}} \leq 1, \end{cases} \quad (8.2)$$

and there is no assumption on the regularity of $\mathbf{g}_{in}(x)$.

We will need to make uses of the analysis of long waves in finite Mach region in the previous sections. We summarize it in the first two lemmas.

Lemma 8.1. *For a given $\mathcal{M} > 1$ there exists $C > 0$ such that for $|x| \leq \mathcal{M}ct$,*

$$\begin{aligned} \|\mathbb{G}_{L;0}(x,t)\|_{L_{\xi}^2} &\leq C \left[\frac{e^{-\frac{|x|^2}{Ct}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{Ct}}}{(1+t)^2} + e^{-t/C} \right] \\ &+ C \begin{cases} \frac{1}{(1+t)^{-1}(|x| + \sqrt{t+1})} & \text{for } |x| \leq ct, \\ 0 & \text{for } |x| \geq ct. \end{cases} \end{aligned} \quad (8.3)$$

Proof. The lemma follows directly from (6.16), (7.8), (7.9) and (7.11). \square

The Green's function applied to non-fluid part, $\mathbb{G}_{L;0}(x,t)\mathbf{P}_1$, has faster decaying rates:

Lemma 8.2. *For a given $\mathcal{M} > 1$ there exists $C > 0$ such that for $|x| \leq \mathcal{M}ct$ it holds*

$$\|\mathbf{P}_0^{iso}\mathbb{G}_{L;0}(x,t)\|_{L_{\xi}^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right), \quad (8.4)$$

$$\|\mathbb{G}_{L;0}\mathbf{P}_1\|_{L_{\xi}^2} \leq C \left[\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^2} + \frac{e^{-\frac{(|x|-ct)^2}{C(t+1)}}}{(1+t)^{5/2}} + e^{-(|x|+t)/C} \right], \quad (8.5)$$

$$\|\mathbf{P}_1\mathbb{G}_{L;0}\mathbf{P}_1\|_{L_{\xi}^2} \leq C \left[\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{5/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(t+1)}}}{(1+t)^3} + e^{-(|x|+t)/C} \right]. \quad (8.6)$$

Proof. First, note that the pairing $\hat{\mathfrak{P}}\mathfrak{A}_1$, (5.43), contains macroscopic

factors in momentum components only. Thus,

$$\begin{cases} \mathbb{P}_0^{iso} \hat{\mathfrak{R}}_1 \equiv 0, \\ \hat{\mathfrak{R}}_1 \mathbb{P}_1 \equiv 0. \end{cases} \quad (8.7)$$

The factor $\frac{1}{(1+t)(|x|+\sqrt{1+t})}$ in (8.3) is due to $\hat{\mathfrak{R}}_1$. Thus, from (8.7), the component $\frac{1}{(1+t)(|x|+\sqrt{1+t})}$ does not show up in the estimates, (8.4), (8.5), and (8.6) for $\|\mathbb{P}_0^{iso} \mathbb{G}_{L;0}\|_{L_\xi^2}$, $\|\mathbb{G}_{L;0} \mathbb{P}_1\|_{L_\xi^2}$, and $\|\mathbb{P}_1 \mathbb{G}_{L;0} \mathbb{P}_1\|_{L_\xi^2}$.

The product $\mathbb{G}_{L;0} \mathbb{P}_1$ forces all pairings in $\mathbb{G}_{L;0} \mathbb{P}_1$ to contain only microscopic components. From (5.31), every microscopic component contains an $\eta \cdot \xi$ factor. This yields a gain of extra factors η and $O(1)|\eta|^2$ in the pairing $\mathbb{G}_{L;0} \mathbb{P}_1$ and $\mathbb{P}_1 \mathbb{G}_{L;0} \mathbb{P}_1$ as compared to the pairings $\hat{\mathfrak{H}}$, $\hat{\mathfrak{C}}$, and $\hat{\mathfrak{R}}_n$. Thus $\mathbb{G}_{L;0} \mathbb{P}_1$ gains an extra derivative in the space variable x , which, for the long waves $\mathbb{G}_{L;0}$, results in the gain of a $(1+t)^{-1/2}$ decaying factor in time; and $\mathbb{P}_1 \mathbb{G}_{L;0} \mathbb{P}_1$ gains a factor $(1+t)^{-1}$. This concludes the lemma. \square

Now we start with the study of the initial value problem (8.1), (8.2). The initial value $\mathfrak{g}_{in} \in L_x^2(L_\xi^2)$.

The Long Wave-Short Wave decomposition can be applied to yield that

$$\|\mathbb{G}_S^t \mathfrak{g}_{in}\|_{L_x^2(L_\xi^2)} \leq O(1) e^{-\nu_0 t} \|\mathfrak{g}_{in}\|_{L_x^2(L_\xi^2)}.$$

We now follow the procedure in Section 4 and write the solution as

$$\mathfrak{g} = \mathbb{O}^t \mathfrak{g}_{in} + \mathbb{A}_k + \mathbb{R}_k.$$

These terms satisfy (4.17)~(4.22). Now, we consider the following identity

$$\mathbb{G}_L^t \mathfrak{g}_{in} + \mathbb{G}_S^t \mathfrak{g}_{in} = \mathbb{O}^t \mathfrak{g}_{in} + \bar{\mathfrak{h}} = \mathbb{O}^t \mathfrak{g}_{in} + \mathbb{A}_k + \mathbb{R}_k.$$

This, (4.22), and $\|\mathbb{G}_L^t \mathfrak{g}_{in}\|_{H_x^k(L_\xi^2)} = O(1)$, (2.5), result in

$$\|\nabla_x^k (\mathbb{G}_S^t \mathfrak{g}_{in} - \mathbb{O}^t \mathfrak{g}_{in} - \mathbb{A}_k)\|_{L_x^2(L_\xi^2)} = \|\nabla_x^k (\mathbb{R}_k - \mathbb{G}_L^t \mathfrak{g}_{in})\|_{L_x^2(L_\xi^2)} = O(1). \quad (8.8)$$

From Lemma 2.6 and (4.19), one has that for some $\nu_2 > 0$

$$\|\mathbb{G}_S^t \mathfrak{g}_{in} - \mathbb{O}^t \mathfrak{g}_{in} - \mathbb{A}_k\|_{L_x^2(L_\xi^2)} \leq O(1) e^{-\nu_2 t}. \quad (8.9)$$

Then, by (8.8) and (8.9) combined with Sobolev's inequality one has that there exists $\nu_2 > 0$ such that

$$\|\mathbb{G}_S^t \mathbf{g}_{in} - \mathbb{O}^t \mathbf{g}_{in} - \mathbf{A}_k\|_{L_x^\infty(L_\xi^2)} \leq O(1) e^{-\nu_2 t}.$$

Again, with (4.19) one has that there exists $\nu_2 > 0$ such that

$$\|\mathbb{G}_S^t \mathbf{g}_{in}\|_{L_x^\infty(L_\xi^2)} \leq O(1) e^{-\nu_2 t}. \quad (8.10)$$

Now, combining (5.41), Lemma 8.1, and (8.10) we have the following theorem:

Theorem 8.3. (Finite Mach Number Region) *For $|x| \leq \mathcal{M}ct$ there exists $C > 0$ such that*

$$\begin{aligned} & \|\mathbb{G}^t \mathbf{g}_{in}(x, t)\|_{L_\xi^2} \\ & \leq C \left[\frac{e^{-\frac{|x|^2}{Ct}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{Ct}}}{(1+t)^2} + \int_0^t \left\{ \tau \iint_{|y|=1} \frac{e^{-\frac{|x-cty|^2}{Ct}}}{(1+t)^{5/2}} d\mathbf{S}_y \right\} d\tau + e^{-t/C} \right] \end{aligned}$$

for any \mathbf{g}_{in} satisfying (8.2).

Next we consider wave structure outside finite Mach Number region $|x| \geq \mathcal{M}ct$. The structure is obtained by the energy estimates which originated from [27] for 1-D problem. Here, we modify it for this 3-D problem.

In order to obtain pointwise estimates from the energy estimates by Sobolev theory, we need some regularity property. Thus, we consider the variable \mathbf{R}_k defined in (4.18) with $k \geq 6$. The condition $k \geq 6$ is for the use of the Sobolev's embedding theorem in \mathbb{R}^3 .

Lemma 8.4. *For each given $k \geq 0$ there exists $C_k > 0$ such that*

$$\|\mathbb{O}^t \mathbf{g}_{in}(x)\|_{L_\xi^2}, \|\mathbf{A}_k(x, t)\|_{L_\xi^2}, \|\mathbf{B}_k(x, t)\|_{L_\xi^2} \leq C_k e^{-(|x|+t)/C_k}.$$

Remark 8.5. This is a consequence of hard sphere collision model that the collision frequency $\nu(\xi) = O(1)|\xi|$; and $D \ll 1$ which is the parameter for \mathbf{K}_1 given in (4.1).

Consider the weight function

$$W(x, t; \Omega, \epsilon) = e^{\epsilon(x \cdot \Omega - \mathcal{M}ct)}$$

in terms of a non-negative parameter ϵ and a direction $\Omega \in \mathbb{S}^2$. Here, the number ϵ is chosen to relate to the constant in Lemma 8.4:

$$\epsilon \leq \frac{1}{2C_k}.$$

Under this condition

$$\|\sqrt{W}\mathbb{O}^t \mathbf{g}_{in}\|_{L_x^2(L_\xi^2)}, \|\sqrt{W}\mathbf{A}_k(\cdot, t)\|_{L_x^2(L_\xi^2)}, \|\sqrt{W}\mathbf{B}_k(\cdot, t)\|_{L_x^2(L_\xi^2)} \leq O(1)e^{-\epsilon \mathcal{M}ct}. \quad (8.11)$$

Now, choose $\mathbf{g} \in SO(3)$ with $\mathbf{g}\Omega = (1, 0, 0)^t$ and consider the following new coordinate system:

$$\begin{cases} \bar{x} = \mathbf{g}(x - \mathcal{M}ct\Omega), \\ \bar{t} = t, \\ \bar{\xi} = \mathbf{g}\xi. \end{cases}$$

Under this new coordinate system,

$$\begin{cases} W = e^{\epsilon \bar{x}^1}, \\ \partial_{\bar{t}} \mathbf{R}_k - \mathcal{M}c \partial_{\bar{x}^1} \mathbf{R}_k + \bar{\xi} \cdot \nabla_{\bar{x}} \mathbf{R}_k - L \mathbf{R}_k = \mathbf{K} \mathbf{B}_k. \end{cases}$$

Now, consider the weighted energy estimate

$$\begin{aligned} & \int_{\mathbb{R}^3} W(\bar{x}^1)(\mathbf{R}_k, \mathbf{K} \mathbf{B}_k) d\bar{x} \\ &= \int_{\mathbb{R}^3} W(\bar{x}^1)(\mathbf{R}_k, \partial_{\bar{t}} \mathbf{R}_k - \mathcal{M}c \partial_{\bar{x}^1} \mathbf{R}_k - L \mathbf{R}_k) d\bar{x} \\ &= \frac{1}{2} \frac{d}{d\bar{t}} \int_{\mathbb{R}^3} W(\bar{x}^1)(\mathbf{R}_k, \mathbf{R}_k) d\bar{x} \\ & \quad + \int_{\mathbb{R}^3} W(\bar{x}^1) [\epsilon(\mathbf{R}_k, (\mathcal{M}c - \bar{\xi}^1) \mathbf{R}_k) - (\mathbf{P}_1 \mathbf{R}_k, L \mathbf{P}_1 \mathbf{R}_k)] d\bar{x}. \end{aligned}$$

Since $\mathcal{M} > 2$,

$$(\mathbf{P}_0 \mathbf{R}_k, (\mathcal{M}c - \bar{\xi}^1) \mathbf{P}_0 \mathbf{R}_k) \geq (\mathcal{M} - 1)c(\mathbf{P}_0 \mathbf{R}_k, \mathbf{P}_0 \mathbf{R}_k) \geq c(\mathbf{P}_0 \mathbf{R}_k, \mathbf{P}_0 \mathbf{R}_k).$$

By choosing ϵ sufficiently small, we have from this and Lemma 2.4 that

$$\frac{1}{2} \int_{\mathbb{R}^3} W(\bar{x}^1)(\mathbf{R}_k, \mathbf{R}_k) d\bar{x} + \frac{\epsilon}{2} \int_{\mathbb{R}^3} (\mathbf{R}_k, \mathbf{R}_k) d\bar{x} \leq O(1) \frac{1}{\epsilon} \|\sqrt{W} \mathbf{B}_k\|_{L_x^2(L_\xi^2)}^2.$$

From this and (8.11), there exists $C > 0$ such that

$$\|\sqrt{W} \mathbf{R}_k\|_{L_x^2(L_\xi^2)} \leq C e^{-\bar{t}/C}. \quad (8.12)$$

From (4.22)

$$\|\nabla_{\bar{x}}^j \mathbf{R}_k(\cdot, t)\|_{L_x^2(L_\xi^2)} \leq C \text{ for } t \geq 0, j \in \{1, \dots, k\}. \quad (8.13)$$

From (8.12) and (8.13), we have, for even number k ,

$$\|\nabla_{\bar{x}}^j (W^{1/4} \mathbf{R}_k)\|_{L_x^2(L_\xi^2)} \leq O(1) e^{-\frac{\bar{t}}{2C}} \text{ for } 1 \leq j \leq k/2. \quad (8.14)$$

On the other hand, the proof for showing (8.12) can be also used to show that (8.14) is true for $j = 0$.

By letting $k = 4$ one can use Sobolev's embedding theorem to show that

$$\sup_{\bar{x} \in \mathbb{R}^3} \|W^{1/4} \mathbf{R}_k(\bar{x}, \bar{t})\|_{L_\xi^2} \leq O(1) e^{-\frac{\bar{t}}{2C}}.$$

We thus conclude that there exists $C > 0$ such that

$$\|\mathbf{R}_k(\bar{x}, \bar{t})\|_{L_\xi^2} \leq C e^{-\frac{\bar{t}}{C}} e^{-\bar{x}^1/C},$$

or

$$\|\mathbf{R}_k\|_{L_\xi^2} \leq C e^{-t/C} e^{-[\Omega \cdot x - \mathcal{M}ct]/C}.$$

Here, the coefficient C is independent of the direction Ω and so, for $|x| \geq \mathcal{M}ct$,

$$\|\mathbf{R}_k\|_{L_\xi^2} \leq C e^{-t/C} e^{-|x| + \mathcal{M}ct}/C}.$$

From this we can conclude that there exists $C > 0$ such that for $|x| > \mathcal{M}ct$

$$\|\mathbf{R}_k\|_{L_\xi^2} \leq C e^{-(t+|x|)/C}. \quad (8.15)$$

From (8.15) and Lemma 8.4, we have the following theorem for wave struc-

ture outside the finite Mach number region:

Theorem 8.6.(Outside Finite Mach Region) *For $|x| > \mathcal{M}ct$ (with $\mathcal{M} \geq 2$) there exists $C > 0$ such that the solution of (8.1) satisfies*

$$\|\mathbf{g}(x, t)\|_{L^2_\xi} \leq Ce^{-(t+|x|)/C}.$$

9. Green's Function.

Consider the initial-value problem for the Green's function

$$\begin{cases} \partial_t \mathbb{G} + \xi \cdot \nabla_x \mathbb{G} = L\mathbb{G}, \\ \mathbb{G}(x, 0, \xi, \xi_0) = \delta(x)\delta(\xi - \xi_0). \end{cases}$$

To study the wave structure of the Green's function, we reduce the situation to the case of the general initial-value problem as in Section 8. For this we construct the kinetic-like waves. The first term is particle wave:

$$h_0(x, t) \equiv \mathbb{S}^t \delta(x)\delta(\xi - \xi_0).$$

$$h_k(x, t) \equiv \int_0^t \mathbb{S}^{t-s} \mathbb{K} h_{k-1}(x, s) ds \text{ for } k \geq 1. \quad (9.1)$$

From direct calculations,

$$\begin{cases} h_0(x, t, \xi) = e^{-\nu(\xi_0)t} \delta(x - \xi_0 t) \delta(\xi - \xi_0), \\ h_1(x, t, \xi) = \int_0^t K(\xi, \xi_0) e^{-\nu(\xi)(t-s) - \nu(\xi_0)s} \delta(x - (t-s)\xi - s\xi_0) ds, \\ h_2(x, t, \xi) = \int_0^t \int_{\mathbb{R}^3} \int_0^{s_1} e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_0)s} K(\xi, \xi_1) K(\xi_1, \xi_0) \\ \qquad \qquad \qquad \delta(x - (t-s_1)\xi - (s_1-s)\xi_1 - s\xi_0) ds d\xi_1 ds_1. \end{cases} \quad (9.2)$$

Both h_0 and h_1 are generalized functions. However, h_2 is regular function due to the extra mixing in the space and as well as in the velocity variables. To analyze these, we start with a definition of scattering path:

Definition 9.1.(Scattering Path) Let $(\xi, \xi_0, \tau_0) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$. We

denote Ξ_{ξ, ξ_0, τ_0} is a scattering path of a particle with velocity ξ_0 which scatters into the velocity ξ at time τ_0 :

$$\Xi_{\xi, \xi_0, \tau_0} \equiv \{(\xi(\tau - \tau_0) + \xi_0\tau_0, \tau) \in \mathbb{R}^3 \times \mathbb{R}^+ : \tau \in [\tau_0, \infty)\}.$$

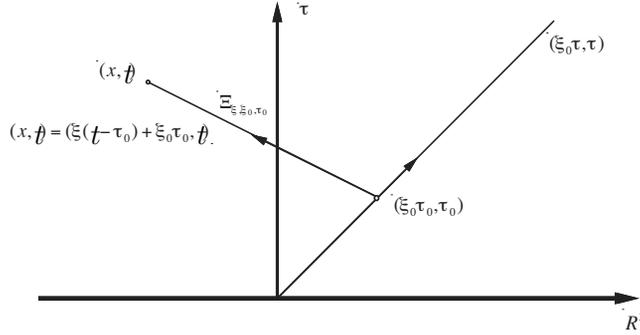


Figure C

With ξ and $\xi_0 \in \mathbb{R}^3$, the path Ξ_{ξ, ξ_0, τ_0} is represented in four dimensional space $\mathbb{R}^3 \times \mathbb{R}^+$ in Figure C. With this, we denote regular point by \mathcal{R} :

$$\mathcal{R} \equiv \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ / \cup_{\tau_0 \geq 0} \{ (x, t, \xi) : (x, t) \in \Xi_{\xi, \xi_0, \tau_0}, \xi \in \mathbb{R}^3 \}.$$

Lemma 9.2. *If $(x, t, \xi) \in \mathcal{R}$, then*

$$x - (t - \tau)\xi - \xi_0\tau \neq 0 \text{ for all } \tau \in \mathbb{R}. \tag{9.3}$$

Proof.

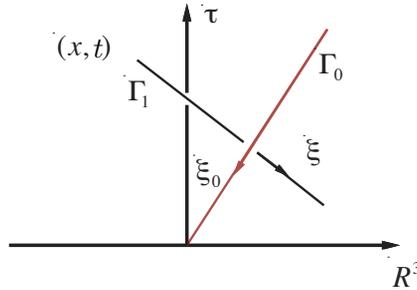


Figure D

This lemma is clear from geometric consideration, Figure D. \square

Lemma 9.3. *If $(x, t, \xi) \in \mathcal{R}$, then*

$$|\mathbf{h}_2(x, t, \xi)| < \infty.$$

Proof. From the representation (9.2), we can integrate the δ function with respect to ξ_1 variable to result in

$$\begin{aligned} & \mathbf{h}_2(x, t, \xi) \\ &= \int_0^t \int_0^{s_1} \frac{e^{-\nu(\xi)(t-s_1) - \nu(\xi_1)(s_1-s) - \nu(\xi_0)s} K(\xi, \xi_1) K(\xi_1, \xi_0)}{(s_1 - s)^3} \Big|_{\xi_1 = \frac{x - (t-s_1)\xi - s\xi_0}{s_1 - s}} ds ds_1. \end{aligned}$$

From (9.3),

$$\min_{s_1 \in [0, t]} |x - (t - s_1)\xi - s_1\xi_0| > 0.$$

Thus there exists $C > 0$ and $\delta > 0$ such that

$$\frac{|x - (t - s_1)\xi - s\xi_0|}{|s_1 - s|} > \frac{C}{|s - s_1|} \text{ whenever } |s_1 - s| < \delta.$$

This and the expression for the collision kernel (2.1) yield

$$\lim_{(s_1 - s) \rightarrow 0} \frac{K(\xi, \xi_1) K(\xi_1, \xi_0)}{(s_1 - s)^3} = 0,$$

whence we have

$$\mathbf{h}_2(x, t, \xi) < \infty \text{ for } (x, t, \xi) \in \mathcal{R}. \quad \square$$

Remark 9.4. By a proper decomposition of δ function, one can show that $\cup_{\substack{\xi \in \mathbb{R}^3, \\ \tau \in \mathbb{R}^+}} \Xi_{\xi, \xi_0, \tau}$ are removable singularity.

Lemma 9.5. *For each $l \geq 3$, there exists $C_l > 0$ such that*

$$\|\mathbf{h}_l(x, t)\|_{L_{\xi, l-3}^\infty} < C_l e^{-(|x|+t)/C}.$$

Proof. From (9.1), this lemma can be proved by induction on the index

$l \geq 3$ of \mathbf{h}_l .

The above two lemmas and the operator \mathbb{S}^t are sufficient to obtain that there exist $C > 0$ and C_2 such that

$$\|\mathbf{h}_2(x, t)\|_{L_\xi^2} < C_2 e^{-(|x|+t)/C}. \quad (9.4)$$

Then, from (9.4) and Lemma 2.2, one has that

$$\|\mathbf{K}\mathbf{h}_2\|_{L_{\xi,0}^\infty} \leq O(1)e^{-(|x|+t)/C}.$$

Substitute this into (9.1) and from (4.5) one can conclude this lemma for $l = 3$. We assume that this lemma is true for $l \leq k$. This assumption combined with Lemma 2.2 yields

$$\|\mathbf{K}\mathbf{h}_k\|_{L_{k-2}^\infty} \leq C_k e^{-(|x|+t)/C}.$$

Then, from (9.1) and (4.5), we can conclude that this lemma is true for $l = k + 1$. \square

Theorem 9.6.(Main Theorem I) *Green's function $\mathbb{G}(x, t)$ as an L_ξ^2 operator-valued function satisfies that there exists $C > 0$ such that*

$$\begin{aligned} \|\mathbb{G}(x, t)\|_{L_\xi^2} &\leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right) \\ &\quad + C \begin{cases} \frac{1}{(1+t)(\sqrt{1+t}+|x|)} & \text{for } |x| \leq ct + \sqrt{1+t}, \\ 0 & \text{for } |x| \geq ct + \sqrt{t}, \end{cases} \end{aligned} \quad (9.5)$$

$$\|\mathbf{P}_0^{iso}\mathbb{G}(x, t)\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right), \quad (9.6)$$

$$\|\mathbb{G}(x, t)\mathbf{P}_1\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^2} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^{5/2}} + e^{-(|x|+t)/C} \right), \quad (9.7)$$

$$\|\mathbf{P}_1\mathbb{G}(x, t)\mathbf{P}_1\|_{L_\xi^2} \leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{5/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^3} + e^{-(|x|+t)/C} \right). \quad (9.8)$$

Proof. Decompose the Green's function $\mathbb{G}(x, t, \xi; \xi_0)$ into

$$\mathbb{G}(x, t, \xi; \xi_0) \equiv \sum_{l=0}^6 \mathbf{h}_l(x, t, \xi) + \mathbf{r}(x, t, \xi).$$

Treat $\sum_{l=0}^6 \mathbf{h}_l$ as an L_ξ^2 operator-valued function. Then, from Lemma 9.5, there exists $C > 0$

$$\left\| \sum_{l=0}^6 \mathbf{h}_l \right\|_{L_\xi^2} \leq C e^{-(|x|+t)/C}. \quad (9.9)$$

The equation for \mathbf{r} is

$$\partial_t \mathbf{r} + \xi \cdot \nabla_x \mathbf{r} - L \mathbf{r} = \mathbf{K} \mathbf{h}_6. \quad (9.10)$$

From Lemmas 2.2 and 9.5, we have

$$\|\mathbf{K} \mathbf{h}_6\|_{L_{\xi,4}^\infty} \leq C e^{-(|x|+t)/C} \text{ for some } C > 0. \quad (9.11)$$

With this condition, (9.11), the analysis for the solution of (8.1) can be applied. Thus, from Theorems 8.3 and 8.6, the solution \mathbf{r} of (9.10) satisfies

$$\begin{aligned} \|\mathbf{r}(x, t)\|_{L_\xi^2} &\leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right) \\ &\quad + C \begin{cases} \frac{1}{(1+t)(\sqrt{1+t} + |x|)} & \text{for } |x| \leq ct + \sqrt{1+t}, \\ 0 & \text{for } |x| \geq ct + \sqrt{t}. \end{cases} \end{aligned} \quad (9.12)$$

Then, follow the argument in Section 7 of [27] for obtaining estimates in $\|\cdot\|_{L_{\xi,4}^\infty}$ from estimates in $\|\cdot\|_{L_\xi^2}$ to yield that

$$\begin{aligned} \|\mathbf{r}(x, t)\|_{L_{\xi,4}^\infty} &\leq C \left(\frac{e^{-\frac{|x|^2}{C(1+t)}}}{(1+t)^{3/2}} + \frac{e^{-\frac{(|x|-ct)^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+t)/C} \right) \\ &\quad + C \begin{cases} \frac{1}{(1+t)(\sqrt{1+t} + |x|)} & \text{for } |x| \leq ct + \sqrt{1+t}, \\ 0 & \text{for } |x| \geq ct + \sqrt{t}. \end{cases} \end{aligned} \quad (9.13)$$

With this estimate for $\|\mathbf{r}\|_{L_{\xi,4}^\infty}$, one can treat the function \mathbf{r} as an L_ξ^2 operator-valued function. Thus, (9.9) and (9.13) result in (9.5).

The procedure for obtaining (9.6), (9.7), and (9.8) are similar. It is omitted. This completes the proof of Main Theorem I. \square

10. Leading Fluid Waves

We finally prove the Main Theorem II. The main remaining step is to study the leading fluid waves. We have from, (5.39), c.f., (5.9), (5.22), (5.28), that the leading fluid waves are

$$\begin{aligned}
\tilde{\mathbb{G}}^0 &\equiv e^{-i|\eta|ct - A_1|\eta|^2t} \mathbf{gE}_1 \otimes \langle \mathbf{gE}_1 | + e^{-A_2|\eta|^2t} \mathbf{gE}_2 \otimes \langle \mathbf{gE}_2 | \\
&\quad + e^{i|\eta|ct - A_1|\eta|^2t} \mathbf{gE}_3 \otimes \langle \mathbf{gE}_3 | + e^{-A_4|\eta|^2t} \mathbf{gE}_4 \otimes \langle \mathbf{gE}_4 | + e^{-A_5|\eta|^2t} \mathbf{gE}_5 \otimes \langle \mathbf{gE}_5 | \\
&= e^{-i|\eta|ct - A_1|\eta|^2t} \mathbf{g} \left(-\sqrt{\frac{5}{2}}\xi^1 + \frac{1}{6}|\xi|^2 \right) \sqrt{\mathbf{M}} \otimes \langle \mathbf{g} \left(-\sqrt{\frac{5}{2}}\xi^1 + \frac{1}{6}|\xi|^2 \right) \sqrt{\mathbf{M}} | \\
&\quad + e^{i|\eta|ct - A_1|\eta|^2t} \mathbf{g} \left(\sqrt{\frac{5}{2}}\xi^1 + \frac{1}{6}|\xi|^2 \right) \sqrt{\mathbf{M}} \otimes \langle \mathbf{g} \left(\sqrt{\frac{5}{2}}\xi^1 + \frac{1}{6}|\xi|^2 \right) \sqrt{\mathbf{M}} | \\
&\quad + e^{-A_2|\eta|^2t} \mathbf{g} \frac{|\xi|^2}{\sqrt{6}} |\sqrt{\mathbf{M}} \otimes \langle \mathbf{g} \frac{|\xi|^2}{\sqrt{6}} \sqrt{\mathbf{M}} | + e^{-A_4|\eta|^2t} \mathbf{g}\xi^2 |\sqrt{\mathbf{M}} \otimes \langle \mathbf{g}\xi^2 \sqrt{\mathbf{M}} | \\
&\quad + e^{-A_4|\eta|^2t} \mathbf{g}\xi^3 |\sqrt{\mathbf{M}} \otimes \langle \mathbf{g}\xi^3 \sqrt{\mathbf{M}} | \\
&\equiv \hat{\mathbb{G}}_1^0 + \hat{\mathbb{G}}_3^0 + \hat{\mathbb{G}}_2^0 + \hat{\mathbb{G}}_4^0 + \hat{\mathbb{G}}_5^0.
\end{aligned} \tag{10.1}$$

Direct computations yield

$$\begin{aligned}
&\hat{\mathbb{G}}_1^0 + \hat{\mathbb{G}}_3^0 \\
&= e^{-A_1|\eta|^2t} \left\{ (e^{-i|\eta|ct} + e^{i|\eta|ct}) \left(\frac{5}{2} \mathbf{g}\xi^1 \sqrt{\mathbf{M}} \otimes \langle \mathbf{g}\xi^1 \sqrt{\mathbf{M}} | + \frac{1}{36} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | \right) \right. \\
&\quad \left. + \sqrt{\frac{5}{2}} \frac{1}{6} (e^{-i|\eta|ct} - e^{i|\eta|ct}) (\mathbf{g}\xi^1 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | + \langle |\xi|^2 \sqrt{\mathbf{M}} \otimes \mathbf{g}\xi^1 \sqrt{\mathbf{M}} |) \right\} \\
&= e^{-A_1|\eta|^2t} \cos(|\eta|ct) \left\{ 5 \sum_{j,k=1}^3 \frac{\eta^j \eta^k}{|\eta|^2} \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^k \sqrt{\mathbf{M}} | + \frac{1}{18} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | \right\} \\
&\quad + e^{-A_1|\eta|^2t} i \frac{\sin(|\eta|ct)}{|\eta|} \left\{ \frac{1}{3} \sqrt{\frac{5}{2}} \sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | + |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle \xi^j \sqrt{\mathbf{M}} | \right\}
\end{aligned} \tag{10.2}$$

From (3.8), we conclude that

$$\begin{aligned}\mathbb{G}_1^0 + \mathbb{G}_3^0 &= \frac{ct}{4\pi} \iint_{|y|=1} H_1(x + cty) dS_y + \frac{1}{4\pi} \iint_{|y|=1} H_2(x + cty) dS_y \\ &\quad + \frac{ct}{4\pi} \iint_{|y|=1} \nabla H_2(x + cty) \cdot y dS_y.\end{aligned}$$

Next we consider

$$\hat{\mathbb{G}}_2^0 = e^{-A_2|\eta|^2 t} \mathbf{g} \mathbf{E}_2 \otimes \langle \mathbf{g} \mathbf{E}_2 |$$

The inverse Fourier transform can be computed directly and we have

$$\mathbb{G}_2^0 = \frac{1}{6} (4\pi A_2 t)^{-3/2} e^{-\frac{|x|^2}{4A_2 t}} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} |$$

Finally we study

$$\mathbb{G}_4^0 + \mathbb{G}_5^0 = (4\pi A_4 t)^{-3/2} e^{-\frac{|x|^2}{4A_4 t}} (\mathbf{g} \xi^2 \sqrt{\mathbf{M}} \otimes \langle \mathbf{g} \xi^2 \sqrt{\mathbf{M}} | + \mathbf{g} \xi^3 \sqrt{\mathbf{M}} \otimes \langle \mathbf{g} \xi^3 \sqrt{\mathbf{M}} |).$$

Take an orthogonal basis α, β of the space orthogonal to the vector η . In the coordinate system with basis $\{\eta/|\eta|, \alpha, \beta\}$, we have $\mathbf{g} \xi^i = \mathbf{g}(\xi)_i$, $\mathbf{g} \xi = (\xi \cdot \frac{\eta}{|\eta|}, \xi \cdot \alpha, \xi \cdot \beta)$, $\xi = (\xi \cdot \frac{\eta}{|\eta|}) \frac{\eta}{|\eta|} + (\xi \cdot \alpha) \alpha + (\xi \cdot \beta) \beta$, and $\xi_* = (\xi_* \cdot \frac{\eta}{|\eta|}) \frac{\eta}{|\eta|} + (\xi_* \cdot \alpha) \alpha + (\xi_* \cdot \beta) \beta$. Thus

$$\begin{aligned}& (\hat{\mathbb{G}}_4^0 + \hat{\mathbb{G}}_5^0) h \\ &= e^{-A_4|\eta|^2 t} \left(\xi \cdot \alpha \sqrt{\mathbf{M}} \int \xi_* \cdot \alpha \sqrt{\mathbf{M}_*} h(\xi_*) d\xi_* + \xi \cdot \beta \sqrt{\mathbf{M}} \int \xi_* \cdot \beta \sqrt{\mathbf{M}_*} d\xi_* \right) \\ &= e^{-A_4|\eta|^2 t} \sqrt{\mathbf{M}} \xi \cdot \left(\int \{(\xi_* \cdot \alpha) \alpha + (\xi_* \cdot \beta) \beta\} \sqrt{\mathbf{M}_*} h(\xi_*) d\xi_* \right) \\ &= e^{-A_4|\eta|^2 t} \sqrt{\mathbf{M}} \xi \cdot \left(\int \left\{ \xi_* - (\xi_* \cdot \frac{\eta}{|\eta|}) \frac{\eta}{|\eta|} \right\} \sqrt{\mathbf{M}_*} h(\xi_*) d\xi_* \right).\end{aligned}\tag{10.3}$$

Thus we have

$$\begin{aligned}\hat{\mathbb{G}}_4^0 + \hat{\mathbb{G}}_5^0 &= (4\pi A_4 t)^{-3/2} e^{-A_4|\eta|^2 t} \\ &\quad \left(\sum_{j=1}^3 \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^j \sqrt{\mathbf{M}} | - \sum_{j,k=1}^3 \xi^j \sqrt{\mathbf{M}} \frac{\eta^j \eta^k}{|\eta|^2} \otimes \langle \xi^k \sqrt{\mathbf{M}} | \right),\end{aligned}$$

and

$$\begin{aligned} \mathbb{G}_4^0 + \mathbb{G}_5^0 &= (4\pi A_4 t)^{-3/2} e^{-\frac{|x|^2}{4A_4 t}} \sum_{j=1}^3 \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^j \sqrt{\mathbf{M}} | \\ &\quad - \sum_{j,k=1}^3 \left[\int_0^t A_4 (4\pi A_4 \tau)^{-3/2} e^{-\frac{|x|^2}{4A_4 \tau}} d\tau \right]_{x_j x_k} \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^k \sqrt{\mathbf{M}} |. \end{aligned}$$

This completes the study of the leading fluid waves and we have $\sum_{j=1}^5 \mathbb{G}_j^0$ is of the form (1.14), as stated in Main Theorem II as stated in the Introduction. The leading particle-like waves have been studied and estimated in Section 9. It remains to show that the remaining long waves decay faster in the fluid wave region $|x| = O(1)ct$. We will only consider the following

$$\begin{aligned} &e^{\sigma_1(\eta)t} \psi_1(\eta) \otimes \langle \psi_1(\eta) | + e^{\sigma_3(\eta)t} \psi_3(\eta) \otimes \langle \psi_3(\eta) | - \hat{\mathbb{G}}_1^0 + \hat{\mathbb{G}}_3^0 \\ &= e^{i|\eta|ct - A_1 |\eta|^2 t} \left(e^{O_1(|\eta|^2)|\eta|^2 t + i|\eta|O_2(|\eta|^2)t} - 1 \right) \psi_1(0) \otimes \langle \psi_1(0) | \\ &\quad + e^{-i|\eta|ct - A_1 |\eta|^2 t} \left(e^{O_1(|\eta|^2)|\eta|^2 t - i|\eta|O_2(|\eta|^2)t} - 1 \right) \psi_3(0) \otimes \langle \psi_3(0) | \\ &\quad + e^{\sigma_1(\eta)t} (\psi_1(\eta) \otimes \langle \psi_1(\eta) | - \psi_1(0) \otimes \langle \psi_1(0) |) \\ &\quad + e^{\sigma_3(\eta)t} (\psi_3(\eta) \otimes \langle \psi_3(\eta) | - \psi_3(0) \otimes \langle \psi_3(0) |) \\ &\equiv \hat{\mathcal{E}}_{1,3}^1 + \hat{\mathcal{E}}_{1,3}^2 + \hat{\mathcal{E}}_{1,3}^3 + \hat{\mathcal{E}}_{1,3}^4. \end{aligned} \tag{10.4}$$

The first two error terms

$$\begin{aligned} &(\hat{\mathcal{E}}_{1,3}^1)^0 + (\hat{\mathcal{E}}_{1,3}^2)^0 \\ &= e^{-A_1 |\eta|^2 t} \left[O_1(|\eta|^2)|\eta|^2 t \cos(|\eta|ct) - \sin(|\eta|ct) O_2(|\eta|^2)|\eta|t \right] \\ &\quad \cdot (\psi_1(0) \otimes \langle \psi_1(0) | + \psi_3(0) \otimes \langle \psi_3(0) |) \\ &\quad + e^{-A_1 |\eta|^2 t} \left[i O_1(|\eta|^2)|\eta|^2 t \sin(|\eta|ct) + \cos(|\eta|ct) O_2(|\eta|^2)|\eta|t \right] \\ &\quad \cdot (\psi_1(0) \otimes \langle \psi_1(0) | - \psi_3(0) \otimes \langle \psi_3(0) |) \\ &= e^{-A_1 |\eta|^2 t} \left(O_1(|\eta|^2)|\eta|^2 t \cos(|\eta|ct) - O_2(|\eta|^2)|\eta|^2 t \frac{\sin(|\eta|ct)}{|\eta|} \right) \\ &\quad \cdot \left(5 \sum_{j,k=1}^3 \frac{\eta^j \eta^k}{|\eta|^2} \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^k \sqrt{\mathbf{M}} | + \frac{1}{3} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | \right) \end{aligned}$$

$$\begin{aligned}
& -e^{-A_1|\eta|^2 t} \left(iO_1(|\eta|^2)|\eta|^2 t \frac{\sin(|\eta|ct)}{|\eta|} + O_2(|\eta|^2)t \cos(|\eta|ct) \right) \frac{1}{3} \sqrt{\frac{5}{2}} \\
& \cdot \frac{1}{3} \sqrt{\frac{5}{2}} \left(\sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} \otimes \langle |\xi|^2 \sqrt{\mathbf{M}} | + |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle \xi^j \sqrt{\mathbf{M}} | \right). \quad (10.5)
\end{aligned}$$

When compared to $\hat{\mathbb{G}}_1^0 + \hat{\mathbb{G}}_3^0$, the term $II \equiv (\hat{\mathcal{E}}_{1,3}^1)^0 + (\hat{\mathcal{E}}_{1,3}^2)^0$ has extra factor of $O(1)|\eta|^2 t \eta_j$ or $O(1)|\eta|^4 t$. Moreover, $\mathbf{P}_0 II = IIP_0 = 0$. The gain of $|\eta|^2 t$ and $\frac{\eta^j \eta^k}{|\eta|^2}$ translates to a gain of t^{-1} decay. Similarly, replacing $\cos(|\eta|ct)$ with $\eta_j \cos(|\eta|ct)$ also translates to a gain of t^{-1} decay. In summary, $(\mathcal{E}_{1,3}^1)^0 + (\mathcal{E}_{1,3}^2)^0$ has extra decaying factor of t^{-1} when compared to $\mathbb{G}_1^0 + \mathbb{G}_3^0$, though with slightly larger base of $e^{-|x|^2/(4A_1+\epsilon)t}$. For the explanation for these decaying properties see Remark below. For the next two remaining terms, we have from

$$\begin{aligned}
(\hat{\mathbb{G}}_{1,3}^3)^0 + (\hat{\mathbb{G}}_{1,3}^4)^0 &= e^{-i|\eta|ct - A_1|\eta|^2 t} (\psi_1(\eta) \otimes \langle \psi_1(\eta) | - \psi_1(0) \otimes \langle \psi_1(0) |) \\
&+ e^{i|\eta|ct - A_1|\eta|^2 t} (\psi_3(\eta) \otimes \langle \psi_3(\eta) | - \psi_3(0) \otimes \langle \psi_3(0) |) \quad (10.6)
\end{aligned}$$

$$\begin{aligned}
\psi_1(\eta) &= a_1^1(|\eta|) \sqrt{\mathbf{M}} + a_1^2(|\eta|) \sum_{j=1}^3 \frac{\eta^j \xi^j}{|\eta|} \sqrt{\mathbf{M}} + a_1^3(|\eta|) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} \\
&= (a_{1,0}^1(|\eta|^2) + i|\eta| a_{1,1}^1(|\eta|^2)) \sqrt{\mathbf{M}} + (a_{1,0}^2(|\eta|^2) + i|\eta| a_{1,1}^2(|\eta|^2)) \sum_{j=1}^3 \frac{\eta^j \xi^j}{|\eta|} \sqrt{\mathbf{M}} \\
&+ (a_{1,0}^3(|\eta|^2) + i|\eta| a_{1,1}^3(|\eta|^2)) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} + \mathbf{g} \mathbf{b}_1(|\eta|). \quad (10.7)
\end{aligned}$$

$$\begin{aligned}
\psi_3(\eta) &= (a_{1,0}^1(|\eta|^2) - i|\eta| a_{1,1}^1(|\eta|^2)) \sqrt{\mathbf{M}} \\
&+ (-a_{1,0}^2(|\eta|^2) + i|\eta| a_{1,1}^2(|\eta|^2)) \sum_{j=1}^3 \frac{\eta^j \xi^j}{|\eta|} \sqrt{\mathbf{M}} \\
&+ (a_{1,0}^3(|\eta|^2) - i|\eta| a_{1,1}^3(|\eta|^2)) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} + \mathbf{g} \mathbf{I} \mathbf{b}_1(-|\eta|), \quad (10.8)
\end{aligned}$$

$$\begin{cases} \psi_1(0) = a_{1,0}^1(0) \sqrt{\mathbf{M}} + a_{1,0}^2(0) \sum_{j=1}^3 \frac{\xi^j \eta^j}{|\eta|} \sqrt{\mathbf{M}} + a_{1,0}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}}, \\ \psi_3(0) = a_{1,0}^1(0) \sqrt{\mathbf{M}} - a_{1,0}^2(0) \sum_{j=1}^3 \frac{\xi^j \eta^j}{|\eta|} \sqrt{\mathbf{M}} + a_{1,0}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}}, \end{cases}$$

that, except for the microscopic terms $\mathbf{g}b_1(|\eta|)$, $\mathbf{g}Ib_1(-|\eta|)$, and macroscopic terms with extra factor of $|\eta|^2$, the main terms are

$$\begin{aligned}
& (\hat{\mathcal{E}}_{1,3}^3)^{00} + (\hat{\mathcal{E}}_{1,3}^4)^{00} \\
= & 2e^{-A_1|\eta|^2 t} i \cos(c|\eta|t) \left[-i\sqrt{\frac{5}{2}} \sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} \otimes \langle a_{1,1}^1(0) \sqrt{\mathbf{M}} \right. \\
& + a_{1,1}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} - i(a_{1,1}^1(0) \sqrt{\mathbf{M}} + a_{1,1}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}}) \\
& \otimes \langle \sqrt{\frac{5}{2}} \sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} + ia_{1,1}^2(0) \sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} \otimes \langle \frac{1}{6} |\xi|^2 \sqrt{\mathbf{M}} | \\
& + i\frac{1}{6} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle a_{1,1}^2(0) \sum_{j=1}^3 \eta^j \xi^j \sqrt{\mathbf{M}} | \\
& + 2e^{-A_1|\eta|^2 t} \frac{\sin(c|\eta|t)}{|\eta|} \left[\sqrt{10} a_{1,1}^2(0) \sum_{j,k=1}^3 \eta^j \eta^k \xi^j \sqrt{\mathbf{M}} \otimes \langle \xi^k \sqrt{\mathbf{M}} | \right. \\
& - |\eta|^2 (a_{1,1}^1(0) \sqrt{\mathbf{M}} + a_{1,1}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} \otimes \langle \frac{1}{6} |\xi|^2 \sqrt{\mathbf{M}} | \\
& \left. + \frac{|\eta|^2}{6} |\xi|^2 \sqrt{\mathbf{M}} \otimes \langle a_{1,1}^1(0) \sqrt{\mathbf{M}} + a_{1,1}^3(0) \frac{1}{\sqrt{6}} (|\xi|^2 - 3) \sqrt{\mathbf{M}} | \right]. \quad (10.9)
\end{aligned}$$

Thus the purely microscopic term in $\mathcal{E}_{1,3}^3 + \mathcal{E}_{1,3}^4$ has extra decay factor of only $t^{-1/2}$ when compared to $\mathbb{G}_1^0 + \mathbb{G}_3^0$. The terms $\mathbf{g}b_1(|\eta|) \otimes \langle \mathbf{g}b(|\eta|) | + \mathbf{g}Ib_1(-|\eta|) \otimes \langle \mathbf{g}Ib_1(-|\eta|) |$ have factor of $|\eta|^2$ and so decays with extra factor of t^{-1} .

Thus $P_1(\mathcal{E}_{1,3}^3 + \mathcal{E}_{1,3}^4)$ and $(\mathcal{E}_{1,3}^3 + \mathcal{E}_{1,3}^4)P_1$ have extra decay factor of t^{-1} .

Remark 10.1. An extra factor of η^α term yields extra decay rate of $t^{-|\alpha|/2}$ in the fluid region $|x| \leq O(1)t$. This is seen by an example in the following: Consider the integral paths in Section 6, Figure B: The main contribution for integrating η^1 is on $\{\eta^1 \mid \text{Im } \eta^1 = \frac{|x|}{2tA_1}, |\text{Re } \eta^1| \leq \kappa_0/2\}$. Write $\eta^1 = \alpha + \beta i$, $\beta = \frac{|x|}{2tA_1}$, $|\alpha| \leq \kappa_0/2$. In the fluid region, $|x| = O(1)t$, we

have

$$\begin{aligned}
& \iint_{B_2} d\eta^2 d\eta^3 \int_{\Gamma(-\frac{\kappa_0}{2\sqrt{2}}, \frac{\kappa_0}{2\sqrt{2}}, \frac{C}{tM_0})} e^{-A_1[|\eta^1|^2+|\eta^2|^2+|\eta^3|^2]t+ix\cdot\eta} O(1)|\eta| d\eta^1 \\
&= e^{-\frac{|x|^2}{4A_1t}} \iint_{B_2} d\eta^2 d\eta^3 \int_{\Gamma^1} e^{-A_1(\eta^1-i\frac{|x|}{2A_1t})^2-A_1(|\eta^2|^2+|\eta^3|^2)t} O(1)|\alpha + \beta i| d\eta^1 \\
&= e^{-\frac{|x|^2}{4A_1t}} \iint_{B_2} d\eta^2 d\eta^3 \int_{-\kappa_0/2}^{\kappa_0/2} e^{-A_1\alpha^2t-A_1((\eta^2)^2+(\eta^3)^2)t} O(1)(|\alpha| + \frac{|x|}{t}) d\alpha \\
&= O(1)t^{-3/2}t^{-1/2}e^{-\frac{|x|^2}{(4A_1+\epsilon)t}}. \tag{10.10}
\end{aligned}$$

The same works for other terms such as the Huygens waves. This completes the estimates (1.15) for the remaining terms, and the proof of Main Theorem II is complete.

References

1. K. Asano, Local solutions to the initial and initial-boundary value problem for the Boltzmann equation with an external force. I., *J. Math. Kyoto Univ.*, **24** (1984), no. 2, 225-238.
2. C. Bardos, F. Golse and C. D. Levermore, The acoustic limit for the Boltzmann equation, *Arch. Ration. Mech. Anal.*, **153** (2000), no. 3, 177-204.
3. Boltzmann, L. (translated by Stephen G. Brush), *Lectures on Gas Theory*, Dover Publications, Inc. New York, 1964.
4. R. E. Caflisch, The fluid dynamic limit of the nonlinear Boltzmann equation, *Commun. Pure Appl. Math.*, **33** (1980), no. 5, 651-666.
5. T. Carleman, Sur La Théorie de l'Équation Intégrodifférentielle de Boltzmann, *Acta Math.*, **60** (1933), 91-142.
6. T. Carleman, *Problèmes mathématiques dans la théorie cinétique des gaz*, (French) Publ. Sci. Inst. Mittag-Leffler. 2 Almqvist & Wiksells Boktryckeri Ab, Uppsala 1957
7. C. Cercignani, R. Illner and M. Pulvirenti, The mathematical theory of dilute gases, *Appl. Math. Sci.*, 106. Springer-Verlag, New York, 1994.
8. C. Cercignani, The Boltzmann equation and its applications, *Appl. Math. Sci.*, 67. Springer-Verlag, New York, 1988
9. L. Desvillettes and C. Villani, On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The Boltzmann equation, *Invent. Math.*, **159** (2005), no. 2, 245-316.
10. R. J. DiPerna and P.-L. Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability, *Ann. Math.*, **130** (1989), no. 2, 321-366.
11. R. Ellis and M. Pinsky, The first and second fluid approximations to the linearized Boltzmann equation, *J. Math. Pures Appl.*, **54** (1975), no. 9, 125-156.

12. H. Grad, Asymptotic theory of the Boltzmann equation, *Rarefied Gas Dynamics*, J. A. Laurmann, Ed. **1**, 26, pp.26-59. Academic Press, New York (1963).
13. H. Grad, Singular and nonuniform limits of solutions of the Boltzmann equation, *Transport Theory* (Proc. Sympos. Appl. Math., New York, 1967), SIAM-AMS Proc., Vol. I (1969), 269-308. Amer. Math. Soc., Providence, R.I.
14. D. Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*, (Teubner, Leipzig), Chap.22.
15. F. Golse, L. Saint-Raymond, The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, *Invent. Math.*, **155** (2004), no. 1, 81-161.
16. Y. Guo, The Boltzmann equation in the whole space, *Indinan Univ. Math. J.*, **53** (2004), no. 4, 1081-1094.
17. D. Hoff, K. Zumbrun, Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves, *Z. Angew. Math. Phys.*, **48** (1997), no. 4, 597-614.
18. S. Kaniel, and M. Shinbrot, The Boltzmann equation. I. Uniqueness and local existence, *Comm. Math. Phys.*, **58** (1978), no. 1, 65-84.
19. T. Kato, *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132 Springer-Verlag New York, Inc., New York 1966.
20. S. Kawashima, A. Matsumura and T. Nishida, On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation, *Comm. Math. Phys.*, **70** (1979), no. 2, 97-124.
21. T.-P. Liu, Interactions of nonlinear hyperbolic waves. *Nonlinear Analysis* (Taipei, 1989), 171-183, World Sci. Publishing, Teaneck, NJ, 1991
22. T.-P. Liu, Pointwise convergence to shock waves for viscous conservation laws, *Comm. Pure Appl. Math.*, **50** (1997), no. 11, 1113-1182.
23. T.-P. Liu and W. Wang, The pointwise estimates of diffusion wave for the Navier-Stokes systems in odd multi-dimensions, *Comm. Math. Phys.*, **196** (1998), no. 1, 145-173.
24. T.-P. Liu, T. Yang and S.-H. Yu, Energy method for Boltzmann equation, *Phys. D* **188** (2004), no. 3-4, 178-192.
25. T.-P. Liu, S.-H. Yu, Continuum shock profiles for discrete conservation laws. I. Construction, *Comm. Pure Appl. Math.*, **52** (1999), no. 1, 85-127.
26. T.-P. Liu and S.-H. Yu, Continuum shock profiles for discrete conservation laws. II. Stability, *Comm. Pure Appl. Math.* **52** (1999), no. 9, 1047-1073.
27. T.-P. Liu and S.-H. Yu, Green function of Boltzmann equation, planar waves, *Comm. Pure Appl. Math.* **57** (2004), no. 12, 1543-1608.
28. T.-P. Liu and S.-H. Yu, Boltzmann equation: Micro-macro decompositions and positivity of shock profiles, *Comm. Math. Phys.*, **246** (2004), no. 1, 133-179.
29. T.-P. Liu and Y. Zeng, Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws, *Mem. Amer. Math. Soc.* **125** (1997), no. 599,
30. T. Nishida, Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation, *Comm. Math. Phys.*, **61** (1978), no. 2, 119-148.
31. Y. Sone, *Kinetic Theory and Fluid Dynamics*, Birkhauser, 2002.

32. Y. Sone, *Molecular Gas Dynamics*, Birkhauser, to appear.
33. S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, *Proc. Japan Acad. Ser. A Math. Sci.*, **50** (1974), 179-184.
34. S. Ukai, Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace, *C. R. Acad. Sci. Paris Ser. A-B*, **282** (1976), no. 6, Ai, A317-A320.
35. S. Ukai, Solutions of the Boltzmann equation. Patterns and waves, *Stud. Math. Appl.*, **18** (1986), 37-96, North-Holland, Amsterdam.
36. S. Ukai, T. Yang and S.-H. Yu, Nonlinear boundary layers of the Boltzmann equation: I. existence, *Comm. Math. Phys.*, **236** (2003), 373-393.
37. S. Ukai, T. Yang and S.-H. Yu, Nonlinear stability of boundary layers of the Boltzmann equation, I. The case $\mathcal{M}^\infty < -1$, *Comm. Math. Phys.*, **244** (2004), no. 1, 99-109.
38. C. Villani, A review of mathematical topics in collisional kinetic theory. Handbook of mathematical fluid dynamics, Vol. I, 71-305, North-Holland, Amsterdam, 2002, 82-02.
39. S.-H. Yu, Hydrodynamic limits with shock waves of the Boltzmann equation, *Comm. Pure Appl. Math.*, **58** (2005), no. 3, 409-443.
40. Y. Zeng, L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow, *Comm. Pure Appl. Math.*, **47** (1994), no. 8, 1053-1082.

Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan (R.O.C.) and

Department of Mathematics, Stanford University, Stanford, CA 94305, U.S.A.

E-mail: liu@math.stanford.edu

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong.

E-mail: mashyu@city.edu.hk