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MULTIPLICITY RESULTS FOR A CRITICAL AND SUBCRITICAL SYSTEM INVOLVING FRACTIONAL p-LAPLACIAN OPERATOR VIA NEHARI MANIFOLD METHOD

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Abstract

In this paper, we study the existence of multiple solutions to some sub-critical and critical systems involving the fractional p-Laplacian operator. Our main tools are based on variational methods. Precisely, the Nehari manifold method combined with fibering map are used in the study of the sub-critical case. While in the critical one, we use the mountain pass theorem combined with Nehari manifold and Brezis-Lieb lemma.

1. Introduction

In this paper, we investigate the multiplicity of nontrivial solutions to the following elliptic system involving the p-fractional Laplacian

$$\begin{cases} (-\triangle)^s_p u(x) = F_u(x,u,v) + \lambda G_u(x,u,v) & \text{in } \Omega, (u,v) > 0, \\ (-\triangle)^s_p v(x) = F_v(x,u,v) + \lambda G_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $0 < s < 1 < p$ and $n > ps$, the exponents p, r and q satisfy $0 < r < 1 < q \le p^*_s$, $p^*_s = \frac{np}{n-sp}$

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is the fractional critical exponent. $F, G : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are smooth functions satisfy further assumption which will be given later. $(-\triangle)_p^s$ is the p-fractional Laplacian operator defined on smooth functions by

$$(-\triangle)_p^s u = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy, x \in \mathbb{R}^n. \quad (1.2)$$

Problem involving fractional Laplace operator has been given considerable attention since the area arises in a quite natural way in many different contexts, such as the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others see for instance [1, 2, 3, 4] and references therein. Recently, the corresponding non-local problems with nonlinear terms involving fractional Laplacian operator have attracted the attention of many researchers, both for their interesting theoretical structure and their concrete applications see for example the papers [6, 31, 28].

In the literature, many papers discuss quasilinear elliptic problem involving p-fractional operator in which the technical approach adopted is based on the Nehari manifold method and fibering maps. We refer to [21, 32, 33, 37] for the subcritical case and to [35, 36, 37] for the critical case. The multiplicity of solutions to problems similar to (1.1) has been investigated in [20, 30]. In particular, in [20], the authors studied the associated Nehari manifold using the fibering maps in order to prove multiplicity result for the following problem

$$\begin{cases} (-\triangle)_p^s u = \lambda |u|^{q-1} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta & \text{in } \Omega, \\ (-\triangle)_p^s v = \mu |v|^{q-1} v + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha} |v|^{\beta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $p < \alpha + \beta < p^* := \frac{np}{n-sp}$, λ and μ are positive parameters.
Giacomoni et al. [42], considered the following system

$$\begin{cases} -\triangle_p u = f_1(x,u,v), \quad u > 0 & \text{in } \Omega, \\ -\triangle_q u = f_2(x,u,v), \quad v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

In [43], Shen and Zhang studied the existence of positive weak solutions for the follwoing p-Laplacian system

$$\begin{cases} -\triangle_p u = \frac{1}{p^*}\frac{\partial F(x,u,v)}{\partial u} + \lambda a(x)|u|^{q-2}u & \text{in } \Omega, \\ -\triangle_p u = \frac{1}{p^*}\frac{\partial F(x,u,v)}{\partial v} + \mu b(x)|u|^{q-2}u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where F is positively homogeneous of degree p^*. Note that for a given function $H\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and for $l > 1$, we say that H is positively homogeneous of degree l, if for all $t > 0$, we have

$$H(x, tu, tv) = t^l H(x, u, v),\ \forall\ (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}.$$

Motivated by the above mentioned works, the main purpose of the present paper is to prove the existence of two nontrival solutions to problem (1.1). To this aim, throughout this paper we assume that the functions F and G are positively homogeneous of degrees q and r respectively, with $(0 < r < 1 < p < q \leq p_s^*)$. Note that, in this case, the so-called Euler identities implies the existence of $\gamma_1, \gamma_2 > 0$, such that, for all $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, we have

$$\begin{cases} F(x, u, v) \leq \gamma_1(|u|^q + |v|^q), \\ G(x, u, v) \leq \gamma_2(|u|^r + |v|^r). \end{cases} \tag{1.3}$$

This paper is organized as follows. In Section 2, we present the variational setting of the main problem and give some notations and preliminaries about Nehari manifold and fibering maps. In Section 3, we present and prove the first existence result of this paper which is devoted to the subcritical and concave case. While, the critical and concave case is presented and proved in Section 4.

2. Variational Setting and Fibering Maps Analysis

In this section, we will present and prove some important results related to the analysis of fibering maps. In order to state our results, we introduce the following functional space

$$X=\left\{u:\mathbb{R}^n\to\mathbb{R} \text{ is measurable}, u\in L^p(\Omega) \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{n+ps}{p}}}\in L^p(\mathcal{D},dxdy)\right\},$$

endowed with the norm

$$\|u\|_X=\left(\|u\|^p_{L^p(\Omega)}+\int_{\mathcal{D}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}},$$

where $\mathcal{D}=\mathbb{R}^{2n}\backslash(\Omega^c\times\Omega^c)$ with $\Omega^c=\mathbb{R}^n\backslash\Omega$.
Let

$$X_0=\{u\in X:u=0 \text{ in } \mathbb{R}^n\backslash\Omega\}, \tag{2.1}$$

equipped with the norm

$$\|u\|_{X_0}=\left(\int_{\mathcal{D}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}}. \tag{2.2}$$

These types of spaces were introduced in [33]. It is readily seen that X_0 is a uniformly convex Banach space and that the embedding $X_0\hookrightarrow L^\sigma(\Omega)$, is continuous for all $1\leq\sigma\leq p_s^*$ and compact for all $1\leq\sigma<p_s^*$.

Through this paper we consider the space $E=X_0\times X_0$, with the norm

$$\|(u,v)\|_E=\left(\|u\|^p_{X_0}+\|v\|^p_{X_0}\right)^{\frac{1}{p}}. \tag{2.3}$$

Associated to the problem (1.1), we define the functional energy $J_\lambda:E\to\mathbb{R}$ as follows:

$$J_\lambda(u,v)=\frac{1}{p}\|(u,v)\|^p_E-\int_\Omega F(x,u,v)dx-\lambda\int_\Omega G(x,u,v)dx. \tag{2.4}$$

One readily has that J_λ is of class C^1. Moreover, for $\lambda>0$ fixed, the critical points of J_λ are exactly the weak solutions to (1.1). For simplicity, in the rest of this paper, we use the following notations:

$$A(u,v)=\|(u,v)\|^p_E,$$
$$B(u,v)=\int_\Omega F(x,u,v)dx,$$

and

$$C(u,v)=\int_{\Omega}G(x,u,v)dx,$$

Then, we can write J_λ as follows

$$J_\lambda(u,v)=\frac{1}{p}A(u,v)-B(u,v)-\lambda C(u,v).$$

A simple calculation shows that $J_\lambda\in C^1(E,\mathbb{R})$, moreover

$$\langle J'_\lambda(u,v),(u,v)\rangle_E=A(u,v)-q\mathcal{B}-\lambda rC(u,v).$$

Since the functional energy J_λ is not bounded from below on E, it is natural to work in a suitable subset of E. A good candidate for an appropriate subset of E, is the so-called Nehari manifold, which is defined by

$$\mathcal{N}_\lambda=\{(u,v)\in E\backslash\{(0,0)\}:\langle J'_\lambda(u,v),(u,v)\rangle_E=0\}.$$

It is not difficult to see that all nontrivial critical points of J_λ lie on $\mathcal{N}_\lambda$. Moreover, $(u,v)\in\mathcal{N}_\lambda$ if and only if

$$A(u,v)-qB(u,v)-\lambda rC(u,v)=0. \tag{2.5}$$

It is useful to understand that the Nehari manifold $\mathcal{N}_\lambda$, is closely associated with the behavior of the fibering maps $\varphi_{u,v}:(0,\infty)\to\mathbb{R}$, defined as

$$\varphi_{u,v}(t)=J_\lambda(tu,tv):=\frac{1}{p}A(u,v)t^p-B(u,v)t^q-\lambda C(u,v)t^r. \tag{2.6}$$

Such maps are introduced by Drabeket al. in [26] and are also discussed in [14, 15].

Lemma 2.1. *Let $t>0$ and $(u,v)\in E\backslash\{(0,0)\}$. Then, $(tu,tv)\in\mathcal{N}_\lambda$ if and only if $\varphi'_{u,v}(t)=0$. In particular, $(u,v)\in\mathcal{N}_\lambda$ if and only if $\varphi'_{u,v}(1)=0$.*

Proof. The result is an immediate consequence of the fact that for all $t>0$ and all $(u,v)\in E\backslash\{(0,0)\}$, we have

$$\varphi'_{u,v}(t)=\frac{1}{t}\langle J'_\lambda(tu,tv),(tu,tv)\rangle_E.$$

□

Lemma 2.2. *Assume that* $(u_0,v_0)\in\mathcal{N}_\lambda$, *is a local minimizer for* J_λ *on* $\mathcal{N}_\lambda$. *Then* (u_0,v_0) *is a critical point of* J_λ.

Proof. If (u_0,v_0) is a local minimizer for J_λ on $\mathcal{N}_\lambda$. Then, (u_0,v_0) is a solution to the following minimization problem

$$\begin{cases}\min\limits_{(u,v)\in\mathcal{N}_\lambda} J_\lambda(u,v)=J_\lambda\left(u_0,v_0\right)\\ \beta(u_0,v_0)=0,\end{cases}$$

where β is defined by

$$\beta(u,v)=A(u,v)-qB(u,v)-\lambda rC(u,v).$$

Hence, the Lagrangian multipliers theorem implies the existence of a real number δ, such that

$$J'(u_0,v_0)=\delta\beta'(u_0,v_0). \tag{2.7}$$

Thus

$$\langle J'(u_0,v_0),(u_0,v_0)\rangle_E=\delta\langle\beta'(u_0,v_0),(u_0,v_0)\rangle_E. \tag{2.8}$$

Since $(u_0,v_0)\in\mathcal{N}_\lambda$, then we have

$$\langle J'(u_0,v_0),(u_0,v_0)\rangle_E=0. \tag{2.9}$$

On the other hand, a simple calculation shows that

$$\langle\beta'(u_0,v_0),(u_0,v_0)\rangle_E=\varphi''_{u_0,v_0}(1)\ .$$

So, by combining Equations (2.8) and (2.9) with the fact that $(u_0,v_0)\notin\mathcal{N}_\lambda^0$, we obtain $\delta=0$. Finally, replacing δ by zero in (2.7), we get $J'_\lambda(u_0,v_0)=0$. □

In order to prove multiplicity of solutions, we split $\mathcal{N}_\lambda$ into the following three subsets

$$\mathcal{N}_\lambda^+=\{(u,v)\in\mathcal{N}_\lambda:\varphi''_{u,v}(1)>0\}=\{(u,v)\in E:\varphi'_{u,v}(1)=0,\ \text{and}\ \varphi''_{u,v}(1)>0\},$$

$$\mathcal{N}_\lambda^-=\{(u,v)\in\mathcal{N}_\lambda:\varphi''_{u,v}(1)<0\}=\{(u,v)\in E:\varphi'_{u,v}(1)=0,\ \text{and}\ \varphi''_{u,v}(1)<0\},$$

$$\mathcal{N}_\lambda^0=\{(u,v)\in\mathcal{N}_\lambda:\varphi''_{u,v}(1)=0\}=\{(u,v)\in E:\varphi'_{u,v}(1)=0,\ \text{and}\ \varphi''_{u,v}(1)=0\}.$$

Note that

$$\varphi'_{u,v}(t) = t^{p-1}A(u,v) - qt^{q-1}\mathcal{B} - \lambda r t^{r-1}C(u,v), \tag{2.10}$$

and

$$\varphi''_{u,v}(t) = (p-1)t^{p-2}A(u,v) - q(q-1)t^{q-2}B(u,v) - \lambda r(r-1)t^{r-2}C(u,v). \tag{2.11}$$

So, from (2.5), we get

$$\begin{aligned}\varphi''_{u,v}(1) &= (p-1)A(u,v) - q(q-1)B(u,v) - \lambda r(r-1)C(u,v)\\ &= q(p-q)B(u,v) + \lambda r(p-r)C(u,v)\\ &= (p-q)A(u,v) + \lambda r(q-r)C(u,v)\\ &= (p-r)A(u,v) - q(q-r)B(u,v). \end{aligned}\tag{2.12}$$

Put

$$\lambda_* = \frac{1}{\gamma_2}\frac{q-p}{r(q-r)}\left(\frac{p-r}{q(q-r)\gamma_1}\right)^{\frac{p-r}{q-p}}\left(S_p|\Omega|^{\frac{p-p_s^*}{p_s^*}}\right)^{\frac{q-r}{q-p}}. \tag{2.13}$$

Then we have the following important result.

Lemma 2.3. *If $\lambda \in (0,\lambda_*)$. Then, for all $(u,v) \in E \setminus \{(0,0)\}$, there exist two positive numbers t_1 and t_2, such that*

$$(t_1u, t_1v) \in \mathcal{N}_\lambda^+, \text{ and } (t_2u, t_2v) \in \mathcal{N}_\lambda^-.$$

Proof. For $(u,v) \in E$, we define the function $\psi_{u,v} : [0,\infty) \to \mathbb{R}$, as follows

$$\psi_{u,v}(t) = t^{p-r}A(u,v) - qt^{q-r}B(u,v). \tag{2.14}$$

By simple calculation, we get

$$\varphi'_{u,v}(t) = t^{r-1}\left(\psi_{u,v}(t) - \lambda r C(u,v)\right), \tag{2.15}$$

and

$$\varphi''_{u,v}(t) = t^r \psi'_{u,v}(t). \tag{2.16}$$

Hence, $(tu,tv) \in \mathcal{N}_\lambda^+$ if and only if $\psi_{u,v}(t) = \lambda r C(u,v)$ and $\psi'_{u,v}(t) > 0$. Moreover, $(tu,tv) \in \mathcal{N}_\lambda^-$ if and only if $\psi_{u,v}(t) = \lambda r C(u,v)$ and $\psi'_{u,v}(t) < 0$.

It is clear that $\psi'_{u,v}(t)=0$ if and only if $t=0$ or

$$t=T:=\left(\frac{(p-r)}{q(q-r)}\frac{A(u,v)}{B(u,v)}\right)^{\frac{1}{q-p}}.$$

Moreover, the table of variations of the function $\psi_{u,v}$, is as follows

t	0		T		∞
$\psi'_{u,v}$		$+$	0	$-$	
$\psi_{u,v}$	0	$\nearrow$	$\psi_{u,v}(T)>0$	$\searrow$	$-\infty$

On the other hand, from (1.3), (4.1) and Hölder's inequality, we have

$$\begin{aligned}
B(u,v) &\le \gamma_1\left(\|u\|^q_{L^q}+\|v\|^q_{L^q}\right)\\
&\le \gamma_1|\Omega|^{\frac{p_s^*-q}{p_s^*}}\left(\|u\|^q_{L^{p_s^*}}+\|v\|^q_{L^{p_s^*}}\right)\\
&\le \gamma_1|\Omega|^{\frac{p_s^*-q}{p_s^*}}S_p^{-\frac{q}{p}}\left(\|u\|^q_{X_0}+\|v\|^q_{X_0}\right)\\
&\le \gamma_1|\Omega|^{\frac{p_s^*-q}{p_s^*}}S_p^{-\frac{q}{p}}\left(\|u\|^p_{X_0}+\|v\|^p_{X_0}\right)^{\frac{q}{p}}\\
&\le \gamma_1 S_p^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}A(u,v)^{\frac{q}{p}}. \qquad (2.17)
\end{aligned}$$

Likewise

$$C(u,v)\le \gamma_2 S_p^{-\frac{r}{p}}|\Omega|^{\frac{p_s^*-r}{p_s^*}}A(u,v)^{\frac{r}{p}}. \qquad (2.18)$$

Therefore

$$\begin{aligned}
&\psi_{u,v}(T)-\lambda rC(u,v)\\
&=\left(\frac{p-r}{q(q-r)}\right)^{\frac{p-r}{q-p}}\left(\frac{q-p}{q-r}\right)A(u,v)^{\frac{q-r}{q-p}}B(u,v)^{-\frac{p-r}{q-p}}-\lambda rC(u,v)\\
&\ge\left(r\gamma_2 S_p^{-\frac{r}{p}}|\Omega|^{\frac{p_s^*-r}{p_s^*}}\right)(\lambda_*-\lambda)A(u,v)^{\frac{r}{p}}.
\end{aligned}$$

So, for $\lambda\in(0,\lambda_*)$, we get

$$\psi_{u,v}(T)>\lambda rC(u,v)>0.$$

Hence, from the table of variations of $\psi_{u,v}$, there exist $t_1 < T < t_2$, such that:

$$\psi_{u,v}(t_1) = \psi_{u,v}(t_2) = \lambda r C(u,v), \psi'_{u,v}(t_1) > 0 \text{ and } \psi'_{u,v}(t_2) < 0. \tag{2.19}$$

Finally, from (2.15) and (2.16), we see that $(t_1 u, t_1 v) \in \mathcal{N}_\lambda^+$, and $(t_2 u, t_2 v) \in \mathcal{N}_\lambda^-$ □

Lemma 2.4. *For all $\lambda \in (0, \lambda_*)$, we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. We proceed by contradiction. Let $(u_0, v_0) \in \mathcal{N}_\lambda^0$. Then, from (2.12), we have

$$(p-r)A(u_0,v_0) - q(q-r)B(u_0,v_0) = 0.$$

So,

$$qB(u_0,v_0) = \frac{p-r}{q-r}A(u_0,v_0).$$

Consequently, we get

$$\begin{aligned} 0 &= A(u_0,v_0) - qB(u_0,v_0) - \lambda r C(u_0,v_0) \\ &= A(u_0,v_0) - \frac{p-r}{q-r}A(u_0,v_0) - \lambda r C(u_0,v_0) \\ &= \frac{q-p}{q-r}A(u_0,v_0) - \lambda r C(u_0,v_0), \end{aligned}$$

which implies that

$$C(u_0,v_0) = \frac{q-p}{\lambda r(q-r)}A(u_0,v_0). \tag{2.20}$$

Now, as in the proof of Lemma 2.3, we can prove that ψ_{u_0,v_0} achieves its maximum at $T_0 = T(u_0,v_0)$, moreover, $\psi_{u_0,v_0}(T_0) > \lambda r C(u_0,v_0) > 0$. On the other hand, from (2.20), we obtain

$$\begin{aligned} &\psi_{u_0,v_0}(T_0) - \lambda r C(u_0,v_0) \\ &= \left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q(q-r)}\right)^{\frac{p-r}{q-p}}\left(\frac{A(u_0,v_0)^{q-r}}{B(u_0,v_0)^{p-r}}\right)^{\frac{1}{q-p}} - \lambda r C(u_0,v_0) \\ &= \left(\frac{p-r}{q(q-r)}\right)^{\frac{p-r}{q-p}}\left(\frac{q-p}{q-r}\right)\left(\frac{p-r}{q(q-r)}\right)^{\frac{r-p}{q-p}} A(u,v)_0 - \lambda r C(u_0,v_0) \end{aligned}$$

$$= \frac{q-p}{\lambda r(q-r)}A(u_0,v_0) - \lambda r C(u_0,v_0) = 0,$$

which is a contradiction. So we conclude that $\mathcal{N}_\lambda^0 = \emptyset$. □

Lemma 2.5. *For all $\lambda \in (0,\lambda_*)$, the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. Let $(u,v) \in \mathcal{N}_\lambda$. Then, from Equation (2.5), we get

$$B(u,v) = \frac{1}{q}(A(u,v) - \lambda r C(u,v)).$$

Therefore, using (2.18), we get

$$\begin{aligned} J_\lambda(u,v) &= \Big(\frac{1}{p} - \frac{1}{q}\Big)A(u,v) - \lambda\frac{q-r}{q}C(u,v) \\ &\geq \Big(\frac{1}{p} - \frac{1}{q}\Big)A(u,v) - \gamma_2 S_p^{-\frac{r}{p}}|\Omega|^{\frac{p_s^*-r}{p_s^*}}A(u,v)^{\frac{r}{p}} \\ &= \Big(\frac{1}{p} - \frac{1}{q}\Big)\|(u,v)\|_E^p - \gamma_2 S_p^{-\frac{r}{p}}|\Omega|^{\frac{p_s^*-r}{p_s^*}}\|(u,v)\|_E^r. \end{aligned}$$

Since, $p > r$, then, J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

Note that, from Lemma 2.4, for any $\lambda \in (0,\lambda_*)$, we have

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-.$$

Moreover, from Lemma 2.5, the following expressions are well defined

$$\alpha_\lambda^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u,v) \text{ and } \alpha_\lambda^+ = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v).$$

3. First Existence Result

In this section, we will state and prove our first existence result which is devoted to the sub-critical case ($q < p_s^*$).

Theorem 3.1. *Let $s \in (0,1), n > ps$. Assume that $0 < r < 1 < p < q < p_s^*$, then there exists $\lambda_* > 0$, such that for all $\lambda \in (0,\lambda_*)$, problem (1.1) has at least two nontrivial solutions.*

In order to prove Theoerem 3.1, we need to prove the following propositions.

Proposition 3.1. *If* $\lambda \in (0, \lambda_*)$ *and* $0 < r < 1 < q < p_s^*$. *Then the functional* J_λ *has a minimizer* (u_λ, v_λ) *in* $\mathcal{N}_\lambda^+$ *satisfying*

$$J_\lambda(u_\lambda, v_\lambda) = \alpha_\lambda^+ < 0.$$

Proof. Assume that $\lambda \in (0, \lambda_*)$ and $0 < r < 1 < q < p_s^*$.
Since J_λ is bounded from below on $\mathcal{N}_\lambda^+$. Then there exists a sequence $\{(u_k, v_k)\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k \to \infty} J_\lambda(u_k, v_k) = \inf_{(u,v) \in \mathcal{N}_\lambda^+} J_\lambda(u, v).$$

From Lemma 2.5, the sequence $\{(u_k, v_k)\}$ is bounded in E. So, up to a subsequence, there exists $(u_\lambda, v_\lambda) \in E$ such that,

$$(u_k, v_k) \rightharpoonup (u_\lambda, v_\lambda), \text{ weakly in } E \text{ as } k \to \infty.$$

Moreover, by [[33], Lemma 8], up to a subsequence, still denoted by (u_k, v_k), we have

$$\begin{cases} (u_k, v_k) \to (u_\lambda, v_\lambda) \text{ strongly in } L^\sigma(\mathbb{R}^n), \ \forall\, 1 \le \sigma < p_s^*, \\ (u_k, v_k) \to (u_\lambda, v_\lambda) \text{ a.e. in } \mathbb{R}^n. \end{cases}$$

Moreover, from [[13], Theorem IV-9], there exist $1 \le \sigma < p_s^*$ and $l_1, l_2 \in L^\sigma(\mathbb{R}^n)$, such that

$$|u_k(x)| \le l_1(x) \text{ and } |v_k(x)| \le l_2(x) \text{ in } \mathbb{R}^n.$$

Now, since we have

$$\begin{cases} B(u_k, v_k) < \gamma_1 \|(u_k, v_k)\|^q_{L^q \times L^q}, \\ C(u_k, v_k) < \gamma_2 \|(u_k, v_k)\|^r_{L^r \times L^r}. \end{cases}$$

Then, from the dominated convergence theorem, we get

$$\begin{cases} B(u_k, v_k) \to B(u_\lambda, v_\lambda), \\ C(u_k, v_k) \to C(u_\lambda, v_\lambda), \end{cases} \text{as } k \to \infty. \tag{3.1}$$

On the other hand, from Lemma 2.3, there exists $t_1 > 0$, such that $(t_1 u_\lambda, t_1 v_\lambda) \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 u_\lambda, t_1 v_\lambda) < 0$. Hence, we obtain

$$\alpha_\lambda^+ := \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v) < 0.$$

Next, we will show that $(u_k, v_k) \to (u_\lambda, v_\lambda)$ strongly in E. If not, then we have

$$\|(u_\lambda, v_\lambda)\|_{X_0} < \lim_{k\to\infty} \inf \|(u_k, v_k)\|_E.$$

Thus, we get

$$\begin{aligned}
\lim_{k\to\infty} \varphi'_{u_k,v_k}(t_1) &= \lim_{k\to\infty} [t_1^{p-1} A(u_k, v_k) - q t_1^q \mathcal{B}_k - \lambda r t_1^r C(u_k, v_k)] \\
&> t_1^{p-1} A(u_\lambda, v_\lambda) - q t_1^q B(u_\lambda, v_\lambda) - \lambda r t_1^r C(u_\lambda, v_\lambda) \\
&= \varphi'_{u_\lambda, v_\lambda}(t_1) = 0.
\end{aligned}$$

That is, $\varphi'_{u_k,v_k}(t_1) > 0$ for k large enough.
Since $(u_k, v_k) \in \mathcal{N}_\lambda^+$, then $\varphi'_{u_k,v_k}(1) = 0$. Moreover, we can see that $\varphi'_{u_k,v_k}(t) < 0$, for $t \in (0, t_1)$. Therefore, we must have $t_1 > 1$. On the other hand, $\varphi_{u_\lambda,v_\lambda}(t)$ is decreasing on $(0, t_1)$, then we get

$$\begin{aligned}
J_\lambda(t_1 u_\lambda, t_1 v_\lambda) := \varphi_{u_\lambda,v_\lambda}(t) &\leq \varphi_{u_\lambda,v_\lambda}(1) := J_\lambda(u_\lambda, v_\lambda) \\
&< \lim_{k\to\infty} J_\lambda(u_k, v_k) \\
&= \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v),
\end{aligned}$$

which is a contradiction. Hence, $(u_k, v_k) \to (u_\lambda, v_\lambda)$ strongly in E. This implies that

$$J_\lambda(u_k, v_k) \to J_\lambda(u_\lambda, v_\lambda) = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v).$$

We deduce that (u_λ, v_λ) is a minimizer of J_λ on $\mathcal{N}_\lambda^+$. □

Proposition 3.2. *Assume that* $0 < r < 1 < q < p_s^*$. *If* $\lambda \in (0, \lambda_*)$, *then* J_λ *has a minimizer* $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ *in* $\mathcal{N}_\lambda^-$, *satisfying*

$$J_\lambda(\widetilde{u}_\lambda, \widetilde{v}_\lambda) = \alpha_\lambda^- > 0.$$

Proof. Since J_λ is bounded from below on $\mathcal{N}_\lambda^-$, then, there exists a sequence $\{(u_k, v_k)\} \subset \mathcal{N}_\lambda^-$, such that

$$\lim_{k\to\infty} J_\lambda(u_k, v_k) = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v).$$

By the same arguments given in the proof of Proposition 3.1, there exists $(\widetilde{u}_\lambda, \widetilde{v}_\lambda) \in E$ such that, up to a subsequence, we have

$$\begin{cases} A(u_k, v_k) \to A(\widetilde{u}_\lambda, \widetilde{v}_\lambda), \\ B(u_k, v_k) \to B(\widetilde{u}_\lambda, \widetilde{v}_\lambda), \\ C(u_k, v_k) \to C(\widetilde{u}_\lambda, \widetilde{v}_\lambda). \end{cases}$$

Moreover, from the analysis of the fibering map $\varphi_{u,v}(t)$, we know that there exists $t_2 > T$ such that $(t_2 u, t_2 u) \in \mathcal{N}_\lambda^-$.

Next, we show that $(u_k, v_k) \to (\widetilde{u}_\lambda, \widetilde{v}_\lambda)$, strongly in E. To this aim, we proceed by contradiction and we assume that

$$\|(\widetilde{u}_\lambda, \widetilde{v}_\lambda)\|_E < \liminf_{k\to\infty} \|(u_k, v_k)\|_E.$$

For $\{(u_k, v_k)\} \subset \mathcal{N}_\lambda^-$, we have

$$J_\lambda(u_k, v_k) > J_\lambda(tu_k, tv_k)' \ \forall\, t > T.$$

So we get

$$\begin{aligned} J_\lambda(t_2\widetilde{u}_\lambda, t_2\widetilde{v}_\lambda) &= \frac{t_2^p}{p} A(u, v)_2 - t_2^q B(u, v)_2 - \lambda t_2^r C(u, v)_2 \\ &< \lim_{k\to\infty} (\frac{t_2^p}{p} A(u_k, v_k) - t_2^q B(u_k, v_k) - \lambda t_2^r C(u_k, v_k)) \\ &= \lim_{k\to\infty} J_\lambda(t_2 u_k, t_2 v_k) \leq J_\lambda(u_k, v_k) = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v), \end{aligned}$$

which is a contradiction. Hence, $(u_k, v_k) \to (\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ strongly in E. Therefore, we deduce that

$$J_\lambda(u_k, v_k) \to J_\lambda(\widetilde{u}_\lambda, \widetilde{v}_\lambda) = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v), k \to \infty.$$

Hence, $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ is a minimizer for J_λ in $\mathcal{N}_\lambda^-$. □

***Proof of Theorem* 3.1.** From Propositions 3.1, there exists a minimizer $(u_\lambda, u_\lambda) \in \mathcal{N}_\lambda^+$, such that

$$J_\lambda(u_\lambda, v_\lambda) = \alpha_\lambda^+ < 0. \tag{3.2}$$

On the other hand, from Proposition 3.2, there exists a minimizer $(\widetilde{u}_\lambda, \widetilde{v}_\lambda) \in \mathcal{N}_\lambda^-$ such that

$$J_\lambda(\widetilde{u}_\lambda, \widetilde{v}_\lambda) = \alpha_\lambda^- > 0. \tag{3.3}$$

Now, from Lemma 2.2, we get that (u_λ, u_λ) and $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ are solutions to problem (1.1). Moreover, from (3.2) and (3.3), we see that these solutions are nontrivial. Finally, the fact that $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, implies that (u_λ, u_λ) and $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ are distinct. This finishes the proof of Theorem 3.1.

4. Second Existence Result

In this section, we will state and prove the second existence result of this paper which is devoted to the critical case ($q = p_s^*$). The main difficulty is that the Sobolev embedding for the critical exponent is not compact. Since the embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^n)$ is not compact, the energy functional does not satisfy the Palais-Smale condition globally. But it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant, that we can denoted by the following expression:

$$S_p = \inf_{u \in X_0 \setminus \{0\}} \frac{||u||_{X_0}^p}{||u||_{L^{p_s^*}}^p}. \tag{4.1}$$

Our second main result of this paper is the following theorem.

Theorem 4.1. *Assume that $s \in (0,1), n > ps$ and $0 < r < 1 < p < q = p_s^*$. If there exist $t_0 > 0$ and $(u_0, v_0) \in E \setminus \{(0,0)\}$ satisfying*

$$\frac{1}{p} A(u_0, v_0) t_0^p - t_0^{p_s^*} B(u_0, v_0) = \frac{s}{n} (p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{sp}}. \tag{4.2}$$

Then there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, problem (1.1) has at least two nontrivial solutions.

Remark 4.1. The condition (4.2) can be guaranteed by Lemma 2.11.

In this section, we will put

$$M = \left(\frac{p-r}{p}\right)\left(\frac{nr(p_s^*-r)}{p_s^* p S_p}\right)^{\frac{r}{p-r}}\left(\gamma_2|\Omega|^{\frac{p_s^*-r}{p_s^*}}\right)^{\frac{p}{p-r}}, \tag{4.3}$$

where S_p is defined in Equation (4.1).

Proposition 4.3. *Assume that $0 < r < 1 < q = p_s^*$. If $\lambda \in (0, \lambda_*)$, then every Palais-Smale subsequence at level c, with*

$$c < \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}}, \tag{4.4}$$

has a convergent subsequence.

Proof. If $\{(u_k, v_k)\}$ is a Palais-Smale sequence at level c, with c satisfying (4.4), then, Lemma 2.5 implies that $\{(u_k, v_k)\}$ is bounded in E. So, up to a sequence, there exists $(u_*, v_*) \in E$ such that

$$(u_k, v_k) \rightharpoonup (u_*, v_*), \quad \text{weakly in } \; E.$$

Moreover, from [[33], Lemma 8] and the Sobolev embedding, we get

$$\begin{cases} (u_k, v_k) \rightharpoonup (u_*, v_*) \;\text{ weakly in }\; L^{p_s^*}(\mathbb{R}^n) \times L^{p_s^*}(\mathbb{R}^n), \\ (u_k, v_k) \to (u_*, v_*) \text{ strongly in }\; L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n),\; \forall\, 1 \le r < p_s^*, \\ (u_k, v_k) \to (u_*, v_*) \text{ a.e. in }\; \mathbb{R}^n \times \mathbb{R}^n. \end{cases}$$

Moreover, from [[13], Theorem IV-9], there exist $\tilde{l}_1, \tilde{l}_2 \in L^r(\mathbb{R}^n)$ such that, for all $1 \le r < p_s^*$, we have

$$|u_k(x)| \le \tilde{l}_1(x) \text{ and } |v_k(x)| \le \tilde{l}_2(x) \text{ in } \mathbb{R}^n,$$

Therefore, by dominated convergence theorem, we have that

$$C(u_k, v_k) \to C(u_*, v_*), \quad \text{as } k \to \infty. \tag{4.5}$$

On the other hand, from Brezis-Lieb Lemma [[40], Lemma 1.32], we obtain

$$\begin{cases} A(u_k, v_k) = A(u, v)(u_k - u_*, v_k - v_*) + A(u, v)_* + o(1), \\ \\ B(u_k, v_k) = B(u, v)(u_k - u_*, v_k - v_*) + B(u, v)_* + o(1). \end{cases} \tag{4.6}$$

By combining (4.5) with (4.6), we get

$$
\begin{aligned}
&\langle J'_\lambda(u_k,v_k),(u_k,v_k)\rangle_E\\
&= A(u_k,v_k)-p_s^*B(u_k,v_k)-\lambda rC(u_k,v_k)\\
&= A(u,v)(u_k-u_*,v_k-v_*)+A(u,v)_*-p_s^*(B(u,v)(u_k-u_*,v_k-v_*)\\
&\quad +B(u,v)_*)-\lambda rC(u_k,v_k)+o(1)\\
&= \langle J'_\lambda(u_*,v_*),(u_*,v_*)\rangle_E+A(u,v)(u_k-u_*,v_k-v_*)\\
&\quad -p_s^*B(u,v)(u_k-u_*,v_k-v_*).
\end{aligned}
$$

Since $\langle J'_\lambda(u_*,v_*),(u_*,v_*)\rangle_E=0$ and $\langle J'_\lambda(u_k,v_k),(u_k,v_k)\rangle_E\to 0$. Then, we have

$$A(u,v)(u_k-u_*,v_k-v_*)\to b \text{ and } p_s^*B(u,v)(u_k-u_*,v_k-v_*)\to b. \quad (4.7)$$

If $b=0$, then the proof is completed. Assume that $b>0$. From (2.17), we get

$$p_s^*B(u,v)(u_k-u_*,v_k-v_*)\le p_s^*\gamma_1 S_p^{-\frac{p_s^*}{p}}\left(A(u,v)(u_k-u_*,v_k-v_*)\right)^{\frac{p_s^*}{p}}$$

which yields to

$$b\ge (p_s^*\gamma_1)^{\frac{-n}{sp_s^*}}S_p^{\frac{n}{ps}}.$$

Therefore, one has

$$
\begin{aligned}
c &= \lim_{k\to\infty}\left(\frac{1}{p}A(u_k,v_k)-B(u,v)_k-\lambda C(u_k,v_k)\right)\\
&= \lim_{k\to\infty}\Big(\frac{1}{p}A(u,v)(u_k-u_*,v_k-v_*)-B(u,v)(u_k-u_*,v_k-v_*)-\lambda C(u_k,v_k)\Big)\\
&\quad -\frac{1}{p}A(u_*,v_*)-B(u_*,v_*)+o(1)\\
&= J_\lambda(u_*,v_*)+b\Big(\frac{1}{p}-\frac{1}{p_s^*}\Big)\\
&\ge J_\lambda(u_*,v_*)+\frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}}S_p^{\frac{n}{ps}}.
\end{aligned}
$$

Since $c<\frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}}S_p^{\frac{n}{ps}}$, then, we obtain $J_\lambda(u_*,v_*)<0$. In particular, u_*

and v_* are nontrivial. Moreover,

$$B(u_*, v_*) > \frac{1}{p} A(u_*, v_*) - \lambda C(u_*, v_*). \tag{4.8}$$

Now, by combining (2.18) with (4.8), we obtain

$$\begin{aligned}
c &= \lim_{k\to\infty} J_\lambda(u_k, v_k) \\
&= \lim_{k\to\infty} \Big[J_\lambda(u_k, v_k) - \frac{1}{p} \langle J'_\lambda(u_k, v_k), (u_k, v_k)\rangle_E \Big] \\
&= \lim_{k\to\infty} \Big[\Big(\frac{p_s^*}{p} - 1\Big)(B(u_*, v_*)(u_k - u_*, v_k - v_*)) + B(u_*, v_*) - \lambda\Big(\frac{p-r}{p}\Big) C(u_k, v_k) \Big] \\
&= \lim_{k\to\infty} \Big[\frac{sp_s^*}{n} (B(u_*, v_*)(u_k - u_*, v_k - v_*) + B(u_*, v_*)) - \lambda\Big(\frac{p-r}{p}\Big) C(u_*, v_*) \Big] \\
&\geq \frac{s}{n} (p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} + \frac{sp_s^*}{n} B(u_*, v_*) - \lambda\Big(\frac{p-r}{p}\Big) C(u_*, v_*) \\
&\geq \frac{s}{n} (p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - \lambda\gamma_2 S_p^{-\frac{r}{p}} |\Omega|^{\frac{p_s^*-r}{p_s^*}} \Big(\frac{p_s^* - r}{p}\Big) (A(u_*, v_*))^{\frac{r}{p}} + \frac{sp_s^*}{np} A(u_*, v_*) \\
&:= \frac{s}{n} (p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - h(A(u, v)^{\frac{1}{p}}),
\end{aligned} \tag{4.9}$$

where

$$h(\eta) = \lambda\gamma_2 S_p^{-\frac{r}{p}} |\Omega|^{\frac{p_s^*-r}{p_s^*}} \Big(\frac{p_s^* - r}{p}\Big) \eta^r - \frac{sp_s^*}{np} \eta^p.$$

We note that the function h attains its maximum at

$$\eta_0 = \left(\frac{\lambda n r (p_s^* - r)\gamma_2}{spp_s^*} S_p^{-\frac{r}{p}} |\Omega|^{\frac{p_s^*-r}{p_s^*}} \right)^{\frac{1}{p-r}}.$$

Moreover, we have

$$h(\eta_0) = \sup_{\eta>0} h(\eta) = M\lambda^{\frac{p}{p-r}}, \tag{4.10}$$

where M is defined in (4.3).

Now, if we combine (4.9) with (4.10), we obtain

$$c \geq \frac{s}{n} (p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}},$$

which is in contradiction with (4.4). Hence, $b = 0$. Therefore, we conclude that $(u_k, v_k) \to (u_*, v_*)$ strongly in E. This completes the proof. □

Proposition 4.4. *There exist* $\lambda^* > 0$, $t_0 > 0$, *and* $(u_0, v_0) \in E$, *such that for all* $\lambda \in (0, \lambda^*)$, *we have*

$$J_\lambda(t_0 u_0, t_0 v_0) \leq \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}}. \tag{4.11}$$

In particular,

$$\alpha_\lambda^- < \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}}. \tag{4.12}$$

Proof. Since $\lambda \in (0, \lambda_*)$, from Equations (2.13) and (4.3) and using the fact that $0 < \frac{1}{r}\left(\frac{p-r}{p_s^*-r}\right)^{\frac{n}{sp}} < 1$, we get

$$\begin{aligned}\frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda_*^{\frac{p}{p-r}} &> \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}} \\ &> \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} \left(1 - \frac{1}{r}\Big(\frac{p-r}{p_s^*-r}\Big)^{\frac{n}{sp}}\right) > 0.\end{aligned}$$

The above inequality implies that

$$\lambda < \lambda_{**} := \left(\frac{s}{nM}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}}\right)^{\frac{p-r}{p}}.$$

From condition (4.2), there exist $t_0 > 0$ and $(u_0, v_0) \in E \backslash \{(0,0)\}$ such that

$$\begin{aligned}J_\lambda(t_0 u_0, t_0 v_0) &= \frac{1}{p}A(u_0, v_0)t_0^p - t_0^q B(u_0, v_0) - \lambda t_0^r C(u_0, v_0) \\ &= \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - \lambda t_0^r C(u_0, v_0). \end{aligned} \tag{4.13}$$

Let

$$\lambda_{***} = \left(\frac{t_0^{r+1} C(u_0, v_0)}{M}\right)^{\frac{p_s^*-r}{r}}, \forall \lambda \in (0, \lambda_{***})$$

we denote

$$-\lambda t_0^r C(u_0, v_0) < -M\lambda^{\frac{p}{p-r}}.$$

Thus, we obtain that (4.11) hold. Finally, we set $\lambda^* = \min\{\lambda_*, \lambda_{**}, \lambda_{***}\}$, by the analysis of fibering map $\varphi_{u,v}(t) = J_\lambda(tu, tv)$, we get

$$\alpha_\lambda^- < \frac{s}{n}(p_s^* \gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}},$$

for $\lambda \in (0, \lambda_2)$. This completes the proof. □

***Proof of Theorem* 2.** Let $\{(u_k^+, v_k^+)\}$ be a minimizer for J_λ in $\mathcal{N}_\lambda^+$, and $\{(u_k^-, v_k^-)\}$ be a minimizer for J_λ in $\mathcal{N}_\lambda^+$. That is

$$J_\lambda(u_k^+, v_k^+) \to \alpha_\lambda^+, J'_\lambda(u_k^+, v_k^+) \to 0, \tag{4.14}$$

and

$$J_\lambda(u_k^-, v_k^-) \to \alpha_\lambda^-, J'_\lambda(u_k^-, v_k^-) \to 0. \tag{4.15}$$

From (4.12), we see that

$$\alpha_\lambda^+ < 0 < \alpha_\lambda^+ < \frac{s}{n}(p_s^*\gamma_1)^{\frac{-n}{sp_s^*}} S_p^{\frac{n}{ps}} - M\lambda^{\frac{p}{p-r}}.$$

So, from Proposition 4.3, up to a subsequence, there exist $(u_\lambda, v_\lambda) \in \mathcal{N}_\lambda^+$ and $(\widetilde{u}_\lambda, \widetilde{v}_\lambda) \in \mathcal{N}_\lambda^-$, such that

$$(u_k^+, v_k^+) \to (u_\lambda, v_\lambda), \text{ and } (u_k^-, v_k^-) \to (\widetilde{u}_\lambda, \widetilde{v}_\lambda).$$

Therefore, from Lemma 2.2, (u_λ, v_λ) and $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ are solution to problem (1.1). Moreover, from equations (3.2) and (3.3), (u_λ, v_λ) and $(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ are nontrivial. Finally, the fact that $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- \neq \emptyset$, implies that these two solutions are distinct. This finishes the proof.

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