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ADJACENCY FOR SPECIAL REPRESENTATIONS OF A WEYL GROUP

GEORGE LUSZTIG

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA.
E-mail: gyuri@math.mit.edu

0. Introduction

0.1. Let G be a connected reductive group over $\mathbf{C}$. Let $\mathcal{U}_G$ be the set of unipotent conjugacy classes of G. Let $C' \in \mathcal{U}_G$ and let $C \in \mathcal{U}_G$ be maximal with the property that $C \subset \bar{C}' - C'$ ($\bar{C}'$ is the closure of C'). A remarkable result of Kraft and Procesi [7], [8] is that when G is a classical group, then either $\dim(C') = \dim(C) + 2$ or the singularity of C' at a point of C is the same as the singularity at 1 of a minimal unipotent class in a smaller reductive group. This result has been recently extended to exceptional groups by Fu, Juteau, Levy and Sommers [3].

In the late 1980's, inspired by [7], [8], I showed (unpublished) that the results of *loc.cit.* have a (weak) analogue in the case where $\mathcal{U}_G$ is replaced by $\mathcal{U}_G^{sp}$ (the set of special unipotent classes of G). The analogues in this case of the pairs C, C' above can be viewed as edges of a graph with set of vertices $\mathcal{U}_G^{sp}$. One feature that was not present in *loc.cit.* (except in type A) is that $\mathcal{U}_G^{sp}$ has an order reversing involution which preserves the graph structure and that if two edges of the graph are interchanged by this involution then at least one of them is associated to a pair C, C' with $\dim(C') = \dim(C) \pm 2$. (see Theorem 0.3).

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This property allows us to construct the graph above purely in terms of the involution above and truncated induction of Weyl group representations, see Theorem 5.4. (Since $\mathcal{U}_G^{sp}$ is naturally in bijection with the set of special representations of the Weyl group W, we formulate our results in terms of W. This has the advantage that our results make sense for any finite Coxeter group.)

0.2. In the remainder of this paper we fix a finite Coxeter group W. Let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations of W over $\mathbf{C}$. In [9] a certain subset $\mathcal{S}_W$ of $\mathrm{Irr}(W)$ was defined assuming that W is a Weyl group, but the same definition can be given in general. (Later, the name of "special representations" was given to the elements of this subset.) Writing $W = \prod_i W_i$ where W_i are irreducible Coxeter groups we have that $\mathcal{S}_W$ consists of all $E = \boxtimes_i E_i$ with $E_i \in \mathcal{S}_{W_i}$ for all i. Let sgn be the sign representation of W. Let $\mathcal{S}_W^{odd} = \{E \in \mathcal{S}_W; E \otimes \mathrm{sgn} \notin \mathcal{S}_W\}$. When W is irreducible of type $E_7/E_8/H_3/H_4$, $\mathcal{S}_W^{odd}$ consists of the $E \in \mathcal{S}_W$ which have dimension 512/4096/4/16 (there are 1/2/1/2 such E); when W is irreducible of type other than E_7, E_8, H_3, H_4, then $\mathcal{S}_W^{odd} = \emptyset$. We set $\mathcal{S}_W^{ord} = \mathcal{S}_W - \mathcal{S}_W^{odd}$. If $E \in \mathcal{S}_W$ we say that E is odd if $E \in \mathcal{S}_W^{odd}$; we say that E is ordinary if $E \in \mathcal{S}_W^{ord}$.

When W is irreducible let $E \mapsto E^\circ$ be the involution of $\mathcal{S}_W$ such that $E^\circ = E \otimes \mathrm{sgn}$ for $E \in \mathcal{S}_W^{ord}$ and $E \mapsto E^\circ$ is the permutation of $\mathcal{S}_W^{odd}$ of order 1/2/1/2 if W is of type $E_7/E_8/H_3/H_4$. When W is not necessarily irreducible and $W = \prod_i W_i$ with W_i irreducible then the involution $E \mapsto E^\circ$ of $\mathcal{S}_W$ is defined by $\boxtimes_i E_i \mapsto \boxtimes_i (E_i^\circ)$ where $E_i \in \mathcal{S}_{W_i}$.

We can identify $\mathcal{S}_W$ with the set $Cell_W$ of two-sided cells of W by the procedure stated in [6, 1.7]. Then the involution $E \mapsto E^\circ$ of $\mathcal{S}_W$ becomes the involution of $Cell(W)$ given by left multiplication by the longest element of W; the partial order $\leq_{LR}$ on $Cell_W$ in [6] becomes a partial order on $\mathcal{S}_W$ denoted by $\leq$. Note that sgn is the unique maximal element for $\leq$. Using [6, 3.3] we see that

(a) $E \leq E'$ (in $\mathcal{S}_W$) implies $E'^\circ \leq E^\circ$.

Let $Cell_W \to \mathbf{N}$ be the function defined in [10, 5.4] (the definition given in *loc.cit.* for W assumed to be a Weyl group is also applicable without this assumption). This can be viewed as a function $E \mapsto a_E, \mathcal{S}_W \to \mathbf{N}$. Using *loc.cit.* we see that

(b) $E \le E'$ (in $\mathcal{S}_W$) implies $a_{E'} \le a_E$.
Let $\hat{\mathcal{S}}_W$ be the set of all $(E,E') \in \mathcal{S}_W \times \mathcal{S}_W$ such that $E \lneqq E'$ and there is no $E'' \in \mathcal{S}_W$ with $E \lneqq E'', E'' \lneqq E'$. For E,E' in $\mathcal{S}_W$ we say that E,E' are *adjacent* if $(E,E') \in \hat{\mathcal{S}}_W$ or $(E',E) \in \hat{\mathcal{S}}_W$; we then write $E - - - E'$. Note that

(c) $E - - - E'$ implies $E^\circ - - - E'^\circ$.
(See (a).)

Writing $W = \prod_i W_i$ where W_i are irreducible Coxeter groups we have that $E - - - E'$ for $E = \boxtimes_i E_i, E' = \boxtimes_i E'_i$ with E_i, E'_i in $\mathcal{S}_{W_i}$ for all i if and only if $E_i - - - E'_i$ (in $\mathcal{S}_{W_i}$) for one i and $E_i = E'_i$ for all other i.

Theorem 0.3. *Assume that W is irreducible. Let (E,E') be in $\hat{\mathcal{S}}_W$.*

(a) *We have either $a_E - a_{E'} = 1$ or $a_{E'^\circ} - a_{E^\circ} = 1$.*
(b) *To E,E' one can associate an irreducible Coxeter group $W' = W_{E,E'}$ contained in W such that $a_E - a_{E'} = h - 1$ where h is the Coxeter number of $W_{E,E'}$.*

0.4. Let E,E',h be as in 0.3(b). The condition in 0.3(b) does not define $W' = W_{E,E'}$ uniquely (unless $h = 2$ or $h = 3$). Nevertheless I believe that in each case there is a natural choice for W'. We describe it below.

If W is of type $A_n, n \ge 1$ then W' is of type A_{h-1}. If W is of type $B_n, n \ge 2$ then h is even and W' is of type $B_{h/2}$. If W is of type $D_n, n \ge 4$ then h is even and W' is of type $B_{h/2}$ or of type $D_{(h+2)/2}$; more precisely, if E (resp. E') is associated to the pair of partitions $1^{n-j}, 1^j$ (resp. $1^{n-j-1}, 1^{j+1}$) of n with $0 \le j < (n-2)/2$, so that $h = 2n - 2 - 4j$, then W' is of type $D_{(h+2)/2} = D_{n-2j}$; in all other cases W' is of type $B_{h/2}$.

If W is a dihedral group of order $2n \ge 8$, then $h \in \{2, n\}$; if $h = 2$ then W' is of type A_1; if $h = n$ then $W' = W$.

If W is of type $E_8, E_7, E_6, F_4, G_2, H_3, H_4$, the various E_1, E'_1 such that $E_1 - - - E'_1$ are listed in §4 in the form $E_1 \overset{W'}{- - -} E'_1$ where $W' = W_{E_1,E'_1}$; in the case where $a_{E_1} - a_{E'_1} = \pm 1$ we omit writing W' (it is of type A_1). In these cases we denote elements $E_1 \in \mathcal{S}_W$ in the form d_a where $d = \dim E_1, a = a_{E_1}$; when W is of type F_4 there may be two E_1 with the same d,a; we denote them by d_a, d'_a.

1. The Polynomials $\tilde{\Pi}_{E,E'}$

1.1. We now assume (until the end of 1.4) that W is the Weyl group of a reductive connected group G over $\mathbf{C}$. Let $\mathcal{U}_G$ be as in 0.1 and let $\tilde{\mathcal{U}}_G$ be the set of pairs $(C, \mathcal{L})$ where $C \in \mathcal{U}_G$ and $\mathcal{L}$ is an irreducible local system on C, equivariant for conjugation by G. Springer's correspondence provides an imbedding $\iota : \mathrm{Irr}(W) \to \tilde{\mathcal{U}}_G$. As stated in [9, no.9], there is a well defined subset $\mathcal{U}_G^{sp}$ of $\mathcal{U}_G$ such that ι defines a bijection of $\mathcal{S}_W$ onto $\{(C, \mathcal{L}) \in \tilde{\mathcal{U}}_G, C \in \mathcal{U}_G^{sp}, \mathcal{L} = \mathbf{C}\}$. This can be viewed as a bijection $\iota' : \mathcal{S}_W \xrightarrow{\sim} \mathcal{U}_G^{sp}$. Let $\le$ be the partial order on $\mathcal{U}_G^{sp}$ obtained by restricting the obvious partial order on $\mathcal{U}_G$ (the unique maximal element is the regular unipotent class). According to [2] or [4], ι' is compatible with the partial orders on $\mathcal{S}_W, \mathcal{U}_G^{sp}$. It follows that the order reversing involution $E \mapsto E^\circ$ on $\mathcal{S}_W$ can be viewed as an order reversing involution of $\mathcal{U}_G^{sp}$. Now the partial order on $\mathcal{U}_G$ has been explicitly computed in all cases. (See [12] and the references there.) Since ι, ι' are explicitly known, this determines the partial order (hence the adjacency relation) on $\mathcal{S}_W$. In the case where W is of type E_8, E_7, E_6, F_4, G_2 the adjacency relation on $\mathcal{S}_W$ is described in §4; from this one can see that 0.3(a) holds in these cases.

1.2. Let u be an indeterminate. Let E, E' be in $\mathrm{Irr}(W)$. Let $(C, \mathcal{L}) = \iota(\mathcal{E})$, $(C', \mathcal{L}') = \iota(\mathcal{E}')$. Assume that $C \subset \bar{C}'$ where $\bar{C}'$ is the closure of C'. For $i \in \mathbf{Z}$ let $n_{E,E',i}$ be the multiplicity of $\mathcal{L}$ in the restriction to C of the i-th cohomology sheaf of the intersection cohomology complex of $\bar{C}'$ with coefficients in $\mathcal{L}'$. It is known that $n_{E,E',i}$ is zero unless $i \in 2\mathbf{N}$; we set $\Pi_{E,E'} = \sum_{i \in 2\mathbf{N}} n_{E,E',i} u^{i/2} \in \mathbf{N}[u]$.

1.3. Assume that W is irreducible. Assume that (E, E') is in $\hat{\mathcal{S}}_W$. If E and E' are ordinary (see 0.2) we set $\tilde{\Pi}_{E,E'} = \Pi_{E,E'}$. If E is ordinary and E' is odd (see 0.2) we set $\tilde{\Pi}_{E,E'} = \Pi_{E,E'} + \Pi_{E,E'_1}$ where $E'_1 \in \mathrm{Irr}(W)$ is in the same two-sided cell as E' and is distinct from E'. If E is odd and E' is ordinary we set $\tilde{\Pi}_{E,E'} = \Pi_{E,E'} + \Pi_{E_1,E'}$ where $E_1 \in \mathrm{Irr}(W)$ is in the same two-sided cell as E and is distinct from E. (Note that at most one of E, E' is odd).

Conjecture 1.4. In the setup of 1.3 we have

$$\tilde{\Pi}_{E,E'} = u^{e_1-1} + u^{e_2-1} + \cdots + u^{e_r-1}$$

where $e_1 \le e_2 \le \cdots \le e_r$ are the exponents of a well defined irreducible Weyl group W' of rank $r \ge 1$.

Using the known tables (in particular those of [1] for type E_6, E_7, E_8) I have verified this for W of type $B_2, B_3, D_4, D_5, D_6, D_7, E_6, E_7, E_8, F_4, G_2$. The resulting W' is described for types E_8, E_7, E_6, F_4, G_2 in §4. We can thus verify 0.3(b) (in the strengthened form 0.4) in these cases. When W is of type A the conjecture can be deduced from [7]; this also verifies 0.3(a) and 0.3(b) (in the strengthened form 0.4) in this case; an alternative proof is given in §3.

1.5. We now assume that W is irreducible but not a Weyl group. In this case the partial order $\le$ on $\mathcal{S}_W$ is linear (it is described in [4]); the adjacency relation is easily described and 0.3(a) is easily verified (see §4 for types H_4, H_3). For any E, E' in $\mathrm{Irr}(W)$ we set $\Pi_{E,E'} = u^{-a_{E'}} P_{E',E}$ where $P_{E',E} \in \mathbf{Q}(u)$ is attached by Geck and Malle [5] to E, E'. It is known that in our case we have $\Pi_{E,E'} \in \mathbf{N}[u]$. Assuming that $(E, E') \in \hat{\mathcal{S}}_W$, we modify $\Pi_{E,E'}$ to a $\tilde{\Pi}_{E,E'}$ by the same procedure as in 1.3. Then conjecture 1.4 can be extended word by word to our case (except that W' is now a finite irreducible Coxeter group, not necessarily a Weyl group). This conjecture is actually true, as can be verified using the tables for $P_{E',E}$ available through CHEVIE. The values of W' for W of type H_4, H_3 are listed in §4. We can thus verify 0.3(b) (in the strengthened form 0.4) in these cases. In the case where W is a finite dihedral group. 0.3(a) and 0.3(b) (in the strengthened form 0.4) are easily verified.

If W is a Weyl group, it is likely that the procedure above (based on [5]) gives the same polynomials as in 1.3.

2. Type B,D

2.1. Let $\epsilon \in \{0,1\}$. Let $r \geq 1$. For $M \in \epsilon + 2\mathbf{N}$ let $\mathfrak{S}_r^M$ be the set of all $(a_1, a_2, \dots, a_M) \in \mathbf{N}^M$ such that $a_1 \leq a_2 \leq \dots \leq a_M$ with no two consecutive equal signs and such that $a_1+a_2+\dots+a_M = r+(M^2-2M+\epsilon)/4$. For example when $\epsilon = 1$ we have $(0,1,1,2,2,\dots,r,r) \in \mathfrak{S}_r^{2r+1}$, $(r) \in \mathfrak{S}_r^1$; when $\epsilon = 0$ we have $(0,1,1,2,2,\dots,r-1,r-1,r) \in \mathfrak{S}_r^{2r}$, $(0,r) \in \mathfrak{S}_r^2$.

We define an equivalence relation on $\mathfrak{S}_r^\epsilon \cup \mathfrak{S}_r^{\epsilon+2} \cup \mathfrak{S}_r^{\epsilon+4} \cup \dots$ by

$$(a_1, a_2, \dots, a_M) \sim (0,0,a_1+1, a_2+1, \dots, a_M+1) \sim (0,0,1,1,a_1+2, a_2+2, \dots, a_M+2) \sim \dots .$$

Let ${}^\epsilon\mathfrak{S}_r$ be the set of equivalence classes.

2.2. If $a_* = (a_1, a_2, \dots, a_M) \in \mathfrak{S}_r^M$ and $t \geq a_M$ let $a_*^{!t}$ be the sequence obtained from $0,0,1,1,2,2,\dots,t,t$ by removing two entries $t-a$ for any a which appears twice in a_* and by removing one entry $t-a$ for any a which appears exactly once in a_*. Now $a_*^{!t}$ has $M' = 2(t+1)-M$ entries whose sum is

$$t(t+1)-tM+r+(M^2-2M+\epsilon)/4 = r+((2t+2-M)^2-2(2t+2-M)+\epsilon)/4$$

so that $a_*^{!t} \in \mathfrak{S}_r^{M'}$. It is easy to see that the equivalence class of $a_*^{!t} \in \mathfrak{S}_r^{2(t+1)-M}$ is independent of t (if $t \geq a_M$). From the definition we see that if a_* is replaced by an equivalent sequence, the equivalence class of $a_*^{!t}$ (with $t \geq a_M$) is unchanged. Hence $a_* \mapsto a_*^{!t}$ defines a map $\xi \mapsto \xi^\bullet$, ${}^\epsilon\mathfrak{S}_r \to {}^\epsilon\mathfrak{S}_r$. Its square is 1.

2.3. For any $p \geq 2$ let $({}^\epsilon\mathfrak{S}_r \times {}^\epsilon\mathfrak{S}_r)_p$ be the subset of ${}^\epsilon\mathfrak{S}_r \times {}^\epsilon\mathfrak{S}_r$ consisting of pairs (ξ, ξ') which can be represented by $(a_*, a'_*) \in \mathfrak{S}_r^M \times \mathfrak{S}_r^M$ such that for some $k < k+1$ in $[1,M]$ and some a we have

$$(a_k, a_{k+1}) = (a, a+p), (a'_k, a'_{k+1}) = (a+1, a+p-1), a_s = a'_s \text{ for } s \neq k, k+1.$$

For any $p \geq 3$ let $({}^\epsilon\mathfrak{S}_r \times {}^\epsilon\mathfrak{S}_r)'_p$ be the subset of ${}^\epsilon\mathfrak{S}_r \times {}^\epsilon\mathfrak{S}_r$ consisting of pairs (ξ, ξ') which can be represented by $(a_*, a'_*) \in \mathfrak{S}_r^M \times \mathfrak{S}_r^M$ such that for some

$k < l$ in $[1, M]$ with $l = k + 2p - 3$ and some a we have

$$(a_k, a_{k+1}, \dots, a_{l-1}, a_l)$$
$$= (a, a{+}1, a{+}2, a{+}2, a{+}3, a{+}3, \dots, a{+}p{-}2, a{+}p{-}2, a{+}p{-}1, a{+}p),$$
$$(a'_k, a'_{k+1}, \dots, a'_{l-1}, a'_l)$$
$$= (a{+}1, a{+}1, a{+}2, a{+}2, \dots, a{+}p{-}2, a{+}p{-}2, a{+}p{-}1, a{+}p{-}1),$$

and $a_s = a'_s$ for $s \notin \{k, k+1, \dots, l-1, l\}$.

From the definitions we see that $(\xi, \xi') \mapsto (\xi'^{\bullet}, \xi^{\bullet})$ is a bijection

$$({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_p \xrightarrow{\sim} ({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)'_p$$

(if $p \geq 3$) and is a bijection

$$({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_2 \xrightarrow{\sim} ({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_2.$$

2.4. When W is of type B_r, $r \geq 1$, we can identify $\mathcal{S}_W$ with ${}^1\mathfrak{S}_r$ as in [9, no.5]. When W is of type D_r, $r \geq 2$, we can identify $\mathcal{S}_W$ with ${}^0\mathfrak{S}_r$ as in [9, no.5] except that each element of ${}^0\mathfrak{S}_r$ of the form $(a_1, a_2, \dots, a_M)$ with $a_1 = a_2 < a_3 = a_4 < a_5 = a_6 < \dots$ corresponds to two representations in $\mathcal{S}_W$. (We use the notation $(a_1, a_2, a_3, a_4, \dots)$ instead of the notation $\binom{a_1\ a_3\ a_5\ \dots}{a_2\ a_4\ a_6\ \dots}$ in [9].) If E, E' in $\mathcal{S}_W$ correspond to ξ, ξ' in ${}^{\epsilon}\mathfrak{S}_r$ then E, E' are adjacent in $\mathcal{S}_W$ if and only if (i) or (ii) below holds:

(i) (ξ, ξ') or (ξ', ξ) belongs to $({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_p$ for some $p \geq 2$;

(ii) (ξ, ξ') or (ξ', ξ) belongs to $({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)'_p$ for some $p \geq 3$.

If (i) holds then $a_E - a_{E'} = \pm 1$; if (ii) holds then $a_E - a_{E'} = \pm(2p-3)$. The involution $E \mapsto E^{\circ}$ of $\mathcal{S}_W$ corresponds to the involution $\xi \mapsto \xi^{\bullet}$ of ${}^{\epsilon}\mathfrak{S}_r$ and that involution interchanges pairs as in (i) with pairs as in (ii) (if $p \geq 3$) or pairs in (i) with pairs in (i) (if $p = 2$); see 2.3. It follows that 0.3 holds in our case.

3. Type A

3.1. Let $r \geq 1$. For $m \in \mathbf{N}$ let $\mathfrak{T}_r^m$ be the set of all $(a_1, a_2, \dots, a_m) \in \mathbf{N}^m$ such that $a_1 < a_2 < \dots < a_m$ and such that $a_1 + a_2 + \dots + a_m = r + m(m-1)/2$.

For example we have $(1,2,\dots,r) \in \mathfrak{T}_r^r$, $(r) \in \mathfrak{T}_r^1$. We define an equivalence relation on $\mathfrak{T}_r^0 \cup \mathfrak{T}_r^1 \cup \mathfrak{T}_r^2 \cup \dots$ by

$$\begin{aligned}(a_1,a_2,\dots,a_m) &\sim (0,a_1+1,a_2+1,\dots,a_m+1)\\ &\sim (0,1,a_1+2,a_2+2,\dots,a_m+2) \sim \dots .\end{aligned}$$

Let $\mathfrak{T}_r$ be the set of equivalence classes.

3.2. If $a_* = (a_1,a_2,\dots,a_m) \in \mathfrak{T}_r^m$ and $t \ge a_m$ let $a_*^{!t}$ be the sequence obtained from $0,1,2,\dots,t$ by removing an entry $t-a$ for any a which appears in a_*. Now $a_*^{!t}$ has $m' = t+1-m$ entries whose sum is

$$t(t+1)/2 - tm + r + m(m-1)/2 = r + (t+1-m)(t-m)/2$$

so that $a_*^{!t} \in \mathfrak{T}_r^{m'}$. It is easy to see that the equivalence class of $a_*^{!t} \in \mathfrak{T}_r^{t+1-m}$ is independent of t (if $t \ge a_m$). From the definition we see that if a_* is replaced by an equivalent sequence, the equivalence class of $a_*^{!t}$ (with $t \ge a_m$) is unchanged. Hence $a_* \mapsto a_*^{!t}$ defines a map $\xi \mapsto \xi^\bullet$, $\mathfrak{T}_r \to \mathfrak{T}_r$. Its square is 1.

3.3. For any $p \ge 3$ let $(\mathfrak{T}_r \times \mathfrak{T}_r)_p$ be the subset of $\mathfrak{T}_r \times \mathfrak{T}_r$ consisting of pairs (ξ,ξ') which can be represented by $(a_*,a'_*) \in \mathfrak{T}_r^m \times \mathfrak{T}_r^m$ such that for some $k < k+1$ in $[1,m]$ and some a we have

$$(a_k,a_{k+1}) = (a,a+p), (a'_k,a'_{k+1}) = (a+1,a+p-1), a_s = a'_s \text{ for } s \ne k,k+1.$$

For any $p \ge 4$ let $(\mathfrak{T}_r \times \mathfrak{T}_r)'_p$ be the subset of $\mathfrak{T}_r \times \mathfrak{T}_r$ consisting of pairs (ξ,ξ') which can be represented by $(a_*,a'_*) \in \mathfrak{T}_r^m \times \mathfrak{T}_r^m$ such that for some $k < l$ in $[1,m]$ with $l = k+p-1$ and some a we have

$$(a_k,a_{k+1},\dots,a_{l-1},a_l) = (a,a+2,a+3,\dots,a+p-3,a+p-2,a+p),$$

$$(a'_k,a'_{k+1},\dots,a'_{l-1},a'_l) = (a+1,a+2,a+3,\dots,a+p-3,a+p-2,a+p-1),$$

and $a_s = a'_s$ for $s \notin \{k,k+1,\dots,l-1,l\}$.

From the definitions we see that $(\xi,\xi') \mapsto (\xi'^\bullet,\xi^\bullet)$ is a bijection

$$(\mathfrak{T}_r \times \mathfrak{T}_r)_p \xrightarrow{\sim} (\mathfrak{T}_r \times \mathfrak{T}_r)'_p$$

(if $p \geq 4$) and is a bijection

$$(\mathfrak{T}_r \times \mathfrak{T}_r)_3 \xrightarrow{\sim} (\mathfrak{T}_r \times \mathfrak{T}_r)_3.$$

3.4. When W is of type A_{r-1}, $r \geq 2$, we can identify $\mathcal{S}_W$ with $\mathfrak{T}_r$ in the standard way; the unit representation corresponds to $(1, 2, \dots, r)$; sgn corresponds to (r).

If E, E' in $\mathcal{S}_W$ correspond to ξ, ξ' in $\mathfrak{T}_r$ then E, E' are adjacent in $\mathcal{S}_W$ if and only if (i) or (ii) below holds:

(i) (ξ, ξ') or (ξ', ξ) belongs to $(\mathfrak{T}_r \times \mathfrak{T}_r)_p$ for some $p \geq 3$;
(ii) (ξ, ξ') or (ξ', ξ) belongs to $(\mathfrak{T}_r \times \mathfrak{T}_r)'_p$ for some $p \geq 4$.

If (i) holds then $a_E - a_{E'} = \pm 1$; if (ii) holds then $a_E - a_{E'} = \pm(p-2)$. The involution $E \mapsto E^\circ$ of $\mathcal{S}_W$ corresponds to the involution $\xi \mapsto \xi^\bullet$ of $\mathfrak{T}_r$ and that involution interchanges pairs as in (i) with pairs as in (ii) (if $p \geq 4$) or pairs in (i) with pairs in (i) (if $p = 3$). It follows that 0.3 holds in our case.

4. Examples

4.1. In this section we assume that W is of type $E_8, E_7, E_6, F_4, G_2, H_3, H_4$. In these cases we describe the pairs (E, E') of adjacent elements of $\mathcal{S}_W$ in the form $E \overset{W'}{---} E'$ where $W' = W_{E,E'}$ is determined by 1.4, 1.5 (when $a_E - a_{E'} = \pm 1$ we omit writing W').

4.2. Type E_8

$35_2 --- 8_1 --- 1_0,\ 1_{120} \overset{E_8}{---} 8_{91} \overset{E_7}{---} 35_{74}$

$210_4 --- 112_3 --- 35_2,\ 35_{74} \overset{B_6}{---} 112_{63} \overset{F_4}{---} 210_{52}$

$567_6 --- 560_5 --- 210_4,\ 210_{52} \overset{B_3}{---} 560_{47} --- 567_{46}$

$1400_7 --- 700_6 --- 560_5,\ 560_{47} \overset{B_3}{---} 700_{42} \overset{G_2}{---} 1400_{37}$

$525_{12} \overset{G_2}{---} 1400_7 --- 567_5,\ 567_{46} \overset{G_2}{---} 1400_{37} --- 525_{36}$

$2240_{10} --- 3240_9 --- 1400_8 --- 1400_7,$

$1400_{37} \overset{G_2}{---} 1400_{32} --- 3240_{31} \overset{B_2}{---} 2240_{18}$

$4096_{11} - - - 2268_{10} - - - 3240_9, 3240_{31} - - - 2268_{30} \overset{A_4}{- - -} 4096_{26}$

$2800_{13} \overset{A_2}{- - -} 4096_{11} - - - 2240_{10}, 2240_{28} \overset{A_2}{- - -} 4096_{26} - - - 2800_{25}$

$6075_{14} - - - 2800_{13} - - - 525_{12}, 525_{36} \overset{F_4}{- - -} 2800_{25} \overset{B_2}{- - -} 6075_{22}$

$4536_{13} - - - 4200_{12} - - - 4096_{11}, 4096_{26} \overset{A_2}{- - -} 4200_{24} - - - 4536_{23}$

$4480_{16} - - - 5600_{15} - - - 6075_{14} - - - 4536_{13},$

$4536_{23} - - - 6075_{22} - - - 5600_{21} \overset{G_2}{- - -} 4480_{16}$

$4200_{15} - - - 2835_{14} - - - 4536_{13}, 4536_{23} - - - 2835_{22} - - - 4200_{21}$

$4480_{16} - - - 4200_{15} - - - 6075_{14}, 6075_{22} - - - 4200_{21} \overset{G_2}{- - -} 4480_{16}$

$2100_{20} \overset{B_3}{- - -} 5600_{15}, 5600_{21} - - - 2100_{20}$

4.3. Type E_7

$27_2 - - - 7_1 - - - 1_0, 1_{63} \overset{E_7}{- - -} 7_{46} \overset{D_6}{- - -} 27_{37}$

$120_4 - - - 56_3 - - - 27_2, 27_{37} \overset{B_4}{- - -} 56_{30} \overset{B_3}{- - -} 120_{25}$

$120_4 - - - 21_3 - - - 27_2, 27_{37} - - - 21_{36} \overset{F_4}{- - -} 120_{25}$

$105_6 - - - 189_5 - - - 120_4, 120_{25} \overset{B_2}{- - -} 189_{22} - - - 105_{21}$

$315_7 - - - 168_6 - - - 189_5, 189_{22} - - - 168_{21} \overset{G_2}{- - -} 315_{16}$

$315_7 - - - 210_6 - - - 189_5, 189_{22} - - - 210_{21} \overset{B_3}{- - -} 315_{16}$

$105_{12} - - - 315_7 - - - 105_6, 105_{21} \overset{G_2}{- - -} 315_{16} - - - 105_{15}$

$168_6 - - - 189_7 - - - 210_6, 210_{21} - - - 189_{20} - - - 168_{21}$

$378_9 - - - 405_8 - - - 189_7, 189_{20} \overset{B_3}{- - -} 405_{15} - - - 378_{14}$

$105_{15} \overset{B_3}{- - -} 420_{10} - - - 378_9, 378_{14} - - - 420_{13} - - - 105_{12}$

$512_{11} - - - 210_{10} - - - 378_9, 378_{14} - - - 210_{13} \overset{A_2}{- - -} 512_{11}$

$512_{11} - - - 420_{10}, 420_{13} \overset{A_2}{- - -} 512_{11}$

4.4. Type E_6

$$20_2 - - - 6_1 - - - 1_0,\ 1_{36} \overset{E_6}{- - -} 6_{25} \overset{A_5}{- - -} 20_{20}$$

$$64_4 - - - 30_3 - - - 20_2,\ 20_{20} \overset{B_3}{- - -} 30_{15} \overset{A_2}{- - -} 64_{13}$$

$$80_7 - - - 81_6 - - - 60_5 - - - 64_4,\ 64_{13} \overset{A_2}{- - -} 60_{11} - - - 81_{10} \overset{B_2}{- - -} 80_7$$

$$80_7 - - - 24_6 \overset{A_2}{- - -} 64_4,\ 64_{13} - - - 24_{12} \overset{G_2}{- - -} 80_7$$

4.5. Type F_4

$$9_2 - - - 4_1 - - - 1_0,\ 1_{24} \overset{F_4}{- - -} 4_{13} \overset{B_2}{- - -} 9_{10}$$

$$12_4 - - - 8_3 - - - 9_2,\ 9_{10} - - - 8_9 \overset{G_2}{- - -} 12_4$$

$$12_4 - - - 8'_3 - - - 9_2,\ 9_{10} - - - 8'_9 \overset{G_2}{- - -} 12_4$$

4.6. Type G_2

$$2_1 - - - 1_0\ 1_6 \overset{G_2}{- - -} 2_1$$

4.7. Type H_4

Let H_2 (resp. Δ) be a dihedral group of order 10 (resp. 20).

$$9_2 - - - 4_1 - - - 1_0,\ 1_{60} \overset{H_4}{- - -} 4_{31} \overset{\Delta}{- - -} 9_{22}$$

$$25_4 - - - 16_3 - - - 9_2,\ 9_{22} \overset{H_2}{- - -} 16_{18} \overset{A_2}{- - -} 25_{16}$$

$$24_6 - - - 36_5 - - - 25_4,\ 25_{16} - - - 36_{15} \overset{\Delta}{- - -} 24_6$$

4.8. Type H_3

$$5_2 - - - 3_1 - - - 1_0,\ 1_{15} \overset{H_3}{- - -} 3_6 - - - 5_5$$

$$4_3 - - - 5_2,\ 5_5 \overset{A_2}{- - -} 4_3$$

5. Truncated Induction and Adjacency

5.1. In this section we assume that W is a Weyl group. Let W_J be a standard parabolic subgroup of W. In [11] a function $j_{W_J}^W$ ("truncated induction") from a certain subset of $\mathrm{Irr}(W_J)$ to $\mathrm{Irr}(W)$ was defined. From [9] we see that $\mathcal{S}_{W_J}$ is contained in the subset on which $j_{W_J}^W$ is defined and that the image of $\mathcal{S}_{W_J}$ under $j_{W_J}^W$ is contained in $\mathcal{S}_W$. Thus we have a well defined map $j_{W_J}^W : \mathcal{S}_{W_J} \to \mathcal{S}_W$. This map preserves the value of the a-function.

5.2. The following result appears in [9, no.6].

(a) *Assume that W is irreducible, $\neq 1$. Let $E \in \mathcal{S}_W$. Then either E or E° is of the form $j_{W_J}^W(E_1)$ where $W_J \subset W$ is as in* 5.1 *with $W_J \neq W$ and $E_1 \in \mathcal{S}_{W_J}$.*

Thus $\mathcal{S}_W$ admits an inductive definition based on the involution $E \mapsto E^\circ$ and on truncated induction from proper parabolic subgroups. In [4] a definition of $\le$ on $\mathcal{S}_W$ in the same spirit (but with the full induction playing a role) was given.

5.3. We want to show that the adjacency relation on $\mathcal{S}_W$ can be described in terms of $E \mapsto E^\circ$ and truncated induction.

We define a subset $\check{\mathcal{S}}_W$ by induction on the rank of W. Writing $W = \prod_i W_i$ where W_i are irreducible Weyl groups we have that $(E, E') \in \check{\mathcal{S}}_W$ for $E = \boxtimes_i E_i, E' = \boxtimes_i E'_i$ with E_i, E'_i in $\mathcal{S}_{W_i}$ for all i if and only if $(E_i, E'_i) \in \check{\mathcal{S}}_{W_i}$ for one i and $E_i = E'_i$ for all other i. If $W = \{1\}$, we have $\check{\mathcal{S}}_W = \emptyset$. If W is of type A_1 then $\check{\mathcal{S}}_W$ consists of $(\mathrm{sgn}, 1)$. Assume now that W is irreducible of rank ≥ 2.

We say that $(E, E') \in \mathcal{S}_W \times \mathcal{S}_W$ is *induced* if there exists $W_J \subset W$ as in 5.1 with $W_J \neq W$ and $(E_1, E'_1) \in \check{\mathcal{S}}_{W_J}$ such that $a_{E_1} - a_{E'_1} = 1$ (a-function relative to W_J) and $E = j_{W_J}^W(E_1)$, $E' = j_{W_J}^W(E'_1)$.

Then $\check{\mathcal{S}}_W$ consists of all $(E, E') \in \mathcal{S}_W \times \mathcal{S}_W$ such that at least one of (i), (ii), (iii), (iv) below are satisfied.

(i) (E, E') is induced;
(ii) (E'°, E°) is induced;
(iii) W is of type B_3 and (E, E') satisfies $a_E = 3, a_{E'} = 2$ (so that $(E, E') = (E'^\circ, E^\circ)$);
(iv) W is of type E_7 and $(E, E') = (512_{11}, 210_{10})$ (notation of 4.3).

This completes the inductive definition of $\check{\mathcal{S}}_W$.

We state the following result.

Theorem 5.4 *We have* $\check{\mathcal{S}}_W = \hat{\mathcal{S}}_W$.

5.5. We show that

(a) $\check{\mathcal{S}}_W \subset \hat{\mathcal{S}}_W$

by induction on the rank of W. We can assume that W is irreducible of rank ≥ 2. Let $(E, E') \in \check{\mathcal{S}}_W$.

Assume first that (E, E') is induced (from $(E_1, E'_1) \in \check{\mathcal{S}}_{W_J}$). Let C, C' in $\mathcal{U}_G^{sp}$ correspond to E, E' as in 1.1; similarly let C_1, C'_1 in $\mathcal{U}_{G_J}^{sp}$ correspond to E_1, E'_1 where G_J is a Levi subgroup of a parabolic subgroup of G. Since $a_{E_1} - a_{E'_1} = 1$ we see that $\dim(C'_1) - \dim(C_1) = 2$; moreover C_1 is contained in the closure of C'_1. By [11], C and C' are induced from C_1 and C'_1, so that C is contained in the closure of C' and $\dim(C') - \dim(C) = 2$; it follows that $E \lneqq E'$ and there is no $E'' \in \mathcal{S}_W$ such that $E \lneqq E'', E'' \lneqq E'$ and $(E, E') \in \hat{\mathcal{S}}_W$.

Next we assume that (E'°, E°) is induced. By the earlier argument we see that $(E'^{\circ}, E^{\circ}) \in \hat{\mathcal{S}}_W$. This implies that $(E, E') \in \hat{\mathcal{S}}_W$, by 0.2(c). If (E, E') is as in 5.3(iv) then $(E, E') \in \hat{\mathcal{S}}_W$ follows from 4.3. If (E, E') is as in 5.3(iii) then $(E, E') \in \hat{\mathcal{S}}_W$ can be verified directly. This proves (a).

5.6. Let $\epsilon \in \{0, 1\}$. In this subsection we verify some properties of a pair $(a_*, a'_*) \in \mathfrak{S}_r^M \times \mathfrak{S}_r^M$ with $M \in \epsilon + 2\mathbf{N}$ (notation in §2) which will be used in the proof of the converse to 5.5(a) in type B_r, D_r.

We say that (a_*, a'_*) is induced if there exists $k \in \{1, 2, \ldots, r\}$ such that

$a_* = (a_1, a_2, \ldots, a_M)$, $a'_* = (a'_1, a'_2, \ldots, a'_M)$,

$(a_1, a_2, \ldots, a_{M-k}, a_{M-k+1} - 1, a_{M-k+2} - 1, \ldots, a_M - 1) \in \mathfrak{S}_{r-k}^M$,

$(a'_1, a'_2, \ldots, a'_{M-k}, a'_{M-k+1} - 1, a'_{M-k+2} - 1, \ldots, a'_M - 1) \in \mathfrak{S}_{r-k}^M$.

If (a_*, a'_*) represents an element of $({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_p$ with $p \geq 3$ then (a_*, a'_*) is clearly induced. If (a_*, a'_*) represents an element of $({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)'_p$ with $p \geq 3$ and $t \gg 0$ then, by 2.3, $({a'_*}^{!t}, {a_*}^{!t})$ represents an element of $({}^{\epsilon}\mathfrak{S}_r \times {}^{\epsilon}\mathfrak{S}_r)_p$ with $p \geq 3$ hence is induced. We show:

(a) *Assume that* (a_*, a'_*) *represent an element of* $({}^\epsilon\mathfrak{S}_r \times {}^\epsilon\mathfrak{S}_r)_p$ *with* $p = 2$. *Then*

- (i) (a_*, a'_*) *is induced or*
- (ii) $({a'_*}^{!t}, {a_*}^{!t})$ *(with* $t \gg 0$*) is induced or*
- (iii) (a_*, a'_*) *is equivalent to one of* $(0,2),(1,1))$ *or* $((0,0,2),(0,0,2))$ *or* $((0,2,2),(1,1,2))$ *or* $((0,0,2,2),(0,1,1,2))$.

We have $a_* = (\dots, a, a+2, \dots), a'_* = (\dots, a+1, a+1, \dots)$ with the same entries in the same position marked by $\dots$. We assume that (a_*, a'_*) is not as in (i), (ii).

If $a_* = (\dots, a, a+2, c_1, \dots), a'_* = (\dots, a+1, a+1, c_1, \dots)$ then $c_1 = a+2$ (if $c_1 \geq a+3$ then (a_*, a'_*) would be induced).

If $a_* = (\dots, a, a+2, c_1, c_2, \dots), a'_* = (\dots, a+1, a+1, c_1, c_2, \dots)$ then $c_2 = a+3$ (if $c_2 \geq a+4$ then (a_*, a'_*) would be induced).

If $a_* = (\dots, a, a+2, c_1, c_2, c_3, \dots), a'_* = (\dots, a+1, a+1, c_1, c_2, c_3, \dots)$ then $c_3 = a+3$ (if $c_3 \geq a+4$ then (a_*, a'_*) would be induced).

If $a_* = (\dots, a, a+2, c_1, c_2, c_3, c_4, \dots), a'_* = (\dots, a+1, a+1, c_1, c_2, c_3, c_4, \dots)$ then $c_4 = a+4$ (if $c_4 \geq a+5$ then (a_*, a'_*) would be induced).

Continuing in this way we see that

$$a_* = (\dots, a, a+2, ***), a'_* = (\dots, a+1, a+1, ***)$$

where $***$ stands for the sequence

$$(a+2, a+3, a+3, a+4, a+4, a+5, a+5, \dots)$$

with $k \geq 0$ terms.

If both a_*, a'_* end with $a+s, a+s$ $(s \geq 3)$ then $({a'_*}^{!(a+s)}, a_*^{!(a+s)})$ is of the form $((b, \dots), (b', \dots))$ with $b > 0, b' > 0$ hence is induced (contradicting our assumption).

If both a_*, a'_* end with $a+s-1, a+s-1, a+s$ $(s \geq 4)$ then $({a'_*}^{!(a+s)}, a_*^{!(a+s)})$ is of the form $((0, b, \dots), (0, b', \dots))$ with $b \geq 2, b' \geq 2$ hence is induced (contradicting our assumption).

If both a_*, a'_* end with $a+2, a+3$ then $({a'_*}^{!(a+3)}, a_*^{!(a+3)})$ is of the form $((0, b, c, \dots), (0, b', c', \dots))$ with $c > b \geq 1, b' \geq 2$, hence is induced (contradicting our assumption).

Thus we have

(b) $(a_*, a'_*) = (\dots, a, a+2), (\dots, a+1, a+1)$ or

(c) $(a_*, a'_*) = (\dots, a, a+2, a+2), (\dots, a+1, a+1, a+2)$.

If (b) holds then $({a'_*}^{!(a+2)}, a_*^{!(a+2)})$ is of the form $((0,0,2,\dots),(0,1,1,\dots))$. The first part of the argument applies to $((0,0,2,\dots),(0,1,1,\dots))$ and shows that it is of the form $((0,0,2),(0,1,1))$ or of the form $((0,0,2,2),(0,1,1,2))$.

If (c) holds then $({a'_*}^{!(a+2)}, a_*^{!(a+2)})$ is of the form $((0,2,\dots),(1,1,\dots))$. The first part of the argument applies to $((0,2,\dots),(1,1,\dots))$ and shows that it is of the form $((0,2),(1,1))$ or of the form $((0,2,2),(1,1,2))$. It follows that (a_*, a'_*) is as in (iii). This proves (a).

5.7. In this subsection we verify some properties of a pair $(a_*, a'_*) \in \mathfrak{T}_r^m \times \mathfrak{T}_r^m$ with $m \in \mathbf{N}$ (notation in §3) which will be used in the proof of the converse to 5.5(a) in type A_{r-1}.

We say that (a_*, a'_*) is induced if there exists $k \in \{1, 2, \dots, r\}$ such that

$a_* = (a_1, a_2, \dots, a_m)$, $a'_* = (a'_1, a'_2, \dots, a'_m)$,

$(a_1, a_2, \dots, a_{m-k}, a_{m-k+1} - 1, a_{m-k+2} - 1, \dots, a_m - 1) \in \mathfrak{T}_{r-k}^m$,

$(a'_1, a'_2, \dots, a'_{m-k}, a'_{m-k+1} - 1, a'_{m-k+2} - 1, \dots, a'_m - 1) \in \mathfrak{T}_{r-k}^m$.

If (a_*, a'_*) represents an element of $(\mathfrak{T}_r \times \mathfrak{T}_r)_p$ with $p \geq 4$ then (a_*, a'_*) is clearly induced. If (a_*, a'_*) represents an element of $(\mathfrak{T}_r \times \mathfrak{T}_r)'_p$ with $p \geq 4$ and $t \gg 0$ then, by 3.3, $({a'_*}^{!t}, {a_*}^{!t})$ (with $t \gg 0$) represents an element of $(\mathfrak{T}_r \times \mathfrak{T}_r)_p$ with $p \geq 4$ hence is induced. We show:

(a) *Assume that (a_*, a'_*) represent an element of $(\mathfrak{T}_r \times \mathfrak{T}_r)_p$ with $p = 3$. Then*

- (i) *(a_*, a'_*) is induced or*
- (ii) *$({a'_*}^{!t}, {a_*}^{!t})$ (with $t \gg 0$) is induced or*
- (iii) *(a_*, a'_*) is equivalent to $((0,3),(1,2))$.*

We have $a_* = (\dots, a, a+3, \dots), a'_* = (\dots, a+1, a+2, \dots)$ with the same entries in the same position marked by $\dots$. We assume that (a_*, a'_*) is not as in (i), (ii).

If $a_* = (\dots, a, a+3, c_1, \dots), a'_* = (\dots, a+1, a+2, c_1, \dots)$ then $c_1 = a+4$ (if $c_1 \geq a+5$ then (a_*, a'_*) would be induced).

If $a_* = (\dots, a, a+3, c_1, c_2, \dots), a'_* = (\dots, a+1, a+2, c_1, c_2, \dots)$ then $c_2 = a+5$ (if $c_2 \geq a+6$ then (a_*, a'_*) would be induced).

Continuing in this way we see that

$$a_* = (\dots, a, a+3, ***),\ a'_* = (\dots, a+1, a+2, ***)$$

where $***$ stands for the sequence $(a+4, a+5, a+6, \dots)$ with $k \geq 0$ terms. If both a_*, a'_* end with $a+s$ $(s \geq 4)$ then $(a'^{!(a+s)}_*, a^{!(a+s)}_*)$ is of the form $((b, \dots), (b', \dots))$ with $b > 0, b' > 0$ hence is induced (contradicting our assumption). Thus we have

(b) $(a_*, a'_*) = ((\dots, a, a+3), (\dots, a+1, a+2))$.

It follows that $(a'^{!(a+3)}_*, a^{!(a+3)}_*)$ is of the form $((0, 3, \dots), (1, 2, \dots))$. The first part of the argument applies to $((0, 3, \dots), (1, 2, \dots))$ and shows that it is of the form $((0, 3), (1, 2))$. It follows that (a_*, a'_*) is as in (iii). This proves (a).

5.8. We show that

(a) $\hat{\mathcal{S}}_W \subset \check{\mathcal{S}}_W$.

We argue by induction on the rank of W. We can assume that W is irreducible. If W has rank ≤ 1 the result is obvious. Assume now that W has rank ≥ 2. If W is of type A_n, $n \geq 2$ then (a) follows from 5.7(a). If W is of type B_r, $r \geq 2$ or D_r, $r \geq 4$ then (a) follows from 5.6(a). (Note that the only pair (a_*, a'_*) which can appear in 5.6(a) is $((0, 2, 2), (1, 1, 2))$ which corresponds to B_3; the other pairs in 5.6(a) appear in lower rank.) If W is of exceptional type then (a) follows from the tables in §4 and the explicit knowledge of truncated induction. This proves (a). Theorem 5.4 is proved.

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