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A PRIORI ESTIMATE AND NUMERICAL STUDY OF SOLUTION FOR A PARABOLIC EQUATION WITH NONLINEAR INTEGRAL CONDITION

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Abstract

In this paper we present a class of parabolic equation with nonlinear nonlocal conditions of second type where we show two part of this study the theoretical part we prove the existence and uniqueness of the solution by energy inequality method. Then the numerical part where we study the consistence and stability of solution.

1. Introduction

The most famous problems are Heat distribution problems which are considered among the ancient problems studied by many researchers, where the study was done on different domain types. When we consider one-dimensional heat conduction problems of a nonhomogeneous we can solve it easily with condition Neumann or Dirichlet or a mix between them like.

$$\begin{cases} \frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = f(x,t), & x,t \in Q \\ u(x,0) = \varphi(x), & 0 \leq x \leq 1 \\ u_x(0,t) = 0, & 0 \leq t \leq T \\ u_x(1,t) = 0. & 0 \leq t \leq T \end{cases}.$$

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However, many researchers have asked how to find a solution to the problem of the distribution of heat in complex domain and with more complex conditions, as an example of some mathematicians taking the integral conditions of their first and second types, which can modeled a lot of problems in different domains like biology, physics, mechanics and technology...

Those conditions are encountered in various applications such as population dynamics, blood-flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral conditions originating from various engineering disciplines are of growing interest. That is a large number of physical phenomena and many problems in modern physics and technology can be described in terms of nonlocal problems, such as problems in partial differential equations with integral conditions. A large number of problems in modern physics and technology are stated using nonlocal conditions for partial differential equations, which are described using integral conditions [3], [4] and [5]. It is however of the first type

$$\int_\Omega u(x,t) = E(t), \quad \int_\Omega k(x,t)u(x,t)dx = 0,$$

where $t \in (0,T)$, $\Omega \subset \mathbb{R}^n$ and k is a given function. Or second type, where the Dirichlet or Neumann condition modelling by integral condition, for example

$$u(x,t)|_{\partial\Omega} = \int k(x,t)u(x,t)dx,$$

can be used when it is impossible to directly measure the sought quantity on the border, its total value or its average is known. To motivate this, we generalized the integral conditions of the second kind to more general ones by making them nonlinear, and this increased the difficulty of the study, especially since the field of study of heat diffusion became more complex. And this is what we focused on in this article, where in the second part we studied the uniqueness and the existence theorical by the functional method and then we search the numerical solution by applying the compact finite difference technique.

2. Formulation and Treatment of the Problem

2.1. Position of the problem

In the rectangular domain $Q = \Omega \times (0, T)$, with $\Omega = (0, 1)$ and $T < \infty$, we consider the following problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial^2 u}{\partial x^2} = f(x,t), & x,t \in Q \\ u(x,0) = \varphi(x), & 0 \leq x \leq 1 \\ u_x(0,t) = \displaystyle\int_0^1 k_0(x,t) g(u(x,t))dx, & 0 \leq t \leq T \\ u_x(1,t) = \displaystyle\int_0^1 k_1(x,t) h(u(x,t))dx & 0 \leq t \leq T \end{cases}. \tag{P1}$$

where f, ϕ, K_0, K_1, g and h are known functions and a is a positive constant, and the function g and h verify the following inequality

$$\|g(x,t,u)\|_{L^2(Q)} \leqslant C_0 \|u\|_{L^2(Q)}, \text{ and } \|h(x,t,u)\|_{L^2(Q)} \leqslant C_1 \|u\|_{L^2(Q)}, \tag{1}$$

C_0 and C_1 are positive constants. We shall assume that the function φ satisfies a compatibility of boundary conditions, i.e.,

$$\begin{aligned} \phi_x(0) &= \int_0^1 K_0(x,0)\, g(\phi(x))\, dx, \\ \phi_x(1) &= \int_0^1 K_1(x,0)\, h(\phi(x))dx. \end{aligned}$$

2.2. A priori estimate (uniqueness of solution)

$$Lu = \mathcal{F}. \tag{2}$$

Where $L = (\mathcal{L}, \ell)$, with domain of definition E consisting of functions $u \in L^2\left(0, T, L^2(\Omega)\right) = L^2(Q)$ such that $u_x \in L^2(Q)$ and u satisfies the nonlocal conditions ; the operator L is considered from E to F where E is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\|u\|_E^2 = \|u\|_{L^2(Q)}^2 + \|u_x\|_{L^2(Q)}^2,$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 + \|\varphi\|_{L^2(\Omega)}^2 ,$$

is finite.

Theorem 1. *For any function $u \in E$, we have the inequality*

$$\|u\|_E \leq c \|Lu\|_F , \tag{3}$$

where c is a positive constant independent of u.

Proof. Assume that a solution of the problem (P1) exists. We multiply the equation of (P1) by u and integrating over Q^τ, where $Q^\tau = \Omega \times (0, \tau)$, we get

$$\int_{Q^\tau} \mathcal{L}u \cdot u \; dxdt \; = \; \int_{Q^\tau} \; f(x,t) \cdot u \; dxdt. \tag{4}$$

Integrating by parts each term of the left-hand side of (4) over Q^τ, $0 < \tau < T$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 u(x,\tau)^2 \; dx + a\int_0^\tau u_x^2 \; dt \\ &= a\int_0^\tau u_x(1,t)\, u(1,t) \; dt - a\int_0^\tau u_x(0,t)\, u(0,t) \; dt + \frac{1}{2}\int_0^1 \varphi^2 \; dx \\ &\quad + \int_{Q^\tau} \; f \cdot u \; dxdt, \end{aligned} \tag{5}$$

By integrating each term over $(0, T)$ and using the Cauchy Schwartz inequality, finally we get :

$$\begin{aligned} \|u(x,\tau)\|_{(0,T,L^2(\Omega))} + a\|u_x\|_{L^2(Q^\tau)}^2 \; \leq \; & \|f\|_{L^2(Q^\tau)}^2 + \|\varphi\|_{L^2(\Omega)}^2 \\ & + (ac_1^2k_1^2 + ac_0^2k_0^2 + a + 1)\, \|u\|_{L^2(Q^\tau)}^2 . \end{aligned}$$

By putting:

$$C' = ac_1^2k_1^2 + ac_0^2k_0^2 + a + 1,$$

and

$$C = \frac{1}{\min\{1, a\}} \exp(C'T)$$

so, we get:

$$\|u\|^2_{C(0,T,L^2(\Omega))} + \|u_x\|^2_{L^2(Q^\tau)} \leq c^2 \left(\|f\|^2_{L^2(Q^\tau)} + \|\varphi\|^2_{L^2(\Omega)}\right). \tag{6}$$

Finally, we obtain the desired inequality, where $c = \sqrt{\frac{\exp(mT)}{\min\{1,a\}}}$. □

Corollary 1. *The solution is unique, if for any function $u \in D(L)$, we have the following estimate :*

$$\|u\|_E \leq c\,\|\mathcal{F}\|_F \tag{7}$$

Proof. Let u_1 and u_2 be two solutions to the problem (P1)

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

and since L is linear we then get :

$$L(u_1 - u_2) = 0,$$

which gives :

$$u_1 = u_2.$$ □

Corollary 2. *the solution of the problem* (P1) *if it exists, it depends continuousely on $\mathcal{F} \in F$.*

2.3. Existence of solution

This section is consecrated to the proof of the existence of the solution of the problem (P1).

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial^2 u}{\partial x^2} = f(x,t), & x,t \in Q \\ u(x,0) = \varphi(x), & 0 \leq x \leq 1 \\ u_x(0,t) = \displaystyle\int_0^1 k_0(x,t)g(u(x,t))dx, & 0 \leq t \leq T \\ u_x(1,t) = \displaystyle\int_0^1 k_1(x,t)h(u(x,t))dx & 0 \leq t \leq T \end{cases}. \tag{P1}$$

Let us consider the following auxiliary problem with homogeneous equation

$$\mathcal{L}w = \frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial x^2} = 0,$$

with initial data

$$\ell w = w(x,0) = \varphi(x),$$

and the second kind nonlinear integral conditions

$$\begin{aligned} w_x(0,t) &= \int_0^1 K_0(x,t)\, g(w(x,t) + y(x,t))dx, \\ w_x(1,t) &= \int_0^1 K_1(x,t)\, h(w(x,t) + y(x,t))dx. \end{aligned}$$

Where the functions g^* and h^* verify :

$$\|g^*(w)\|_{L^2(Q)} \leqslant b_1 \|w\|_{L^2(Q)} + b_2, \text{ and } \|h^*(w)\|_{L^2(Q)} \leqslant b_3 \|w\|_{L^2(Q)} + b_4,$$
$$b_1, b_2, b_3 \text{ and } b_4 \text{ are positive constants.}$$

Then the auxiliary problem with homogeneous equation becomes :

$$\begin{cases} \mathcal{L}w = \dfrac{\partial w}{\partial t} - a\dfrac{\partial^2 w}{\partial x^2} = 0, & x,t \in Q \\ \ell w = w(x,0) = \varphi(x), & x \in (0,1) \\ w_x(0,t) = \displaystyle\int_0^1 K_0(x,t)\, g^*(w(x,t))\, dx, & t \in (0,T) \\ w_x(1,t) = \displaystyle\int_0^1 K_1(x,t)\, h^*(w(x,t))\, dx. & t \in (0,T) \end{cases} . \tag{P2}$$

If u is a solution of problem (P1) and w is a solution of problem (P2), then $y = u - w$ satisfies the following problem :

$$\begin{cases} \mathcal{L}y = \dfrac{\partial y}{\partial t} - a\dfrac{\partial^2 y}{\partial x^2} = f(x,t), & x,t \in Q \\ \ell y = y(x,0) = 0, & x \in (0,1) \\ y_x(0,t) = 0, & t \in (0,T) \\ y_x(1,t) = 0, & t \in (0,T) \end{cases} . \tag{P3}$$

To show the existence of solutions of the problem (P2), it is enough to transform the problem to the nonlinear ordinary differential equation.

For that we integrate the equation of (P2) over Ω then, we obtain

$$\int_0^1 w_t dx - a\int_0^1 w_{xx} dx = 0,\ \forall x \in \Omega;$$

so

$$\int_0^1 \left[w_t - a(K_1(x,t)\, h^*(w(x,t))\, dx + K_0(x,t)\, g^*(w(x,t)))\right] dx = 0,$$

then, we obtain

$$\int_0^1 (w_t - F(t, w(x,t)))dx = 0, \tag{8}$$

where

$$aK_1(x,t)\, h^*(w(x,t)) - aK_0(x,t)g^*(w(x,t)) = F(t, w(x,t)).$$

So, it is clear that there exists a function ψ verify that

$$w_t - F(t, w(x,t)) = \psi(x,t),\ \text{where}\ \int_0^1 \psi(x,t)\, dx = 0.$$

Thus, we have

$$w_t = G(t, w(t)),$$

where

$$G(t, w(t)) = F(t, w(x,t)) + \psi(x,t).$$

G is a Carathodory mapping, then by applying the theorem of existence and uniqueness we get that $w \subset W^{1,1}$ and by applying the Nemytskii mappings in Lebesgue spaces we get that w_t in $L^2[0,T]$

According to these results, we deduce that the problem (P2) admits a unique solution.

Therefore it remains to solve and prove that the problem (P3) has a unique strong solution. Let the following auxiliary problem with homogeneous conditions

$$\begin{cases} \mathcal{L}y = \dfrac{\partial y}{\partial t} - a\dfrac{\partial^2 y}{\partial x^2} = f\,(x,t)\,, & x,t \in Q \\ \ell y = y(x,0) = 0, & x \in (0,1) \\ y_x\,(0,t) = 0, & t \in (0,T) \\ y_x\,(1,t) = 0, & t \in (0,T) \end{cases} . \tag{P3}$$

Theorem 2. *For any function $y \in E$, we have the inequality*

$$\|y\|_E \leq c\,\|Ly\|_F\,, \tag{9}$$

where c is a positive constant independent of y.

Proof. Assume that a solution of the problem (P3) exists. We multiply the equation of (P3) by y and integrating over Q^τ, where $Q^\tau = \Omega \times (0,\tau)$, we get

$$\int_{Q^\tau} y_t \cdot y - a\int_{Q^\tau} y_{xx} \cdot y = \int_{Q^\tau} f(x,y) \cdot y, \tag{10}$$

Integrating by parts each term of the left-hand side of (10) over Q^τ, $0 < \tau < T$, by using lemma 1 of Gronwall, we obtain

$$\|y\|^2_{L^\infty(0,T;\,L^2(\Omega))} + \|y_x\|^2_{L^2(Q)} \leq c^2\,\|f\|^2_{L^2(Q)}\,.$$

Finally, we obtain the desired inequality, where $c = \sqrt{\frac{\exp(T)}{\min\{1,2a\}}}$. □

Corollary 3. *If for any function $u \in D\,(L)$, we have the following estimate:*

$$\|u\|_E \leq C\,\|\mathcal{F}\|_F\,,$$

then the solution of the problem $(P3)$ if it exists, it is unique.

Proof. Let u_1 and u_2 be two solutions of the problem $(P3)$:

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

and since L is linear according to(9)

$$\|u_1 - u_2\|^2_E \leq c\,\|0\|^2_F = 0,$$

which gives

$$u_1 = u_2.$$

□

2.3.1. Study of the existence of the solution of problem P3

Proposition 3. *The operator L of E in F has a closure.*

Proof. Let $\{y_n\} \in D(L)$ be a sequence, such as :

$$y_n \longrightarrow 0 \text{ in } E,$$

and

$$Ly_n \longrightarrow (f;\varphi) \text{ in } F, \tag{11}$$

it must be demonstrated that

$$f \equiv 0 \text{ and } \varphi \equiv 0.$$

The convergence of y_n towards 0 in E implies :

$$y_n \longrightarrow 0 \text{ in } D^{'}(Q). \tag{12}$$

According to the continuity of the derivation of $D^{'}(Q)$ in $D^{'}(Q)$, the relation (12) involved :

$$\mathcal{L}y_n \longrightarrow 0 \text{ in } D^{'}(Q), \tag{13}$$

Otherwise, the convergence of $\mathcal{L}y_n$ towards f in $L^2(Q)$ generates :

$$\mathcal{L}y_n \longrightarrow f \text{ in } D^{'}(Q). \tag{14}$$

By virtue of the uniqueness of the limit in $D^{'}(Q)$, we calculate from (13) and (14) that :

$$f = 0.$$

then, it is generated from (11) that :

$$\ell y_n \longrightarrow \varphi \text{ in } L^2(\Omega).$$

on the other hand :

$$\begin{aligned}\|y_n\|_E^2 &= \|y_n\|^2_{C(0,T,\ L^2(\Omega))} + \|\partial_x y_n\|^2_{L^2(Q)} \\ \|y_n\|_E^2 &\geq \|y_n(x,0)\|^2_{L^2(\Omega)}.\end{aligned}$$

by crossing the limit, we find :

$$\lim_{n\to+\infty} \|y_n\|_E^2 \geq \|\varphi(x)\|^2_{L^2(\Omega)},$$

Since $u_n \longrightarrow 0$ in E then $\|y_n\|_E^2 \longrightarrow 0$ in E, we find :

$$\|\varphi(x)\|^2_{L^2(\Omega)} \leq 0,$$

from where $\varphi = 0$. □

Definition 4. The solution of the equation

$$\overline{L}u = \mathcal{F},$$

is said to be a strong generalized solution of the problem (P3).

• The theorem (1) is valid for a strong generalized solution, i.e., we have the inequality :

$$\|u\|_E \leq K \left\|\overline{L}u\right\|_F \ \ \forall u \in D(\overline{L}). \tag{15}$$

consequently this last inequality entails the following corollaries :

Corollary 4. *The solution of the problem* (P3) *if it exists, it is unique and depends continuously on* $\mathcal{F} \in F$

Corollary 5. *The set of values* $R(\overline{L})$ *of the operator* $\overline{L}$ *is equal to* $\overline{R(L)}$.

Proof. We can proof this corollary easy □

Theorem 5. *The solution of* (P3) *is exist.*

We must prove that $R(L)$ is dense in F for everything $y \in E$ and for all $\mathcal{F} = (f, \varphi) \in F$.

Let $\overline{L}$ the closure of L, and $D(\overline{L})$ the definition domain of $\overline{L}$

In order to prove the existence of the solution enough to proof Ly is surjective.

According to the density of $\overline{L}$ we have $\overline{R(L)} = F$. Then, we obtain $R(L)^{\perp} = \{0\}_F$.

We have

$$\begin{aligned} R(L)^{\perp} &= \{w \in F, \langle w, \mathcal{F}\rangle_F = 0, \forall \mathcal{F} \in R(L)\} \\ &= \Big\{(w, w_0) \in L^2(Q), \langle w, f\rangle_{L^2(Q)} + \langle w_0, \varphi\rangle_{L^2(Q)} = 0, \forall f \in L^2(Q), \\ &\qquad \forall \varphi \in L^2(Q)\Big\} \end{aligned}$$

and

$$D_0(L) = \{y \in E,\ y(x,0) = 0\}.$$

Then, we get

$$w_0 = 0.$$

It remains to demonstrate that $w = 0$.

We have

$$\langle w, Ly\rangle_{L^2(Q)} = \int_0^1 \int_0^T wLy = 0.$$

We pose $w = y$, we obtain

$$\int_0^1 \int_0^T y(y_t - a\Delta y) = \int_0^1 \int_0^T y \cdot y_t - a \int_0^1 \int_0^T y \cdot y_{xx} = 0.$$

Then

$$\int_0^1 \int_0^T y \cdot y_t = a \int_0^1 \int_0^T y \cdot y_{xx}.$$

By integrating par parts, we get

$$\frac{1}{2} \int_0^1 y^2(x,T) = -a \int_0^T y_x^2 \leq 0.$$

Finaly, we get

$$y = 0 \Longrightarrow w = 0.$$

3. The Numerical Study of the Main Problem

For the numerical solution of the considered problem (P1) we apply the compact finite difference technique. First, we simplify the presentation of the interval $[0,1]$ in M by taking $\Delta x = \frac{1}{M}$ and the interval $[0,T]$ in N by taking $\Delta t = \frac{1}{N}$. By u_i^n we denote the approximation to u at the i^{th} grid-point and n^{th} time step, the grid point (x_i, t_n) are given by : $x_i = i\Delta x$, $i = 0, 1, \ldots, M$. $t_n = n\Delta t$, $n = 0, \ldots, N$. $u_i^n = u(\ i\Delta x, n\Delta t)$. The notations $u_i^n, f_i^n, g(u_i^n)\ , h(u_i^n), \left(\frac{\partial g(u)}{\partial u}\right)_i^n$ and $\left(\frac{\partial h(u)}{\partial u}\right)_i^n$ are used for approximations of $u(x_i,t_n), f(x_i,t_n), g(u(x_i,t_n)), h(u(x_i,t_n)), \frac{\partial g(u(x_i,t_n))}{\partial u}$ and $\frac{\partial h(u(x_i,t_n))}{\partial u}$ respectively. By using the finite difference scheme and by multiplying the operator $\left(1 + \frac{(\Delta x)^2}{12}\delta_x^2\right)$, we obtain :

$$\delta_t u_i^n + \frac{(\Delta x)^2}{12}\delta_x^2(\delta_t u_i^n) - a\delta_x^2 u_i^n = f_i^n + \frac{(\Delta x)^2}{12}\delta_x^2 f_i^n$$

We put $r = a\frac{\Delta t}{(\Delta x)^2}$ the scheme is written as follows :

$$\begin{aligned} &\left(\frac{1}{12} - r\right)u_{i+1}^n + \left(\frac{5}{6} + 2r\right)u_i^n + \left(\frac{1}{12} - r\right)u_{i-1}^n \\ &= \frac{1}{12}u_{i+1}^{n-1} + \frac{5}{6}u_i^{n-1} + \frac{1}{12}u_{i-1}^{n-1} + \frac{\Delta t}{12}(f_{i+1}^n + 10f_i^n + f_{i-1}^n) \qquad (16) \end{aligned}$$

We still have to determine two unknowns u_0^n et u_M^n, for this we approximate the integrals conditions numerically by the composite Simpson rule (We have chosen this approximation because it is of the same order of precision which requires the number of sub-intervals to be even $M = 2i$):

$$\begin{aligned} u_x(0,t_n) &= \int_0^1 k_0(x,t_n)g(u(x,t_n))dx \\ &= \frac{\Delta x}{3}\begin{pmatrix} k_0(x_0,t_n)g(u_0^n) + 4\sum_{i=1}^{\frac{M}{2}} k_0(x_{2i-1},t_n)g(u_{2i-1}^n) \\ +2\sum_{i=1}^{\frac{M}{2}-1} k_0(x_{2i},t_n)g(u_{2i}^n) + k_0(x_M,t_n)g(u_M^n) \end{pmatrix} \end{aligned}$$

Then :

$$3u_x(0,t_n) - \Delta x k_0(x_0,t_n)g(u_0^n) - \Delta x k_0(x_M,t_n)g(u_M^n)$$

$$= \Delta x\left(4\sum_{i=1}^{\frac{M}{2}} k_0(x_{2i-1},t_n)g(u_{2i-1}^n) + 2\sum_{i=1}^{\frac{M}{2}-1} k_0(x_{2i},t_n)g(u_{2i}^n)\right) \quad (17)$$

And :

$$\begin{aligned} u_x(1,t_n) &= \int_0^1 k_1(x,t_n)h(u(x,t_n))dx \\ &= \frac{\Delta x}{3}\left(\begin{array}{l} k_1(x_0,t_n)h(u_0^n) + 4\sum_{i=1}^{\frac{M}{2}} k_1(x_{2i-1},t_n)h(u_{2i-1}^n) \\ +2\sum_{i=1}^{\frac{M}{2}-1} k_1(x_{2i},t_n)h(u_{2i}^n) + k_1(x_M,t_n)h(u_M^n) \end{array}\right) \end{aligned}$$

then :

$$\begin{aligned} &3u_x(1,t_n) - \Delta x k_1(x_0,t_n)h(u_0^n) - \Delta x k_1(x_M,t_n)h(u_M^n) \\ &= \Delta x\left(4\sum_{i=1}^{\frac{M}{2}} k_1(x_{2i-1},t_n)h(u_{2i-1}^n) + 2\sum_{i=1}^{\frac{M}{2}-1} k_1(x_{2i},t_n)h(u_{2i}^n)\right) \end{aligned} \quad (18)$$

By using the linearization technique will be developed to overcome this difficulty. Using Taylors series expansion of the nonlinear terms $g(u_i^n) = g(u(x_i,t_n))$ and $h(u_i^n) = h(u(x_i,t_n))$, we obtain :

$$g(u_i^n) = g(u_i^{n-1}) + \left(\frac{\partial g(u)}{\partial u}\right)_i^{n-1}(u_i^n - u_i^{n-1}) + \dots \quad (19)$$

$$h(u_i^n) = h(u_i^{n-1}) + \left(\frac{\partial h(u)}{\partial u}\right)_i^{n-1}(u_i^n - u_i^{n-1}) + \dots \quad (20)$$

Then, we get :

$$a_0^n u_0^n + a_1^n u_1^n + a_2^n u_2^n + a_3^n u_3^n + a_4^n u_4^n + \dots + a_{M-1}^n u_{M-1}^n + a_M^n u_M^n = L_M^n, \quad (21)$$

where

$$\begin{cases} a_0^n = -25 - 4(\Delta x)^2 k_0(x_0,t_n)\left(\frac{\partial g(u)}{\partial u}\right)_0^{n-1}, \\ a_1^n = 48 - 16(\Delta x)^2 k_0(x_1,t_n)\left(\frac{\partial g(u)}{\partial u}\right)_1^{n-1}, \\ a_2^n = -36 - 8(\Delta x)^2 k_0(x_2,t_n)\left(\frac{\partial g(u)}{\partial u}\right)_2^{n-1}, \\ a_3^n = 16 - 16(\Delta x)^2 k_0(x_3,t_n)\left(\frac{\partial g(u)}{\partial u}\right)_3^{n-1}, \\ a_4^n = -3 - 8(\Delta x)^2 k_0(x_4,t_n)\left(\frac{\partial g(u)}{\partial u}\right)_4^{n-1}, \\ a_{2i-1}^n = -16(\Delta x)^2 k_0(x_{2i-1},t_n)\left(\frac{\partial g(u)}{\partial u}\right)_{2i-1}^{n-1} u_{2i-1}^n \quad ; i = 3,\dots,\frac{M}{2}, \\ a_{2i}^n = -8(\Delta x)^2 k_0(x_{2i},t_n)\left(\frac{\partial g(u)}{\partial u}\right)_{2i}^{n-1} u_{2i}^n \quad ; i = 3,\dots,\frac{M}{2}-1, \end{cases} \tag{22}$$

and

$$\begin{aligned} L_M^n \ = \ & 16(\Delta x)^2 \sum_{i=1}^{M} k_1(x_{2i-1},t_n)\Big[h(u_{2i-1}^{n-1}) - \Big(\frac{\partial h(u)}{\partial u}\Big)_{2i-1}^{n-1} u_{2i-1}^{n-1}\Big] \\ & +8(\Delta x)^2 \sum_{i=1}^{M-1} k_1(x_{2i},t_n)\Big[h(u_{2i}^{n-1}) - \Big(\frac{\partial h(u)}{\partial u}\Big)_{2i}^{n-1} u_{2i}^{n-1}\Big] \\ & +4(\Delta x)^2 \Big[k_1(x_0,t_n)\Big[h(u_0^{n-1}) - \Big(\frac{\partial h(u)}{\partial u}\Big)_0^{n-1} u_0^{n-1}\Big]\Big] \\ & +4(\Delta x)^2 \Big[k_1(x_{2M},t_n)\Big[h(u_{2M}^{n-1}) - \Big(\frac{\partial h(u)}{\partial u}\Big)_{2M}^{n-1} u_{2M}^{n-1}\Big]\Big], \end{aligned} \tag{23}$$

then, we have :

$$\begin{aligned} b_0^n u_0^n + \cdots + b_{2M-4}^n u_{2M-4}^n + b_{2M-3}^n u_{2M-3}^n + b_{2M-2}^n u_{2M-2}^n \\ + b_{2M-1}^n u_{2M-1}^n + b_{2M}^n u_{2M}^n = \gamma_M^n, \end{aligned} \tag{24}$$

where

$$\begin{cases} b_0^n = -4(\Delta x)^2 k_1(x_0,t_n)\left(\frac{\partial h(u)}{\partial u}\right)_0^{n-1}, \\ b_{2M-4}^n = 3-8(\Delta x)^2 k_1(x_{2M-4},t_n)\left(\frac{\partial h(u)}{\partial u}\right)_{2M-4}^{n-1}, \\ b_{2M-3}^n = -16-16(\Delta x)^2 k_1(x_{2M-3},t_n)\left(\frac{\partial h(u)}{\partial u}\right)_{2M-3}^{n-1}, \\ b_{2M-2}^n = 36-8(\Delta x)^2 k_1(x_{2M-2},t_n)\left(\frac{\partial g(u)}{\partial u}\right)_{2M-2}^{n-1}, \\ b_{2M-1}^n = -48-16(\Delta x)^2 k_1(x_{2M-1},t_n)\left(\frac{\partial h(u)}{\partial u}\right)_{2M-1}^{n-1}, \\ b_{2M}^n = 25-4(\Delta x)^2 k_1(x_{2M},t_n)\left(\frac{\partial g(u)}{\partial u}\right)_{2M}^{n-1}, \\ b_{2i-1}^n = -16(\Delta x)^2 k_1(x_{2i-1},t_n)\left(\frac{\partial h(u)}{\partial u}\right)_{2i-1}^{n-1} \quad ,i=1,\ldots,M-2, \\ b_{2i}^n = -8(\Delta x)^2 k_1(x_{2i},t_n)\left(\frac{\partial h(u)}{\partial u}\right)_{2i}^{n-1} \quad ,i=1,\ldots,M-2, \end{cases} \tag{25}$$

and

$$\begin{aligned} \gamma_M^n \;=\; & 16(\Delta x)^2\sum_{i=1}^{M} k_1(x_{2i-1},t_n)\Big[h(u_{2i-1}^{n-1})-\Big(\frac{\partial h(u)}{\partial u}\Big)_{2i-1}^{n-1}u_{2i-1}^{n-1}\Big] \\ & +8(\Delta x)^2\sum_{i=1}^{M-1} k_1(x_{2i},t_n)\Big[h(u_{2i}^{n-1})-\Big(\frac{\partial h(u)}{\partial u}\Big)_{2i}^{n-1}u_{2i}^{n-1}\Big] \\ & +4(\Delta x)^2\Big[k_1(x_0,t_n)\Big[h(u_0^{n-1})-\Big(\frac{\partial h(u)}{\partial u}\Big)_0^{n-1}u_0^{n-1}\Big]\Big] \\ & +4(\Delta x)^2\Big[k_1(x_{2M},t_n)\Big[h(u_{2M}^{n-1})-\Big(\frac{\partial h(u)}{\partial u}\Big)_{2M}^{n-1}u_{2M}^{n-1}\Big]\Big]. \end{aligned} \tag{26}$$

Combining (21),(23), with (16) yields an $(M+1)\times(M+1)$ linear system of equations. We write the system in the matrix from

$$A^nU^{n+1}=B^n,$$

which

$$A^n=\begin{pmatrix} a_0^n & a_1^n & a_2^n & a_3^n & a_4^n & \cdots & a_{M-4}^n & a_{M-3}^n & a_{M-2}^n & a_{M-1}^n & a_M^n \\ \frac{1}{12}-r & \frac{5}{6}+2r & \frac{1}{12}-r & 0 & \cdots & \cdots & \cdots & \cdots & \cdot & \cdot & 0 \\ \ddots & \ddots & \ddots & \ddots & & & & & & & \vdots \\ 0 & & & \ddots & & & & 0 & \frac{1}{12}-r & \frac{5}{6}+2r & \frac{1}{12}-r \\ b_0^n & b_1^n & b_2^n & b_3^n & b_4^n & \cdots & b_{2M-4}^n & b_{2M-3}^n & b_{2M-2}^n & b_{2M-1}^n & b_{2M}^n \end{pmatrix},$$

$$U^{n+1} = \begin{pmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix},$$

$$B^n = \begin{pmatrix} L_M^n \\ L_1^n \\ \vdots \\ L_{2M-1}^n \\ \gamma_M^n \end{pmatrix},$$

where $a_0^n, \dots, a_M^n$, $b_0^n, \dots, b_M^n$, L_M^n and γ_M^n are the coefficients in (22), (25), and (26) respectively.

4. Numerical Experiments

To test the above algorithm described in Section 3.3 , we use two examples with known analytical solutions as follows:

Example 1. The first test example to be solved is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \left(1 + \pi^2\right) \exp(t) \cos(\pi x), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{27}$$

with the initial condition

$$u(x, 0) = \cos(\pi x), \qquad 0 < x < 1, \tag{28}$$

and the nonlinear nonlocal boundary conditions

$$u_x(0, t) = \int_0^1 \sin(\pi x) u^3(x, t)\, dx, \qquad 0 < t \leq T, \tag{29}$$

$$u_x(1, t) = \int_0^1 \sin(\pi x) u^5(x, t)\, dx, \qquad 0 < t \leq T. \tag{30}$$

The analytic solution is

$$u(x, t) = \cos(\pi x) \exp(t). \tag{31}$$

In Table 1 we present results with $h = 0.05, 0.005$ and $k = 0.4$ using the finite difference formulate for $x = 0.1$ and $t = 0.01, 0.02, 0.03, \ldots, 0.1$. Table 2 gives the maximum errors of the numerical solutions experimental order of convergence. The maximum error is defined as follows

$$Er = \|u - u_{hk}\|_\infty = max_{0\leq k\leq N}\{max_{0\leq i\leq M}|u(x_i, t_k) - u_i^k|\},$$

and the experiment order convergence for the scheme is calculated using the formula :

$$order = \frac{ln(Er(h_{i-1})/Er(h_i))}{ln(h_{i-1}/h_i)}.$$

Table 1: Some numerical results at $x=0.1$ for $h=0.05$ and $h=0.005$ for Example 1.

t_i	exact	CBES $h = 0.05$	CBES $h = 0.005$
0.01	0.96061479	0.96061503	0.96061479
0.02	0.97026913	0.97026948	0.97026913
0.03	0.98002050	0.98002092	0.98002050
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.1	1.05108000	1.05108060	1.05108000

Table 2: The maximum errors and experiment order of convergence for Example 1.

M	N	maximum errors	order
4	40	3.88×10^{-4}	
8	640	$2.50 \cdot 10^{-5}$	3.953
16	10240	$1.57 \cdot 10^{-6}$	3.995
32	163840	$9.83 \cdot 10^{-8}$	3.997

From the table it is clear that the results are in good agreement as compared with the exact ones. Moreover, the new scheme is fourth order accurate in space . Figure 1 illustrates the exact solution and an approximate solution of Example 1 by CBES.

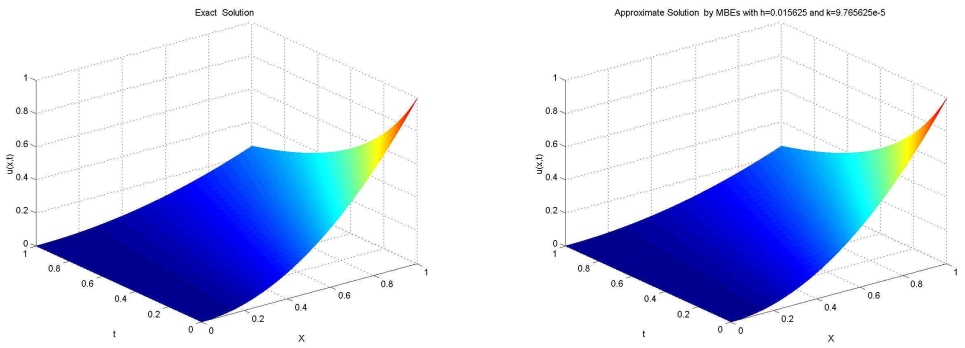

Figure 1: (a) Exact and (b) Approximate Solution by CBES for Example 1.

Example 2. The second test example to be solved is

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=2t-2\pi(3x^2-3x)\cos(2\pi(x^3-\frac{3}{2}x^2))$$
$$+4\pi(3x^2-3x)^2\sin(2\pi(x^3-\frac{3}{2}x^2)),\ 0<x<1,\ 0<t\leq T, \quad (32)$$

with the initial condition

$$u\left(x,0\right)=\sin\left(2\pi(x^3-\frac{3}{2}x^2)\right),\qquad 0<x<1, \quad (33)$$

and the nonlocal boundary conditions

$$u_x\left(0,t\right)=\int_0^1 2\pi(3x^2-3x)\cos(2\pi(x^3-\frac{3}{2}x^2))e^{u(x,t)}dx,\qquad 0<t\leq T, \quad (34)$$
$$u_x\left(1,t\right)=\int_0^1 2\pi(3x^2-3x)\cos(2\pi(x^3-\frac{3}{2}x^2))\frac{1}{1+u\left(x,t\right)}dx,\quad 0<t\leq T, \quad (35)$$

The analytic solution is

$$u\left(x,t\right)=\sin\left(2\pi(x^3-\frac{3}{2}x^2)\right)+t^2. \quad (36)$$

In Table 3 we present results with for $h=0.05$ and $h=0.005$ and $r=0.4$ using the finite difference formulate discussed in Section 2 for $x=0.1$ and $t=0.01;0.02;0.03;\ldots;0.1$. Table 4 gives the maximum errors of the numerical solutions.

Table 3: Some numerical results at $x = 0.1$ for $h = 0.05$ and $h = 0.005$.

t_i	exact	CBES $h = 0.05$	CBES $h = 0.005$
0.01	0.30911699	0.30912416	0.30911707
0.02	0.30941699	0.3094286	0.30941711
0.03	0.30991699	0.30993220	0.30991715
⋮	⋮	⋮	⋮
0.1	0.31901699	0.31904755	0.31901730

Table 4: The maximum errors and experiment order of convergence for example 2
.

M	N	maximum errors	order
4	40	4.749373×10^{-3}	
8	640	$2.949950 \cdot 10^{-4}$	4.008
16	10240	$1.840864 \cdot 10^{-5}$	4.002
32	163840	$1.150093 \cdot 10^{-6}$	4.0005

Figure 2 illustrate the exact solution and an approximate solution of Example 2 by CBES.

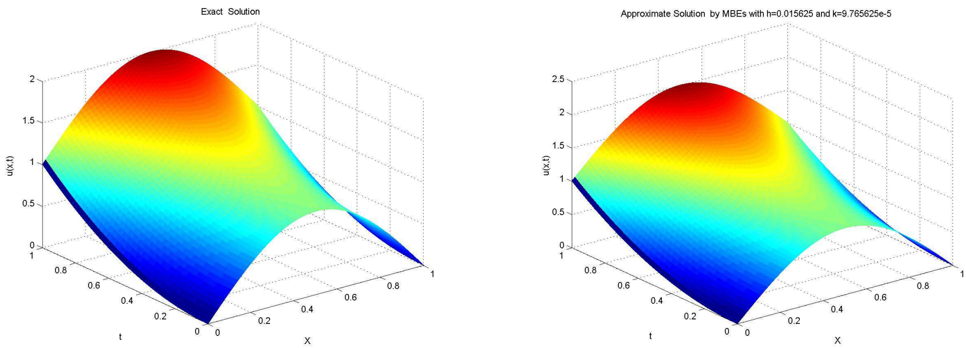

Figure 2: (a) Exact and (b) Approximate Solution by CBES for Example 2.

From the table it is clear that the results are in good agreement as compared with the exact ones.

5, Conclusion

The study of heat diffusion phenomena has attracted the attention of many scientists for many years because of their great importance in our daily lives, but what aroused our interest in studying this equation in a more complex field defined by nonlinear integral conditions of second type, as we were able to simulate the solution as we look forward to studying more problems complicated.

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