PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS FOR NEUTRAL NONLINEAR COUPLED VOLTERRA INTEGRO-DIFFERENTIAL SYSTEMS WITH TWO VARIABLE DELAYS

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Abstract

In this paper, we study the existence of periodic and asymptotically periodic solutions of neutral nonlinear coupled Volterra integro-differential systems. We furnish sufficient conditions for the existence of such solutions. Krasnoselskii’s fixed point theorem is used in this analysis.

1. Introduction

The study of the existence of periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Some contributions on the existence of periodic solutions for differential equations have been made (see \textsuperscript{[1]}-\textsuperscript{4}, \textsuperscript{19}, \textsuperscript{20}). On the other hand,
the concept of asymptotic periodicity is more general than periodicity and from an applied perspective, asymptotically periodic systems describe world more realistically and accurately than periodic ones, we can see [8], [9], [12], [26], [27], for more details.

In 1928 Volterra [25] noted that many physical problems were being modeled by integral and integro-differential equations. Today we see that such models have application in several branches of applied science, such as control theory, mathematical biology, viscoelasticity, nuclear reactors, many other areas, and for this reason this type of equation has received much attention in recent years, (see for example [5], [7], [10], [11], [14]-[21], [23]). Motivated by the papers [6], [13], [22] and the references therein and by using Krasnoselskii’s fixed point theorem, in this paper, we study the existence of periodic and asymptotically periodic solutions of the following system of coupled neutral nonlinear Volterra integro-differential equations with two delays

\[
\begin{align*}
\dot{x}(t) &= h_1(t) x(t) + G_1(t, x(t), y(t), x(t - \tau_1(t)), y(t - \tau_2(t))) \\
&\quad + c_1(t) x'(t - \tau_1(t)) + \int_{-\infty}^{t} a_1(t, s) f_1(x(s), y(s)) \, ds, \\
\dot{y}(t) &= h_2(t) y(t) + G_2(t, x(t), y(t), x(t - \tau_1(t)), y(t - \tau_2(t))) \\
&\quad + c_2(t) y'(t - \tau_2(t)) + \int_{-\infty}^{t} a_2(t, s) f_2(x(s), y(s)) \, ds,
\end{align*}
\]

(1.1)

where the functions \(h_i, c_i, a_i, i = 1, 2\) are assumed to be continuous in their arguments throughout the paper. The functions \(G_i(t, x, y, z, w), i = 1, 2\) is continuous, periodic in \(t\) and Lipschitz continuous in \(x, y, z\) and \(w\), \(f_i(x, y), i = 1, 2\) is continuous and Lipschitz continuous in \(x\) and \(y\), and for some positive constants \(N_j\) and \(R_j\), \(j = 1, 4\) we have

\[
|G_1(t, y_1, y_2, y_3, y_4) - G_1(t, x_1, x_2, x_3, x_4)| \leq \sum_{j=1}^{4} N_j |y_j - x_j|,
\]

\[
|G_2(t, y_1, y_2, y_3, y_4) - G_2(t, x_1, x_2, x_3, x_4)| \leq \sum_{j=1}^{4} R_j |y_j - x_j|,
\]

and for some positive constants \(d_j\) and \(q_j\), \(j = 1, 2\) we have

\[
|f_1(y_1, y_2) - f_1(x_1, x_2)| \leq \sum_{j=1}^{2} d_j |y_j - x_j|,
\]

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we also assume that $G_1(t, 0, 0, 0, 0) = G_2(t, 0, 0, 0, 0) = f_1(0, 0) = f_2(0, 0) = 0$.

We assume that there exists a positive real number $T$, such that
\[
\begin{align*}
  h_i(t + T) &= h_i(t), \quad c_i(t + T) = c_i(t), \\
  a_i(t + T, s + T) &= a_i(t, s), \quad \tau_i(t + T) = \tau_i(t), \quad i = 1, 2,
\end{align*}
\]
(1.2)
with $c_i$ continuously differentiable, $\tau_i$ twice continuously differentiable and $\tau_i(t) \geq \tau_i^* > 0$ for $i = 1, 2$. To have a well behaved mapping we must assume that
\[
\tau_i'(t) \neq 1, \quad \int_0^T h_i(s) \, ds \neq 0, \quad i = 1, 2.
\]
(1.3)

Define $P_T = \{(\varphi, \psi) : (\varphi, \psi)(t + T) = (\varphi, \psi)(t)\}$, where both $\varphi$ and $\psi$ are real valued continuous functions on $\mathbb{R}$. Then $P_T$ is a Banach space when endowed with the maximum norm
\[
\|(x, y)\| = \max \left\{ \max_{t \in [0, T]} |x(t)|, \max_{t \in [0, T]} |y(t)| \right\}.
\]

**Definition 1.1.** A function $x$ is called asymptotically $T$-periodic if there exist two functions $x_1$ and $x_2$ such that $x_1$ is $T$-periodic, $\lim_{t \to \infty} x_2(t) = 0$ and $x(t) = x_1(t) + x_2(t)$ for all $t$.

**Lemma 1.2.** Assume (1.2) and (1.3). If $x, y \in P_T$, then $x$ and $y$ is a solution of (1.1) if and only if
\[
\begin{align*}
  x(t) &= \frac{c_1(t) x(t - \tau_1(t))}{1 - \tau_1(t)} - \int_t^{t + T} e^{\int_u^{t + T} h_1(s) \, ds} \, \frac{1}{1 - e^{\int_0^t h_1(s) \, ds}} \, \tau_1(u) \, x(u - \tau_1(u)) \, du \\
  &\quad + \int_t^{t + T} e^{\int_u^{t + T} h_1(s) \, ds} \, \frac{1}{1 - e^{\int_0^t h_1(s) \, ds}} \, G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du \\
  &\quad + \int_t^{t + T} e^{\int_u^{t + T} h_1(s) \, ds} \, \frac{1}{1 - e^{\int_0^t h_1(s) \, ds}} \, a_1(u, s) \int_{-\infty}^u f_1(x(s), y(s)) \, ds \, du,
\end{align*}
\]
(1.4)
Consequently, we have

\[ y(t) = \frac{c_2(t) y(t - \tau_2(t)) - \int_t^{t+T} e^{\int_0^{t+T} h_2(s)ds} \tau_2(u) y(u - \tau_2(u)) \, du}{1 - \tau_2(t)} + \int_t^{t+T} \frac{e^{\int_0^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} G_2(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du + \int_t^{t+T} \frac{e^{\int_0^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} \int_u^{\infty} a_2(u, s) f_2(x(s), y(s)) \, ds \, du, \]

where

\[ r_1(u) = \frac{(c_1'(u) - c_1(u) h_1(u))(1 - \tau_1'(u)) + \tau_1''(u) c_1(u)}{(1 - \tau_1'(u))^2}, \]

and

\[ r_2(u) = \frac{(c_2'(u) - c_2(u) h_2(u))(1 - \tau_2'(u)) + \tau_2''(u) c_2(u)}{(1 - \tau_2'(u))^2}. \]

**Proof.** Let \( x, y \in P_T \) be a solution of (1.1). Next we multiply both sides of the first equation in (1.1) by \( e^{-\int_0^T h_1(s)ds} \) and then integrate from \( t \) to \( t + T \), to obtain

\[
\int_t^{t+T} [x(u)e^{-\int_0^u h_1(s)ds}]' \, du
= \int_t^{t+T} e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du + \int_t^{t+T} e^{-\int_0^u h_1(s)ds} c_1(u) x'(u - \tau_1(u)) \, du + \int_t^{t+T} e^{-\int_0^u h_1(s)ds} a_1(u, s) f_1(x(s), y(s)) \, ds \, du.
\]

Consequently, we have

\[
x(t + T)e^{-\int_0^{t+T} h_1(s)ds} - x(t)e^{-\int_0^T h_1(s)ds}
= \int_t^{t+T} e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du + \int_t^{t+T} e^{-\int_0^u h_1(s)ds} c_1(u) x'(u - \tau_1(u)) \, du + \int_t^{t+T} e^{-\int_0^u h_1(s)ds} a_1(u, s) f_1(x(s), y(s)) \, ds \, du.
\]
Multiply both sides with \(e^{\int_t^{t+T} h_1(s)ds}\) and using the fact that \(x(t+T) = x(t)\) and \(e^{\int_t^{t+T} h_1(s)ds} = e^{\int_t^{t+T} H_1(s)ds}\), we obtain

\[
x(t) = \int_t^{t+T} \frac{e^{\int_t^{t+T} h_1(s)ds}}{1-e^{\int_t^{t+T} h_1(s)ds}} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u))) \, du \\
+ \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} C_1(u) x'(u-\tau_1(u)) \, du \\
+ \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} \int_t^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du.
\]  

(1.8)

Letting

\[
\int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} C_1(u) x'(u-\tau_1(u)) \, du \\
= \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} C_1(u) \left(1 - \frac{\tau_1'(u)}{\tau_1'(u)}\right) x'(u-\tau_1(u)) \, du.
\]

Performing an integration by parts, we get

\[
\int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} C_1(u) x'(u-\tau_1(u)) \, du \\
= \left[ C_1(u) x(u-\tau_1(u)) e^{\int_t^{t+T} h_1(s)ds} \right]_t^{t+T} - \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} \tau_1(u) x(u-\tau_1(u)) \, du \\
= C_1(t) x(t-\tau_1(t)) \left(1 - \frac{e^{\int_t^{t+T} h_1(s)ds}}{1-e^{\int_t^{t+T} h_1(s)ds}}\right) - \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} \tau_1(u) x(u-\tau_1(u)) \, du,
\]

(1.9)

where \(\tau_1\) is given by (1.6). Substituting (1.9) into (1.8), we obtain

\[
x(t) = \frac{C_1(t) x(t-\tau_1(t))}{1-\tau_1'(t)} - \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} \frac{\tau_1(u) x(u-\tau_1(u))}{1-e^{\int_t^{t+T} h_1(s)ds}} \, du \\
+ \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} G_1(u, x(u), y(u), x(u-\tau_1(u)), y(u-\tau_2(u))) \, du \\
+ \int_t^{t+T} e^{\int_t^{t+T} h_1(s)ds} \int_t^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du.
\]

(1.10)

The proof of (1.10) is similar and hence we omit it.
2. Periodic Solutions

Lemma 2.1 ([24]). Let $\mathbb{M}$ be a bounded closed convex nonempty subset of a Banach space $(\mathcal{S}, \| \cdot \|)$. Suppose that $A$ and $B$ map $\mathbb{M}$ into $\mathcal{S}$ such that

(i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,
(ii) $A$ is compact and continuous,
(iii) $B$ is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Let $\gamma_i(t) = \frac{\alpha_i(t)}{\alpha_i(t)}$, $i = 1, 2$, we assume that $\sup_{t \in [0, T]} |\gamma_i(t)| = \mu_i < 1$, and let $\mu = \max \{ \mu_1, \mu_2 \}$. Let $\beta_i$, $i = 1, 2$ be positive constants such that $0 < \mu_i + \beta_i < 1$. Moreover, we assume the existence of positive constants $M_i$, $K_i$, $\alpha_i$, $L_i$ and $\theta_i$, $i = 1, 2$ such that

\[ |f_1(x, y)| \leq M_1, \]
\[ |f_2(x, y)| \leq M_2, \]
\[ |G_1(t, x, y, z, w)| \leq K_1, \quad |G_2(t, x, y, z, w)| \leq K_2, \]
\[ \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_1(s)ds}}{1 - e^{\int_{0}^{T} h_1(s)ds}} \left| \int_{-\infty}^{u} |a_1(u, s)| ds du \right| \leq \alpha_1, \]
\[ \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_2(s)ds}}{1 - e^{\int_{0}^{T} h_2(s)ds}} \left| \int_{-\infty}^{u} |a_2(u, s)| ds du \right| \leq \alpha_2, \]
\[ \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_1(s)ds}}{1 - e^{\int_{0}^{T} h_1(s)ds}} \left| r_1(u) \right| du \leq \beta_1, \]
\[ \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_2(s)ds}}{1 - e^{\int_{0}^{T} h_2(s)ds}} \left| r_2(u) \right| du \leq \beta_2, \]
\[ \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_1(s)ds}}{1 - e^{\int_{0}^{T} h_1(s)ds}} du \leq L_1, \quad \int_{t}^{t+T} \frac{e^{\int_{u}^{t+T} h_2(s)ds}}{1 - e^{\int_{0}^{T} h_2(s)ds}} du \leq L_2, \]

and

\[ \int_{-\infty}^{u} |a_1(u, s)| ds \leq \theta_1, \quad \int_{-\infty}^{u} |a_2(u, s)| ds \leq \theta_2. \]
Set
\[
M = \max \left\{ \frac{L_1 K_1 + \alpha_1 M_1}{1 - \mu_1 - \beta_1}, \frac{L_2 K_2 + \alpha_2 M_2}{1 - \mu_2 - \beta_2} \right\}.
\]

We define subset \( \Omega_{x,y} \) of \( P_T \) as follows
\[
\Omega_{x,y} = \{(x, y) : (x, y) \in P_T \text{ with } \| (x, y) \| \leq M \}.
\]
Then \( \Omega_{x,y} \) is a bounded, closed and convex subset of \( P_T \). Now for \( (x, y) \in \Omega_{x,y} \) we can define an operator \( E : \Omega_{x,y} \to P_T \) by
\[
E(x, y)(t) = (E_1(x, y)(t), E_2(x, y)(t)),
\]
where
\[
E_1(x, y)(t) = \frac{c_1(t)x(t - \tau_1(t))}{1 - \tau'_1(t)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} r_1(u)x(u - \tau_1(u))du
\]
\[
+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u)))du
\]
\[
+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_1(s)ds}}{1 - e^{\int_0^T h_1(s)ds}} \int_{-\infty}^{u} a_1(u, s)f_1(x(s), y(s))ds du,
\]
and
\[
E_2(x, y)(t) = \frac{c_2(t)y(t - \tau_2(t))}{1 - \tau'_2(t)} - \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} r_2(u)y(u - \tau_2(u))du
\]
\[
+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} G_2(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u)))du
\]
\[
+ \int_t^{t+T} \frac{e^{\int_u^{t+T} h_2(s)ds}}{1 - e^{\int_0^T h_2(s)ds}} \int_{-\infty}^{u} a_2(u, s)f_2(x(s), y(s))ds du.
\]

To apply Lemma 2.1, we need to construct two mappings, one is a contraction and the other is continuous and compact. Therefore, we state
As \( (2.11) \) as

\[
E_1 (x, y) (t) = B_1 (x, y) (t) + A_1 (x, y) (t),
\]

where \( B_1, A_1 : \Omega_{x,y} \to P_T \) are given by

\[
B_1 (x, y) (t) = \frac{c_1 (t) x (t - \tau_1 (t))}{1 - \tau'_1 (t)},
\]

and

\[
A_1 (x, y) (t) = - \int_t^{t+T} e^{\int_u^{t+T} h_1 (s) ds} \frac{1}{1 - e^{\int_0^T h_1 (s) ds}} \tau_1 (u) x (u - \tau_1 (u)) du
\]

\[
+ \int_t^{t+T} e^{\int_u^{t+T} h_1 (s) ds} \frac{1}{1 - e^{\int_0^T h_1 (s) ds}} G_1 (u, x (u), y (u), x (u - \tau_1 (u)), y (u - \tau_2 (u))) du
\]

\[
+ \int_t^{t+T} e^{\int_u^{t+T} h_1 (s) ds} \frac{1}{1 - e^{\int_0^T h_1 (s) ds}} \int_{-\infty}^u a_1 (u, s) f_1 (x (s), y (s)) ds du.
\]

And we state \( (2.12) \) as

\[
E_2 (x, y) (t) = B_2 (x, y) (t) + A_2 (x, y) (t),
\]

where \( B_2, A_2 : \Omega_{x,y} \to P_T \) are given by

\[
B_2 (x, y) (t) = \frac{c_2 (t) y (t - \tau_2 (t))}{1 - \tau'_2 (t)},
\]

and

\[
A_2 (x, y) (t) = - \int_t^{t+T} e^{\int_u^{t+T} h_2 (s) ds} \frac{1}{1 - e^{\int_0^T h_2 (s) ds}} \tau_2 (u) y (u - \tau_2 (u)) du
\]

\[
+ \int_t^{t+T} e^{\int_u^{t+T} h_2 (s) ds} \frac{1}{1 - e^{\int_0^T h_2 (s) ds}} G_2 (u, x (u), y (u), x (u - \tau_1 (u)), y (u - \tau_2 (u))) du
\]

\[
+ \int_t^{t+T} e^{\int_u^{t+T} h_2 (s) ds} \frac{1}{1 - e^{\int_0^T h_2 (s) ds}} \int_{-\infty}^u a_2 (u, s) f_2 (x (s), y (s)) ds du.
\]

Now for \((x, y) \in \Omega_{x,y}\) we can define the operators \(B, A : \Omega_{x,y} \to P_T\) by

\[
B (x, y) (t) = (B_1 (x, y) (t), B_2 (x, y) (t))
\]

and

\[
A (x, y) (t) = (A_1 (x, y) (t), A_2 (x, y) (t)).
\]
Observe that, since the functions $G_i(t, x_1, x_2, x_3, x_4)$, $i = 1, 2$ is Lipschitz continuous in $x_1, x_2, x_3, x_4$ and $f_i(x_1, x_2)$, $i = 1, 2$ is Lipschitz continuous in $x_1, x_2$ we have

$$|G_1(t, x_1, x_2, x_3, x_4)| = |G_1(t, x_1, x_2, x_3, x_4) - G_1(t, 0, 0, 0)| + |G_1(t, 0, 0, 0)|$$

$$\leq \sum_{j=1}^{4} N_j |x_j|,$$

$$|G_2(t, x_1, x_2, x_3, x_4)| = |G_2(t, x_1, x_2, x_3, x_4) - G_2(t, 0, 0, 0)| + |G_2(t, 0, 0, 0)|$$

$$\leq \sum_{j=1}^{4} R_j |x_j|,$$

$$|f_1(x_1, x_2)| = |f_1(x_1, x_2) - f_1(0, 0)| + |f_1(0, 0)|$$

$$\leq \sum_{j=1}^{2} d_j |x_j|,$$

and

$$|f_2(x_1, x_2)| = |f_2(x_1, x_2) - f_2(0, 0)| + |f_2(0, 0)|$$

$$\leq \sum_{j=1}^{2} q_j |x_j|.$$

Theorem 2.2. Suppose (1.2), (1.3) and (2.1)-(2.9) hold. Suppose that

$$\beta_1 + L_1 \sum_{j=1}^{4} N_j + \alpha_1 \sum_{j=1}^{2} d_j \leq 1, \text{ and } \beta_2 + L_2 \sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j \leq 1.$$

Then (1.1) has a $T$-periodic solution.

Proof. In order to prove that (1.1) has a $T$-periodic solution, we shall make sure that $A$ and $B$ satisfy the conditions of Lemma 2.1. For all $(x, y) \in \Omega_{x,y}$,
we have \((x, y)(t + T) = (x, y)(t)\) and \(||(x, y)|| \leq M\). Now let us discuss \(B(x, y) + A(x, y)\). We have

\[
B_1(x, y)(t + T) = \frac{c_1(t + T) x(t + T - \tau_1(t + T))}{1 - \tau_1'(t + T)} = \frac{c_1(t) x(t - \tau_1(t))}{1 - \tau_1'(t)} = B_1(x, y)(t),
\]

and

\[
A_1(x, y)(t + T)
\]

\[
= - \int_{t + T}^{t + T + 2T} e^{\int_u^{t + T} h_1(s)ds} r_1(u) x(u - \tau_1(u)) du
\]

\[
+ \int_{t + T}^{t + T + 2T} e^{\int_u^{t + T} h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]

\[
+ \int_{t + T}^{t + T + 2T} e^{\int_u^{t + T} h_1(s)ds} \int_{-\infty}^{u} a_1(u, s) f_1(x(s), y(s)) ds du
\]

\[
= \int_t^{t + T} e^{\int_u^{t + T} h_1(s)ds} r_1(u) x(u - \tau_1(u)) du
\]

\[
+ \int_t^{t + T} e^{\int_u^{t + T} h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]

\[
+ \int_t^{t + T} e^{\int_u^{t + T} h_1(s)ds} \int_{-\infty}^{u} a_1(u, s) f_1(x(s), y(s)) ds du
\]

\[
= A_1(x, y)(t).
\]

Then \(E_1(x, y)(t + T) = E_1(x, y)(t)\). In a similar way we can easily show that \(E_2(x, y)(t + T) = E_2(x, y)(t)\). Therefore, \(E(x, y)(t + T) = E(x, y)(t)\).

For any \((x, y) \in \Omega_{x,y}\), we will show that \(|E(x, y)(t)| \leq M\). In view of the above estimates, we have

\[
|B_1(x, y)(t)| \leq \left| \frac{c_1(t)}{1 - \tau_1'(t)} \right| |x(t - \tau_1(t))| \leq \mu_1 M,
\]

and

\[
|A_1(x, y)(t)| \leq \int_t^{t + T} \left| \frac{e^{\int_s^{t + T} h_1(s)ds}}{1 - e^{\int_s^{t} h_1(s)ds}} \right| |r_1(u)| |x(u - \tau_1(u))| du
\]
As a consequence of (2.10)

\[
\frac{L_1 K_1 + \alpha_1 M_1}{1 - \mu_1 - \beta_1} \leq M,
\]

so,

\[
L_1 K_1 + \alpha_1 M_1 \leq (1 - \mu_1 - \beta_1) M.
\]

This implies that

\[
|E_1(x, y)(t)| \leq \mu_1 M + \beta_1 M + L_1 K_1 + \alpha_1 M_1 \\
\leq \mu_1 M + \beta_1 M + (1 - \mu_1 - \beta_1) M = M.
\]

In a similar way we can easily show that

\[
|E_2(x, y)(t)| \leq M.
\]

Thus, \(E\) maps \(\Omega_{x,y}\) into itself, i.e. \(E(\Omega_{x,y}) \subseteq \Omega_{x,y}\). We will now show that \(A\) is continuous. Let \(\{(x_n, y_n)\}\) be a sequence in \(\Omega_{x,y}\) such that

\[
\lim_{n \to \infty} \|(x_n, y_n) - (x, y)\| = 0.
\]

Since \(\Omega_{x,y}\) is closed, we have \((x, y) \in \Omega_{x,y}\). Then by the definition of \(A\) we have

\[
\|A(x_n, y_n) - A(x, y)\| \\
= \max_{t \in [0, T]} \left( \max_{t \in [0, T]} |A_1(x_n, y_n)(t) - A_1(x, y)(t)|, \max_{t \in [0, T]} |A_2(x_n, y_n)(t) - A_2(x, y)(t)| \right),
\]

in which

\[
|A_1(x_n, y_n)(t) - A_1(x, y)(t)|
\]
we have

Thus

By a similar argument we can easily argue that

Thus

This result proves that $A$ is continuous.

We now have to show that $A$ is compact. For $n \in \mathbb{N}$, let $(x_n, y_n) \in \Omega_{x,y}$, we have

$$
\left| A_1 \left( x_n, y_n \right) (t) \right| \\
\leq \int_t^{t+T} \left| e^{\int_t^{t+T} h_1(s)ds} \right| |r_1(u)||x_n(u - \tau_1(u)) - x(u - \tau_1(u))| \, du \\
+ \int_t^{t+T} \left| e^{\int_t^{t+T} h_1(s)ds} \right| G_1(u, x_n(u), y_n(u), x_n(u - \tau_1(u)), y_n(u - \tau_2(u))) \, du \\
+ \int_t^{t+T} \left| e^{\int_t^{t+T} h_1(s)ds} \right| \int_{-\infty}^{u} |a_1(u, s)||f_1(x_n(s), y_n(s)) - f_1(x(s), y(s))| \, ds \, du,
$$

the continuity of $G_1$ and $f_1$ along with the Lebesgue dominated convergence theorem implies that

$$
\lim_{n \to \infty} \max_{t \in [0,T]} |A_1 \left( x_n, y_n \right)(t) - A_1 (x, y)(t)| = 0.
$$

$$
\lim_{n \to \infty} \max_{t \in [0,T]} |A_2 \left( x_n, y_n \right)(t) - A_2 (x, y)(t)| = 0.
$$

Thus

$$
\lim_{n \to \infty} \|A( x_n, y_n) - A (x, y)\| = 0.
$$

This result proves that $A$ is continuous.
In a similar way we can easily show that

\[|A_2(x_n, y_n)(t)| \leq \left( \beta_2 + L_2 \sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j \right) M \leq M.\]

Thus

\[\|A(x_n, y_n)\| \leq M.\]

If we calculate \((A(x_n, y_n))'(t)\), then

\[
(A_1(x_n, y_n))'(t) \\
\leq G_1(t, x_n(t), y_n(t), x_n(t - \tau_1(t)), y_n(t - \tau_2(t))) - r_1(t) x_n(t - \tau_1(t)) \\
+ \int_{-\infty}^{t} a_1(t, s) f_1(x_n(s), y_n(s)) ds + h_1(t) \\
\times \left[ \int_{t}^{t+T} \frac{e^{t+T} h_1(s) ds}{1 - e^{t+T} h_1(s) ds} G_1(u, x_n(u), y_n(u), x_n(u - \tau_1(u)), y_n(u - \tau_2(u))) du \\
- \int_{t}^{t+T} \frac{e^{t+T} h_1(s) ds}{1 - e^{t+T} h_1(s) ds} r_1(u) x_n(u - \tau_1(u)) du \\
+ \int_{t}^{t+T} \frac{e^{t+T} h_1(s) ds}{1 - e^{t+T} h_1(s) ds} \int_{-\infty}^{u} a_1(u, s) f_1(x(s), y(s)) ds du \right] \\
= G_1(t, x_n(t), y_n(t), x_n(t - \tau_1(t)), y_n(t - \tau_2(t))) - r_1(t) x_n(t - \tau_1(t)) \\
+ \int_{-\infty}^{t} a_1(t, s) f_1(x_n(s), y_n(s)) ds + h_1(t) A_1(x_n, y_n)(t).\]

Hence, for some positive constant \(D_1\), we obtain

\[
|(A_1(x_n, y_n))'(t)| = |G_1(t, x_n(t), y_n(t), x_n(t - \tau_1(t)), y_n(t - \tau_2(t)))| + |r_1(t)| |x_n(t - \tau_1(t))| \\
+ \int_{-\infty}^{t} |a_1(t, s)| |f_1(x_n(s), y_n(s))| ds + |h_1(t)| |A_1(x_n, y_n)(t)| \\
\leq \left[ \sum_{j=1}^{4} N_j + \theta_1 \sum_{j=1}^{2} d_j + \theta_3 + \theta_4 \right] M \leq D_1,
\]

where \(\sup_{t \in [0, T]} |r_1(t)| = \theta_3\), \(\sup_{t \in [0, T]} |h_1(t)| = \theta_4\). In a similar way we can
show for some positive constant $D_2$ that

$$\left| (A_2(x_n, y_n))'(t) \right| \leq \left[ \sum_{j=1}^{4} R_j + \theta_2 \sum_{j=1}^{2} q_j + \theta_5 + \theta_6 \right] M \leq D_2,$$

where $\sup_{t \in [0,T]} |r_2(t)| = \theta_5$, $\sup_{t \in [0,T]} |h_2(t)| = \theta_6$. Thus

$$\|(A(x_n, y_n))'\| \leq D,$$

where $D = \max(D_1, D_2)$. Thus, the sequence $(A(x_n, y_n))$ is uniformly bounded and equi-continuous. The Arzela-Ascoli theorem implies that there exists a subsequence $(A(x_{n_k}, y_{n_k}))$ of $(A(x_n, y_n))$ converges uniformly to a continuous $T$-periodic function $(x^*, y^*)$. Thus, $A$ is compact.

For all $(x_1, y_1), (x_2, y_2) \in \Omega_{x,y}$

$$|B_1(x_1, y_1)(t) - B_1(x_2, y_2)(t)| = \left| \frac{c_1(t)x_1(t - \tau_1(t))}{1 - \tau_1'(t)} - \frac{c_1(t)x_2(t - \tau_1(t))}{1 - \tau_1'(t)} \right|$$

$$\leq \mu_1 |x_1(t - \tau_1(t)) - x_2(t - \tau_1(t))|,$$

hence $B_1$ is contraction because $\mu_1 < 1$. In a similar way we can easily show that

$$|B_2(x_1, y_1)(t) - B_2(x_2, y_2)(t)| \leq \mu_2 |y_1(t - \tau_2(t)) - y_2(t - \tau_2(t))|,$$

hence $B_2$ is contraction because $\mu_2 < 1$. Then

$$|B(x_1, y_1)(t) - B(x_2, y_2)(t)|$$

$$= \max \{|B_1(x_1, y_1)(t) - B_1(x_2, y_2)(t)|, |B_2(x_1, y_1)(t) - B_2(x_2, y_2)(t)|\},$$

this implies that

$$\|B(x_1, y_1) - B(x_2, y_2)\| \leq \mu \|(x_1, y_1), (x_2, y_2)\|.$$

Hence $B$ is contraction.

Thus, the conditions of Lemma 2.1 are satisfied and there is a $(x, y) \in \Omega_{x,y}$, such that $(x, y) = A(x, y) + B(x, y)$. \qed
In the next theorem we relax condition (2.2).

**Theorem 2.3.** Suppose (1.2), (1.3), (2.1) and (2.3)–(2.9) hold. Suppose that

\[ \beta_1 + L_1 \sum_{j=1}^{4} N_j + \alpha_1 \sum_{j=1}^{2} d_j \leq 1, \quad \text{and} \quad \beta_2 + L_2 \sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j \leq 1. \]

In addition, we assume the existence of continuous nondecreasing function \( W_2 \) such that

\[ |f_2(x,y)| \leq f_2(|x|,y) \leq Q_2 W_2(|x|), \quad (2.19) \]

for some positive constant \( Q_2 \), and for \( u > 0 \) we ask that

\[ \frac{W_2(u)}{u} \leq \frac{1 - \mu_2 - \beta_2 - L_2 K_2}{\alpha_2 Q_2}. \quad (2.20) \]

Then (1.1) has a \( T \)-periodic solution.

**Proof.** Set

\[ M = \max \left\{ \frac{L_1 K_1 + \alpha_1 M_1}{1 - \mu_1 - \beta_1}, \frac{L_2 K_2 + \alpha_2 Q_2 W_2(M)}{1 - \mu_2 - \beta_2} \right\}. \quad (2.21) \]

Note that due to (2.20) we have

\[ M \geq \frac{L_2 K_2 + \alpha_2 Q_2 W_2(M)}{1 - \mu_2 - \beta_2}, \]

and hence (2.20) is well defined. For any \((x,y) \in \Omega_{x,y}\), we have by the proof of the previous theorem that

\[ |E_1(x,y)(t)| \leq M. \]

Thus

\[ |B_2(x,y)(t)| \leq \left| \frac{c_2(t)}{1 - \tau_2(t)} \right| |y(t - \tau_2(t))| \leq \mu_2 M, \]

and

\[ |A_2(x,y)(t)| \]
In addition, we assume the existence of continuous nondecreasing function $\beta$.

Theorem 2.4. Suppose $1.2$, $1.3$, $2.2$ and $2.3$, $2.9$ hold. Suppose that

$$
\beta_1 + L_1 \sum_{j=1}^{4} N_j + \alpha_1 \sum_{j=1}^{2} d_j \leq 1 \quad \text{and} \quad \beta_2 + L_2 \sum_{j=1}^{4} R_j + \alpha_2 \sum_{j=1}^{2} q_j \leq 1.
$$

In addition, we assume the existence of continuous nondecreasing function $\beta$. The rest of the proof follows along the lines of the proof of Theorem 2.2.

As a consequence of (2.3),

$$
\frac{L_2 K_2 + \alpha_2 Q_2 W_2 (M)}{1 - \mu_2 - \beta_2} \leq M,
$$

so,

$$
L_2 K_2 + \alpha_2 Q_2 W_2 (M) \leq (1 - \mu_2 - \beta_2) M.
$$

This implies that

$$
|E_2(x,y)(t)| \leq \mu_2 M + \beta_2 M + L_2 K_2 + \alpha_2 Q_2 W_2 (M)
\leq \mu_2 M + \beta_2 M + (1 - \mu_2 - \beta_2) M = M.
$$

The rest of the proof follows along the lines of the proof of Theorem 2.2.
\( W_1 \) such that
\[
|f_1(x, y)| \leq f_1(x, |y|) \leq Q_1 W_1(|y|),
\]
for some positive constant \( Q_1 \), and for \( u > 0 \) we ask that
\[
W_1(u) \leq \frac{1 - \mu_1 - \beta_1 - \frac{L_1 K_1}{M}}{\alpha_1 Q_1}. \tag{2.23}
\]
Then (1.1) has a \( T \)-periodic solution.

**Proof.** Set
\[
M = \max \left\{ \frac{L_1 K_1 + \alpha_1 Q_1 W_1(M)}{1 - \mu_1 - \beta_1}, \frac{L_2 K_2 + \alpha_2 M_2}{1 - \mu_1 - \beta_1} \right\}. \tag{2.24}
\]
Note that due to (2.23) we have
\[
M \geq \frac{L_1 K_1 + \alpha_1 Q_1 W_1(M)}{1 - \mu_1 - \beta_1},
\]
and hence (2.24) is well defined. For any \((x, y) \in \Omega_{x,y}\), we have by the proof of the previous theorem that
\[
|E_2(x, y)(t)| \leq M.
\]
Thus
\[
|B_1(x, y)(t)| \leq \left| \frac{c_1(t)}{1 - T_1(t)} \right| |x(t - \tau_1(t))| \leq \mu_1 M,
\]
and
\[
|A_1(x, y)(t)|
\leq \int_t^{t+T} \left| e^{\int_u^{t+T} p(t)ds} \right| |r_1(u)| |x(u - \tau_1(u))| du
+ \int_t^{t+T} \left| e^{\int_u^{t+T} p(t)ds} \right| |G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u)))| du
+ \int_t^{t+T} \left| e^{\int_u^{t+T} p(t)ds} \right| \int_{-\infty}^{t} |a_1(u, s)| f_1(x(s), |y(s)|) ds du
\]
As a consequence of (2.24)
\[ L_1 K_1 + \alpha_1 Q_1 W_1 (M) \leq (1 - \mu_1 - \beta_1) M. \]

This implies that
\[ |E_1(x, y)(t)| \leq \mu_1 M + \beta_1 M + L_1 K_1 + \alpha_1 Q_1 W_1 (M) \]
\[ \leq \mu_1 M + \beta_1 M + (1 - \mu_1 - \beta_1) M = M. \]

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

3. Asymptotic Stability of Periodic Solutions

In this section, we show that under mild conditions one obtains asymptotically periodic solutions. We do not assume the periodicity condition on the functions \( a_1, a_2, c_1, c_2, \tau_1, \tau_2, G_1 \) and \( G_2 \), we only assume \( h_1 \) and \( h_2 \) are \( T \)-periodic, and
\[ \int_0^T h_1(s)ds = 0, \quad \int_0^T h_2(s)ds = 0. \] (3.1)
Since \( h_1 \) and \( h_2 \) are \( T \)-periodic, there are constants \( m_k \) and \( M_k^* \), \( k = 1, 2 \) such that
\[ m_1 \leq e^{\int_0^T h_1(s)ds} \leq M_1^*, \quad m_2 \leq e^{\int_0^T h_1(s)ds} \leq M_2^*. \] (3.2)
Furthermore, we assume that there are positive numbers \( V_1 \) and \( V_2 \) such that
\[ \int_t^{\infty} \int_{-\infty}^u |a_1(u, s)|dsdu \leq V_1, \quad \int_t^{\infty} \int_{-\infty}^u |a_2(u, s)|dsdu \leq V_2. \] (3.3)
In addition, we suppose that
\[
\sup_{t \in \mathbb{R}} \frac{c_1(t)}{1 - \tau_1'(t)} = \mu_1^*, \quad \sup_{t \in \mathbb{R}} \frac{c_2(t)}{1 - \tau_2'(t)} = \mu_2^*, \quad \max \{\mu_1^*, \mu_2^*\} = \mu^* < 1,
\]
and
\[
\lim_{t \to \infty} \frac{c_1(t)}{1 - \tau_1'(t)} = 0, \quad \lim_{t \to \infty} \frac{c_2(t)}{1 - \tau_2'(t)} = 0, \tag{3.4}
\]
Finally, we make the assumption that
\[
\lim_{t \to \infty} \int_t^\infty \int_{-\infty}^u |a_1(u, s)| \, ds \, du = 0, \quad \lim_{t \to \infty} \int_t^\infty \int_{-\infty}^u |a_2(u, s)| \, ds \, du = 0, \tag{3.5}
\]
and for positive constants \(M_k^*, \, k = 3, 6\) we ask that
\[
\int_t^\infty |r_1(u)| \, du \leq M_3^*, \quad \int_t^\infty |r_2(u)| \, du \leq M_4^*, \tag{3.8}
\]
and
\[
\int_t^\infty \frac{e^{\int_0^t h_1(s) \, ds}}{e^{\int_0^t h_1(s) \, ds}} u \, du \leq M_5^*, \quad \int_t^\infty \frac{e^{\int_0^t h_2(s) \, ds}}{e^{\int_0^t h_2(s) \, ds}} u \, du \leq M_6^*. \tag{3.9}
\]
Finally, we make the assumption that
\[
1 - \mu_1^* - M_1^* M_3^* m_1^{-1} > 0, \tag{3.10}
\]
and
\[
1 - \mu_2^* - M_2^* M_4^* m_2^{-1} > 0. \tag{3.11}
\]

**Theorem 3.1.** Assume (1.2), (1.3). Then \(x\) and \(y\) is a solution of (1.1) if and only if
\[
x(t) = p_t e^{\int_0^t h_1(s) \, ds} + \frac{c_1(t)}{1 - \tau_1'(t)} x(t - \tau_1(t)) + \int_t^\infty \frac{e^{\int_0^t h_1(s) \, ds}}{e^{\int_0^t h_1(s) \, ds}} r_1(u) x(u - \tau_1(u)) \, du
\]
\[
- \int_t^\infty \frac{e^{\int_0^t h_1(s) \, ds}}{e^{\int_0^t h_1(s) \, ds}} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du
\]
\[
- \int_t^\infty \frac{e^{\int_0^t h_1(s) \, ds}}{e^{\int_0^t h_1(s) \, ds}} \int_{-\infty}^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du, \tag{3.12}
\]
and
\[
y(t) = \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t)y(t - \tau_2(t))}{1 - \tau_2'(t)} + \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^t h_2(s)ds}} r_2(u) y(u - \tau_2(u)) du
\]
\[
- \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^t h_2(s)ds}} G_2(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]
\[
- \int_t^\infty \frac{e^{\int_0^t h_2(s)ds}}{e^{\int_0^t h_2(s)ds}} \int_{-\infty}^u a_2(u, s) f_2(x(s), y(s)) ds du, \quad (3.13)
\]
where $r_1$ and $r_2$ are given by (1.6) and (1.7).

**Proof.** Let $(x, y)$ be a solution of (1.1). Next we multiply both sides of the first equation in (1.1) by $e^{-\int_0^t h_1(s)ds}$, and then integrate from $t$ to $\infty$, to obtain
\[
\int_t^\infty \left[ x(u) e^{-\int_0^u h_1(s)ds} \right]' du
\]
\[
= \int_t^\infty e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]
\[
+ \int_t^\infty e^{-\int_0^u h_1(s)ds} c_1(u) x'(u - \tau_1(u)) du
\]
\[
+ \int_t^\infty e^{-\int_0^u h_1(s)ds} \int_{-\infty}^u a_1(u, s) f_1(x(s), y(s)) ds du.
\]
Consequently, we have
\[
\rho_1^* - x(t) e^{-\int_0^t h_1(s)ds}
\]
\[
= \int_t^\infty e^{-\int_0^u h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]
\[
+ \int_t^\infty e^{-\int_0^u h_1(s)ds} c_1(u) x'(u - \tau_1(u)) du
\]
\[
+ \int_t^\infty e^{-\int_0^u h_1(s)ds} \int_{-\infty}^u a_1(u, s) f_1(x(s), y(s)) ds du.
\]
Multiply both sides with $e^{\int_0^t h_1(s)ds}$, we obtain
\[
x(t) = \rho_1^* e^{\int_0^t h_1(s)ds}
\]
\[
- \int_t^\infty \frac{e^{\int_0^u h_1(s)ds}}{e^{\int_0^t h_1(s)ds}} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) du
\]
Performing an integration by parts, we get

\[- \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} c_1(u) x'(u - \tau_1(u)) \, du \]

\[= \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} \int_0^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du. \tag{3.14} \]

Letting

\[\int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} c_1(u) x'(u - \tau_1(u)) \, du \]

\[= \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} \int_0^u \frac{c_1(u)}{1 - \tau'_1(u)} (1 - \tau'_1(u)) x'(u - \tau_1(u)) \, du. \]

Performing an integration by parts, we get

\[\int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} c_1(u) x'(u - \tau_1(u)) \, du \]

\[\quad = \left[ \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} \frac{c_1(u) x(u - \tau_1(u))}{1 - \tau'_1(u)} \right]_t^{\infty} - \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} r_1(u) x(u - \tau_1(u)) \, du \]

\[= \rho_{1*} e^{\int_0^u h_1(s) ds} - \frac{c_1(t)}{1 - \tau'_1(t)} x(t - \tau_1(t)) - \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} r_1(u) x(u - \tau_1(u)) \, du, \tag{3.15} \]

where \( r_1 \) is given by (1.6). Substituting (3.15) into (3.14), we obtain

\[x(t) = \rho_1 e^{\int_0^u h_1(s) ds} + \frac{c_1(t) x(t - \tau_1(t))}{1 - \tau'_1(t)} + \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} r_1(u) x(u - \tau_1(u)) \, du \]

\[- \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du \]

\[- \int_t^\infty \frac{e^{\int_s^t h_1(u) du}}{e^{\int_0^u h_1(s) ds}} \int_0^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du, \]

where \( \rho_1 = \rho_1^* - \rho_{1*}^* \). The proof of (3.13) is similar and hence we omit it. \( \Box \)

**Theorem 3.2.** Suppose that (2.1), (2.2) and (3.1) - (3.11) hold. Then system

(1.1)

has asymptotically \( T \)-periodic solution \((x, y)\) satisfying

\[x(t) = x_1(t) + x_2(t), \; y(t) = y_1(t) + y_2(t),\]
where
\[ x_1(t) = \rho_1 e^{\int_0^t h_1(s) \, ds}, \quad y_1(t) = \rho_2 e^{\int_0^t h_2(s) \, ds}, \quad t \in \mathbb{R}, \]
for arbitrary fixed nonzero constants \( \rho_1, \rho_2 \) and
\[ \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} y_2(t) = 0. \]

**Proof.** Define
\[ P^*_T = \{ (\varphi, \psi) : \varphi = \varphi_1 + \varphi_2, \quad \psi = \psi_1 + \psi_2, \quad \varphi(t) + T, \quad (\varphi_1, \psi_1)(t + T) = (\varphi_1, \psi_1)(t), \]
and \( (\varphi_2, \psi_2)(t) \to (0,0) \) as \( t \to \infty \), \}
where both \( \varphi \) and \( \psi \) are real valued bounded continuous functions on \( \mathbb{R} \). Then \( P^*_T \) is a Banach space when endowed with the maximum norm
\[
\| (x, y) \| = \max \left\{ \sup_{t \in \mathbb{R}} |x(t)|, \sup_{t \in \mathbb{R}} |y(t)| \right\}.
\]
We define a subset \( \Omega^*_{x,y} \) of \( P^*_T \) as follows. For a constant \( V^* \) to be defined later in the proof, let
\[
\Omega^*_{x,y} = \{ (x, y) \in P^*_T \text{ with } \| (x, y) \| \leq V^* \}.
\]
Then \( \Omega^*_{x,y} \) is a bounded, closed and convex subset of \( P^*_T \). Now for \( (x, y) \in \Omega^*_{x,y} \) we can define an operator \( F : \Omega^*_{x,y} \to P^*_T \) by
\[
F(x, y)(t) = (F_1(x, y)(t), F_2(x, y)(t)),
\]
where
\[
F_1(x, y)(t) = \rho_1 e^{\int_0^t h_1(s) \, ds} + \frac{c_1(t) x(t - \tau_1(t))}{1 - \tau_1'(t)}
\]
\[ + \int_t^\infty e^{\int_0^t h_1(s) \, ds} \int_0^u A_1(\tau_1(s), x(s)) \, ds \, du
\]
\[ - \int_0^\infty e^{\int_0^t h_1(s) \, ds} \int_0^u A_1(\tau_1(s), x(s)) \, ds \, du,
\]
and
\[
F_2(x, y)(t) = \rho_2 e^{\int_0^t h_2(s) \, ds} + \frac{c_2(t) y(t - \tau_1(t))}{1 - \tau_2'(t)}
\]
\[ + \int_t^\infty e^{\int_0^t h_2(s) \, ds} \int_0^u A_2(\tau_2(s), y(s)) \, ds \, du
\]
\[ - \int_0^\infty e^{\int_0^t h_2(s) \, ds} \int_0^u A_2(\tau_2(s), y(s)) \, ds \, du,
\]
\[
(3.16)
\]
and

\[
F_2(x, y)(t) = \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t) y(t - \tau_2(t))}{1 - \tau'_2(t)} \\
+ \int_t^\infty e^{\int_0^t h_2(s)ds} \rho_2(u) y(u - \tau_2(u)) \, du \\
- \int_t^\infty e^{\int_0^t h_2(s)ds} G_2(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du \\
- \int_t^\infty e^{\int_0^t h_2(s)ds} F_2(u, x(s), y(s)) \, ds \, du.
\]

(3.17)

We will show that the mapping \( F \) has a fixed point in \( \Omega^*_{x,y} \). To apply Lemma 2.1 we need to construct two mappings, one is a contraction and the other is continuous compact. Therefore, we state (3.12) as

\[
F_1(x, y)(t) = B_3(x, y)(t) + A_3(x, y)(t),
\]

where \( B_3, A_3 : \Omega^*_{x,y} \rightarrow P_T^* \) are given by

\[
B_3(x, y)(t) = \rho_1 e^{\int_0^t h_1(s)ds} + \frac{c_1(t) x(t - \tau_1(t))}{1 - \tau'_1(t)},
\]

and

\[
A_3(x, y)(t) = \int_t^\infty e^{\int_0^t h_1(s)ds} \rho_1(u) x(u - \tau_1(u)) \, du \\
- \int_t^\infty e^{\int_0^t h_1(s)ds} G_1(u, x(u), y(u), x(u - \tau_1(u)), y(u - \tau_2(u))) \, du \\
- \int_t^\infty e^{\int_0^t h_1(s)ds} \int_{-\infty}^u a_1(u, s) f_1(x(s), y(s)) \, ds \, du.
\]

And we state (3.13) as

\[
F_2(x, y)(t) = B_4(x, y)(t) + A_4(x, y)(t),
\]

where \( B_4, A_4 : \Omega^*_{x,y} \rightarrow P_T^* \) are given by

\[
B_4(x, y)(t) = \rho_2 e^{\int_0^t h_2(s)ds} + \frac{c_2(t) y(t - \tau_2(t))}{1 - \tau'_2(t)},
\]
and

\[ A_4 (x, y) (t) = \int_t^\infty e^{\int_u^t b_2(s)ds} e^{\int_u^t h_2(s)ds} r_2 (u) y (u - \tau_2 (u)) du \]

\[ - \int_t^\infty e^{\int_u^t b_2(s)ds} G_2 (u, x(u), y(u), x(u - \tau_1 (u)), y(u - \tau_2 (u))) du \]

\[ - \int_t^\infty e^{\int_u^t b_2(s)ds} \int_{-\infty}^u a_2 (u, s) f_2 (x (s), y (s)) ds du. \]

Now for \((x, y) \in \Omega^*_x, y\) we can define the operators \(B^*, A^* : \Omega^*_x, y \to P^*_1\) by

\[ B^* (x, y) (t) = (B_3 (x, y) (t), B_4 (x, y) (t)), \]

\[ A^* (x, y) (t) = (A_3 (x, y) (t), A_4 (x, y) (t)). \]

Set \(V^* = \max \{ V_1^*, V_2^* \}\), where

\[ V_1^* = \frac{K_1 M_5^* m_1^{-1} + M_1^* M_1 V_1 m_1^{-1} + \rho_1 M_1^*}{1 - \mu_1 - M_1^* M_2^* m_1}, \]

\[ V_2^* = \frac{K_2 M_5^* m_2^{-1} + M_2^* M_2 V_2 m_1^{-1} + \rho_2 M_2^*}{1 - \mu_2 - M_2^* M_2^* m_2}. \]

We note that \(V^*\) is well defined due to (3.10) and (3.11). First, we demonstrate that \(F \left( \Omega^*_x, y \right) \subseteq \Omega^*_x, y\). If \((x, y) \in \Omega^*_x, y\), then by (3.10) we have

\[ \left| F_1 (x, y) (t) - \rho_1 e^{\int_0^t h_1(s)ds} \right| \]

\[ \leq \left| \frac{c_1 (t)}{1 - \tau_1 (t)} \right| |x (t - \tau_1 (t))| + \int_t^\infty e^{\int_u^t h_1(s)ds} |r_1 (u)| |x (u - \tau_1 (u))| du \]

\[ + \int_t^\infty e^{\int_u^t h_1(s)ds} |G_1 (u, x(u), y(u), x(u - \tau_1 (u)), y(u - \tau_2 (u)))| du \]

\[ + \int_t^\infty e^{\int_u^t h_1(s)ds} \int_{-\infty}^u |a_1 (u, s)| |f_1 (x (s), y (s))| ds du, \]

\[ \leq \mu_1 V^* + M_1^* M_5^* m_1^{-1} V^* + K_1 M_5^* m_1^{-1} + M_1^* M_1 V_1 m_1^{-1}, \]

and in a similar way we have

\[ \left| F_2 (x, y) (t) - \rho_2 e^{\int_0^t h_2(s)ds} \right| \leq \mu_2 V^* + M_2^* M_5^* m_2^{-1} V^* + K_2 M_5^* m_2^{-1} + M_2^* M_2 V_2 m_2^{-1}. \]
This implies that

\[ |F_1 (x, y) (t)| \leq \mu_1^* V^* + M_1^* M_1^* m_1^* V^* + K_1 M_1^* m_1^* + M_1^* V_1 m_1^* + \rho_1 M_1^* \leq V^*, \]

and

\[ |F_2 (x, y) (t)| \leq \mu_2^* V^* + M_2^* M_2^* m_2^* V^* + K_2 M_2^* m_2^* + M_2^* V_2 m_2^* + \rho_2 M_2^* \leq V^*. \]

Hence, \( F (\Omega_{x,y}^*) \subseteq \Omega_{x,y}^* \) as desired. The work to show that \( A^* \) is completely continuous and \( B^* \) is a contraction is similar to the corresponding work in Theorem 2.2 and hence we omit it here. Therefore, by Krasnoselskii's fixed point theorem, there exists a fixed point \((x, y) \in \Omega_{x,y}^* \) such that

\[ F (x, y) (t) = (F_1 (x, y) (t), F_2 (x, y) (t)) = (x (t), y (t)). \]

By Theorem 3.1 we know that this fixed point is a solution of (1.1).

For an arbitrary fixed point \((x, y) \in \Omega_{x,y}^* \) of \( F \), we obtain from (3.4)-(3.7)

\[ \lim_{t \to \infty} |x (t) - x_1 (t)| = \lim_{t \to \infty} |F_1 (x, y) (t) - x_1 (t)| = 0, \]

and

\[ \lim_{t \to \infty} |y (t) - y_1 (t)| = \lim_{t \to \infty} |F_2 (x, y) (t) - y_1 (t)| = 0. \]

By letting

\[
x_2 (t) = \frac{c_1 (t) x (t - \tau_1 (t))}{1 - \tau_1^* (t)} + \int_{t}^{\infty} \frac{e^h_0 h_1 (s) ds}{\int_{t}^{\infty} e^h_0 h_1 (s) ds} t_1 (u) x (u - \tau_1 (u)) du
\]

\[
- \int_{t}^{\infty} \frac{e^h_0 h_1 (s) ds}{\int_{t}^{\infty} e^h_0 h_1 (s) ds} G_1 (u, x (u), y (u), x (u - \tau_1 (u)), y (u - \tau_2 (u))) du
\]

\[
- \int_{t}^{\infty} \frac{e^h_0 h_1 (s) ds}{\int_{t}^{\infty} e^h_0 h_1 (s) ds} \int_{-\infty}^{u} a_1 (u, s) f_1 (x (s), y (s)) ds du,
\]

and

\[
y_2 (t) = \frac{c_2 (t) y (t - \tau_2 (t))}{1 - \tau_2^* (t)} + \int_{t}^{\infty} \frac{e^h_0 h_2 (s) ds}{\int_{t}^{\infty} e^h_0 h_2 (s) ds} t_2 (u) y (u - \tau_2 (u)) du
\]

\[
- \int_{t}^{\infty} \frac{e^h_0 h_2 (s) ds}{\int_{t}^{\infty} e^h_0 h_2 (s) ds} G_2 (u, x (u), y (u), x (u - \tau_1 (u)), y (u - \tau_2 (u))) du
\]
\[- \int_{t}^{\infty} \frac{e_{f_{1}}^{t} h_{2}(s)}{e_{f_{0}}^{t} h_{2}(s)} ds \int_{-\infty}^{u} a_{2}(u, s) f_{2}(x(s), y(s)) ds du.\]

We see that \((x, y)\) given by
\[
x(t) = x_{1}(t) + x_{2}(t), \quad y(t) = y_{1}(t) + y_{2}(t),
\]
is an asymptotically \(T\)-periodic solution of (1.1). Note that by (3.4)-(3.7)
\[
\lim_{t \to \infty} \left| x_{2}(t) \right| \leq V^{*} \lim_{t \to \infty} \left| \frac{c_{1}(t)}{1 - \tau_{1}(t)} \right| + V^{*} \lim_{t \to \infty} \int_{t}^{\infty} \frac{e_{f_{0}}^{t} h_{1}(s)}{e_{f_{0}}^{t} h_{1}(s)} |r_{1}(u)| du
\]
\[
+ K_{1} \lim_{t \to \infty} \int_{t}^{\infty} \frac{e_{f_{0}}^{t} h_{1}(s)}{e_{f_{0}}^{t} h_{1}(s)} d(u)
\]
\[
+ M_{1} \lim_{t \to \infty} \int_{t}^{\infty} \frac{e_{f_{0}}^{t} h_{1}(s)}{e_{f_{0}}^{t} h_{1}(s)} \int_{-\infty}^{u} |a_{1}(u, s)| ds du,
\]
Hence
\[
\lim_{t \to \infty} x_{2}(t) = 0.
\]
Similarly
\[
\lim_{t \to \infty} y_{2}(t) = 0.
\]
Finally, we show that \(x_{1}\) and \(y_{1}\) are \(T\)-periodic. From (3.1), one can see
\[
x_{1}(t + T) = c_{1} e_{f_{0}}^{t+T} h_{1}(s) ds
\]
\[
= c_{1} e_{f_{0}}^{t} h_{1}(s) ds e_{f_{1}}^{t+T} h_{1}(s) ds
\]
\[
= c_{1} e_{f_{0}}^{t} h_{1}(s) ds e_{f_{0}}^{t} h_{1}(s) ds
\]
\[
= c_{1} e_{f_{0}}^{t} h_{1}(s) ds
\]
\[
= x_{1}(t).
\]
Similarly, \(y_{1}\) is \(T\)-periodic. \(\Box\)

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References


