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THE ABSENCE OF ARBITRAGE PROPERTY IN MIXED FRACTIONAL BROWNIAN MOTION SETTING

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Abstract

In this paper, we investigate the condition full support property (CFS) for the stochastic process $S_t = R_t + \int_0^t \phi_s dM_s^H$, where M_s^H is a mixed fractional Brownian motion, R_t a continuous adapted process, and (ϕ_t) an elementary predictable process. The problem of the absence of arbitrage for this process is treated.

1. Introduction

The conditional full support(CFS) property which is a mere condition on asset prices indicating that at any time the asset price path can continue arbitrarily close to any given path with positive conditional probability, was introduced by Guasoni et al. (2008, [7]) for the fractional Brownian motion, the only H-self similar Gaussian precess with stationary increments with the arbitrary Hurst parameter $H \in (0, 1)$. This result was generalized in Cherny (2008,[3]), who proved that any Brownian moving average satisfies the (CFS) property with respect to ordinary Brownian motion. In 2009, Pakkanan [11] established the condition of the desired property for a general class of stochastic processes. After that, this notion was investigated for

Gaussian processes with stationary increments by Gasbarra (2011,[6]).

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In contrast to the considerable monographs that deal with fractional Brownian motion at a great expense, we find and see a rarity in those on the mixed fractional Brownian motion (MFBM) introduced by Patrick Cheridito (2001,[12]) and defined as a linear combination between the Brownian motion and the independent fractional Brownian motion of parameter $H \in (0,1)$. This process (MFBM) is not a semimartingale if $H \in (0, \frac{1}{2}) \bigcup (\frac{1}{2}, \frac{3}{4}]$, and it is not also a Markov process.

The purpose of this work is to study the properties and characteristics of the conditional full support property for asset prices driven by the Mixed fractional Brownian motion process.

The plan of this paper is organized as follows. In section 2, we collect some preliminaries of the consistent price system and the conditional full support needed to establish our main result. In section 3, we give some useful basic concepts on the mixed fractional Brownian motion. In section 4, we supply our main result on the conditional full support for the stochastic process $S_t = R_t + \int_0^t \phi_s dM_s$, where M_s^H is a Mixed fractional Brownian motion, and we deal with the absence of arbitrage opportunities.

2. Preliminaries

Definition 1. A Wiener process B_t (standard Brownian Motion) is a stochastic process with the following properties:

1. $B_0 = 0$.
2. Non-overlapping increments are independent: $\forall 0 \leq t < T \leq s < S$, the increments $B_T - B_t$ and $B_S - B_s$ are independent random variables.
3. $\forall\ 0 \leq t < s$ the increment $B_s - B_t$ is a normal random variable, with zero mean and variance $s - t$.
4. $w \in \Omega$, the path $t \to B_t(w)$ is a continuous function.

Definition 2. For $H \in (0,1)$, a fractional Brownian motion of Hurst parameter H is a centered and continuous Gaussian process, denoted by $(B_t^H)_{t\geq 0}$ with the covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t-s \mid^{2H}).$$

Remark 1. Trivially, when $H = \frac{1}{2}$, the fBm is the standard Brownian motion.

2.1. Basic properties of fractional Brownian motion

According to [2], the fractional Brownian motion, B_t^H, of Hurst parameter $H \in (0,1)$ has the following properties:

- Selfsimilarity: for all $a > 0$, $(B_{at}^H) \stackrel{d}{=} (a^H B_t^H)$.
- Stationary increments: for all $h > 0$, $(B_{t+h}^H - B_h^H) \stackrel{d}{=} B_t^H$.
- Holder Continuity: For $H \in (0,1)$, the sample paths of fBm are a.s. Hölder continuous of an order strictly less than H.
- Differentiability: For $H \in (0,1)$, the fBm sample path B^H is not differentiable. Indeed, for every $t_0 \in [0,\infty]$,

$$\mathbb{P}\left(\lim_{t\to t_0}\sup\left|\frac{B_t^H - B_{t_0}^H}{t-t_0}\right| = \infty\right) = 1.$$

- Bounded variation: The fBm is of an unbounded variation, i.e.

$$\sup_{t_i}\sum_i \mid B_{t_{i+1}}^H - B_{t_i}^H \mid = \infty.$$

- The fBm is not a semimartingale for $H \neq \frac{1}{2}$: the fact that the fBm is not a semimartingale implies that we are not able to integrate with respect to it, as we usually do in the classical stochastic calculus. Effectively, the most general class of integrators are semimartingales.
- Long-range dependence: The fractional Brownian motion B_t^H is long-range dependent for $H \in (\frac{1}{2}, 1)$.

2.2. Representation of fBm on a finite interval

There are some representations of the fBm as a Wiener integral defined on an interval, e.g. commonly taken as $[0,T]$:

For the representation on the real line, for a one-sided fBm $(B_t^H)_{0\leq t\leq T}$, we have a general formula

$$B_t^H = \int_0^t K_H(t,s)dB_s, \qquad t \in [0,T],$$

where $(B_t)_{0\leq t\leq T}$ is a one-sided standard Brownian motion.

Let B^H be a fractional Brownian motion with a parameter $H \in (0,1)$. According to [8], the fBm admits a representation as a Wiener integral of the form

$$B^H = \int_0^t K_H(t,s)dB_s, \text{ Levy-Hida representation},$$

where $B = (B_t)_{t\in T}$ is a Wiener process, and $K_H(t,s)$ is the kernel

$$K_H(t,s) = d^H(t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} F_1\left(\frac{t}{s}\right),$$

d_H being a constant and

$$F_1(z) = d_H\left(\frac{1}{2} - H\right) \int_0^{z-1} \theta^{H-\frac{3}{2}}(1-(\theta+1)^{H-\frac{1}{2}})d\theta.$$

If $H > \frac{1}{2}$, the kernel K_H has the simpler expression

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $t > s$ and $c_H = (\frac{H(H-1)}{\beta(2-2H, H-\frac{1}{2})})^{\frac{1}{2}}$. As the process B^H is a fBm, then

$$\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = R_H(t,s).$$

Further, the kernel K_H satisfies the condition:

$$\frac{\partial K_H}{\partial t}(t,s) = d_H\left(H - \frac{1}{2}\right)(\frac{s}{t})^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

2.3. Representations of the fBm on $\mathbb{R}$

According to [8], we present some representations of the fBm as a Wiener integral (i.e. w.r.t Brownian motion):

$$B_t^H = C\int_{\mathbb{R}} K_H(t,u)dB_u,$$

where C is a standardized constant.

- Moving average representation

  The FBm can be represented as an integral with respect to a standard Brownian motion on the whole real line. Let $(B_s)_{s\in\mathbb{R}}$ be a standard Brownian motion. Then

$$B_t^H = \frac{1}{C(H)}\int_{\mathbb{R}}[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}]dB_s,$$

  where $C(H) > 0$ an explicit normalizing constant.

- Harmonizable representation

  This is another representation which uses the complex-valued Brownian motion (but the fBm is real-valued). In fact, for a fBm $(B_t^H)_{t\in\mathbb{R}}$, we have

$$B_t^H = \frac{1}{C_1(H)}\int_{\mathbb{R}}\frac{e^{itx}-1}{ix} \mid x\mid^{-(H-\frac{1}{2})} d\tilde{B}_x, \;\; t\in\mathbb{R},$$

  where $(\tilde{B}_t)_{t\in\mathbb{R}}$ is a complex Brownian measure and

$$C_1(H) = \left(\frac{\Pi}{H\Gamma(2H)\sin(H\Pi)}\right)^{1/2}.$$

  Let us note that the complex Brownian measure on $\mathbb{R}$ can be splitted as $\tilde{B} = B_1 + iB_2$ and is such that $B_1(A) = B_1(-A), B_2(A) = -B_2(-A)$ and $\mathbb{E}(B_1(A))^2 = \frac{|A|}{2}, \forall A \in \mathcal{B}(\mathbb{R})$. We also call this representation, the spectral representation.

3. Condition Full Support and Stochastic Integral

Definition 3. Let $\mathbf{F}$ be a separable metric space and $\mu : \mathfrak{B}(F) \to [0,1]$ be a Borel probability measure; the support of μ, noted by $supp(\mu)$, is the unique minimal closed set $A \subset \mathbf{F}$ such that $\mu(A) = 1$.

Let $(X_t)_{t\in[0,T]}$ be a continuous stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be its natural filtration.

Moreover, let $C_x([u,v], I)$ be the space of function $f \in C([u,v], \mathbf{I})$ such that $f(u) = x \in \mathbf{I}$.

As usual, we equip the spaces $C([u,v], \mathbf{I})$ and $C_x([u,v], \mathbf{I})$, $x \in \mathbf{I}$, with the uniform topologies.

Definition 4. [10] We say that the process X has Conditional Full Support (CFS) with respect to the filtration $\mathbb{F}$, or briefly $\mathbb{F}$-CFS, if

(1) X is adapted to $\mathbb{F}$.

(2) For all $t \in [0,T]$ and $\mathbf{P}$-almost all $\omega \in \Omega$,

$$supp(law[(X_u)_{u\in[t,T]} \mid \mathcal{F}_t](\omega)) = C_{X_t(\omega)}([t,T], \mathbf{I}).$$

Definition 5 ([10])**.** Let $\varepsilon > 0$. An ε-consistent price system to X is a pair $(\tilde{X}, \mathbf{Q})$, where $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ and $\tilde{X}$ is a $\mathbf{Q}$-martingale in the filtration $\mathcal{F}$, such that $\frac{1}{1+\varepsilon} \leq \frac{\tilde{X}_i(t)}{X_i(t)} \leq 1+\varepsilon$, almost surely for all $t \in [0,T]$ and $i = 1, ..., n$.

The result on absence of arbitration is presented in the following Theorem:

Theorem 1 ([7])**.** *Let X_t be an $\mathbb{R}^d_+$-valued, continuous adapted process satisfying CFS. Then X admits an ε-consistent pricing system for all $\varepsilon > 0$.*

Next, we present some results about CFS for the Brownian motion:

$$Z_t := R_t + \int_0^t \phi_s dB_s, \qquad t \in [0,T],$$

where R is a continuous process, the integrator B is a Brownian motion, and the integrand ϕ satisfies some varying assumptions. We take into consideration two cases:

1. Independent integrands and Brownian integrators.

Theorem 2 ([7])**.** *Let us define*

$$Z_t := R_t + \int_0^t \phi_s dB_s, \qquad t \in [0,T].$$

Suppose that

(1) $(R_t)_{t\in[0,T]}$ *is a continuous process,*

(2) $(\phi_t)_{t\in[0,T]}$ *is a measurable process s.t.* $\int_0^T \phi_s^2 ds < \infty$ *a.s,*

(3) $(B_t)_{t\in[0,T]}$ *is a standard Brownian motion independent of* R *and* ϕ.

If we have,

$$meas(t \in [0,T] : \phi_t = 0) = 0 \qquad \mathbf{P} - a.s, (meas : Lebesgue \quad measure),$$

then Z *has CFS.*

2. Progressive integrands and Brownian integrators.

Let's note that the assumption about independence between B and (R, ϕ) cannot be dispensed with, in general, without imposing additional conditions. Namely, if,

$$R_t = 1; \phi_t := e^{B_t - \frac{1}{2}t}; t \in [0,T],$$

then $Z = \phi = \xi(B)$, the Doleans exponential of B, which is strictly positive, does not have CFS, if the process is considered in $\mathbb{R}$.

Theorem 3 ([7]). *Suppose that*

(1) $(X_t)_{t\in[0,T]}$ *is a continuous process,*

(2) R *and* ϕ *are progressive* $[0,T] \times C([0,T])^2 \to \mathbb{R}$,

(3) ε *is a random variable,*

(4) *and* $\mathcal{F}_t = \sigma\varepsilon, X_s, B_s : s \in [0,t], t \in [0,T]$.

If W *is an* $\mathcal{F}_t \in [0,T]$*-Brownian motion and*

(1) $\mathbb{E}[e^{\lambda} \int_0^T \phi_s^{-2} ds] < \infty$ *for all* $\lambda > 0$,

(2) $\mathbb{E}[e^{2} \int_0^T \phi_s^{-2} h_s^2 ds] < \infty$, *and*

(3) $\int_0^T \phi_s^2 ds \leq \bar{K}$ *a.s for some constant* $\bar{K} \in (0, \infty)$,

then the process

$$Z_t = \varepsilon + \int_0^t R_s ds + \int_0^t \phi_s dB_s, \qquad t \in [0,T]$$

has CFS.

4. Mixed fractional Brownian motion

The mixed fractional Brownian motion of parameter H is a stochastic process that has been introduced by [12], used in mathematical finance, in the modelling of some arbitrage-free and complete markets. In this part, we present some stochastic properties of this process.

It is well-known that the fractional Brownian motion of Hurst parameter $H \in]0;1[$ is a centered Gaussian process $B^H = B_t^H, t \geq 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t - s \mid^{2H}).$$

If $H = 1/2$, B^H is the ordinary Brownian motion denoted by $B = B_t, t \geq 0$.

Let us take a and b as two real constants such that $(a, b) \neq (0, 0)$.

Definition 6. A mixed fractional Brownian motion (MFBM) of parameters a, b, and H is a process $M^H = M_t^H(a, b); t \geq 0 = M_t^H; t \geq 0$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+ \quad M_t^H = M_t^H(a, b) = aB_t + bB_t^H,$$

where the two processes $(B_t^H)_{t\in\mathbb{R}_+}$ and $(B_t)_{t\in\mathbb{R}_+}$ are independent.

According to [10], the mixed fractional Brownian motion $(M_t^H(a, b))_{t\in\mathbb{R}_+}$ has the following properties:

- M^H is a centered Gaussian process.
- For all $t \in \mathbb{R}_+$, $\mathbb{E}((M_t^H(a, b))^2) = a^2 t + b^2 t^{2H}$.
- $\forall s \in \mathbb{R}_+$, $\forall t \in \mathbb{R}_+$,

$$Cov(M_t^H(a, b), M_s^H(a, b)) = a^2(t \wedge s) + \frac{1}{2}[b^2(t^{2H} + s^{2H} - \mid t - s \mid^{2H})],$$

  where $t \wedge s = 1/2(t + s - \mid t - s \mid)$.
- The increments of the MFBM are stationary.
- For all $H \in]0;1[\backslash\{1/2\}, a \in \mathbb{R}$ and $b \in \mathbb{R}\backslash\{0\}$, $(M_t^H(a, b))_{t\in\mathbb{R}_+}$ is not a Markovian process.

- For any $h>0, M_{ht}^H(a,b) \triangleq M_t^H(ah^{1/2}, bh^H)$. This property is called the mixed-self-similarity.
- For all $a\in\mathbb{R}$ and $b\in\mathbb{R}\backslash\{0\}$, the increments of $(M_t^H(a,b))_{t\in\mathbb{R}_+}$ are long-range dependent if and only if $H>1/2$.
- $\forall\ s\in\mathbb{R}_+, \forall t\in\mathbb{R}_+, \forall h\in\mathbb{R}_+, 0<h\le t-s$, the correlation coefficient $\rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H)$ is given as

$$\begin{aligned}&\rho(M_{t+h}^H - M_t^H, M_{s+h}^H - M_s^H)\\&\quad=\frac{b^2}{2(a^2h+b^2h^{2H})}\big[(t-s+h)^{2H}-2(t-s)^{2H}+(t-s-h)^{2H}\big].\end{aligned}$$

  For all $a\in\mathbb{R}$ and $b\in\mathbb{R}\backslash\{0\}$, the increments of $(M_t^H(a,b))_{t\in\mathbb{R}_+}$ are positively correlated if $1/2<H<1$, uncorrelated if $H=1/2$, and negatively correlated if $0<H<1/2$.

- For all $T>0$ and $\gamma<1/2\wedge H$, the MFBM has a modification which sample paths have a Hölder-continuity, with order γ, on the interval $[0,T]$.
- For all $\alpha\in]0,1/2\wedge H[$, the sample paths of the MFBM are almost surely α-differentiable at every $t_0\ge 0$, and

$$\forall t_0\ge 0,\quad \mathbb{P}\{d^\alpha M_{t_0}^H=0\}=1.$$

- For all $\alpha\in]1/2\wedge H;1[$, the sample paths of the MFBM are nowhere α-differentiable, almost surely.

5. Main Result

The main goal of this work is to present conditions that imply the conditional full support (CFS) property, introduced by [7], for the stochastic process:

$$S_t = R_t + \int_0^t \phi_s dM_s^H,$$

where M_s^H is a mixed fractional Brownian motion.

Our main result is given in the following Theorem.

Theorem 4. *Let us consider the process*

$$S_t = R_t + \int_0^t \phi_s dM_s^H,$$

where

- $(R_t)_{t\in[0,T]}$ *is a continuous adapted process,*
- $(\phi_t)_{t\in[0,T]}$ *is an elementary predictable s.t.* $\int_0^T \phi_s^2 ds < \infty$ *a.s.,*
- $(M_t^H)_{t\in[0,T]}$ *is a mixed fractional Brownian motion independent of* R *and* ϕ.

If we have

$$meas(t \in [0,T] : \phi_t = 0) = 0, \qquad P - a.s.(meas : Lebesgue\ measure),$$

then S *has CFS.*

Proof. As the mixed fractional Brownian motion is a Gaussian process, then for the proof of our theorem, we can follow the same steps of Proposition 4.2 given in [7].

Let

$$S(t) = \int_0^t \phi_s dM_s^H.$$

It is sufficient to prove that the conditional law $P(S_{|_{[v,T]}} \mid \mathcal{F}_v)$ has full support on $C_{S_v}([v,\mu],\mathbb{R})$ almost surely. Then, it will be enough to show this property on an interval, where ϕ is constant with respect to time (and thus continuous). Therefore, we can take T small enough such that ϕ has the form $\phi(t) = \zeta$ on $[v,T]$, where $\zeta \neq 0$ and it is $\mathcal{F}_v$-measurable.

In our paper, we have $(M_t^H)_{t\in[0,T]}$ is a mixed fractional Brownian motion, then

$$M_t^H = M_t^H(a,b) = aB_t + bB_t^H.$$

Without loss of generality, we can study the property of CFS for the mfBm for $a = b = 1$; we have

$$S(t) = \int_0^t \phi_s dM_s^H$$

$$= \int_0^t \phi_s d[B_s^H + B_s] \quad (B_s^H \textit{ and } B_s \textit{ are independent})$$
$$= \int_0^t \phi_s dB_s^H + \int_0^t \phi_s dB_s.$$

Remark 2. The independence does not concern the components of the Gaussian process S^H but it concerns the couple (B^H, B).

Next, we have to prove that

$$S(t) = \int_v^t \phi_s K_H(t,s) dB_s + \int_v^t \phi_s dB_s$$

has full support on $C_0([v,T],\mathbb{R})$.

Theorem 3 in [5] states that the topological support of a continuous Gaussian process is equal to the norm closure of its reproducing kernel Hilbert space.

In our case, the support of $S(t)$ is

$$\mathbb{H} := \Big\{ f \in C_0([v,T],\mathbb{R}) : f(t) = \int_v^t \phi(s) K_H(t,s) g(s) ds + \int_v^t \phi(s) g(s) ds, \\ \text{for some } \mathrm{g} \in \mathrm{L}^2[\mathrm{v}, \mathrm{T}] \Big\}.$$

Thus, it is sufficient to show that $\mathbb{H}$ is norm-dense in $C_0([v,T],\mathbb{R})$.

To ensure this condition, we define the Liouville fractional integral operator for any $f \in L^1[a,b]$ and $\alpha > 0$,

$$(I_{a^-}^{\alpha} g)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t g(s)(t-s)^{\alpha-1} ds, \qquad a \le t \le b,$$

and represent the kernel operator K_H for the fractional Brownian motion:

$$(K_H g)(t) := \int_0^t K_H(t,s) g(s) ds, \qquad t \in [0,T].$$

We can treat this problem following these two cases:

1. Case $H < \frac{1}{2}$. Let

$$\big(K_H(g\phi)\big)(t) := \zeta \int_v^t K_H(t,s)g(s)ds, \qquad g \in L^2[0,T],\ t \in [0,T].$$

Using Theorem 2.1 in [5], we have

$$\big(K_H(g\phi)\big) = I_{0^+}^{2H}\Big(s^{\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H}\big(s^{H-\frac{1}{2}}(g\phi)(s)\big)\Big).$$

For any v, the argument needs to be split into two steps:

Step 1. We rely on the following Lemma:

Lemma 1 ([7]). *If $g \in C_0[v,T]$, then $L_1 g \in C_0([v,T]$, where*

$$(L_1 g)(t) = (I_{0^+}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}g(s)))(t).$$

Moreover, $L_1 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has a dense range (with respect to the uniform norm).

As we have $\varphi \in \mathcal{C}_0[v,T]$, then $L_1\varphi\phi \in \mathcal{C}_0([v,T])$, where

$$(L_1\varphi\phi)(t) = (I_{0^+}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}(g\phi)(s)))(t).$$

Let as recall the identity for $a,b > 0$,

$$\int_0^t (t-u)^{a-1}u^{b-1}du = \mathcal{C}(a,b)t^{a+b-1},$$

where $\mathcal{C}(a,b) \neq 0$ is a constant.

And define, for a fixed $\alpha > 0$:

$$\varphi(s) := \frac{(s-v)^\alpha}{\zeta s^{H-\frac{1}{2}}}.$$

Then, we obtain, for $t \in [v,T]$

$$(L_1\varphi\phi)(t) = \frac{\zeta}{\Gamma(\frac{1}{2}-H)}\int_v^t (t-s)^{-H-\frac{1}{2}}\varphi(s)s^{H-\frac{1}{2}}ds + \int_v^t \zeta\varphi(s)ds$$

$$= \frac{1}{\Gamma(\frac{1}{2}-H)} \int_v^t (t-s)^{-H-\frac{1}{2}} (s-v)^{\alpha} ds + \int_v^t (s-v)^{\alpha} ds$$
$$\leq \int_0^{t-v} u^{\alpha} (t-v-u)^{-H-\frac{1}{2}} du + \int_0^{t-v} u^{\alpha} du$$
$$= \mathcal{C}(\frac{1}{2}-H, \alpha+1)(t-v)^{\alpha-H+\frac{1}{2}} + \mathcal{C}(1,\alpha+1)(t-v)^{\alpha+1}.$$

By varying α, we find that $(t-v)^n \in \mathrm{Im}(L_1)$ for $n \geq 1$ and the Stone-Weierstrass theorem guarantees that $\mathrm{Im}(L_1)$ is dense in $\mathcal{C}_0[v,T]$.

Step 2. For this step, the following lemma is needed :

Lemma 2 ([10])**.** *If $g \in C_0[v,T]$, then $L_2 g \in C_0([v,T]$, where*

$$(L_2 g)(t) = (I_{0^+}^{2H}(s^{\frac{1}{2}-H} g(s)))(t),$$

and $L_2 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has a dense range.

Since the restriction of K_H to $C_0[v,T]$ is exactly $L_2 \circ L_1$, we may conclude that $K_H : C_0[v,T] \longrightarrow C_0[v,T]$ has a dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v,T]$.

2. Case $H \geq \frac{1}{2}$.

Following [5], we have the following representation:

$$K_H(g\phi) = I_{0^+}^{1}(s^{H-\frac{1}{2}} I_{0^+}^{H-\frac{1}{2}}(s^{\frac{1}{2}-H}(g\phi))).$$

Then, for the proof, two steps are considered

Step 1. We have $g \in C_0[v,T]$, then $L_3 g\phi \in C_0([v,T]$, where

$$(L_3(g\phi))(t) = \zeta(I_{0^+}^{1}(s^{H-\frac{1}{2}} g(s)))(t).$$

Moreover, $L_1 : C_0[v,T] \longrightarrow C_0[v,T]$ is continuous and has a dense range (with respect to the uniform norm), where $\mathcal{C}(a,b) \neq 0$ is a constant.

Define, for a fixed $\alpha > 0$,

$$g(s) := \frac{(s-v)^{\alpha}}{\zeta s^{\frac{1}{2}-H}},$$

Then, we obtain, for $t \in [v, T]$,

$$
\begin{aligned}
(L_3 g\phi) &= \frac{\zeta}{\Gamma(H-\frac{1}{2})} \int_v^t (t-s)^{H-\frac{3}{2}} g(s) s^{\frac{1}{2}-H} ds + \int_v^t \zeta g(s) ds \\
&= \frac{1}{\Gamma(H-\frac{1}{2})} \int_v^t (t-s)^{H-\frac{3}{2}} (s-v)^{\alpha} ds + \int_v^t (s-v)^{\alpha} ds \\
&\leq \int_0^{t-v} u^{\alpha} (t-v-u)^{H-\frac{3}{2}} du + \int_0^{t-v} u^{\alpha} du \\
&= \mathcal{C}(H-\frac{1}{2}, \alpha+1)(t-v)^{\alpha+H-\frac{1}{2}} + \mathcal{C}(1, \alpha+1)(t-v)^{\alpha+1}.
\end{aligned}
$$

By varying α, we find that $(t-v)^n \in \mathrm{Im}(L_3)$, for $n \geq 1$, and the Stone-Weierstrass theorem guarantees that $\mathrm{Im}(L_3)$ is dense in $\mathcal{C}_0[v, T]$.

Step 2. We have $g \in C_0[v, T]$, then $L_4 g\phi \in C_0([v, T]$, where

$$
(L_4(g\phi)(t) = \zeta(I^1_{0^+}(s^{H-\frac{1}{2}}(g\phi)(s)))(t),
$$

and $L_4 : C_0[v, T] \longrightarrow C_0[v, T]$ is continuous and has a dense range in $C_0[v, T]$.

Since the restriction of K_H to $C_0[v, T]$ is exactly $L_4 \circ L_3$, we may conclude that $K_H : C_0[v, T] \longrightarrow C_0[v, T]$ has a dense range and, a fortiori, $\mathbb{H}$ is norm-dense in $C_0[v, T]$.

Then we can check that there exists the property of CFS for the process (S_t) and there are the consistent price systems. □

6. Conclusion

This article deals the absence of the arbitration opportunity by using the conditional full support property applied to processes with irregular trajectories such as the mixed fractional Brownian motion. As a perspective, we can study this property for Lévy fractional Brownian motion and mixed fractional Brownian motion on which the processes B^H_t and B_t are dependent.

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