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ON SOLUTIONS OF HYBRID TIME FRACTIONAL HEAT PROBLEM

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Abstract

In this research, the analytic solution of hybrid fractional differential equation with non-homogenous Dirichlet boundary conditions in one dimension is established. Since non-homogenous initial boundary value problem involves hybrid fractional order derivative, it has classical initial and boundary conditions. By means of separation of variables method and the inner product defined on $L^2[0,l]$, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville. Illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

1. Introduction

Since mathematical models including fractional derivatives play a vital role fractional derivatives draw a growing attention of many researchers in various branches of sciences. Therefore there are many different fractional derivatives such as Caputo, Riemann-Liouville, Atangana-Baleanu. However these fractional derivatives do not satisfy most important properties of ordinary derivative which leads to many difficulties to analyze or obtain the solution of fractional mathematical models. As a result many scientists focus on defining new fractional derivatives to cover the setbacks of the defined ones. Moreover the success of mathematical modelling of systems or processes depends on the fractional derivative, it involves, since the correct

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choice of the fractional derivative allows us to model the real data of systems or processes accurately.

In order to the define new fractional derivatives, various methods exists and these ones are classified based on their features and formation such as non-local fractional derivatives and local fractional derivatives. The constant proportional Caputo hybrid operator (CPCHO) is a newly defined fractional derivative which is a combination of the Caputo derivative and the proportional derivative and is defined as:

$$\begin{aligned} {}_0^{CPC}D_t^{\alpha} f(t) =& \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(K_1(\alpha) f(\tau) + K_0(\alpha) f'(\tau)\big) (t-\tau)^{-\alpha} d\tau \\ =& K_1(\alpha) {}_0^{RL}I_t^{1-\alpha} f(t) + K_0(\alpha) {}_0^{C}D_t^{\alpha} f(t). \end{aligned} \tag{1}$$

where the functions K_0 and K_1 satisfy certain properties in terms of limit [2]. The domain of this operator contains functions f on positive reals such that f and its derivative f' are locally L^1 functions. Moreover ${}_0^{RL}I_t^{\alpha}$ and ${}_0^{C}D_t^{\alpha}$ represent RiemannLiouville integral and Caputo derivative, respectively.Note that this hybrid fractional operator can be enounced as a linear combination of the Caputo fractional derivative and the RiemannLiouville fractional integral.

In this study, we focus on obtaining the solution of following fractional diffusion equation including various CPCHO by making use of the separation of variables method (SVM):

$$ {}_0^{CPC}D_t^{\alpha} u(x,t) = \gamma^2 u_{xx}(x,t), \tag{2}$$

$$ u(0,t) = u_0, u(l,t) = u_1, \tag{3}$$

$$ u(x,0) = f(x) \tag{4}$$

where $0 < \alpha < 1, 0 \leq x \leq l, 0 \leq t \leq T, \gamma \in \mathbb{R}$, u_0 and u_1 are constants. Here we use the following forms of the proportional derivatives:

$$ {}_0^{CPC}{D_\alpha}^1 f(t) = (1-\alpha)\ {}_0^{RL}I_t^{1-\alpha} f(t) + \alpha {}_0^{C}D_t^{\alpha} f(t), \tag{5}$$

$$ {}_0^{CPC}{D_\alpha}^2 f(t) = \left(1-\alpha^2\right)\ {}_0^{RL}I_t^{1-\alpha} f(t) + \alpha^2 {}_0^{C}D_t^{\alpha} f(t). \tag{6}$$

From a physical aspect, the intrinsic nature of the physical system can be reflected to the mathematical model of the system by using fractional derivatives. Therefore the solution of the fractional mathematical model is in excellent agreement with the predictions and experimental measurement of it. The systems whose behaviour is non-local can be modelled better by fractional mathematical models and the degree of its non-locality can be arranged by the order of fractional derivative. In order to analyze the diffusion in a non-homogenous medium that has memory effects it is better to analyze the solution of the fractional mathematical model for this diffusion. As a result in order to model a process, the correct choices of fractional derivative and its order must be determined.

In this study, hybrid fractional derivative is used to model diffusion problems as in the case of hybrid fractional derivative, models including hybrid fractional derivatives gives better results than models including integer order derivatives. In the mathematical modelling of diffusion problem for different matters such as liquid, gas and temperature, the suitable fractional order α is chosen, since the diffusion coefficient γ^2 depends on the order α of fractional derivative [3]. This mathematical modelling describe the behaviour of matter in a phase. There are many published work on the diffusion of various matters in science especially in fluid mechanics and gas dynamics [14, 1, 5, 6, 15, 7, 8, 9, 10, 11, 12, 13]. From this aspect, analysis of this problem plays an important role in application. Moreover sub-diffusion cases for which $0 < \alpha < 1$ are under consideration. The solution of the fractional mathematical model of sub-diffusion cases behaves much slower than the solution of the integer-order mathematical model unlike fractional mathematical model for super-diffusion.

The main goal of this study is to establish the analytic solution of following time fractional differential equations with non-homogenous Dirichlet boundary and initial condition.

$$ {}_{0}^{CPC}D_t^{\alpha} u\left(x,t\right) = \gamma^2 u_{xx}\left(x,t\right), \tag{7} $$

$$ u\left(0,t\right) = u_0, u\left(l,t\right) = u_1, \tag{8} $$

$$ u\left(x,0\right) = f(x) \tag{9} $$

where $0 < \alpha < 1, 0 \leq x \leq l, 0 \leq t \leq T, \gamma \in \mathbb{R}$, u_0 and u_1 are constants. Before investigating the solution of the problem (7)-(9), let us define the function $v\left(x,t\right)$ which homogenizes the boundary conditions (8) as follows:

$$ v\left(x,t\right) = u\left(x,t\right) + \frac{x}{l}\left(u_0 - u_1\right) - u_0. \tag{10} $$

Via (10), the problem (7)-(9) turns into the following problem (11)-(13).

$$ {}_0^{CPC}D_t^{\alpha} v(x,t) = \gamma^2 v_{xx}(x,t), \tag{11} $$

$$ v(0,t) = 0, v(l,t) = 0, \tag{12} $$

$$ v(x,0) = f(x) + \frac{x}{l}(u_0 - u_1) - u_0 \tag{13} $$

where $0 < \alpha < 1, \gamma \in \mathbb{R}, 0 \le x \le l,\ 0 \le t \le T$, u_0 and u_1 are constants.

2. Main Results

The analytic form of the solution for the problem (11)-(13) is established by employing the well known method SVM.

$$ v(x,t;\alpha) = X(x)T(t;\alpha) \tag{14} $$

where $0 \le x \le l, 0 \le t \le T$.
Utilizing (14) in (11) and some arrangement leads to the following:

$$ \frac{{}_0^{CPC}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2. \tag{15} $$

Taking the right hand side of equation (15) and related boundary conditions (12) into account the following problem is obtained:

$$ X''(x) + \lambda^2 X(x) = 0, \tag{16} $$

$$ X(0) = X(l) \tag{17} $$

which has the solution of the following form:

$$ X(x) = e^{rx}. \tag{18} $$

As a result the following characteristic equation is reached:

$$ r^2 + \lambda^2 = 0. \tag{19} $$

Case 1. If $\lambda = 0$, the solutions of the equation (19) are two coincident roots $r_1 = r_2$ which cause to the solution of the problem (16)-(17) in the following form:

$$X(x) = k_1 x + k_2, \tag{20}$$

The first boundary condition yields

$$X(0) = k_2 = 0 \tag{21}$$

which leads to the following solution

$$X(x) = k_1 x. \tag{22}$$

Similarly second boundary condition leads to

$$X(l) = k_1 l = 0 \Rightarrow k_1 = 0 \tag{23}$$

which implies that $X(x) = 0$ which implies that there is not any solution for $\lambda = 0$.

Case 2. If $\lambda > 0$, the solutions of the equation (19) are two distinct real roots r_1, r_2 which cause to the solution of the problem (16)-(17) in the following form:

$$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{24}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \tag{25}$$

which leads to the following solution

$$X(x) = c_1 \left(e^{r_1 x} - e^{r_2 x}\right). \tag{26}$$

Similarly second boundary condition leads to

$$X(l) = c_1 \left(e^{r_1 l} - e^{r_2 l}\right) = 0. \tag{27}$$

Since $e^{r_1 l} \neq e^{r_2 l}$, the equation (27) is satisfied if and only if $c_1 = 0 = c_2$ which implies that $X(x) = 0$ which implies that there is not any solution for $\lambda > 0$.

Case 3. If $\lambda < 0$, the solutions of the equation (19) are two complex roots which cause to the solution of the problem (16)-(17) in the following form:

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x). \tag{28}$$

By making use of the first boundary condition we have

$$X(0) = c_1 = 0. \tag{29}$$

Hence the solution becomes

$$X(x) = c_2 \sin(\lambda x), \tag{30}$$

Similarly last boundary condition leads to

$$X(l) = c_2 \sin(\lambda l) = 0 \tag{31}$$

which implies that

$$\sin(\lambda l) = 0 \tag{32}$$

which yields the following eigenvalues

$$\lambda_n = \frac{w_n}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \dots \tag{33}$$

where $w_n = n\pi$ satisfy the equation $sin(w_n) = 0$.
As a result the solution is obtained as follows:

$$X_n(x) = c_2 \sin\left(w_n\left(\frac{x}{l}\right)\right), n = 1, 2, 3, \dots \tag{34}$$

The second equation in (15) for eigenvalue λ_n yields the ordinary differential equation below:

$$\frac{{}_0^{CPC}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = -\gamma^2\lambda_n^2 \tag{35}$$

which yields the following solution [2]

$$T_n(t;\alpha) = E^1_{\alpha,1,\ 1}\left(\frac{-\gamma^2\lambda_n^2}{K_0(\alpha)}t^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)}t\right), n = 0, 1, 2, 3, \dots \tag{36}$$

where a bivariate Mittag-Leffler function $E^{(\gamma)}_{\alpha,\ \beta,\ \kappa}(x,y)$ proposed by Özarslan and Kürt [4], is represented in double power series as follows:

$$E^{(\gamma)}_{\alpha,\beta,\ \kappa}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\kappa s)}\ \frac{x^r}{r!}\ \frac{y^{\kappa s}}{s!},$$
$$\alpha,\beta,\ \gamma\in\mathbb{C},\ Re(\alpha),\ Re(\beta),\ Re(\kappa)>0.$$

The solution for every eigenvalue λ_n is constructed as

$$v_n(x,t;\alpha)=X_n(x)\,T_n(t;\alpha)=E^{1}_{\alpha,1,\ 1}\left(\frac{-\gamma^2\lambda_n^2}{K_0(\alpha)}t^{\alpha},\frac{-K_1(\alpha)}{K_0(\alpha)}t\right)\sin\left(w_n\left(\frac{x}{l}\right)\right),$$
$$n=0,1,2,3,\ldots \tag{37}$$

which leads to the following general solution

$$v(x,t;\alpha)=\sum_{n=0}^{\infty}A_n\sin\left(w_n\left(\frac{x}{l}\right)\right)\ E^{1}_{\alpha,1,\ 1}\left(\frac{-\gamma^2\lambda_n^2}{K_0(\alpha)}t^{\alpha},\frac{-K_1(\alpha)}{K_0(\alpha)}t\right). \tag{38}$$

Note that it satisfies boundary condition and fractional differential equation.

The coefficients of general solution are established by taking the following initial condition into account:

$$v(x,0)=f(x)+\frac{x}{l}(u_0-u_1)-u_0=\sum_{n=0}^{\infty}A_n\sin\left(w_n\left(\frac{x}{l}\right)\right). \tag{39}$$

The coefficients A_n for $n=0,1,2,3,\ldots$ determined by the help of inner product defined on $L^2[0,l]$:

$$A_n=\frac{2}{l}\Big[\int_0^l\sin\left(\frac{k\pi x}{l}\right)f(x)dx+(u_0-u_1)\int_0^l\sin\left(\frac{k\pi x}{l}\right)\frac{x}{l}dx$$
$$-u_0\int_0^l\sin\left(\frac{k\pi x}{l}\right)dx\Big]. \tag{40}$$

Substituting (40) in (38) leads to the solution of the problem (11)-(13). By making use of (10) and this solution, we obtain the general solution of the problem (7)-(9).

3. Illustrative Example

Let the following mathematical problem be considered:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \\ &u\left(0,t\right) = 1, u\left(2,t\right) = 1 \\ &u\left(x,0\right) = -\sin\left(\pi x\right) + 1 \end{aligned} \tag{41}$$

whose solution is given in the following form:

$$u\left(x,t\right) = -\sin\left(\pi x\right)e^{-\pi^2 t} + 1 \tag{42}$$

where $0 \leq x \leq 2, 0 \leq t \leq T$.
Now the following time fractional form of above problem is taken in hand:

$${}_{0}^{CPC}D_t^{\alpha}u\left(x,t\right) = u_{xx}\left(x,t\right), \tag{43}$$

$$u\left(0,t\right) = 1, u\left(2,t\right) = 1, \tag{44}$$

$$u\left(x,0\right) = -\sin\left(\pi x\right) + 1 \tag{45}$$

where $0 < \alpha < 1, 0 \leq x \leq 2,\ 0 \leq t \leq T$.
To make the boundary conditions (44) homogenous, we apply the transformation

$$v\left(x,t\right) = u\left(x,t\right) - 1 \tag{46}$$

to the above problem which leads to the following fractional heat-like problem

$${}_{0}^{CPC}D_t^{\alpha}v\left(x,t\right) = v_{xx}\left(x,t\right), \tag{47}$$

$$v\left(0,t\right) = 0, v\left(2,t\right) = 0, \tag{48}$$

$$v\left(x,0\right) = -\sin\left(\pi x\right) \tag{49}$$

where $0 < \alpha < 1, 0 \leq x \leq 2,\ 0 \leq t \leq T$.
The method SVM yields the following equations:

$$\frac{{}_{0}^{CPC}D_t^{\alpha}\left(T\left(t;\alpha\right)\right)}{T\left(t;\alpha\right)} = \frac{X''\left(x\right)}{X\left(x\right)} = -\lambda^2. \tag{50}$$

Taking the right hand side of equation (50) and related boundary conditions (48) into account the following problem is obtained:

$$X''(x) + \lambda^2 X(x) = 0, \tag{51}$$

$$X(0) = X(2) = 0. \tag{52}$$

The representation of the solution for the eigenvalue problem (51)-(52) is obtained as

$$X_n(x) = \sin\left(\frac{n\pi x}{2}\right), n = 1, 2, 3, \ldots \tag{53}$$

The second equation in (50) for every eigenvalue λ_n yields the following equation:

$$\frac{{}_{0}^{CPC}D_t^{\alpha}\left(T(t;\alpha)\right)}{T(t;\alpha)} = -\lambda^2 \tag{54}$$

which has the following solution

$$T_n(t;\alpha) = E_{\alpha,1,\ 1}^{1}\left(\frac{-\left(\frac{n^2\pi^2}{4}\right)}{K_0(\alpha)}t^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)}t\right), n = 0, 1, 2, 3, \ldots \tag{55}$$

For each eigenvalue λ_n, we obtain the following solution:

$$v_n(x,t;\alpha) = E_{\alpha,1,\ 1}^{1}\left(\frac{-\left(\frac{n^2\pi^2}{4}\right)}{K_0(\alpha)}t^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)}t\right)\sin\left(\frac{n\pi x}{2}\right), n = 0, 1, 2, 3, \ldots \tag{56}$$

and hence Superposition Principle leads to the following sum:

$$v(x,t;\alpha) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) E_{\alpha,1,\ 1}^{1}\left(\frac{-\left(\frac{n^2\pi^2}{4}\right)}{K_0(\alpha)}t^{\alpha}, \frac{-K_1(\alpha)}{K_0(\alpha)}t\right). \tag{57}$$

Utilizing the $L^2[0,2]$ inner product and initial condition (45) allow us to determine the coefficients A_n as follows:

$$v(x,0) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right). \tag{58}$$

The coefficients A_n for $n = 0, 1, 2, 3, \ldots$ are determined by the help of the inner product as follows:

$$A_n = -\int_0^2 \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) dx. \tag{59}$$

For $n \neq 2, A_n = 0$. $n = 2$ we get

$$\begin{aligned} A_2 &= -\int_0^2 \sin^2(\pi x) dx = -\int_0^2 \left(\frac{1}{2} - \frac{\cos(2\pi x)}{2}\right) dx \\ &= -\left(\frac{x}{2} - \frac{\sin(2\pi x)}{4}\right)\Bigg|_{x=0}^{x=2} = -1. \end{aligned} \tag{60}$$

Thus

$$v(x, t; \alpha) = -\sin(\pi x) E^1_{\alpha,1,\ 1}\left(\frac{-\pi^2}{K_0(\alpha)} t^\alpha, \frac{-K_1(\alpha)}{K_0(\alpha)} t\right). \tag{61}$$

By making use of (46) and the solution (61), we obtain the general solution of the problem (43)-(45) as follows:

$$u(x, t; \alpha) = -\sin(\pi x) E^1_{\alpha,1,\ 1}\left(\frac{-\pi^2}{K_0(\alpha)} t^\alpha, \frac{-K_1(\alpha)}{K_0(\alpha)} t\right) + 1. \tag{62}$$

The accuracy of the obtained solution is checked by substituting $\alpha = 1$ into (62) which leads to the solution of the problem (41).
Particularly solution of the problem (43)-(45) have the following form for the specific functions K_0 and K_1:

Case 1: For $K_0(\alpha) = \alpha, K_1(\alpha) = 1 - \alpha$, the solution becomes

$$u(x, t; \alpha) = -\sin(\pi x) E^1_{\alpha,1,\ 1}\left(\frac{-\pi^2}{\alpha} t^\alpha, \frac{\alpha - 1}{\alpha} t\right) + 1. \tag{63}$$

Case 2: For $K_0(\alpha) = \alpha^2, K_1(\alpha) = 1 - \alpha^2$, the solution becomes

$$u(x, t; \alpha) = -\sin(\pi x) E^1_{\alpha,1,\ 1}\left(\frac{-\pi^2}{\alpha^2} t^\alpha, \frac{\alpha^2 - 1}{\alpha^2} t\right) + 1. \tag{64}$$

The graphics of solutions for Case 1, Case 2 and Problem (41) in 2D are given in Fig. 1-4 for various values of α.

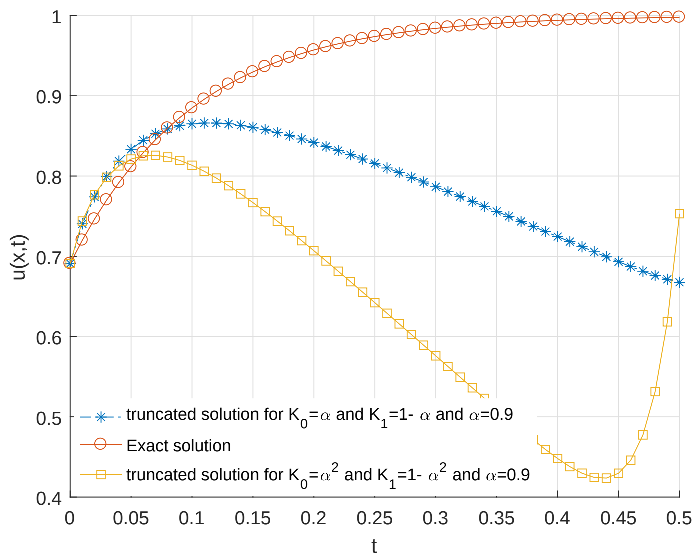

Figure 1: The graphics of solutions for Example for different functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ in 2D at $x = 0.1$ and for $\alpha = 0.9$.

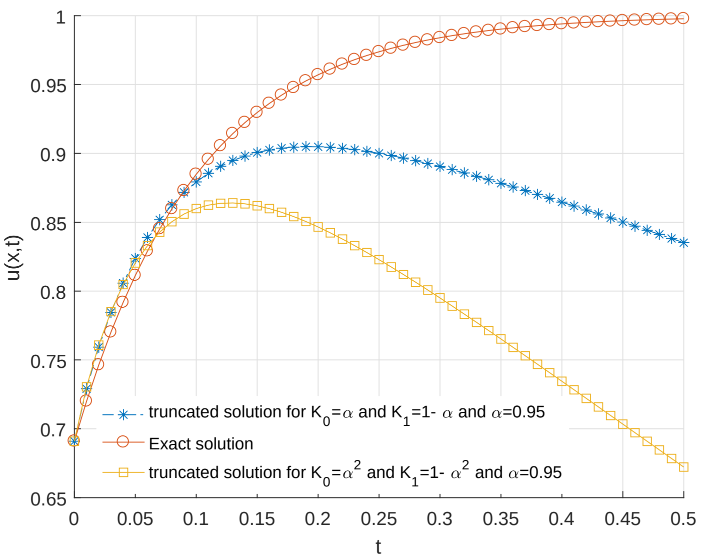

Figure 2: The graphics of solutions for Example for different functions $K_0\left(\alpha\right)$ and $K_1\left(\alpha\right)$ in 2D at $x = 0.1$ and for $\alpha = 0.95$.

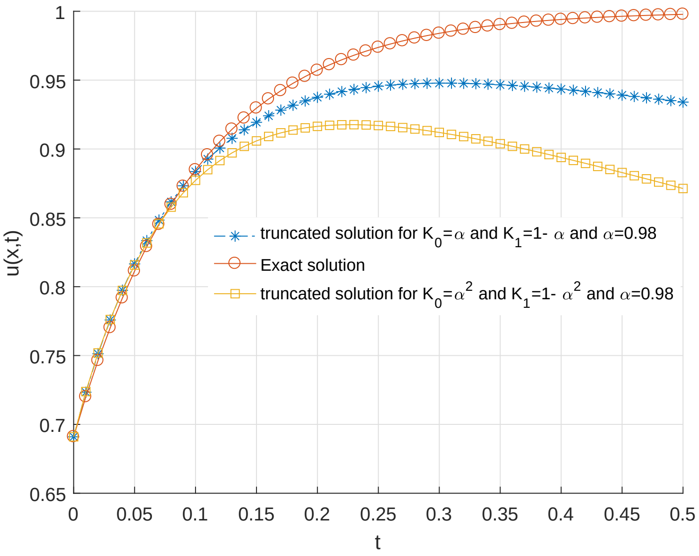

Figure 3: The graphics of solutions for Example for different functions $K_0(\alpha)$ and $K_1(\alpha)$ in 2D at $x = 0.1$ and for $\alpha = 0.98$.

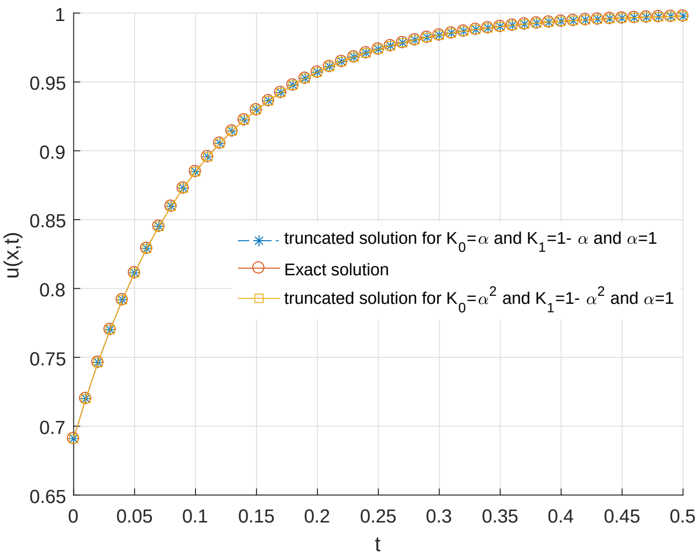

Figure 4: The graphics of solutions for Example for different functions $K_0(\alpha)$ and $K_1(\alpha)$ in 2D at $x = 0.1$ and for $\alpha = 1$.

4. Conclusion

The solution of the mathematical problem with hybrid time fractional derivative is constructed by the method SVM in terms of bivariate Mittag-Leffler function. Besides the accuracy of the solution is tested by taking $\alpha = 1$ in the solution which leads to the solution of the mathematical problem with ordinary derivative. As a result the illustrative example indicates that the method SVM plays an influence role in the construction of mathematical problems including fractional derivatives.

Based on the analytic solution, we reach the conclusion that diffusion processes decays with time until initial condition is reached when α is less than a certain value of α for Case 1 but diffusion processes decays with time for all values of α between 0 and 1 for Case 2. As α tends to 0, the rate of decaying increases. This implies that in the mathematical model for diffusion of the matter which has small diffusion rate the value of α must be close to 0. This model can account for various diffusion processes of various methods.

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