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POSITIVE CONJUGACY CLASSES IN WEYL GROUPS

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1. Let W be a Weyl group. In this paper we introduce the notion of positive conjugacy class of W. This generalizes the notion of elliptic regular conjugacy class in the sense of Springer [9].

Let $w \mapsto |w|$ be the length function on W. Let $S = \{s \in W; |s| = 1\}$.

Let v be an indeterminate. Recall that the Iwahori-Hecke algebra of W is the associative $\mathbf{Q}(v)$-algebra H which, as a $\mathbf{Q}(v)$-vector space has basis $\{T_w; w \in W\}$ and has multiplication given by $T_w T_{w'} = T_{ww'}$ if $|ww'| = |w| + |w'|$ and $(T_s + 1)(T_s - v^2) = 0$ if $s \in S$; note that T_1 is the unit element of H. This is a split semisimple algebra. Let $\mathbf{q} = v^2$.

For w, w' in W let $N^{w,w'}$ be the trace of the $\mathbf{Q}(v)$-linear map $H \to H$, $h \mapsto T_w h T_{w'^{-1}}$. We have $N^{w,w'} \in \mathbf{Z}[\mathbf{q}]$ and

(a) $$N^{w,w'} = \sum_{E \in \mathrm{Irr}W} \mathrm{tr}(T_w, E_v)\mathrm{tr}(T_{w'}, E_v)$$

where $\mathrm{Irr}W$ is the set of irreducible $\mathbf{Q}[W]$-modules up to isomorphism and for $E \in \mathrm{Irr}W$, E_v denotes the corresponding simple H-module (which in [8, 3.3] is denoted by $E(v^2)$). Note that when v is specialized to 1, H becomes the group algebra $\mathbf{Q}[W]$ of W and $N^{w,w'}$ specializes to $N^{w,w'}(1)$, the number of elements $y \in W$ such that $wy = yw'$. In particular, if $w \in W$, $N^{w,w}$ specializes to n^w, the order of the centralizer of w in W; thus the polynomial $N^{w,w}$ can be viewed as a $\mathbf{q}$-analogue of the number n^w.

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If C is a conjugacy class in W we denote by C_{min} the set of all $y \in C$ such that $C \to \mathbf{N}$, $w \mapsto |w|$, reaches its minimum at y. By a result of Geck and Pfeiffer [4, 3.2.9], for any $E \in \mathrm{Irr}W$, $w \mapsto \mathrm{tr}(T_w, E_v)$ is constant on C_{min}. Using this and (a) we see that $w \mapsto N^{w,w}$ is constant on C_{min}. We say that C is *positive* if $C \neq \{1\}$ and for some/any $w \in C_{min}$ we have $N^{w,w} \in \mathbf{N}[\mathbf{q}]$. (We then also say that any element $w \in C_{min}$ is positive.)

2. For any $f \in \mathbf{Z}[\mathbf{q}]$ we write $f = \sum_{i \geq 0} f_i \mathbf{q}^i$ where $f_i \in \mathbf{Z}$. For $w \in W$ let S_w be the set of all $s \in S$ such that s appears in some/any reduced expression for w. Let $\mathcal{L}_w = \{s \in S; |sw| < |w|\}$, $\mathcal{R}_w = \{s \in S; |ws| < |w|\}$. For a, a' in W we write $T_a T_{a'} = \sum_{b \in W} \phi(a, a', b) T_b$ where $\phi(a, a', b) \in \mathbf{Z}[\mathbf{q}]$. We show:

(a) $\phi(a, a', b)_{|a|} \geq 0$, $\phi(a, a', b)_i = 0$ *for* $i > |a|$. *If* $\phi(a, a', b)_{|a|} \neq 0$ *then either* $a' = b$, $S_a \subset \mathcal{L}_{a'}$ *or* $|b| < |a'|$. *If* $a' = b$, $S_a \subset \mathcal{L}_{a'}$ *then* $\phi(a, a', b)_{|a|} = 1$.

We argue by induction on $|a|$. When $|a| = 0$ the result is obvious. Assume now that $|a| \geq 1$. We write $a = a_1 s$ where $s \in S$, $|a_1| = |a| - 1$. If $|sa'| = |a'| + 1$ then $\phi(a, a', b) = \phi(a_1, sa', b)$ and the induction hypothesis shows that $\phi(a, a', b)_i = 0$ for $i \geq |a|$. Since $s \in S_a$, $s \notin \mathcal{L}_{a'}$, we see that the desired result holds. Next we assume that $|sa'| = |a'| - 1$. Then $\phi(a, a', b) = \mathbf{q}\phi(a_1, sa', b) + (\mathbf{q} - 1)\phi(a_1, a', b)$. From the induction hypothesis we see that $\phi(a, a', b)_i = 0$ if $i > |a|$, that $\phi(a, a', b)_{|a|} = \phi(a_1, sa', b)_{|a_1|} + \phi(a_1, a', b)_{|a_1|} \geq 0$ and that if $\phi(a, a', b)_{|a|} \neq 0$ then either $\phi(a_1, sa', b)_{|a_1|} \neq 0$ or $\phi(a_1, a', b)_{|a_1|} \neq 0$, so that we are in one of the cases (i)-(iv) below.

(i) $sa' = b$,

(ii) $|b| < |sa'|$;

(iii) $a' = b$, $S_{a_1} \subset \mathcal{L}_{a'}$;

(iv) $|b| < |a'|$.

In case (i), (ii), (iii) we have $|b| < |a'|$; in case (iii) have $a' = b$ and $S_a \subset \mathcal{L}_{a'}$ (since $S_a = S_{a_1} \cup \{s\}$), as desired. Now assume that $a' = b$, $S_a \subset \mathcal{L}_{a'}$. It remains to show that $\phi(a_1, sa', b)_{|a_1|} + \phi(a_1, a', b)_{|a_1|} = 1$. By the induction hypothesis we have $\phi(a_1, a', b)_{|a_1|} = 1$ (since $S_{a_1} \subset \mathcal{L}_{a'}$) $\phi(a_1, sa', b)_{|a_1|} = 0$ (since $sa' \neq b$ and $|b| \not< |sa'|$). This completes the proof.

The following result is proved in the same way as (a).

(b) *$\phi(a,a',b)_{|a'|}\ge 0$, $\phi(a,a',b)_i=0$ for $i>|a'|$. If $\phi(a,a',b)_{|a'|}\ne 0$ then either $a=b$, $S_{a'}\subset\mathcal{R}_{a'}$ or $|b|<|a|$. If $a=b$, $S_{a'}\subset\mathcal{R}_{a'}$ then $\phi(a,a',b)_{|a'|}=1$.*
For a,a',a'' in W we have $T_aT_{a'}T_{a''}=\sum_{b\in W}f(a,a',a'',b)T_b$ where $f(a,a',a'',b)\in\mathbf{Z}[\mathbf{q}]$. Let $n=|a|+|a''|$. We show:

(c) *$f(a,a',a'',a')_n\ge 0$, $f(a,a',a'',a')_i=0$ for $i>n$. If $f(a,a',a'',a')_n\ne 0$ then $S_a\subset\mathcal{L}_{a'}$ and $S_{a''}\subset\mathcal{R}_{a'}$. Conversely, if $S_a\subset\mathcal{L}_{a'}$ and $S_{a''}\subset\mathcal{R}_{a'}$ then $f(a,a',a'',a')_n=1$.*
We have $f(a,a',a'',a')=\sum_{c\in W}\phi(a,a',c)\phi(c,a'',a')$. Hence for $i\ge 0$ we have $f(a,a',a'',a')_i=\sum_{c\in W;j\ge 0,j'\ge 0,j+j'=i}\phi(a,a',c)_j\phi(c,a'',a')_{j'}$. Using (a) and (b) in the last sum we can take $j\ge|a|,j'\ge|a''|$. Hence if $i>n=|a|+|a''|$ then $f(a,a',a'',a')_i=0$ and
$f(a,a',a'',a')_n=\sum_{c\in W}\phi(a,a',c)_{|a|}\phi(c,a'',a')_{|a''|}\ge 0$. Assume now that $f(a,a',a'',a')_n\ne 0$. Then in the last sum we can assume that

$$c=a',\ S_a\subset\mathcal{L}_{a'}\text{ or }|c|<|a'|\text{ and }a'=c,\ S_{a''}\subset\mathcal{R}_c\text{ or }|a'|<|c|$$

Thus we can assume that $c=a'$, $S_a\subset\mathcal{L}_{a'}$ and $S_{a''}\subset\mathcal{R}_{a'}$ and using again (a), (b) we have $f(a,a',a'',a')_n=1$.

For w,w' in W we set $n=|w|+|w'|$; we show:

(d) *$N^{w,w'}_n=\sharp(a'\in W;S_w\subset\mathcal{L}_{a'},S_{w'}\subset\mathcal{R}_{a'})>0$, $N^{w,w'}_i=0$ for $i>n$.*
We have $N^{w,w'}=\sum_{a'\in W}f(w,a',w',a')$ and the result follows from (c). (We use that if $a'=w_0$, the longest element of W then $S_w\subset\mathcal{L}_{a'}=S$, $S_{w'}\subset\mathcal{R}_{a'}=S$.)

From (d) we deduce:

(e) *Let w,w',n be as in* (d). *Assume that either $S_w=S$ or $S_{w'}=S$. then $N^{w,w'}_n=1$.*
Indeed, if $a'\in W$ satisfies $S\subset\mathcal{L}_{a'}$ or $S\subset\mathcal{R}_{a'}$ then $a'=w_0$.

We state the following result.

(f) *Let w,w',n be as in* (d). *For $i=0,1,\dots,n$ we have $N^{w,w'}_i=(-1)^nN^{w,w'}_{n-i}$.*
Let $^-\colon\mathbf{Q}(v)\to\mathbf{Q}(v)$ be the field automorphism such that $\bar v=v^{-1}$. For $E\in\mathrm{Irr}W$ let $E^\dagger\in\mathrm{Irr}W$ be the tensor product of E with the sign representation of W. It is known that for $w\in W$ we have

$$\mathrm{tr}(T_w,E^\dagger_v)=(-v^2)^{|w|}\mathrm{tr}(T^{-1}_{w^{-1}},E_v)=(-v^2)^{|w|}\overline{\mathrm{tr}(T_w,E_v)}.$$

It follows that

$$
\begin{aligned}
N^{w,w'} &= \sum_{E\in \mathrm{Irr}W} \mathrm{tr}(T_w, E_v^\dagger)\mathrm{tr}(T_{w'}, E_v^\dagger) \\
&= \sum_{E\in \mathrm{Irr}W} (-v^2)^{|w|}\overline{\mathrm{tr}(T_w, E_v)}(-v^2)^{|w'|}\overline{\mathrm{tr}(T_{w'}, E_v)} \\
&= \sum_{E\in \mathrm{Irr}W} (-v^2)^{n}\overline{\mathrm{tr}(T_w, E_v)\mathrm{tr}(T_{w'}, E_v)}.
\end{aligned}
$$

We see that

$$N^{w,w'} = (-\mathbf{q})^n \overline{N^{w,w'}}$$

and (f) follows.

3. We now assume that W is irreducible. Let $\nu = |w_0|$ where w_0 is the longest element of W. An element $w \in W$ (or its conjugacy class) is said to be *elliptic* if its eigenvalues in the reflection representation of W are all $\neq 1$. For any $d \in \{2, 3, 4, \dots\}$ let C^d be the set of all elliptic elements $w \in W$ which have order d and are regular in the sense of Springer [9]. It is known [9] that C^d is either empty or a single conjugacy class in W. Let $\mathcal{D} = \{d \in \{2, 3, \dots\}; C^d \neq \emptyset\}$. It is known [9] that if $d \in \mathcal{D}$ and $w \in C^d_{min}$ then $|w| = 2\nu/d$. Let h be the Coxeter number of W. We have $h \in \mathcal{D}$.

According to [9], the set $\mathcal{D}$ is as follows:

Type $A_n (n \geq 1)$: $\mathcal{D} = \{n + 1\}$.

Type $B_n (n \geq 2)$: $\mathcal{D} = \{d \in \{2, 4, 6, \dots\}; 2n/d = \text{integer}\}$.

Type D_n (n even, $n \geq 4$):

$\mathcal{D} = \{d \in \{2, 4, 6, \dots\}; (2n - 2)/d = \text{odd integer or } 2n/d = \text{integer}\}$.

Type D_n (n odd, $n\geq 5$): $\mathcal{D}=\{d\in\{2, 4, 6, \dots\}; (2n-2)/d=\text{odd integer}\}$.

Type E_6: $\mathcal{D} = \{3, 6, 9, 12\}$.

Type E_7: $\mathcal{D} = \{2, 6, 14, 18\}$.

Type E_8: $\mathcal{D} = \{2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30\}$.

Type F_4: $\mathcal{D} = \{2, 3, 4, 6, 8, 12\}$.

Type G_2: $\mathcal{D} = \{2, 3, 6\}$.

We note the following properties:

(a) If $2 \in \mathcal{D}$, $d \in \mathcal{D}$ is even and $w \in C^d_{min}$ then $w^{d/2} = w_0$, $(d/2)|w| = |w_0|$ hence $T_w^{d/2} = T_{w_0}$.

(b) If $d = h$, $w \in C^d_{min}$ then $T^d_w = T^2_{w_0}$.

(c) If $d \in \mathcal{D}$, $h/d \in \mathbf{N}$ and $y \in C^h_{min}$ then $y^{h/d} \in C^d_{min}$ and $(h/2)|y| = |y^{h/d}|$ hence $T_y^{h/d} = T_{y^{h/d}}$.

The equation $w^{d/2} = w_0$ in (a) holds by examining the characteristic polynomial of w and $w^{d/2}$ in the reflection representation of W; then (a) follows. The equality in (b) can be deduced from [1, Ch.V,§6, Ex.2]. The equation $w^{h/d} \in C^d_{min}$ in (c) holds by examining the characteristic polynomial of w and $w^{h/d}$ in the reflection representation of W; then (c) follows.

For any $E \in \mathrm{Irr}W$ we define $a_E \in \mathbf{N}$ as in [8, 4.1]. Let $\tilde{a}_E = \nu - a_E + a_{E^\dagger}$.

(d) $T^2_{w_0} = v^{2\tilde{a}_E} 1 : E_v \to E_v$.
This can be deduced from [8, (5.12.2)]; a closely related statement was first proved by Springer, see [4, 9.2.2].

We show:

(e) *Let $E \in \mathrm{Irr}W$ and let $d \in \mathcal{D}, w \in C^d_{min}$. Then all eigenvalues of $T_w : E_v \to E_v$ (in an algebraic closure of $\mathbf{Q}(v)$) are roots of 1 times $v^{2\tilde{a}_E/d}$.*
If d is as in (a) then the result follows from (a) and (d). If $d = h$ then the result follows from (b) and (d). If d, y are as in (c) then the result follows from (c) and the previous sentence. From the description of $\mathcal{D}$ for various types we see that if $d \in \mathcal{D}$ is not as in (a) then it is as in (c). This proves (e). (A closely related result can be found in [4, 9.2.5].)

From (e) we deduce:

(f) *In the setup of* (e), $\mathrm{tr}(T_w, E_v)$ *equals* $v^{2\tilde{a}_E/d}\mathrm{tr}(w, E)$*; this is* 0 *if* $2\tilde{a}_E/d \notin \mathbf{Z}$.
(The idea of the proof leading to (f) appeared in [8, p.320].) Using (f) and 1(a) we deduce:

(g) *If $d \in \mathcal{D}$, $w \in C^d_{min}$, then*

$$N^{w,w} = \sum_{E \in \mathrm{Irr}W} \mathbf{q}^{2\tilde{a}_E/d} \mathrm{tr}(w, E)^2.$$

In particular, we have $N^{w,w} \in \mathbf{N}[\mathbf{q}]$ *and* w *is positive.*

4. Using 1(a) and the CHEVIE package [5] one can find a list of positive conjugacy classes in W (assumed to be irreducible of low rank). I thank Gongqin Li for help with programming in GAP. The list of positive conjugacy classes in W which are not regular elliptic for W of type $E_6, E_7, E_8, F_4, G_2, B_5, B_6$ is as follows. (We specify a conjugacy class by the characteristic polynomial of one of its elements in the reflection representation. We denote by Φ_k the k-th cyclotomic polynomial; thus $\Phi_2 = \mathbf{q}+1$, $\Phi_3 = \mathbf{q}^2+\mathbf{q}+1$, etc.)

Type E_6: none.

Type E_7: $\Phi_{12}\Phi_6\Phi_2, \Phi_{10}\Phi_6\Phi_2, \Phi_{10}\Phi_2^3, \Phi_8\Phi_4\Phi_2,$ $\Phi_4^2\Phi_2^3$.

Type E_8: $\Phi_{18}\Phi_6, \Phi_{18}\Phi_2^2, \Phi_9\Phi_3, \Phi_{14}\Phi_2^2$.

Type F_4: none.

Type G_2: none.

Type B_5: $\Phi_8\Phi_2, \Phi_6\Phi_2^2, \Phi_4^2\Phi_2, \Phi_2\Phi_4\Phi_6$.

Type B_6: $\Phi_{10}\Phi_2^2, \Phi_8\Phi_4, \Phi_8\Phi_2^2, \Phi_6\Phi_2^3$.

In each of these examples any positive element of W is elliptic; we expect this to be true in general. The example of B_6 suggests that if W is of type B_n with $2n = 4+8+\cdots+4k$, then an element of W with cycle type $(4)(8)\dots(4k)$ might be positive.

Remark. In a first version of this paper, the fourth conjugacy class listed above for type B_5 was omitted by mistake. I thank Jean Michel for pointing this out.

5. Let $\mathbf{k}$ be an algebraic closure of the finite field F_q with q elements. Let G be a connected reductive group over $\mathbf{k}$ with a fixed F_q-split rational structure and whose Weyl group is W. Let $F: G \to G$ be the corresponding Frobenius map. For $w \in W$ let X_w be the variety of Borel subgroups B of G such that B and $F(B)$ are in relative position w, see [2, 1.3]. The finite group $G^F = \{g \in G; F(g) = g\}$ acts on X_w by conjugation. For w, w' in W we denote by $X_{w,w'} = G^F\backslash(X_w \times X_{w'})$ the space of G^F-orbits for the diagonal action of G^F on $X_w \times X_{w'}$. Now $(B, B') \mapsto (F(B), F(B'))$ induces a map $X_{w,w'} \to X_{w,w'}$ (denoted again by F) which is the Frobenius map for an F_q-rational structure on $X_{w,w'}$. By [7, 3.8] for any integer $e \geq 1$ we have

(a) $$\sharp(\xi \in X_{w,w'}; F^e(\xi) = \xi) = N^{w,w'}(q^e).$$

6. In the remainder of this paper we assume that G in no.5 is simply connected and W is irreducible. In the case where w is a Coxeter element of minimal length of W, the left hand side of 5(a) (with $w = w'$) has been computed in [6, p.158]. This gives the following formulas for $N^{w,w}$.

Type $A_n (n \geq 1)$: $\mathbf{q}^{2n} + \mathbf{q}^{2n-2} + \cdots + \mathbf{q}^2 + 1$.

Type $B_n (n \geq 2)$: $\mathbf{q}^{2n} + 2\mathbf{q}^{2n-2} + 2\mathbf{q}^{2n-4} + \cdots + 2\mathbf{q}^2 + 1$.

Type $D_n (n \geq 4)$: $\mathbf{q}^{2n} + \mathbf{q}^{2n-2} + 2\mathbf{q}^{2n-4} + 2\mathbf{q}^{n-6} + \cdots + 2\mathbf{q}^4 + \mathbf{q}^2 + 1$.

Type E_6: $\mathbf{q}^{12} + \mathbf{q}^{10} + 2\mathbf{q}^8 + 4\mathbf{q}^6 + 2\mathbf{q}^4 + \mathbf{q}^2 + 1$.

Type E_7: $\mathbf{q}^{14} + \mathbf{q}^{12} + 2\mathbf{q}^{10} + 4\mathbf{q}^8 + 2\mathbf{q}^7 + 4\mathbf{q}^6 + 2\mathbf{q}^4 + \mathbf{q}^2 + 1$.

Type E_8: $\mathbf{q}^{16} + \mathbf{q}^{14} + 2\mathbf{q}^{12} + 4\mathbf{q}^{10} + 2\mathbf{q}^9 + 10\mathbf{q}^8 + 2\mathbf{q}^7 + 4\mathbf{q}^6 + 2\mathbf{q}^4 + \mathbf{q}^2 + 1$.

Type F_4: $\mathbf{q}^8 + 2\mathbf{q}^6 + 6\mathbf{q}^4 + 2\mathbf{q}^2 + 1$.

Type G_2: $\mathbf{q}^4 + 4\mathbf{q}^2 + 1$.

Let $\mathcal{N}_G$ be the variety consisting of all pairs (g, g') where g runs through the standard Steinberg cross section of the set of regular elements of G and g' is an element in the centralizer of g in G modulo the centre of G. (This variety, introduced in [6, p.158], makes sense even if $\mathbf{k}$ is replaced by the complex numbers. It plays a role in [3] where it is called the *universal centralizer*.) According to [6, p.158], the number of F_q-rational points of $\mathcal{N}_G$ is equal to $N^{w,w}(q)$ hence it is given by the formulas above with $\mathbf{q} = q$.

7. Let C be a conjugacy class of W. For $w \in C$, the part of weight j of the i-th l-adic cohomology space with compact support $H^i_c(X_w, \bar{\mathbf{Q}}_l)$ is a direct sum $\oplus_\rho V^i_{\rho,j} \otimes \rho$ where ρ runs over the unipotent representations of G^F (up to isomorphism) and $V^i_{\rho,j}$ are finite dimensional $\bar{\mathbf{Q}}_l$-vector spaces in such a way that the G^F-action is only through the action on ρ and the Frobenius action is only through an action on $V^i_{\rho,j}$ (where it is multiplication by $q^{j/2}\lambda_\rho$ with

λ_ρ a root of 1 independent of w, i, j, and the parity of j is independent of w, i, see [7, 3.9], [8]). Using the Grothendieck-Lefschetz fixed point formula, from 5(a) we deduce for any $e \geq 1$:

$$N^{w,w}(q^e) = \sum_{i,i',j,j',\rho} (-1)^{i+i'} \dim(V^i_{\rho,j}) \dim(V^{i'}_{\rho^*,j'}) q^{je/2} q^{j'e/2}$$

where ρ^* is the dual of ρ and we have used that $\lambda_{\rho^*} = \lambda_\rho^{-1}$. This implies

(a) $$N^{w,w} = \sum_{i,i',j,j',\rho} (-1)^{i+i'} \dim(V^i_{\rho,j}) \dim(V^{i'}_{\rho^*,j'}) v^{j+j'}.$$

If we assume that

(b) the G^F-modules $H^i_c(X_w, \bar{\mathbf{Q}}_l)$, dual of $H^{i'}_c(X_w, \bar{\mathbf{Q}}_l)$ are disjoint for any i, i' such that $i \neq i' \bmod 2$
then from (a) we could deduce that $N^{w,w} \in \mathbf{N}[\mathbf{q}]$. Hence if we assume further that $w \in C_{min}, C \neq \{1\}$ it would follow that C is positive.

We conjecture that, conversely, if C is positive and $w \in C_{min}$ then (b) holds. It is also likely that in this case,

(c) the G^F-modules $H^i_c(X_w, \bar{\mathbf{Q}}_l)$, $H^{i'}_c(X_w, \bar{\mathbf{Q}}_l)$ are disjoint for any i, i' such that $i \neq i' \bmod 2$.
This disjointness property holds when w is as in §6, see [6].

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