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SOLVABILITY OF A SOLUTION AND CONTROLLABILITY FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we establish the existence and uniqueness of solutions for a nonlinear fractional differential equation with nonlocal boundary conditions. We employ Schauder fixed point theorem to study the existence of a solution of the problem. We also use the Banach fixed point theorem to study the existence of a unique solution. Finally, we provide examples to illustrate our results. Thus, we study the null-controllability for the fractional differential equation with constraints on the control. The main tool used to solve the problem of existence and convergence is an observability inequality of Carleman type, which is "adapted" to the constraints. We then apply the obtained results to the sentinels theory of Lions.

1. Existence and Uniqueness for Nonlinear Fractional Differential Equations

1.1. Preliminaries

Firstly, we study the existence and uniqueness of the solution for a following fractional differential equation :

$$ {}^{c}D^{\alpha}y+f\left(y\right)+\,{}^{c}D^{\alpha-1}g\left(y\right)=0, 0<t<1 \text{ and } y\left(\varsigma\right)=0,\ {}^{c}D^{p}y\left(1\right)=\mu y\left(\eta\right). \quad (1)$$

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where $1 < \alpha < 2, 0 < p < 1$, and $0 < \eta < \varsigma < 1$. The differential operator is Caputo fractional derivative. Suppose that the functions f and g are increasing. Then according to the Schauder fixed point theorem and Banach contraction principal the solution exists and is unique (see [11]).

Let $X = \mathcal{C}([0,1])$ be the Banach space of $\mathbb{R}$ valued continuous functions on $[0,1]$ endowed with the norm

$$\|y\|_X = \max_{t\in[0,1]} |y(t)|,$$

and we consider a closed bounded subset of X

$$U = \{y \in X,\ y(t) \geq 0, \quad t \in [0,1]\}.$$

Definition 1 ([9])**.**

1. The fractional integral of order α for a function $y : [0,\infty) \longrightarrow \mathbb{R}$ can be written as
$$I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \tag{2}$$
provided the right side is point wise defined on $\mathbb{R}^+$.
2. The Caputo derivative of order α for a function $y : [0,\infty) \longrightarrow \mathbb{R}$ can be written as
$$^cD^{\alpha} y(t) = I^{n-\alpha} y^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds, \tag{3}$$
where $n = [\alpha] + 1$, provided the right side is point wise defined on $\mathbb{R}^+$.
3. Let $(X, \|.\|)$ be a Banach space and $T : X \longrightarrow X$. The operator T is a contraction operator if there is an $k \in (0,1)$ such that $y, v \in X$ imply
$$\|Ty - Tv\| \leq k \|y - v\|.$$

The following lemmas give some properties of fractional integrals.

Lemma 1 ([10])**.**

1. *Let $\alpha, \beta > 0$, Then the following relation hold*
$$^cD^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \quad R(\beta) \geq n. \tag{4}$$

2. *For a function $y \in \mathcal{C}^n[0,1]$ and $\alpha > 0$, the following relation hold*

$$I^{\alpha}\left({}^{c}D^{\alpha}y\left(t\right)\right) = y\left(t\right) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \quad c_i \in \mathbb{R}. \qquad (5)$$

3. *Let $y \in \mathcal{C}^n[0,1]; f, g \in \mathcal{C}([0,1] \times [0,+\infty), [0,+\infty))$, y'' and $\frac{\partial g}{\partial t}$ exist, y is a solution of problem (1) if and only if y is a solution of the integral equation*

$$\begin{aligned} y\left(t\right) =& \frac{t-\varsigma}{\Delta}[I^{\alpha-p}f\left(1,y\left(1\right)\right) + I^{1-p}g\left(1,y\left(1\right)\right) \\ &+ \mu I^{\alpha}\left(f\left(\varsigma,y\left(\varsigma\right)\right) - f\left(\eta,y\left(\eta\right)\right)\right) + \mu I\left(g\left(\varsigma,y\left(\varsigma\right)\right) - g\left(\eta,y\left(\eta\right)\right)\right)] \\ &+ I^{\alpha}f\left(\varsigma,y\left(\varsigma\right)\right) + Ig\left(\varsigma,y\left(\varsigma\right)\right) - I^{\alpha}f\left(t,y\left(t\right)\right) - Ig\left(t,y\left(t\right)\right). \qquad (6) \end{aligned}$$

where $\Delta = \frac{1}{\Gamma(2-p)} + \mu\varsigma - \mu\eta > 0$.

Now, we state the fixed point theorems which enable us to prove the existence and uniqueness of a solution of (1).

Theorem 1 ([5]).

1. *Let U be a nonempty closed convex subset of a Banach space X and $T: X \longrightarrow X$ be a contraction operator. Then there is a unique $y \in U$ with $Ty = y$.*
2. *Let U be a nonempty closed convex subset of a Banach space X and $T: X \longrightarrow X$ be a continuous compact operator. Then T has a fixed point in U.*

1.1. Main results

Now, we consider the results of existence and uniqueness problem of the FDE (1). Assume that the following growth conditions hold

(A1) The functions $f, g : [0,1] \times [0,+\infty) \to \mathbb{R}^+$ be continuous.

(A2) There exists a positive constants M and N such that for $y(t) \in U$

$$\max\left\{f\left(t,y(t)\right), t \in [0,1]\right\} \leq M,$$

$$\max\left\{g\left(t,y(t)\right), t \in [0,1]\right\} \leq N.$$

(A3) For $t \in [0,1]$ and $y, v \in X$ there exist a positive real numbers $k_1, k_2 < 1$ such that

$$|f(t,y) - f(t,v)| \leq k_1 \|y - v\|,$$
$$|g(t,y) - g(t,v)| \leq k_2 \|y - v\|,$$

and we use the following notations

$$\lambda_1 = \frac{\varsigma^{\alpha} + 1}{\Gamma(\alpha+1)} + \left|\frac{1-\varsigma}{\Delta}\right| \left(\frac{1}{\Gamma(\alpha - p + 1)} + \frac{\mu}{\Gamma(\alpha+1)}(\varsigma^{\alpha} + \eta^{\alpha})\right).$$
$$\lambda_2 = \frac{1-\varsigma}{\Gamma(2-p)\Delta} + \frac{\mu(\varsigma+\eta)(1-\varsigma)}{\Delta} + \varsigma + 1.$$

Theorem 2.

1. *Assume that* (A1), (A2) *hold. Then the problem* (1) *has a solution.*
2. *Assume that* (A1), (A2) *and* (A3) *are satisfied and the following inequality holds* $(\lambda_1 k_1 + \lambda_2 k_2) < 1$.

Then the FDE (1) has a unique solution $y \in U$.

1.3. Examples

In this subsection, we present some examples to illustrate the main results.

Example 1. We consider the problem :

$$^{c}D^{3/2}y + f(y) + {}^{c}D^{1/2}g(y) = 0,\ 0<t<1 \text{ and } y(\varsigma) = 0,\ {}^{c}D^{1/2}y(1) = \mu y(\eta). \tag{7}$$

In this problem we have $\alpha = \frac{3}{2}, p = \frac{1}{2}, \mu = \frac{1}{10}, \eta = \frac{1}{3}$ and $\varsigma = \frac{1}{2}$; we take

$$f(y) = \left(e^{t} - 1\right) + \frac{e^{-t}}{2}\left(\frac{y(t)}{1+y(t)}\right),$$
$$g(y) = \frac{1}{20}\left(\frac{1+y(t)}{3+y(t)}\right),$$

we obtain that

$$k_1 = \frac{1}{2}, k_2 = \frac{1}{10}, \lambda_1 = 1.2857 \text{ and } \lambda_2 = 2.0452.$$

So, $\lambda_1 k_1 + \lambda_2 k_2 = 0.8473 < 1$; then by Theorem 2, problem (7) has a unique solution.

Example 2. We consider the problem :

$$^{c}D^{5/2}y+f\left(y\right)+{}^{c}D^{3/2}g\left(y\right)=0,\ 0<t<1 \text{ and } y\left(\varsigma\right)=0,\ {}^{c}D^{1/2}y\left(1\right)=\mu y\left(\eta\right). \tag{8}$$

Here $\alpha = \frac{5}{2}, p = \frac{1}{2}, \mu = \frac{1}{15}, \eta = \frac{1}{5}$ and $\varsigma = \frac{1}{4}$.

We take

$$f\left(y\right) = t^3 \sin y(t),$$
$$g\left(y\right) = \frac{t}{\sqrt{5}(1+t)}\left(1 + t\cos y(t)\right),$$

we get

$$k_1 = \frac{1}{3}, k_2 = \frac{1}{\sqrt{5}}, \lambda_1 = 0.5219 \text{ and } \lambda_2 = 1.7395.$$

So, $\lambda_1 k_1 + \lambda_2 k_2 = 0.9506 < 1$; by Theorem 2, we conclude that problem (8) has a unique solution.

Conclusion 1. *As a preliminary summary, we study the existence and uniqueness of solutions for nonlinear fractional differential equations with the Caputo derivative. We were able to give an integral representation of our problem. Schauder fixed point theorem was the key of our analysis to establish existence of positive solutions. However, adding an extra condition, we succeeded to obtain a unique solution by using Banach fixed point theorem.*

2. Null-controllability for the Fractional Differential Equations

2.1. Introduction

Let Ω open subset bounded of $\mathbb{R}^n$ with $\partial\Omega = \Gamma$ of class $\mathcal{C}^2$.

Let $\omega \subset \Omega$ non empty. We pose $\mathcal{Q} = \Omega \times (0,1), \Sigma = \Gamma \times (0,1)$.

Let the problem adjoint :

$$-{}^{c}D^{\alpha}q+f'\left(y_0\right)q-{}^{c}D^{\alpha-1}g\left(y_0\right)q=h+v\chi_{\omega} \text{ in } \mathcal{Q},\ q=0 \text{ on } \Sigma,\ q\left(1\right)=0 \text{ in } \Omega, \tag{9}$$

where $f'(y_0)$ denotes the derivative of f at point y_0 and where y_0 is the solution of the problem

$$^cD^\alpha y_0 + f(y_0) + {}^cD^{\alpha-1} g(y_0) = \xi \text{ in } \mathcal{Q},\ y_0 = 0 \text{ on } \Sigma,\ y_0(0) = y^0 \text{ in } \Omega.$$

With

$$h \in L^2(\mathcal{Q}), \tag{10}$$

and χ_ω denotes the characteristic function of ω. Then, $\forall v \in L^2(\omega \times (0,1))$, the problem (9) admits has unique solution q.

We suppose that

$$\mathcal{K} \text{ is of finite dimension.} \tag{11}$$

Let $\mathcal{K}^\perp$ the orthogonal of $\mathcal{K}$ in $L^2(\omega \times (0,1))$. So we ask the question: look for a function $v \in L^2(\omega \times (0,1))$ such that

$$\left| \begin{array}{l} v \in \mathcal{K}^\perp \\ q(0) = 0 \text{ in } \Omega. \\ \|v\|_{L^2(\omega\times(0,1))} = \min. \end{array} \right. \tag{12}$$

What is called a problem of null controllability with the constraints on the control v.

In this work we use the variational method to establish the existence of an optimal control and the penalization method to characterize it. So we pose the following hypothesis

$$\begin{cases} \nexists\ k \in \mathcal{K} \text{ such that} \\ k \in L^2\left(0,1; H^1(\omega)\right) \text{ with } {}^cD^\alpha k + f'(y_0)\, k + {}^cD^{\alpha-1} g(y_0)\, k = 0 \text{ in } \omega \times (0,1). \end{cases} \tag{13}$$

Then, we introduce a weight function θ which will be precisely defined in the following Lemma 2, but which -for instance- is such that

$$h \in L^2(\mathcal{Q}) \text{ and } \theta h \in L^2(\mathcal{Q}). \tag{14}$$

We can now formulate our main result :

Theorem 3. *Under the previous hypothesis* (10), (11), (13) *and* (14), *and* $\forall \omega \subset \Omega, \exists! v$ *solution of the problem* (9) *and* (12), *of minimum norm in* $L^2(\Omega \times (0,1))$.

To prove the Theorem 3 we use the inequality of observability. So let

$$\begin{aligned} L &= {}^cD^\alpha + f'(y_0) + {}^cD^{\alpha-1} g(y_0), \\ \mathcal{V} &= \left\{ \rho \in \mathcal{C}^\infty\left(\overline{\mathcal{Q}}\right), \rho = 0 \text{ on } \Sigma \right\}, \end{aligned} \tag{15}$$

and $P =$ the orthogonal projection operator of $L^2(\omega \times (0,1))$ into $\mathcal{K}$.

Let $a_{\theta,P}(\cdot,\cdot)$ defined by :

$$a_{\theta,P}(\rho, \widehat{\rho}) = \int_0^1 \int_\Omega L_\rho L_{\widehat{\rho}} dxdt + \int_0^1 \int_\omega (\rho - P\rho)(\widehat{\rho} - P\widehat{\rho})\, dxdt. \tag{16}$$

According to (13) and Lemma 2 next, this bilinear form is a scalar product on $\mathcal{V}$.

Let $V_{\theta,P}$ be the Hilbert space, completed of $\mathcal{V}$ for the scalar product $a_{\theta,P}(\rho, \widehat{\rho})$ and the associated norm.

We give now the characterization of optimal control by the optimality system. More specifically, all of the functions v such that $(9)-(12)$ hold (admissible controls), is not empty and we immediately see that it is a closed convex set of $L^2(\omega \times (0,1))$. Therefore, there is a unique $\widehat{v}$ of minimal norm in $L^2(\omega \times (0,1))$. Now le $\widehat{q}$ the unique associated solution such that $(9)-(12)$ hold.

Theorem 4. *Under the hypotheses of the Theorem* 3, *the couple* $(\widehat{v}, \widehat{q})$ *is the solution of the optimal system* $(9)-(12)$ *if and only if there is a function* $\widehat{\rho}$ *such that* $(\widehat{v}, \widehat{q}, \widehat{\rho})$ *is the solution of the optimal system* $(22)-(24)$.

2.2. Characterization of optimal control

We study in this work the existence and the characterization of an optimal control for the problem $(9)-(12)$ which is a problem of null-controllability with linear constraints on the control. The main result is as follows :

Theorem 5. *We suppose* (10), (11), (13) *and* (14). *So for all open non empty* ω *of* Ω, *there is a control* v *solution of the problem* (9)$-$(12). *In addition, there is a single control* $\widehat{v}$ *of minimum norm in* $L^2(\Omega \times (0,1))$.

The proof of Theorem 5 goes through several stages which essentially use the following lemma.

Lemma 2. *We suppose* (10), *then there is a "weight" function* θ *checking* $\theta > 0$, θ *of class* $\mathcal{C}^2$ *on* $\mathcal{Q}$, $\frac{1}{\theta}$ *bounded on* $\mathcal{Q}$ *and there is a constant* $C > 0$ *such that*

$$\int_0^1 \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dxdt \leq C \left(\int_0^1 \int_\Omega |L_\rho|^2 \, dxdt + \int_0^1 \int_\omega |\rho - P\rho|^2 \, dxdt \right), \forall \rho \in \mathcal{V}. \quad (17)$$

The proof of the lemma is based on three arguments: The following classic observability inequality :

$$\int_0^1 \int_\Omega \frac{1}{\theta^2} |\rho|^2 \, dxdt \leq C \left(\int_0^1 \int_\Omega |L_\rho|^2 \, dxdt + \int_0^1 \int_\omega |\rho|^2 \, dxdt \right), \forall \rho \in \mathcal{V}.$$

The compactness of the operator P ensured here by the finite dimension of $\mathcal{K}$ and finally the continuity of P.

As for the demonstration of Theorem 5, here are some indications. The second member of (17) induces on the space $\mathcal{V}$ defined in (15) the scalar product (16) which allows to construct Hilbert space $V_{\theta,P}$ supplemented by $\mathcal{V}$ for the norm associated with (16). In this framework, the linear form $\rho \longmapsto \int_0^1 \int_\Omega h\rho dxdt$ is continuous on $V_{\theta,P}$ and this, thanks to (17) and the hypothesis (14) on h. Consequently, the Lax-Milgram theorem ensures the existence of a unique ρ_θ in $V_{\theta,P}$ solution of the variational problem :

$$a_{\theta,P}(\rho_\theta, \rho) = \int_0^1 \int_\Omega h\rho dxdt, \forall \rho \in V_{\theta,P}.$$

We set $v_\theta = -\left(\rho_\theta - P\rho_\theta \chi_\omega\right) \chi_\omega$ and $q_\theta = L\rho_\theta$. Then the couple (v_θ, q_θ) is a solution to the problem of null controllability (9) and (12). This establishes the first part of Theorem 5.

We can therefore speak of the set of controls v solutions of the controllability problem (9) and (12). This set is not empty. It is convex and closed in $L^2(0,1;L^2(\omega))$. Consequently, there exists a single optimal control

$\widehat{v}$ of minimum norm in $L^2(0,1;L^2(\omega))$. Which establishes the second part of Theorem 5; and therefore completes his demonstration.

We now characterize the optimal control $\widehat{v}$ of Theorem 5. Let $\widehat{q}$ be the only element associated with $\widehat{v}$ such that the pair $(\widehat{v},\widehat{q})$ satisfies (9) and (12). We characterize $(\widehat{v},\widehat{q})$ by a system of optimality using the penalization method. More precisely, for $\epsilon > 0$ we introduce the penalized function J_ϵ defined by :

$$\begin{aligned} J_\epsilon(v,q) =& \frac{1}{2}\|v\|^2_{L^2(\omega\times(0,1))} \\ &+ \frac{1}{2\epsilon}\left\|-{}^cD^\alpha q + f'(y_0)\,q -{}^c D^{\alpha-1}g(y_0)\,q - h - v\chi_\omega\right\|^2_{L^2(\Omega\times(0,1))}, \end{aligned}$$

where the couples (v,z) are such that

$$\begin{aligned} v\in\mathcal{K}^\perp, -{}^cD^\alpha q + f'(y_0)\,q -{}^c D^{\alpha-1}g(y_0)\,q \in L^2(\Omega\times(0,1)), \\ q=0 \text{ on } \Sigma,\ q(1)=0,\ q(0)=0 \text{ in } \Omega. \end{aligned} \tag{18}$$

The problem of optimal control

$$\inf\left\{J_\epsilon(v,q),\ (v,q) \text{ checked } (18)\right\}, \tag{19}$$

admits a unique solution (v_ϵ,q_ϵ) that we characterize by a system of optimality.

Proposition 1. *We place ourselves under the hypotheses of Theorem* 5*. The couple* (v_ϵ,q_ϵ) *is the optimal solution of the problem* (19) *if and only if there exists a function* ρ_ϵ *such as the triplet* $(v_\epsilon,q_\epsilon,\rho_\epsilon)$ *or solution of the optimality system*

$$\begin{aligned} -{}^cD^\alpha q_\epsilon + f'(y_0)\,q_\epsilon - {}^cD^{\alpha-1}g(y_0)\,q_\epsilon = h + v_\epsilon\chi_\omega + \epsilon\rho_\epsilon \\ \text{in } \mathcal{Q},\ q_\epsilon=0 \text{ on } \Sigma,\ q_\epsilon(1)=0 \text{ in } \Omega, \end{aligned} \tag{20}$$

$$\begin{aligned} q_\epsilon(0)=0 \text{ in } \Omega,\ {}^cD^\alpha\rho_\epsilon + f'(y_0)\,\rho_\epsilon + {}^cD^{\alpha-1}g(y_0)\,\rho_\epsilon = 0 \\ \text{in } \mathcal{Q},\ \rho_\epsilon=0 \text{ on } \Sigma,\ v_\epsilon=(\rho_\epsilon - P\rho_\epsilon\chi_\omega)\,\chi_\omega. \end{aligned} \tag{21}$$

Remark 1. On the other hand, we have no information on $\rho_\epsilon(0)$ and $\rho_\epsilon(1)$; we nevertheless obtain the convergence of the triplet $(v_\epsilon,q_\epsilon,\rho_\epsilon)$ when $\epsilon\longrightarrow 0$ towards a triplet $(\widehat{v},\widehat{q},\widehat{\rho})$ characteristic of the optimal solution of the problem (9) and (12).

More specifically, we have

Theorem 6. *We place ourselves under the hypotheses of Theorem* 5. *The couple* $(\widehat{v}, \widehat{q})$ *is the optimal solution of the problem* (9) *and* (12) *if and only s' there exists a function* $\widehat{\rho}$ *such that the triplet* $(\widehat{v}, \widehat{q}, \widehat{\rho})$ *is a solution of the optimality system*

$$\widehat{v} \in \mathcal{K}^{\perp}, \widehat{q} \in \mathcal{C}\left(0,1;L^2(\Omega)\right) \cap L^2\left(0,1;H_0^1(\Omega)\right), \widehat{\rho} \in V_{\theta,P}, \tag{22}$$

$$-{}^cD^{\alpha}\widehat{q} + f'(y_0)\,\widehat{q} - {}^c D^{\alpha-1}g(y_0)\,\widehat{q} = h + \widehat{v}\chi_{\omega}$$
$$\textit{in } \mathcal{Q},\ \widehat{q} = 0 \textit{ on } \Sigma,\ \widehat{q}(1) = 0 \textit{ in } \Omega, \tag{23}$$

$$\widehat{q}(0) = 0 \textit{ in } \Omega,\ {}^cD^{\alpha}\widehat{\rho} + f'(y_0)\,\widehat{\rho} + {}^cD^{\alpha-1}g(y_0)\,\widehat{\rho} = 0$$
$$\textit{in } \mathcal{Q},\ \widehat{\rho} = 0 \textit{ on } \Sigma,\ \widehat{v} = (\widehat{\rho} - P\widehat{\rho}\chi_{\omega})\,\chi_{\omega}. \tag{24}$$

2.3. Applications to discriminating sentinel

We consider in a first step an equation of state which, here, is given by the following evolution system

$${}^cD^{\alpha}y + f(y) + {}^cD^{\alpha-1}g(y) = \xi + \lambda\widehat{\xi} \text{ in } \mathcal{Q},\ y = 0 \text{ on } \Sigma,\ y(0) = y^0 + \tau\widehat{y}^0 \text{ in } \Omega. \tag{25}$$

Where the data in (25) are incomplete in the following sense : the functions ξ and y^0 are known with ξ in $L^2(\mathcal{Q})$ and y^0 in $L^2(\Omega)$. On the other hand, the terms $\lambda\widehat{\xi}$ and $\tau\widehat{y}^0$ are not known. We suppose that

$$\left\|\widehat{\xi}\right\|_{L^2(\mathcal{Q})} \leq 1, \left\|\widehat{y}^0\right\|_{L^2(\Omega)} \leq 1 \text{ and } \lambda, \tau \in \mathbb{R} \text{ are quite small.}$$

We then assume that there is a unique solution $y = y(x,t;\lambda,\tau) = y(\lambda,\tau) \in L^2\left(0,1,H_0^1(\Omega)\right) \cap L^{\infty}\left(0,1,L^2(\Omega)\right)$.

We then give in a second step an observation y_{obs} of y on a non-empty open $\mathcal{O}$ of Ω, that is : $y_{obs} = m_0 + \sum_{i=1}^{N} \beta_i m_i$ where the functions $m_0, m_1, ..., m_N$ are known in $L^2(\mathcal{O} \times (0,1))$, but the real coefficients β_i are not known. We suppose that the β_i are "small" and that the functions m_i are linearly independent.

Finally, in a third step, we consider a functional $\mathcal{S}$ to be determined from a function h_0 of $L^2(\mathcal{O}\times(0,1))$ and a non-empty open set ω such as $\omega\subset\mathcal{O}\subset\Omega$. More precisely, for a control function $u\in L^2(\omega\times(0,1))$, we pose

$$\mathcal{S}(\lambda,\tau)=\int_0^1\int_{\mathcal{O}} h_0 y(x,t;\lambda,\tau)\,dxdt+\int_0^1\int_{\omega} u y(x,t;\lambda,\tau)\,dxdt. \tag{26}$$

Definition 2. We say that $\mathcal{S}$ is the discriminating sentinel defined by h_0,ω and $\mathcal{O}$ if there exists a control u such that the couple $(u,\mathcal{S})$ satisfies the following three conditions :

$$\int_0^1\int_{\mathcal{O}} h_0 m_i dxdt+\int_0^1\int_{\omega} u m_i dxdt=0,\quad 1\le i\le N, \tag{27}$$

$$\frac{\partial\mathcal{S}}{\partial\tau}(0,0)=0,\quad\forall\widehat{z}^0, \tag{28}$$

$$\|u\|_{L^2(\omega\times(0,1))}=\min. \tag{29}$$

Remark 2. The case $\omega=\mathcal{O}$ corresponds to the original notion of sentinels as introduced by Lions in [8] for an observation and a control of supports in a same open $\omega=\mathcal{O}$. We therefore propose in the previous definition a generalization of the concept of sentinels in the case of observation and control of supports in two separate open $\omega\neq\mathcal{O}$.

The existence of a control u, and therefore of a sentinel $\mathcal{S}$, is in fact equivalent to a problem of null-controllability with constraints on the control. To see it, we transform the conditions (27) and (28). For condition (27), we consider the vector subspace of $L^2(\omega\times(0,1))$ generated by the functions $m_i\chi_\omega$. Let $\mathcal{K}$ be this space, then there exists k_0 unique in $\mathcal{K}$ such that

$$\int_0^1\int_{\mathcal{O}} h_0 m_i dxdt+\int_0^1\int_{\omega} k_0 m_i dxdt=0,\quad 1\le i\le N.$$

If therefore, we denote $\mathcal{K}^\perp$ the orthogonal additional of $\mathcal{K}$ in $L^2(\omega\times(0,1))$, then the condition (27) is equivalent to $u-k_0=v\in\mathcal{K}^\perp$. We then transform the condition (28) by returning on the one hand to the definition of $\frac{\partial\mathcal{S}}{\partial\tau}(0,0)$ and by introducing on the other hand adjoint q. We then show that the search for a control u such that the pair $(u,\mathcal{S})$ satisfies (27)–(29)

is equivalent to the search for a control v such that the couple (v,q) is a solution of the following system :

$$\begin{cases} v\in\mathcal{K}^{\perp}, -{}^{c}D^{\alpha}q+f'(y_0)\,q-{}^{c}D^{\alpha-1}g(y_0)\,q=h_0\chi_{\mathcal{O}}+k_0\chi_{\omega}+v\chi_{\omega}\text{ in }\mathcal{Q},\\ q=0\text{ on }\Sigma,\ q(1)=0\text{ in }\Omega,\ q(0)=0\text{ in }\Omega,\ \|v\|_{L^2(\omega\times(0,1))}=\min. \end{cases} \tag{30}$$

We recognize in the problem (30) the problem (9) and (12) with $h = h_0\chi_{\mathcal{O}} + k_0\chi_{\omega}$.

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