

MORSE INEQUALITIES AND EMBEDDINGS FOR CR MANIFOLDS WITH CIRCLE ACTION

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Abstract

In this paper, we would like to make a report on Morse inequalities and embeddings for CR manifolds with transversal CR circle action. The results are contained in [24], [23], [21], [20] and [25]. Furthermore, in the last section we will give an explicit example for S^1 -equivariant Szegő kernels and their expansion on the sphere S^3 in \mathbb{C}^2 with respect to a family of transversal CR S^1 -actions.

1. Introduction

Let X be a real smooth manifold of dimension $2n - 1$, $n \geq 2$, together with a Lie group action of S^1 . A CR structure (of codimension one or hypersurface type) on X is a complex subbundle $T^{1,0}X$ of rank $n - 1$ of the complexified tangent space CTX which is formally Frobenius integrable and satisfies $T^{1,0}X \cap \overline{T^{1,0}X} = \{0\}$. Given a Lie group action of S^1 we say that this S^1 -action is *CR* if it preserves $T^{1,0}X$ and *transversal* if its infinitesimal generator is transversal to the real part of $T^{1,0}X \oplus \overline{T^{1,0}X}$ at every point.

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We start with a summary of results. In [24], we study the Fourier components of Kohn-Rossi cohomology on compact CR manifolds with transversal CR S^1 -action. By using the Kohn's L^2 -estimate on CR manifold, we show that without Levi-form assumptions the Fourier components of Kohn-Rossi cohomology are always finite dimensional. Then by studying the asymptotic properties of the Fourier components of the Szegő kernel function we get the asymptotic property of the dimension of the Fourier components of the Kohn-Rossi cohomology. A corollary of our results is that under the condition that the CR manifold is weakly pseudoconvex everywhere and strongly pseudoconvex at one point, the CR manifold admits many S^1 -equivariant CR functions.

In [21], we study the equivariant embedding of a compact strongly pseudoconvex CR manifold with a transversal CR circle action. We develop an asymptotic expansion for the Fourier components of the Szegő kernels concerning CR functions which only lie in the positive Fourier components of the space of CR functions. Making use of the Szegő kernel asymptotic expansion, we can construct enough S^1 -equivariant CR functions by which the CR manifold can be CR embedded into some \mathbb{C}^N . Moreover, in the last section of this paper, we give an example to describe explicitly such Szegő kernel asymptotic expansion with respect to the positive Fourier components of the space of CR functions on the unit sphere S^3 in \mathbb{C}^2 .

It is well known that there are no compact Levi-flat CR manifolds in the complex Euclidean space. Thus, a natural question is whether the Levi-flat CR manifolds can CR embedded into the projective space by means of CR sections of a CR line bundle of positive curvature. This is the analogue of the *Kodaira embedding theorem* from complex geometry. In Kodaira's original proof of the Kodaira embedding theorem (see [34]), the Kodaira vanishing theorem is used to show that there are enough holomorphic sections with values in the large tensor powers of a holomorphic positive line bundle. In [23], we use the Szegő kernel asymptotic expansion method developed by Hsiao and Marinescu [19] to establish Morse inequalities on CR manifolds with a transversal CR circle action when such CR manifolds admit rigid CR line bundles. On such CR manifold, without any assumption on the positivity of the Levi-form we showed that a positive rigid CR line bundle is always big.

In [25], we prove that a certain weighted Fourier-Szegő kernel admits a full asymptotic expansion and by using these asymptotics, we show that if X admits a transversal CR locally free S^1 -action and there is a rigid positive CR line bundle L over X , then X can be CR embedded into the projective space *without any positivity assumptions* on the Levi form. In particular, when X is Levi-flat and admits a transversal CR S^1 -action, we improve the regularity in the Kodaira embedding theorem of Ohsawa and Sibony [35] to C^∞ .

2. Morse Inequalities and Embeddings for CR Manifolds with Circle Action, I: without line bundle

The problem of embedding CR manifolds attracts a lot of attention. A classical result of L. Boutet de Monvel [5] tells us that any compact strongly pseudoconvex CR manifold of real dimension greater or equal to five can be globally CR embedded into \mathbb{C}^N , for some $N \in \mathbb{N}$. The classical example of non-embeddable three dimensional strongly pseudoconvex CR manifold appears implicitly in the non-fillable example of pseudoconcave manifold by Grauert [16], Andreotti-Siu [2] and Rossi [36] and was explicitied by Burns [7]. In [30] Lempert proved that a compact strongly pseudoconvex three dimensional CR manifold which admits a transversal CR circle action can always be CR embedded into the complex Euclidean space. Lempert proved that such CR manifold can always bound a strongly pseudoconvex complex surface. Then by Kohn's result [28] the Cauchy-Riemann operator $\bar{\partial}_b$ as an operator between L^2 spaces will have closed range. Thus by L. Boutet de Monvel [5] and Kohn [29] the CR manifold will be CR embedded into some complex Euclidean space. In [21], under the same condition as in [30], we showed that the functions in the CR embedding map can only come from the Fourier components of the space of CR functions. For further information on the embedding problem for CR manifolds we refer the reader to [12, 14, 15] and the references therein. Now we describe the results in [21].

Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n - 1$, $n \geq 2$, where $T^{1,0}X$ is the given CR structure on X . That is, $T^{1,0}X$ is a subbundle of the complexified tangent bundle $\mathbb{C}TX$ of rank $n - 1$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$.

We assume that X admits a S^1 -action: $S^1 \times X \rightarrow X$, $(e^{i\theta}, x) \mapsto e^{i\theta} \circ x$. Here, we use $e^{i\theta}$ ($0 \leq \theta < 2\pi$) to denote the S^1 -action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 -action given as follows

$$(Tu)(x) = \frac{\partial}{\partial \theta} \left(u(e^{i\theta} \circ x) \right) \Big|_{\theta=0}, u \in C^\infty(X). \quad (2.1)$$

Definition 2.1. We say that the S^1 -action is CR if

$$[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X), \quad (2.2)$$

where $[\cdot, \cdot]$ is the Lie bracket between smooth vector fields on X . Furthermore, we say that the S^1 -action is transversal if for each $x \in X$,

$$\mathbb{C}T(x) \oplus T_x^{1,0}(X) \oplus T_x^{0,1}X = \mathbb{C}T_xX. \quad (2.3)$$

For $x \in X$, we say that the period of x is $\frac{2\pi}{q}$, $q \in \mathbb{N}$, if $e^{i\theta} \circ x \neq x$ for every $0 < \theta < \frac{2\pi}{q}$ and $e^{i\frac{2\pi}{q}} \circ x = x$. For each $q \in \mathbb{N}$, put

$$X_q = \left\{ x \in X; \text{the period of } x \text{ is } \frac{2\pi}{q} \right\} \quad (2.4)$$

and set $p = \min \{q \in \mathbb{N}; X_q \neq \emptyset\}$. Without loss of generality, we assume $p = 1$. It is well-known that if X is connected, then X_1 is an open and dense subset of X (see [13] and also the appendix in [24]). We denote $X_{\text{reg}} := X_1$. We call $x \in X_{\text{reg}}$ a regular point of the S^1 -action. Let X_{sing} be the complement of X_{reg} .

We assume throughout this paper that $(X, T^{1,0}X)$ is a compact connected CR manifold with a transversal CR locally free S^1 -action and we denote by T the global vector field induced by the S^1 -action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real 1-form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

Definition 2.2. For $x \in X$, the Levi-form \mathcal{L}_x associated with the CR structure is the Hermitian quadratic form on $T_x^{1,0}X$ defined as follows. For any $U, V \in T_x^{1,0}X$, pick $\mathcal{U}, \mathcal{V} \in C^\infty(X, T^{1,0}X)$ such that $\mathcal{U}(x) = U$, $\mathcal{V}(x) = V$. Set

$$\mathcal{L}_x(U, \bar{V}) = \frac{1}{2i} \langle [\mathcal{U}, \bar{\mathcal{V}}](x), \omega_0(x) \rangle \quad (2.5)$$

where $[,]$ denotes the Lie bracket between smooth vector fields. Note that \mathcal{L}_x does not depend on the choice of \mathcal{U} and \mathcal{V} .

Definition 2.3. The CR structure on X is called pseudoconvex at $x \in X$ if \mathcal{L}_x is positive semidefinite. It is called strongly pseudoconvex at x if \mathcal{L}_x is positive definite. If the CR structure is (strongly) pseudoconvex at every point of X , then X is called a (strongly) pseudoconvex CR manifold. In particular, we call X a Levi-flat CR manifold if the Levi-form vanishes everywhere.

Let $\bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator where $\Omega^{0,q}(X)$ is the space of smooth $(0, q)$ -forms on X defined as in [8]. Set $\Omega_m^{0,q}(X) = \{u \in \Omega^{0,q}(X) : L_T u = imu\}$ where L_T is the Lie derivative along direction T . From (2.2) and (2.3) one has $L_T \bar{\partial}_b = \bar{\partial}_b L_T$. Then it follows that $\bar{\partial}_b : \Omega_m^{0,q}(X) \rightarrow \Omega_m^{0,q+1}(X)$. In [24] we define

$$H_{b,m}^q(X) := \frac{\text{Ker } \bar{\partial}_b : \Omega_m^{0,q}(X) \rightarrow \Omega_m^{0,q+1}(X)}{\text{Im } \bar{\partial}_b : \Omega_m^{0,q-1}(X) \rightarrow \Omega_m^{0,q}(X)} \quad (2.6)$$

for $m \in \mathbb{Z}$. In particular, $H_{b,m}^0(X) = \{u \in C^\infty(X) : Tu = imu, \bar{\partial}_b u = 0\}$. We call $H_{b,m}^0(X)$ the m -th Fourier components of the space of CR functions.

2.1. Morse inequalities

From Kohn's L^2 -estimate it is easy to see that $\dim H_{b,m}^q(X)$ is always finite dimensional without any assumption on the Levi-form. In [24], we studied the Morse inequalities for $\dim H_{b,m}^q(X)$ as $m \rightarrow \infty$. We have the following results.

Theorem 2.4. *Let X be a compact connected CR manifold with a transversal CR S^1 -action. Assume that $\dim_{\mathbb{R}} X = 2n - 1, n \geq 2$. Then for every $q = 0, 1, 2, \dots, n - 1$, we have*

$$\dim H_{b,m}^q(X) \leq \frac{m^{n-1}}{2\pi^n} \int_{X(q)} |\det \mathcal{L}_x| dv_X(x) + o(m^{n-1}), m \rightarrow \infty. \quad (2.7)$$

Here, $X(q)$ is a subset of X where the Levi-form is nondegenerate and has exactly q -negative eigenvalues. $\det \mathcal{L}_x$ is the multiplication of the eigenvalues of the Levi-form with respect to a given S^1 -invariant Hermitian metric on X .

Theorem 2.5. *Let X be a compact connected CR manifold with a transversal CR S^1 -action. Assume that $\dim_{\mathbb{R}} X = 2n - 1, n \geq 2$. For every $q = 0, 1, 2, \dots, n - 1$, as $m \rightarrow \infty$, we have*

$$\sum_{j=0}^q (-1)^{q-j} \dim H_{b,m}^j(X) \leq \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^q (-1)^{q-j} \int_{X^{(j)}} |\det \mathcal{L}_x| dv_X(x) + o(m^{n-1}). \quad (2.8)$$

In particular, when $q = n - 1$, as $m \rightarrow \infty$, we have the asymptotic Riemann-Roch theorem

$$\sum_{j=0}^{n-1} (-1)^j \dim H_{b,m}^j(X) = \frac{m^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^j \int_{X^{(j)}} |\det \mathcal{L}_x| dv_X(x) + o(m^{n-1}). \quad (2.9)$$

Recently, analogues to the Atiyah-Singer index theorem in complex geometry a CR index theorem is established in [9].

Corollary 2.6. *Let X be a compact connected CR manifold with a transversal CR S^1 -action. Assume that $\dim_{\mathbb{R}} X = 2n - 1, n \geq 2$. If X is weakly pseudoconvex and strongly pseudoconvex at a point, then as $m \rightarrow \infty$*

$$\dim H_{b,m}^0(X) \approx m^{n-1}, \quad \dim H_{b,m}^q(X) = o(m^{n-1}) \text{ for } q \geq 1. \quad (2.10)$$

In the main results above, we only consider the Morse inequalities for the positive Fourier component $\dim H_{b,m}^q(X)$ as $m \rightarrow \infty$. In fact, we also have the Morse inequalities for the negative Fourier component $H_{b,m}^q(X)$ as $m \rightarrow -\infty$. More precisely, we have

Theorem 2.7. *Let X be a compact connected CR manifold with a transversal CR S^1 -action. Assume that $\dim_{\mathbb{R}} X = 2n - 1, n \geq 2$. For every $q = 0, 1, 2, \dots, n - 1$, as $m \rightarrow -\infty$, we have*

$$\begin{aligned} \dim H_{b,m}^q(X) &\leq \frac{|m|^{n-1}}{2\pi^n} \int_{X^{(n-1-q)}} |\det \mathcal{L}_x| dv_X(x) + o(|m|^{n-1}), \\ \sum_{j=0}^q (-1)^{q-j} \dim H_{b,m}^j(X) & \\ &\leq \frac{|m|^{n-1}}{2\pi^n} \sum_{j=0}^q (-1)^{q-j} \int_{X^{(n-1-j)}} |\det \mathcal{L}_x| dv_X(x) + o(|m|^{n-1}). \end{aligned} \quad (2.11)$$

In particular, when $q = n - 1$, as $m \rightarrow -\infty$, we have the asymptotic Riemann-Roch theorem

$$\sum_{j=0}^{n-1} (-1)^j \dim H_{b,m}^j(X) = \frac{|m|^{n-1}}{2\pi^n} \sum_{j=0}^{n-1} (-1)^j \int_{X^{(n-1-j)}} |\det \mathcal{L}_x| dv_X(x) + o(|m|^{n-1}). \quad (2.12)$$

From Theorem 2.4, Theorem 2.5 and Theorem 2.7, we deduce

Theorem 2.8. *Let X be a compact connected CR manifold of real dimension $2n-1$ with a transversal CR S^1 -action. Let $q \in \{0, 1, \dots, n-1\}$. Assume that the Levi form of X has q non-positive and $n-1-q$ non-negative eigenvalues everywhere. Then*

$$\begin{aligned} \dim H_{b,m}^j(X) &= o(m^{n-1}), \text{ as } m \rightarrow \infty, \text{ for } j \neq q \\ \dim H_{b,m}^j(X) &= o(|m|^{n-1}), \text{ as } m \rightarrow -\infty, \text{ for } j \neq n-1-q. \end{aligned} \quad (2.13)$$

If moreover the Levi-form is non-degenerate at some point, then

$$\begin{aligned} \dim H_{b,m}^q(X) &\approx m^{n-1}, \text{ as } m \rightarrow \infty \\ \dim H_{b,m}^{n-1-q}(X) &\approx |m|^{n-1}, \text{ as } m \rightarrow -\infty \\ \dim H_b^q(X) &= \infty, \dim H_b^{n-1-q}(X) = \infty. \end{aligned} \quad (2.14)$$

In particular, if X is weakly pseudoconvex and strongly pseudoconvex at a point, then

$$\dim H_{b,m}^{n-1}(X) \approx |m|^{n-1} \text{ as } m \rightarrow -\infty$$

and in particular $\dim H_b^{n-1}(X) = \infty$. Moreover, $\dim H_{b,m}^q(X) = o(|m|^{n-1})$ as $m \rightarrow -\infty$ for $q \leq n-2$.

Recently, when X is only assumed to be weakly pseudoconvex the estimate $\dim H_{b,m}^q(X) = o(|m|^{n-1})$ when $m \rightarrow -\infty$ is improved by Wang and Zhou [38, Theorem 1.3].

2.2. Szegő kernel expansion and embeddings of CR manifolds

In this section, we describe the results in [21]. From Corollary 2.6, we know that $\dim H_{b,m}^0(X) \approx m^{n-1}$ as $m \rightarrow \infty$ when X is strongly pseudo-

convex. Hence, the space of CR functions which lie in the positive Fourier components of the space of CR functions is very large and thus a natural question is that whether X can be CR embedded into some \mathbb{C}^N by CR functions which lie in the positive Fourier components. In [21] we give an affirmative answer to this question. More precisely, we proved

Theorem 2.9. *Let X be a compact connected strongly pseudoconvex CR manifold with a locally free transversal CR S^1 -action. Then X can be CR embedded into some complex Euclidean space by the CR functions which lie in the positive Fourier components of the space of CR functions.*

The main machinery to establish Theorem 2.9 are Szegő kernel expansion methods. In [21], making use of the complex stationary phase formula due to Melin-Sjöstrand [33] we have the following asymptotic expansion of the Fourier components of the Szegő kernel.

Theorem 2.10. *Let X be a compact strongly pseudoconvex CR manifold with a transversal CR circle action. For $x_0 \in X_{\text{reg}}$, let (z, θ, φ) be canonical coordinates centered at x_0 and defined on a canonical patch $D_1 = \{(z, \theta) : |z| < \varepsilon_1, |\theta| < \pi\}$. Set $D = \{(z, \theta) \in \mathbb{C}^{n-1} \times \mathbb{R} : |z| < \varepsilon, |\theta| < \frac{\pi}{2}\} \Subset D_1$. Then on $D \times D$, we have*

$$S_m(x, y) \equiv \frac{1}{2\pi} e^{im(x_{2n-1} - y_{2n-1} + \Phi(z, w))} \hat{b}(z, w, m) \bmod O(m^{-\infty}),$$

where

$$\begin{aligned} \hat{b}(z, w, m) &\sim \sum_{j=0}^{\infty} m^{n-1-j} \hat{b}_j(z, w) \quad \text{in } S_{\text{loc}}^{n-1}(1, \tilde{D} \times \tilde{D}), \\ \hat{b}_j(z, w) &\in C^\infty(\tilde{D} \times \tilde{D}), \quad j = 0, 1, 2, \dots, \\ \hat{b}_0(z, z) &= \pi^{-(n-1)} |\det \mathcal{L}_x|, \quad x = (z, 0), \quad \forall z \in \tilde{D}. \end{aligned} \tag{2.15}$$

Here, we set $\tilde{D} = \{z \in \mathbb{C}^{n-1} : |z| < \varepsilon\}$. In particular, we have

$$S_m(x, x) \equiv \frac{1}{2\pi} \hat{b}(z, z, m) \bmod O(m^{-\infty}). \tag{2.16}$$

Theorem 2.11. *Let X be a compact strongly pseudoconvex CR manifold with a transversal CR circle action. Assume $x_0 \in X_k, k > 1$. Let D_1, D*

and (z, θ, φ) be as above with $\frac{\pi}{k} - \epsilon$ replaced by δ_1 . For $k \nmid m$ we have $S_m(x, x_0) = 0$ for all $x \in D$. For $k \mid m$ we have

$$S_m(x, x_0) \equiv \frac{k}{2\pi} e^{im(x_{2n-1} + \Phi(z, 0))} \hat{b}(z, 0, m) \bmod O(m^{-\infty}) \quad (2.17)$$

on D . In particular, given $k \mid m$ and $x = x_0$, we have

$$S_m(x_0, x_0) = \frac{k}{2\pi} \hat{b}(0, 0, m) + O(m^{-\infty})$$

and

$$\hat{b}(0, 0, m) \sim \hat{b}_0(0, 0) m^{n-1} + \hat{b}_1(0, 0) m^{n-2} + \dots$$

in the sense that for any $N \in \mathbb{N}_0$ there exists $C_N > 0$ independent of m such that

$$\left| \hat{b}(0, 0, m) - \sum_{j=0}^N \hat{b}_j(0, 0) m^{n-1-j} \right| \leq C_N m^{n-2-N}$$

holds for all $m \in \mathbb{N}$.

In [20], we give a complete description of the diagonal expansion for the m -th Fourier component of the Szegő kernel function as $m \rightarrow +\infty$. We also obtain explicit formulas for the first three coefficients of such an expansion.

3. Morse Inequalities and Embeddings for CR Manifolds with Circle Action, II: with line bundle

The well-known Grauert-Riemenschneider conjecture states that a semi-positive holomorphic line bundle on a compact complex manifold is big if it is positive at least at one point. This conjecture was first solved by Siu [37] by using the $\bar{\partial}$ -equation method. Motivated by Siu's solution of Grauert-Riemenschneider conjecture, Demailly [10] gave another proof of this conjecture by establishing the holomorphic Morse inequalities. The original proof of the holomorphic Morse inequalities was based partly on Siu's techniques and partly on an extension of Witten's analytic proof for the standard Morse inequalities. Natural problems [11] are to extend the Morse inequalities to cohomology associated to other operators than the $\bar{\partial}$ -operator and to non-compact complex manifolds. Getzler [17] extended the Morse inequalities for $H_b^q(X, L^k)$ on a compact strongly pseudoconvex CR manifold X for

$1 \leq q \leq n-2$ where L is a CR line bundle over X . Recently, Hsiao-Marinescu [19] established the Morse inequalities for $H_b^q(X, L^k)$ on a general CR manifold which satisfies $Y(q)$ condition. Both Getzler's and Hsiao-Marinescu's work depend on the sign of the Levi-form on CR manifolds satisfying some algebraic conditions. So their work can not be used on Levi-flat CR manifolds which play an important part in CR geometry. In [23] we studied the Morse inequalities on CR manifolds which admit rigid CR line bundles. For Morse inequalities on non-compact complex manifolds which will be not focused on in this paper we refer the reader to [4, 26, 31, 32] and the references therein.

In our work [23] we do not need any restrictions on the behavior of the Levi-form of the CR manifold. From the Morse inequalities we have established, we prove that a rigid CR line bundle is big if it is positive. In order to describe this work, we first recall the definition of rigid CR line bundles.

3.1. Rigid CR vector bundle

In this subsection we use the notations from [22].

Definition 3.1. Let $D \subset X$ be an open set. We say that a function $u \in C^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^\infty(X)$ is Cauchy-Riemann (CR for short) if $\bar{\partial}_b u = 0$. We say that $u \in C^\infty(X)$ is rigid CR if $\bar{\partial}_b u = 0$ and $Tu = 0$.

The following definitions for CR vector bundles can be found in [27].

Definition 3.2. A complex vector bundle (E, π, X) over X is called CR vector bundle if

- (i) E is a CR manifold,
- (ii) $\pi: E \rightarrow X$ is a CR submersion,
- (iii) $E \oplus E \ni (\xi_1, \xi_2) \rightarrow \xi_1 + \xi_2 \in E$ and $\mathbb{C} \times E \ni (\lambda, \xi) \rightarrow \lambda\xi \in E$ are CR maps.

A smooth section $s \in \Gamma(U, E)$ defined on an open set $U \subset X$ is called CR section if the map $s: U \rightarrow E$ is CR.

Let (E_1, π_1, X) and (E_2, π_2, X) be two CR vector bundles over X . A map $F: E_1 \rightarrow E_2$ is called a CR bundle isomorphism if F is a C^∞ -diffeomorphism such that F, F^{-1} are CR maps, $\pi_2 \circ F = \pi_1$ and F is fiberwise linear.

Given a CR vector bundle (E, π, X) we find (see [27]) the linear partial differential operator $\bar{\partial}_b^E: \Gamma(X, E) \rightarrow \Gamma(X, E \otimes T^{*0,1}X)$ satisfying

- (a) $\bar{\partial}_b^E(f \cdot s) = s\bar{\partial}_b(f) + f\bar{\partial}_b^E(s)$ for all $f \in C^\infty(X)$ and $s \in \Gamma(X, E)$,
- (b) $s \in \Gamma(U, E)$ is a CR section if and only if $\bar{\partial}_b^E s = 0$.

Definition 3.3. A CR vector bundle (E, π, X) of rank r is called locally CR trivializable if for any point $p \in X$ there exists an open neighborhood $U \subset X$ such that $E|_U$ is CR vector bundle isomorphic to the trivial CR vector bundle $U \times \mathbb{C}^r$.

The following lemma is well-known.

Lemma 3.4. *Let (E, π, X) be a CR vector bundle. The following are equivalent:*

- (i) (E, π, X) is locally CR trivializable,
- (ii) For any $p \in X$ there exists a smooth frame $\{f_1, \dots, f_r\}$ of $E|_U$ on an open neighborhood $U \subset X$ around p such that $f_1, \dots, f_r: U \rightarrow E$ are CR sections.

Remark 3.5. Let $\{f_1, \dots, f_r\}$ be a frame of $E|_U$ for some open set $U \subset X$. Then $\{f_1, \dots, f_r\}$ is called CR frame if any $f_k, 1 \leq k \leq r$, is a CR section. Given two CR frames of $E|_U$ we find by (a) and (b) that the corresponding transition matrix is CR in the sense that any entry is a CR function.

Definition 3.6. Let $(X, T^{1,0}X)$ be a CR manifold and let $T \in \Gamma(X, TX)$ be a CR vector field. A CR bundle lift of T to (E, π, X) is a linear partial differential operator $T^E: \Gamma(X, E) \rightarrow \Gamma(X, E)$ (with smooth coefficients) such that

- (i) $T^E(f \cdot s) = T(f) \cdot s + fT^E(s)$ for all $f \in C^\infty(X)$ and $s \in \Gamma(X, E)$,
- (ii) $[T^E, \bar{\partial}_b^E] = 0$.

In order to define $[T^E, \bar{\partial}_b^E]$ we need to define T^E on $(0, 1)$ forms with values in E first. But this definition follows immediately from the fact that any $w \in \Gamma(X, E \otimes T^{*0,1}X)$ locally can be written $w = \sum_{j=1}^r f_j \otimes \omega^j$ where $\{\omega^j\}$ are $(0, 1)$ -forms and $\{f_j\}$ are local frames of E and that T is defined also for $(0, q)$ -forms using the Lie derivative.

Definition 3.7. Let $(X, T^{1,0}X)$ be a CR manifold and let $T \in \Gamma(X, TX)$ be a CR vector field. A CR vector bundle (E, π, X) of rank r over X with a CR bundle lift T^E of T is called rigid CR (with respect to T^E) if for every point $p \in X$ there exists an open neighborhood U around p and a CR frame $\{f_1, \dots, f_r\}$ of $E|_U$ with $T^E(f_j) = 0$ for $1 \leq j \leq r$.

A section $s \in \Gamma(X, E)$ is called a rigid CR section if $T^E s = 0$ and $\bar{\partial}_b^E s = 0$. The frame $\{f_j\}_{j=1}^r$ in Definition 3.7 is called a rigid CR frame of $E|_U$. Note that it follows from Lemma 3.4 that any rigid CR vector bundle is locally CR trivialisable.

Remark 3.8. In [24] and [25] we gave a definition of rigid CR vector bundle by assuming that the transition functions of a chosen local trivializing frame are rigid and CR. This definition is actually equivalent to the one given in Definition 3.7 (see Lemma 3.9).

Lemma 3.9. *Let (E, π, X) be CR vector bundle over a CR manifold $(X, T^{1,0}X)$ of codimension d and let $T \in \Gamma(X, TX)$ be a CR vector field. The following are equivalent:*

- (i) *T has a CR bundle lift T^E such that (E, π, X) is rigid CR with respect to T^E .*
- (ii) *There exist an open cover $\{U_j\}_{j \in \mathbb{N}}$ of X and CR frames $\{f_1^j, \dots, f_r^j\}$ for $E|_{U_j}$, $j \in \mathbb{N}$, such that the corresponding transition matrices are rigid CR in the sense that any entry is a rigid CR function.*

Lemma 3.10. *Let $(X, T^{1,0}X)$ be a CR manifold with a transversal CR S^1 -action. Let T be the infinitesimal generator of the S^1 -action. Let (E, π, X) be a locally CR trivialisable CR vector bundle of rank $r = 1$. Assume that T^E is a CR bundle lift of T to (E, π, X) . Then (E, π, X) is rigid CR. More precisely, for any $p \in X$ there exist an open neighborhood $U \subset X$ around p and a CR frame $\{f\}$ of $E|_U$ with $T^E(f) = 0$.*

Definition 3.11. Let E be a rigid vector bundle over X . Let $\langle \cdot | \cdot \rangle_E$ be a Hermitian metric on E . We say that $\langle \cdot | \cdot \rangle_E$ is a rigid Hermitian metric if for every local rigid frame f_1, \dots, f_r of E , we have $T\langle f_j | f_k \rangle_E = 0$, for every $j, k = 1, 2, \dots, r$.

In order to simplify the notation we will denote by $\bar{\partial}_b, T$ the operators $\bar{\partial}_b^E, T^E$ where E is any rigid CR vector bundle on X . Consider a locally CR trivializable CR line bundle L over X with a CR bundle lift of T . By Lemma 3.10 we find that L is rigid CR with respect to that bundle lift. Hence there exists an open covering $(U_j)_{j=1}^N$ and a family of rigid CR trivializing frames $\{s_j\}_{j=1}^N$ with each s_j defined on U_j and the transition functions between different rigid CR frames are rigid CR functions. Let L^k be the k -th tensor power of L . Then $\{s_j^k\}_{j=1}^N$ is a family of rigid CR trivializing frames on each U_j . Let $\bar{\partial}_b^{L^k} : \Omega^{0,q}(X, L^k) \rightarrow \Omega^{0,q+1}(X, L^k)$ be the tangential Cauchy-Riemann operator. Since L^k is rigid CR we have $\bar{\partial}_b f = \bar{\partial}_b f_j \otimes s_j^k, Tf = (Tf_j) \otimes s_j^k$ for any $f = f_j \otimes s_j^k \in \Omega^{0,q}(X, L^k)$ and

$$T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,q}(X, L^k). \tag{3.1}$$

Let h^L be a Hermitian fiber metric on L . The local weight of h^L with respect to a local rigid CR trivializing section s of L^L over an open subset $D \subset X$ is the function $\Phi \in C^\infty(D, \mathbb{R})$ for which

$$|s(x)|_{h^L}^2 = e^{-2\Phi(x)}, x \in D. \tag{3.2}$$

We denote by Φ_j the weight of h^L with respect to s_j .

Definition 3.12. Let L be a rigid CR line bundle and let h^L be a Hermitian metric on L . The curvature of (L, h^L) is the the Hermitian quadratic form $R^L = R^{(L, h^L)}$ on $T^{1,0}X$ defined by

$$R_p^L(U, V) = \langle d(\bar{\partial}_b \Phi_j - \partial_b \Phi_j)(p), U \wedge \bar{V} \rangle, \quad U, V \in T_p^{1,0}X, \quad p \in U_j. \tag{3.3}$$

Due to [19, Proposition 4.2], R^L is a well-defined global Hermitian form, since the transition functions between different frames s_j are annihilated by T .

Definition 3.13. We say that (L, h^L) is positive if the associated curvature R_x^L is positive definite at every $x \in X$.

3.2. Morse inequalities for CR manifolds with circle action

Let $(X, T^{1,0}X)$ be a compact connected CR manifold of dimension $2n-1$, $n \geq 2$. Let L be a rigid CR line bundle over X . For every $u \in \Omega^{0,q}(X, L^k)$, we can define $Tu \in \Omega^{0,q}(X, L^k)$ in a natural way since locally we can find rigid CR frames such that the transition functions are rigid and CR. Moreover, we have

$$T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,q}(X, L^k), \quad (3.4)$$

where $\bar{\partial}_b : \Omega^{0,q}(X, L^k) \rightarrow \Omega^{0,q+1}(X, L^k)$ denotes the tangential Cauchy-Riemann operator. For every $m \in \mathbb{Z}$, the m -th Fourier component of $\bar{\partial}_b$ cohomology is given by

$$H_{b,m}^q(X, L^k) := \frac{\text{Ker } \bar{\partial}_b : \Omega_m^{0,q}(X, L^k) \rightarrow \Omega_m^{0,q+1}(X, L^k)}{\text{Im } \bar{\partial}_b : \Omega_m^{0,q-1}(X, L^k) \rightarrow \Omega_m^{0,q}(X, L^k)}. \quad (3.5)$$

Without any Levi curvature assumption, for any $m \in \mathbb{Z}$ and any $q = 0, 1, 2, \dots, n-1$, we have

$$\dim H_{b,m}^q(X, L^k) < \infty. \quad (3.6)$$

Fix $\lambda \geq 0$ and set $H_{b,\leq\lambda}^q(X, L^k) := \bigoplus_{m \in \mathbb{Z}, |m| \leq \lambda} H_{b,m}^q(X, L^k)$. In this work, we study the asymptotic behavior of the space $H_{b,\leq k\delta}^q(X, L^k)$ and its partial Szegő kernel. Our main results are the following

Theorem 3.14. *For k large and for every $q = 0, 1, 2, \dots, n-1$, we have*

$$\dim H_{b,\leq k\delta}^q(X, L^k) \leq (2\pi)^{-n} \frac{(-1)^q}{(n-1)!} k^n \int_X \int_{\mathbb{R}_{x,q} \cap [-\delta, \delta]} (i\mathcal{R}_x^L + 2si\mathcal{L}_x)^{n-1} \wedge (-\omega_0) + o(k^n), \quad (3.7)$$

where \mathcal{R}_x^L denotes the curvature of L , \mathcal{L}_x denotes the Levi form of X , ω_0 is the unique global non-vanishing real one form determined by $\langle \omega_0, u \rangle = 0$, $\forall u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$ and

$$\mathbb{R}_{x,q} := \{s \in \mathbb{R} : \mathcal{R}_x^L + 2s\mathcal{L}_x \text{ has exactly } q \text{ negative and } n-1-q \text{ positive eigenvalues}\}. \quad (3.8)$$

Note that $\mathcal{R}_x^L, \mathcal{L}_x \in T_x^{*1,0}X \wedge T_x^{*0,1}X$ (see Definition 2.2). Hence, $(\mathcal{R}_x^L + 2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x))$ is a global $2n-1$ form on X . Any Hermitian fiber

metric h^L on L induces a curvature \mathcal{R}^L . It is easy to see that the integral in (3.7) does not depend on the choice of Hermitian fiber metric of L .

Theorem 3.15. *For k large and for every $q = 0, 1, 2, \dots, n - 2$, we have*

$$\begin{aligned} & \sum_{j=0}^q (-1)^{q-j} \dim H_{b, \leq k\delta}^j(X, L^k) \\ & \leq (2\pi)^{-n} \frac{k^n}{(n-1)!} (-1)^q \sum_{j=0}^q \int_X \int_{\mathbb{R}_{x,j} \cap [-\delta, \delta]} (i\mathcal{R}_x^L + i2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x)) ds + o(k^n), \end{aligned} \tag{3.9}$$

and when $q = n - 1$, we have asymptotic Riemann-Roch-Hirzebruch theorem

$$\begin{aligned} & \sum_{j=0}^{n-1} (-1)^j \dim H_{b, \leq k\delta}^j(X, L^k) \\ & = (2\pi)^{-n} \frac{k^n}{(n-1)!} \sum_{j=0}^{n-1} \int_X \int_{\mathbb{R}_{x,j} \cap [-\delta, \delta]} (i\mathcal{R}_x^L + i2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x)) ds + o(k^n). \end{aligned} \tag{3.10}$$

Assume that \mathcal{R}^L is positive. If $\delta > 0$ is small enough then $\mathbb{R}_{x,j} \cap [\delta, \delta] = \emptyset$, $\forall x \in X$ and for every $j = 1, 2, \dots, n - 1$. From this observation, (3.7) and (3.10), we conclude that

$$\begin{aligned} & \dim H_{b, \leq k\delta}^0(X, L^k) \\ & = (2\pi)^{-n} \frac{1}{(n-1)!} k^n \int_X \int_{\mathbb{R}_{x,0} \cap [-\delta, \delta]} (i\mathcal{R}_x^L + i2s\mathcal{L}_x)^{n-1} \wedge (-\omega_0(x)) ds + o(k^n). \end{aligned}$$

Hence, $\dim H_{b, \leq k\delta}^0(X, L^k) \approx k^n$. We conclude that

Theorem 3.16. *If L is a positive rigid CR line bundle, then L is big, that is, $\dim H_b^0(X, L^k) \gtrsim k^n$ when $k \gg 1$.*

3.3. Kodaira embedding theorems for CR manifolds with circle action

Ohsawa and Sibony [35] constructed for every $\kappa \in \mathbb{N}$ a CR projective embedding of class C^κ of a Levi-flat CR manifold by using $\bar{\partial}$ -estimates. A natural question is whether we can improve the regularity to $\kappa = \infty$. Adachi [1] showed that the answer is no, in general. The analytic difficulty of

this problem comes from the fact that the Kohn Laplacian is not hypoelliptic on Levi flat manifolds. In [25] we give an affirmative answer to this question under the condition that the CR manifold admits a transversal CR S^1 -action and a rigid positive CR line bundle. Now we describe the results in [25].

For every $m \in \mathbb{Z}$, let $L_m^2(X, L^k) \subset L^2(X, L^k)$ be the completion of $C_m^\infty(X, L^k)$ with respect to the Hermitian inner product $(\cdot | \cdot)$ induced by the S^1 -invariant Hermitian metric on X and the rigid Hermitian fiber metric h^L for L . Let

$$Q_{m,k}^{(0)} : L^2(X, L^k) \rightarrow L_m^2(X, L^k) \quad (3.11)$$

be the orthogonal projection with respect to $(\cdot | \cdot)$. We recall the weighted Fourier Szegő projection defined in [25]. Fix $\delta > 0$ and a function

$$\tau_\delta \in C_0^\infty((-\delta, \delta)), \quad 0 \leq \tau_\delta \leq 1, \quad \tau_\delta = 1 \text{ on } \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]. \quad (3.12)$$

Let $F_{k,\delta} : L^2(X, L^k) \rightarrow L^2(X, L^k)$ be the bounded operator given by $F_{k,\delta}(u) = \sum_{m \in \mathbb{Z}} \tau_\delta\left(\frac{m}{k}\right) Q_{m,k}^{(0)}(u), \forall u \in L^2(X, L^k)$. For every $\lambda > 0$, we consider the partial Szegő projector $\Pi_{k,\leq\lambda} : L^2(X, L^k) \rightarrow H_{b,\leq\lambda}^0(X, L^k)$ which is the orthogonal projection on the space of equivariant CR functions of degree less than λ . The weighted Fourier-Szegő operator is defined as follows.

$$P_{k,\delta} := F_{k,\delta} \circ \Pi_{k,\leq k\delta} \circ F_{k,\delta} : L^2(X, L^k) \rightarrow H_{b,\leq k\delta}^0(X, L^k). \quad (3.13)$$

The Schwartz kernel of $P_{k,\delta}$ with respect to dv_X is the smooth function $P_{k,\delta}(x, y) \in L_x^k \otimes (L_y^k)^*$ satisfying

$$(P_{k,\delta}u)(x) = \int_X P_{k,\delta}(x, y)u(y) dv_X(y), \quad u \in L^2(X, L^k). \quad (3.14)$$

Let $P_{k,\delta,s} : L_{\text{comp}}^2(D) \rightarrow L^2(D)$ be the localization of $P_{k,\delta}$ with respect to a local rigid frame s of L over an open subset $D \subset X$. Here, $L_{\text{comp}}^2(D)$ is a subspace of $L^2(D)$ with elements having compact support in D . Let $P_{k,\delta,s}(x, y) \in C^\infty(D \times D)$ be the Schwartz kernel of $P_{k,\delta,s}$ with respect to dv_X , defined as in (3.14). Then we can describe the structure of the localized Fourier-Szegő kernel $P_{k,\delta,s}(x, y)$.

Theorem 3.17. *Let X be a compact CR manifold with a transversal CR locally free S^1 -action and let L be a positive rigid CR line bundle on X .*

Consider a point $p \in X$ and a canonical coordinate neighborhood $(D, x = (x_1, \dots, x_{2n-1}))$ centered at $p = 0$. Let s be a local rigid CR frame of L on D and set $|s|_h^2 = e^{-2\Phi}$. Fix $\delta > 0$ small enough and $D_0 \Subset D$. Then

$$P_{k,\delta,s}(x, y) = \int_{\mathbb{R}} e^{ik\varphi(x,y,t)} g(x, y, t, k) dt + O(k^{-\infty}) \text{ on } D_0 \times D_0, \quad (3.15)$$

where $\varphi \in C^\infty(D \times D \times (-\delta, \delta))$ is a phase function such that for some constant $c > 0$ we have

$$\begin{aligned} d_x \varphi(x, y, t)|_{x=y} &= -2\text{Im} \bar{\partial}_b \Phi(x) - t\omega_0(x), \\ d_y \varphi(x, y, t)|_{x=y} &= 2\text{Im} \bar{\partial}_b \Phi(x) + t\omega_0(x), \quad \text{Im} \varphi(x, y, t) \geq c|z - w|^2, \\ (x, y, t) &\in D \times D \times (-\delta, \delta), \quad x = (z, x_{2n-1}), \quad y = (w, y_{2n-1}), \\ \text{Im} \varphi(x, y, t) + \left| \frac{\partial \varphi}{\partial t}(x, y, t) \right|^2 &\geq c|x - y|^2, \quad (x, y, t) \in D \times D \times (-\delta, \delta), \\ \varphi(x, y, t) = 0 \text{ and } \frac{\partial \varphi}{\partial t}(x, y, t) = 0 &\text{ if and only if } x = y, \end{aligned} \quad (3.16)$$

and $g(x, y, t, k) \in S_{\text{loc}}^n(1; D \times D \times (-\delta, \delta)) \cap C_0^\infty(D \times D \times (-\delta, \delta))$ is a symbol with expansion

$$g(x, y, t, k) \sim \sum_{j=0}^{\infty} g_j(x, y, t) k^{n-j} \text{ in } S_{\text{loc}}^n(1; D \times D \times (-\delta, \delta)), \quad (3.17)$$

and for $x \in D_0$ and $|t| < \delta$ we have

$$g_0(x, x, t) = (2\pi)^{-n} |\det(R_x^L + 2t\mathcal{L}_x)| |\tau_\delta(t)|^2. \quad (3.18)$$

Corollary 3.18. *In the conditions of Theorem 3.17 we have as $k \rightarrow \infty$,*

$$P_{k,\delta}(x, x) \sim \sum_{j=0}^{\infty} k^{n-j} b_j(x) \text{ in } S_{\text{loc}}^n(1; X) \quad (3.19)$$

where $b_j(x) \in C^\infty(X)$, $j = 0, 1, 2, \dots$, and

$$b_0(x) = (2\pi)^{-n} \int_{\mathbb{R}} |\det(R_x^L + 2t\mathcal{L}_x)| |\tau_\delta(t)|^2 dt, \quad (3.20)$$

with $\tau_\delta \in C_0^\infty(\mathbb{R})$ introduced in (3.12).

In [25], we give an example to show that the Szegő kernel for $H_{b, \leq k\delta}^0(X)$,

L^k) does not admit an asymptotic expansion as $k \rightarrow \infty$. Thus, the weighted Szegő kernel is necessary for the existence of an expansion.

We define now the Kodaira map. Consider an open set $D \subset X$ which is S^1 -invariant and let $s : D \rightarrow L$ be a local rigid CR trivializing section on D . For any $u \in C^\infty(X, L^k)$ we write $u(x) = s^k(x) \otimes \tilde{u}(x)$ on D , with $\tilde{u} \in C^\infty(D)$. Let $\{f_j\}_{j=1}^{d_k}$ be an orthonormal basis of $H_{b, \leq k\delta}^0(X, L^k)$ with respect to $(\cdot | \cdot)$ such that $f_j \in H_{b, m_j}^0(X, L^k)$ and set $g_j = F_{k, \delta} f_j$, $1 \leq j \leq d_k$. The Kodaira map is defined on D by

$$\begin{aligned} \Phi_{k, \delta} : D &\longrightarrow \mathbb{C}\mathbb{P}^{d_k-1}, \\ x &\longmapsto [F_{k, \delta} f_1, \dots, F_{k, \delta} f_{d_k}] := [\tilde{g}_1(x), \dots, \tilde{g}_{d_k}(x)], \text{ for } x \in D. \end{aligned} \tag{3.21}$$

Theorem 3.19. *Let $(X, T^{1,0}X)$ be a compact CR manifold with a transversal CR locally free S^1 -action. Assume there is a rigid positive CR line bundle L over X . Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $k(\delta)$ so that for $k > k(\delta)$ and any orthonormal basis $\{f_j\}_{j=1}^{d_k}$ of $H_{b, \leq k\delta}^0(X, L^k)$ with respect to $(\cdot | \cdot)$ such that the map $\Phi_{k, \delta}$ introduced in (3.21) is a smooth CR embedding.*

In [18], it was shown that if X admits a transversal CR locally free S^1 -action and there is a rigid positive CR line bundle L over X , then X can be CR embedded into projective space under the assumption that condition $Y(0)$ holds on X . In Theorem 3.19 we remove the Levi curvature assumption $Y(0)$ used in [18].

As a consequence of Theorem 3.19 we obtain an embedding result for Levi-flat CR manifolds.

Corollary 3.20. *Let X be a compact Levi-flat CR manifold. Assume that X admits a transversal CR locally free S^1 -action and a positive rigid CR line bundle. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there exists $k(\delta)$ so that for $k > k(\delta)$ the map $\Phi_{k, \delta}$ introduced in (3.21) is a C^∞ CR embedding of X in $\mathbb{C}\mathbb{P}^{d_k-1}$ which is S^1 -equivariant with respect to weighted diagonal actions.*

4. Equivariant Szegő Kernels on S^3

In this section we will give an explicit example for S^1 -equivariant Szegő kernels and their expansion. We will study the 3-sphere S^3 as the boundary of the open unit ball B^2 in \mathbb{C}^2 together with a family of CR S^1 -actions. On the one hand for each of these actions we have to construct a metric on S^3 satisfying several properties (see the definition of rigid Hermitian metric). We will do this in Section 4.1 and we will also calculate the determinant of the Levi form (see Lemma 4.7) there. On the other hand we will compute the Szegő kernel for positive Fourier coefficients in such settings explicit by constructing an orthonormal basis for the function spaces in question (see Section 4.2, Theorem 4.10). In Section 4.3 we will discuss the example in context of the general theory as presented in Section 2.2.

A point in \mathbb{C}^2 or S^3 is always denoted by $z = (z_1, z_2)$.

4.1. Setting

Let $X = S^3 = \{\rho(z) := |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ be the 3-sphere together with the CR structure given by $T^{1,0}X = \mathbb{C}TX \cap T^{1,0}\mathbb{C}^2 = \mathbb{C}Z$ where

$$Z_z = \gamma(z)^{-1} \left(\bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2} \right)$$

for $z \in X$ and γ is a smooth non vanishing function defined on \mathbb{C}^2 . Moreover, let $\ell: X \rightarrow \mathbb{C}^2$ denote the inclusion map. For $n \in \mathbb{Z}$ consider the holomorphic S^1 -action $\tilde{\mu}: S^1 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $(e^{i\theta}, z) \mapsto (e^{i\theta}z_1, e^{in\theta}z_2)$. Then $\tilde{\mu}$ restricts to a CR action on X which we will denote by μ . One has

$$T_z := \frac{\partial}{\partial \theta} \mu(e^{i\theta}, z)|_{\theta=0} = i \left(z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + n \left(z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right) \right)$$

for $z \in X$ and T (resp. Z) can be extended in an obvious way to a vector field on \mathbb{C}^2 also denoted by T (resp. Z). We further assume that $|\gamma|_X$ is μ -invariant. The following theorem describes crucial properties for the CR S^1 -action μ on X for several n (see Definition 2.1 for the definition of transversal CR S^1 -action).

Theorem 4.1. *One has that μ is:*

- (i) *locally free* $\Leftrightarrow n \neq 0$
- (ii) *globally free* $\Leftrightarrow n \in \{\pm 1\}$
- (iii) *transversal* $\Leftrightarrow n > 0$

Proof. For $n \neq 0$ one has that $T_z = 0$ implies $z = 0 \notin X$. On the other hand $T_{(0,1)} = 0$ when $n = 0$ which proves (i). In order to prove (ii) one observe that for $z = (0, z_2) \in X$, $\mu(e^{i\theta}, z) = z$ if and only if $n\theta \in 2\pi\mathbb{Z}$ and for $z = (z_1, z_2) \in X$ such that $z_1 \neq 0$ one has $\mu(e^{i\theta}, z) = z$ if and only if $\theta \in 2\pi\mathbb{Z}$. For the third part we define a 1-form α on \mathbb{C}^2 by

$$\alpha_z = \frac{i}{2} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2).$$

Then $\alpha \neq 0$ in a neighbourhood around X and since $\alpha(\text{grad}\rho)|_X = 0$ (where ρ is a defining function for X) one has that $\ell^*\alpha$ defines a non vanishing 1-form on X . One has $\alpha(Z) = \alpha(\bar{Z}) = 0$ and $\alpha(T)_z = |z_1|^2 + n|z_2|^2$. Thus, for $n > 0$ one obtains $\alpha(T) > 0$ which implies $\mathbb{C}T \cap T^{1,0}X \oplus T^{0,1}X = 0$. Given $n \leq 0$ set $z_1 = \sqrt{-n/(1-n)}$, $z_2 = \sqrt{1/(1-n)}$ and $z = (z_1, z_2)$. Then $|z|^2 = 1$ and

$$\alpha(T)_z = \frac{-n}{1-n} + \frac{n}{1-n} = 0 = \alpha(Z)_z = \alpha(\bar{Z})_z.$$

Since $\ell^*\alpha_z \neq 0$ and the linear independency of Z and \bar{Z} one has $T_z \in T_z^{1,0}X \oplus T_z^{0,1}X$. \square

Remark 4.2. For $|n| > 1$ one can write $X = X_{\text{reg}} \dot{\cup} X_n$ where $X_{\text{reg}} = \{z \in X | z_1 \neq 0\}$.

Given $m \in \mathbb{Z}$, $m \geq 0$, consider the space

$$\mathbb{C}[z_1, z_2]_m := \text{span}_{\mathbb{C}} \left(\{z \mapsto z_1^l z_2^k \mid l, k \geq 0, m = l + nk\} \right).$$

Lemma 4.3. *We have that the linear map $\ell^* : \mathbb{C}[z_1, z_2]_m \rightarrow H_{b,m}^0(X)$ is well defined and injective.*

Proof. Since $k, l \geq 0$ one has $(z \mapsto z_1^l z_2^k)|_X \in H_b^0(X)$ as the restriction of a holomorphic function and $2\pi z_1^l z_2^k = \int_0^{2\pi} (e^{i\theta} z_1)^l (e^{in\theta} z_2)^k e^{-im\theta} d\theta$ for all $z \in X$ if and only if $m = l + nk$ (see (2.6) for the definition of $H_{b,m}^0(X)$). Hence

$\ell^*\mathbb{C}[z_1, z_2]_m \subset H_{b,m}^0(X)$. From the Bochner's extension theorem it follows that every smooth function in $H_b^0(X)$ extends uniquely to a function in $H^0(B^2) \cap C^\infty(\overline{B^2})$ where $H^0(B^2)$ denotes the space of holomorphic functions on B^2 . Hence for $f \in \mathbb{C}[z_1, z_2]_m$ with $\ell^*f = 0$ we must have $f = 0$ which proves the injectivity of ℓ^* restricted to $\mathbb{C}[z_1, z_2]_m$. \square

As a direct consequence of Lemma 4.3 we have

$$\dim(\ell^*\mathbb{C}[z_1, z_2]_m) = \begin{cases} \lfloor \frac{m}{n} \rfloor + 1, & \text{for } n \geq 1, \\ \infty, & \text{else,} \end{cases}$$

and hence $\dim H_{b,m}^0(X) = \infty$ in the case $n \leq 0$.

Remark 4.4. We observe the importance of having a transversal CR S^1 -action for $H_{b,m}^0(X)$ being finite dimensional.

Lemma 4.5. *Given $n \geq 1$ we have $\ell^*\mathbb{C}[z_1, z_2]_m = H_{b,m}^0(X)$.*

Proof. By Lemma 4.3 we have $\ell^*\mathbb{C}[z_1, z_2]_m \subset H_{b,m}^0(X)$. On the other hand choose an arbitrary $f \in H_{b,m}^0(X)$. By the Bochner's extension theorem we find a uniquely determined $F \in H^0(B^2) \cap C^\infty(\overline{B^2})$ such that $F|_X = f$. Put

$$\tilde{F}(z) = \frac{1}{2\pi} \int_0^{2\pi} F(\tilde{\mu}(e^{i\theta}, z)) e^{-im\theta} d\theta \text{ for all } z \in \overline{B^2}.$$

Then $\tilde{F} \in H^0(B^2) \cap C^\infty(\overline{B^2})$ and for any $z \in X$ we have

$$\tilde{F}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\tilde{\mu}(e^{i\theta}, z)) e^{-im\theta} d\theta = f(z)$$

which implies $\tilde{F}|_X = f$. By the uniqueness of the extension we find $F = \tilde{F}$ and hence

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} F(\tilde{\mu}(e^{i\theta}, z)) e^{-im\theta} d\theta \text{ for all } z \in B^2. \quad (4.1)$$

Since F is holomorphic on B^2 we can write $F(z) = \sum_{l,k \geq 0} a_{l,k} z_1^l z_2^k$ in a small neighbourhood around 0. Using (4.1) we find that $a_{l,k} = 0$ for $m \neq l + nk$, i.e. only finite many $a_{l,k}$'s are different from 0. From the identity theorem for holomorphic functions we find $F = \hat{F}|_{\overline{B^2}}$ with $\hat{F} \in \mathbb{C}_m[z_1, z_2]$ and $\ell^*\hat{F} = F|_X = f$. \square

From now on we assume $n \geq 1$. Since μ is transversal we find that a global frame for $\mathbb{C}TX$ is given by (Z, \bar{Z}, T) , where Z (resp. \bar{Z}) is a frame for $T^{1,0}X$ (resp. $T^{0,1}X$). We want to construct an S^1 -invariant Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that

$$(*) \quad T^{1,0}X \perp T^{0,1}X, \quad T \perp (T^{1,0}X \oplus T^{0,1}X), \quad \langle T | T \rangle = 1,$$

$$(**) \quad \langle u | v \rangle \text{ is real if } u, v \text{ are real tangent vectors.}$$

We do so by defining (Z, \bar{Z}, T) to be an orthonormal frame. Then, $(*)$ and $(**)$ are satisfied. Moreover, the assumptions on γ and the construction of Z imply

$$d\mu(e^{i\theta}, \cdot)_z Z_z = \lambda(e^{i\theta}, z) Z_{\mu(e^{i\theta}, z)}$$

for some smooth function λ on $S^1 \times X$ with $|\lambda| \equiv 1$. Thus, the metric is rigid. Note that for the S^1 -actions considered in this example, any rigid Hermitian metric which satisfies $(*)$ and $(**)$ can be obtained in this way.

For $z \neq 0$ we define

$$\alpha_z = \frac{\gamma(z)}{|z_1|^2 + n|z_2|^2} (nz_2 dz_1 - z_1 dz_2) \in T_z^{1,0*} \mathbb{C}^2$$

and

$$\tilde{\omega}_z = -\frac{i}{2(|z_1|^2 + n|z_2|^2)} (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2).$$

Furthermore, we set $Z^* = \ell^* \alpha$, $\bar{Z}^* = \ell^* \bar{\alpha}$ and $\omega_0 = \ell^* \tilde{\omega}$.

Lemma 4.6. $(Z^*, \bar{Z}^*, -\omega_0)$ is the dual frame for (Z, \bar{Z}, T) .

Proof. A direct calculation shows $\omega_0(Z) = \omega_0(\bar{Z}) = 0$, $\omega_0(T) = -1$, $Z^*(T) = \bar{Z}^*(T) = 0$, $Z^*(\bar{Z}) = \bar{Z}^*(Z) = 0$ and $Z^*(Z) = \bar{Z}^*(\bar{Z}) = 1$. \square

Using this lemma we can compute the Levi form \mathcal{L} and its determinant:

Lemma 4.7. *One has*

$$|\det \mathcal{L}_z| = \frac{1}{2} \frac{|\gamma(z)|^{-2}}{|z_1|^2 + n|z_2|^2}.$$

Proof. Consider

$$\mathcal{L}_z = \frac{i}{2} d\omega_0|_{T_z^{1,0}X \times T_z^{0,1}X}$$

$$\begin{aligned}
 &= \frac{1}{2(|z_1|^2 + n|z_2|^2)} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) \Big|_{T_z^{1,0}X \times T_z^{0,1}X} \\
 &= \frac{1}{2} \frac{|\gamma(z)|^{-2}}{|z_1|^2 + n|z_2|^2} Z_z^* \wedge \bar{Z}_z^*. \quad \square
 \end{aligned}$$

We define an orientation on X by saying (Z, \bar{Z}, T) is an oriented frame. Then the volume form of X is given by

$$dV_X = -\frac{i}{2}(Z^* + \bar{Z}^*) \wedge (Z^* - \bar{Z}^*) \wedge (-\omega_0) = -iZ^* \wedge \bar{Z}^* \wedge \omega_0 = -i\ell^*(\alpha \wedge \bar{\alpha} \wedge \tilde{\omega}).$$

In the next section we need to compute several integrals on X . Thus, it is useful to have the following expression,

Lemma 4.8. *One has*

$$\begin{aligned}
 (\alpha \wedge \bar{\alpha} \wedge \tilde{\omega})_z = -\frac{i}{2} \left(\frac{|\gamma(z)|}{|z_1|^2 + n|z_2|^2} \right)^2 & \left((z_1 d\bar{z}_1 - \bar{z}_1 dz_1) \wedge dz_2 \wedge d\bar{z}_2 \right. \\
 & \left. + ndz_1 \wedge d\bar{z}_1 \wedge (z_2 d\bar{z}_2 - \bar{z}_2 dz_2) \right).
 \end{aligned}$$

Proof. One calculates

$$\begin{aligned}
 & \frac{2i(|z_1|^2 + n|z_2|^2)^3}{|\gamma(z)|^2} (\alpha \wedge \bar{\alpha} \wedge \tilde{\omega})_z \\
 &= (n^2|z_2|^2 dz_1 \wedge d\bar{z}_1 + |z_1|^2 dz_2 \wedge d\bar{z}_2 - nz_2 \bar{z}_1 dz_1 \wedge \bar{z}_2 - nz_1 \bar{z}_2 dz_2 \wedge \bar{z}_1) \\
 & \quad \wedge (z_1 d\bar{z}_1 - \bar{z}_1 dz_1 + z_2 d\bar{z}_2 - \bar{z}_2 dz_2) \\
 &= z_2 n^2 |z_2|^2 dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - \bar{z}_2 n^2 |z_2|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 + z_1 |z_1|^2 d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\
 & \quad - \bar{z}_1 |z_1|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_2 + nz_2 |z_1|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 - n\bar{z}_1 |z_2|^2 dz_1 \wedge dz_2 \wedge d\bar{z}_2 \\
 & \quad - n\bar{z}_2 |z_1|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 + nz_1 |z_2|^2 d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\
 &= (|z_1|^2 + n|z_2|^2) (z_1 d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 - \bar{z}_1 dz_1 \wedge dz_2 \wedge d\bar{z}_2 \\
 & \quad + nz_2 dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - n\bar{z}_2 dz_1 \wedge d\bar{z}_1 \wedge dz_2). \quad \square
 \end{aligned}$$

4.2. Computation of the Szegő kernel

Recall that we assume $n \geq 1$. In this section we will construct an orthonormal basis for $H_{b,m}^0(X)$. Therefore we choose $\gamma \in C^\infty(\mathbb{C}^2)$ (see Section

4.1) such that

$$\gamma(z) = \sqrt{|z_1|^2 + n|z_2|^2}$$

on X . Then all the assumptions on γ stated in Section 4.1 are satisfied.

Fix $m \geq 0$. For $0 \leq k \leq \lfloor \frac{m}{n} \rfloor$ define $s_k \in H_{b,m}^0(X)$ by

$$s_k(z) = \sqrt{a_k} z_1^{m-nk} z_2^k, \quad a_k = \frac{m + (1-n)k + 1}{4\pi^2} \binom{m + (1-n)k}{k}. \quad (4.2)$$

One has the following lemma which we will prove in the end of this section.

Lemma 4.9. *The set $\{s_0, s_1, \dots, s_{\lfloor \frac{m}{n} \rfloor}\}$ is an orthonormal basis for $H_{b,m}^0(X)$.*

Using this lemma we can write down the Szegő kernel (see Section 2.2).

Theorem 4.10. *Fix $n \in \mathbb{N}$, $n \geq 1$. For the metric on X constructed in Section 4.1 with γ chosen as above and any $m \geq 0$ the Szegő kernel $S_m \in C^\infty(X \times X)$ for $H_{b,m}^0(X)$ is given by*

$$S_m(z, w) = \frac{1}{4\pi^2} \sum_{k=0}^{\lfloor \frac{m}{n} \rfloor} \binom{m + (1-n)k}{k} (m + (1-n)k + 1) (z_1 \bar{w}_1)^{m-nk} (z_2 \bar{w}_2)^k.$$

In the following we will prove Lemma 4.9.

Proof of Lemma 4.9. Consider the map

$$\begin{aligned} \psi: (0, 1) \times (0, 2\pi)^2 &\rightarrow X \\ (r, s, t) &\mapsto (re^{is}, \sqrt{1-r^2}e^{it}). \end{aligned}$$

Then for any $f \in C^\infty(X)$ one has

$$\int_X f dV_X = \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \psi^*(f dV_X),$$

i.e. we have to compute $\psi^*(dV_X)$. We write down

$$\psi^* dz_1 = e^{is}(dr + ir ds), \quad \psi^* dz_2 = e^{it} \left(-\frac{r}{\sqrt{1-r^2}} dr + i\sqrt{1-r^2} dt \right).$$

Thus,

$$\psi^*(dz_1 \wedge d\bar{z}_1) = -2irdr \wedge ds, \quad \psi^*(dz_2 \wedge d\bar{z}_2) = 2irdr \wedge dt$$

and

$$\psi^*(z_1 d\bar{z}_1 - \bar{z}_1 dz_1) = -2ir^2 ds, \quad \psi^*(z_2 d\bar{z}_2 - \bar{z}_2 dz_2) = -2i(1-r^2)dt.$$

Using this we get

$$\begin{aligned} \psi^*((z_1 d\bar{z}_1 - \bar{z}_1 dz_1) \wedge dz_2 \wedge d\bar{z}_2) &= -4r^3 dr \wedge ds \wedge dt \\ \psi^*(ndz_1 \wedge d\bar{z}_1 \wedge (z_2 d\bar{z}_2 - \bar{z}_2 dz_2)) &= 4n(-r+r^3)dr \wedge ds \wedge dt, \end{aligned}$$

which leads to

$$\begin{aligned} \psi^*(dV_X) &= -\frac{1}{2} \left(\frac{|\gamma(re^{is}, \sqrt{1-r^2}e^{it})|}{r^2+n(1-r^2)} \right)^2 (-4r^3+4nr^3-4nr)dr \wedge ds \wedge dt \\ &= 2r \frac{|\gamma(re^{is}, \sqrt{1-r^2}e^{it})|^2}{r^2+n(1-r^2)} dr \wedge ds \wedge dt \\ &= 2rdr \wedge ds \wedge dt \end{aligned}$$

where for the last line we used that $(\gamma \circ \psi)(r, s, t) = \sqrt{r^2+n(1-r^2)}$. Now we compute

$$\begin{aligned} \int_X s_k \bar{s}_l dV_X &= \sqrt{a_k a_l} \int_0^1 \int_0^{2\pi} \int_0^{2\pi} r^{2m-n(k+l)} \sqrt{1-r^2}^{k+l} e^{in(l-k)s} e^{i(k-l)t} 2rdrdsdt \\ &= \begin{cases} 0, & \text{for } k \neq l, \\ 4\pi^2 a_k \int_0^1 (r^2)^{m-nk} (1-r^2)^k 2rdr, & \text{for } k = l. \end{cases} \end{aligned}$$

This shows directly that the s_k are pairwise orthogonal. In order to prove $\|s_k\| = 1$, $0 \leq k \leq \lfloor m/n \rfloor$ we set

$$I(k, m-nk) := \int_0^1 (r^2)^{m-nk} (1-r^2)^k 2rdr$$

for $0 \leq k \leq \lfloor m/n \rfloor$ and observe for $k > 0$

$$\begin{aligned} I(k, m-nk) &= \int_0^1 r^{m-nk} (1-r)^k dr \\ &= \left[\frac{r^{m-nk+1} (1-r)^k}{m-nk+1} \right]_0^1 + \frac{k}{m-nk+1} \int_0^1 r^{m-nk+1} (1-r)^{k-1} dr \\ &= \frac{k}{m-nk+1} I(k-1, m-nk+1) \end{aligned}$$

and $I(0, m - nk + k) = (m - nk + k + 1)^{-1}$. By induction one gets

$$I(k, m - nk) = \left(\binom{m - nk + k}{k} (m - nk + k + 1) \right)^{-1} = (4\pi^2 a_k)^{-1}$$

which finishes the proof of Lemma 4.9. \square

4.3. Discussion of the general theory in context of the example

For $n \geq 1$ we have that the CR S^1 -action μ on the compact CR manifold $X = S^3$ is transversal (see Theorem 4.1). We also constructed a rigid Hermitian metric such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{1,0}X \oplus T^{0,1}X)$, $\langle T|T \rangle = 1$ and $\langle u|v \rangle$ is real if u, v are real tangent vectors in Section 4.1. Theorem 4.10 provides an expression for the Szegő kernel:

$$S_m(z, w) = \frac{1}{4\pi^2} \sum_{k=0}^{\lfloor \frac{m}{n} \rfloor} \binom{m + (1-n)k}{k} (m + (1-n)k + 1) (z_1 \bar{w}_1)^{m-nk} (z_2 \bar{w}_2)^k. \quad (4.3)$$

From Lemma 4.7 and its proof we find that the CR structure is strictly pseudoconvex and that the determinant of the Levi form is given by

$$|\det \mathcal{L}_z| = \frac{1}{2} \frac{1}{(|z_1|^2 + n|z_2|^2)^2}.$$

On the one hand, all the assumptions for applying Theorem 2.10 or 2.11 are satisfied. On the other hand we have an explicit expression for the Szegő kernel. We will now study the expression in several cases and compare it to the results stated in Theorem 2.10 and Theorem 2.11 in that specific case.

In the case $n = 1$ one has $X_{\text{reg}} = X$ and (4.3) simplifies to

$$S_m(z, w) = \frac{1}{2} \cdot \frac{m+1}{2\pi^2} (z_1 \bar{w}_1 + z_2 \bar{w}_2)^m.$$

Because of $|\det \mathcal{L}_z| = \frac{1}{2}$, one observes that

$$S_m(z, z) = \frac{m+1}{2\pi^2} \cdot \frac{1}{2} = \frac{1}{2\pi} \left(\frac{1}{\pi} |\det \mathcal{L}_z| m^1 + \frac{1}{\pi} |\det \mathcal{L}_z| m^0 \right)$$

which verifies Theorem 2.10 and shows that the leading term of the expansion

of $S_m(z, z)$ coincides with the term stated in (2.16). Given $n \geq 2$ one considers the following two cases:

For $z \in X_n$ and $w \in X$ one has

$$S_m(z, w) = \begin{cases} 0, & \text{for } n \nmid m, \\ \left(\frac{m}{n} + 1\right) \frac{(z_2 \bar{w}_2)^{\frac{m}{n}}}{4\pi^2}, & \text{else,} \end{cases}$$

and $|\det \mathcal{L}_z| = 1/(2n^2)$. Thus, for $z \notin X_{\text{reg}}$

$$S_m(z, z) = \frac{m+n}{2\pi^2} \frac{\chi_{m,n}}{2n^2} = \frac{\chi_{m,n}}{2\pi} \left(\frac{1}{\pi} |\det \mathcal{L}_z| m^1 + \frac{n}{\pi} |\det \mathcal{L}_z| m^0 \right)$$

where $\chi_{m,n} = n$ for $n \mid m$ and $\chi_{m,n} = 0$ otherwise, which coincides with the behaviour of the Szegő kernel expansion on $X \setminus X_{\text{reg}}$ predicted in Theorem 2.11.

By way of comparison, for $z, w \in X$ with $|z_1| = 1$ (which implies $z \in X_{\text{reg}}$) one finds

$$S_m(z, w) = \frac{m+1}{4\pi^2} (z_1 \bar{w}_1)^m$$

and $|\det \mathcal{L}_z| = 1/2$ which leads to

$$S_m(z, z) = \frac{m+1}{2\pi^2} \cdot \frac{1}{2} = \frac{1}{2\pi} \left(\frac{1}{\pi} |\det \mathcal{L}_z| m^1 + \frac{1}{\pi} |\det \mathcal{L}_z| m^0 \right),$$

i.e. $S_m(z, z)$ has an asymptotic expansion as described in Theorem 2.10.

Vice versa, one can apply Theorem 2.10 in order to deduce results on the asymptotic behavior of certain expressions in elementary analysis. Let $m, n \geq 1$ be integer and $a \in (0, 1)$ a real number. Consider the expression

$$A(m, n, a) := \sum_{k=0}^{\lfloor \frac{m}{n} \rfloor} \binom{m + (1-n)k}{k} (m + (1-n)k + 1) (1-a)^{m-nk} a^k.$$

Choosing $z \in X_{\text{reg}}$ satisfying $|z_2|^2 = a$ one finds that $A(m, n, a) = 4\pi^2 S_m(z, z)$ holds. Thus, using Theorem 2.10 we can study the behavior of $A(m, n, a)$ for $m \rightarrow \infty$. One has for example

$$\lim_{m \rightarrow \infty} m^{-1} A(m, n, a) = (1 + (n-1)a)^{-2}.$$

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