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SOLVABILITY AND BLOW-UP OF SOLUTIONS OF SEMI LINEAR PARABOLIC PROBLEM WITH INTEGRAL CONDITION OF SECOND TYPE

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Abstract

In this paper, semi-linear parabolic equation with integral boundary condition of second type is investigated. The existence, uniqueness and Blow-up of weak solutions in finite time are established. The proof is proceeds in two steps; using the variable separation method for the solvability of the linear cas and applying an iterative process and a priori estimate, we prove the existence, uniqueness of the weak solution of the semi-linear problem. Finally, we study a blow-up of solution in finite time for a super-linear problem by using eigen functions method introduced by Kaplan.

1. Introduction

Many natural phenomena and modern problems of physics, mechanics, biology and technology can be modeled by partial differential equations (PDEs) with non-local conditions. Indeed, thanks to the modeling of these phenomena through partial differential equations that we have been able to

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understand the role of this or that parameter, end above all, obtain forecasts that are sometimes extremely precise. Partial differential equations were probably formulated for the first time during the birth of rational mechanics during the 17st century (Newton, Leibniz, ...). Then the "catalog" of partial differential equations has been enriched as the science development and in particular of physics. If we only have to remember a few names, we must cite that of Euler, then those of Navier and Stokes, for the equations of fluid mechanics, those of Fourier for the heat equation, of Maxwell for those of electromagnetism, of Schrodinger and Heisenberg for the equations of quantum mechanics, and of course that of Einstein for the PDEs of the theory of relativity. A giant leap was made by L. Schwartz when he gave birth to the theory of distributions (around the 1950s), and at least comparable progress is due to L. Hormander for the development of pseudo differential calculus (in the early 1970s). It is certainly good to bear in mind that the study of PDEs remains a very active field of research at the start of the 21st century.

The study of partial differential equations is at the interface of many scientific problems, simple models in physics or in population dynamics naturally lead to nonlinear partial differential equations. Indeed, most of the phenomena of physics or engineering sciences are non-linear and a modeling by linear equations risks in some cases to erase events that the linear equations can not take into account.

When the modeled phenomenon is not stationary, the mathematical model is usually represented by parabolic or hyperbolic evolution equations. The typical example of parabolic equations is the heat equation

$$\forall x \in \Omega, \ \frac{\partial u\,(x,t)}{\partial t} = D\ \Delta u(x,t) + \frac{P}{\rho C},$$

or Δ is the Laplacian operator, D is the coefficient of thermal diffusivity and P is the volumetric heat source.

Mathematical modeling of problems with integral conditions is encountered in plasma physics (particle diffusion processes in a turbulent plasma) [29], heat transmission theory ([4], [7]−[8], [11], [20], [21]−[22], [36], [37]), thermo-elasticity ([23], [38], [27]−[28]), some technological processes [24], middle oscillations [13], groundwater dynamics [25], [30], moisture spread [25], chemical engineering [6], Semiconductor [3], demographic models [5] and in mathematical problems in biology [26]. Also, the integral conditions are also used for the inverse problems of the theory of thermal conduction ([9]−[10], [12], [14]−[17], [18]).

So boundary conditions of integral type, it is however of the first type

$$\int_0^1 u(x,t) = E(t), \quad \int_0^1 k(x,t)u(x,t)dx = 0, \text{ where } k \text{ is a given function.}$$

or second type

$$u(0,t) = \int_0^1 k(x,t)u(x,t)dx, \forall t \in (0,T),$$
$$u(1,t) = \int_0^1 k(x,t)u(x,t)dx, \forall t \in (0,T),$$

can be used when it is impossible to measure directly the quantity sought on the border, its total value or its average is known.

As the first question to ask in the theoretical study is to know if, for a non-linear evolution equation with initial conditions and boundary conditions, there exists at least one local solution and if it is unique in the case considered, these problems have been solved for a large class of nonlinear evolution equations by a series of useful methods and theories which have been developed, particularly since the 1960s, such as the Faedo Galerkin method (Compactness method), the fixed point method, the semi-group method and the monotonous iterative method. (For more details on these methods consult [1]).

Some results of blow up solution of parabolic equation have been obtained in [31]−[32]−[35], where f is positive and superlinear with Dirichlet conditions, on the other hand in [33] they used the methods extend to a wide variety of problems of the form

$$u_t = f(u)\mathcal{L}u,$$

where $f(u)$ is a nonlinear function of u, and where $\mathcal{L}$ is a second-order linear differential operator. And One of the interesting blow up of solutions to singular parabolic equations with nonlinear sources is studyed by N. T. Duy and A. N. Dao in [34].

In our article we treat for a first time the blow up solution in finite time of parabolic equation where $f = u^p, p \geq 1$, with second type integral boundry condition. Or the main purpose of this paper is to study the existence and the uniqueness of the weak solution of the mixed semi-linear problem for an equation of the heat with an integral condition of second type, where we start by studying the non-local linear problem by the variable separation method , then we apply an iterative process based on the results obtained by the linear problem, to demonstrate the existence and the uniqueness of the weak solution of the semi-linear problem. Then we use a slight variant of the eigenfunction method (Kaplan Methote) introduced by Kaplan in [19] to find the maximum time of the existence of the solution.

2. Formulation of the Semi-linear Problem (P_1)

Let $Q = \left\{(x,t) \in \mathbb{R}^2 \text{ with : } 0 < x < 1 \text{ and } 0 < t < T\right\}.$

Consider the following semi-linear problem :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u(x,t) = f\left(x,t,u\right) & \forall\left(x,t\right) \in \overline{Q} \\ u\left(x,0\right) = \varphi\left(x\right) & \forall x \in (0,1) \\ \frac{\partial u}{\partial x}(0,t) = 0 & \forall t \in (0,T) \\ u(1,t) = \displaystyle\int_0^1 u(x,t)dx & \forall t \in (0,T) \end{cases}, \qquad (P_1)$$

with the following condition :

Condition 1: Since the function $f \in L^2\left(\Omega\right)$ is lipschitz, that is, there is a positive constant k as

$$\left\|f\left(x,t,u_1\right) - f\left(x,t,u_2\right)\right\|_{L^2(Q)} \leq k\left(\left\|u_1 - u_2\right\|_{L^2(Q)}\right) \qquad \forall\ u_1, u_2 \in L^2\left(Q\right).$$

3. The Study of the Associated Linear Problem

3.1. Position of the associated linear problem (P_2)

In the rectangular area $Q = (0,1) \times (0,T)$, with $T < \infty$, we consider the following linear problem :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u(x,t) = f(x,t) & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,1) \\ \frac{\partial u}{\partial x}(0,t) = 0 & \forall t \in (0,T) \\ u(1,t) = \int_0^1 u(x,t)dx & \forall t \in (0,T) \end{cases}, \qquad (P_2)$$

whose parabolic equation is given as follows :

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \Delta u(x,t) = f(x,t),$$

with the initial condition

$$\ell u = u(x,0) = \varphi(x), \qquad x \in (0,1),$$

Neumann boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t \in (0,T),$$

and the integral condition of the second type

$$u(1,t) = \int_0^1 u(x,t)\,dx, \qquad t \in (0,T).$$

3.2. Resolution of problem (P_2) by the variable separation method

Let the following associated homogeneous linear problem :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u(x,t) = 0 & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,1) \\ \frac{\partial u}{\partial x}(0,t) = 0 & \forall t \in (0,T) \\ u(1,t) = \int_0^1 u(x,t)dx & \forall t \in (0,T) \end{cases}. \qquad (P_3)$$

Let

$$u(x,t) = X(x)T(t). \qquad (3.1)$$

3.2.1. Find $X(x)$

Replacing (3.1) in (P_3), we obtain

$$\begin{cases} T^{'}X - X^{''}T &= 0 \\ X(x)T(0) &= \varphi(x) \\ X^{'}(0)T(t) &= 0 \\ X(1)T(t) &= \int_0^1 X(x)T(t) \end{cases}.$$

So we get for $\lambda > 0$ that

$$\frac{X^{''}}{X} = \frac{T^{'}}{T} = -\lambda,$$

which gives a Sturm-Liouville problem

$$\begin{cases} X^{''}(x) + \lambda X(x) &= 0 \\ X^{'}(0) &= 0 \\ X(1) &= \int_0^1 X(x)dx \end{cases}. \qquad (P_3^{'})$$

Then the solution of the previous problem is given by

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

or A and B are two real arbitrary.

Using the Neumann condition, we find

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x,$$

as

$$X'(0) = 0,$$

then

$$B = 0.$$

Hence

$$X(x) = A\cos\sqrt{\lambda}x.$$

On the one hand, we use the Dirichlet condition, we get

$$X(1) = A\cos\sqrt{\lambda}, \tag{3.2}$$

and on the other hand

$$\begin{aligned}\int_0^1 X(x)dx &= \int_0^1 A\cos\sqrt{\lambda}x dx \\ &= \frac{A}{\sqrt{\lambda}}\sin\sqrt{\lambda}x\Big|_0^1 \\ &= \frac{A}{\sqrt{\lambda}}\sin\sqrt{\lambda}. \end{aligned} \tag{3.3}$$

From (3.2)−(3.3) we obtain the eigenvalues by the following equation :

$$\sqrt{\lambda} = \tan\sqrt{\lambda}.$$

3.2.2. Find $T(t)$

According to the superposition theorem, we pose :

$$u(x,t) = \sum_{n\geq 0} X_n(x)\cdot T_n(t). \tag{3.4}$$

replacing (3.4) in (P_2), he comes :

$$\sum_{n\geq 0}\left(T_n' + \lambda T_n\right)\cdot\cos\left(\sqrt{\lambda}x\right) = \sum_{n\geq 0}\cos\left(\sqrt{\lambda_n}x\right)\cdot f_n(t), \qquad \forall n\in\mathbb{N},$$

which implies

$$T_n' + \lambda T_n = f_n(t). \tag{3.5}$$

As

$$\begin{aligned} u(x,0) &= \sum_{n\geq 0}\cos\left(\sqrt{\lambda_n}x\right)T_n(0) \\ &= \varphi(x) \\ &= \sum_{n\geq 0}\varphi_n\cdot\cos\left(\sqrt{\lambda_n}x\right), \end{aligned}$$

then

$$\varphi_n = \int_0^1 \varphi(x) \cdot \cos\left(\sqrt{\lambda_n} x\right) dx,$$

hence

$$T_n(0) = \varphi_n. \tag{3.6}$$

We can then solve in a simple way the ODE (3.5)−(3.6) using the constant variation method, so it comes

$$T_n(t) = \varphi_n e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda \tau} f_n(\tau)\, d\tau.$$

Hence, the explicit solution of the problem (P_3) is given as follows :

$$\sum_{n \geq 0} \left(A_n \cos \sqrt{\lambda} x\right) \cdot \left(\varphi_n e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda \tau} \cdot f_n(\tau)\, d\tau\right).$$

4. Solvability of the Weak Solution of the Semi-linear Problem (P_1)

This section is devoted to the proof of the existence and the uniqueness of the solution of the problem (P_1):

We consider the following auxiliary problem with the homogeneous equation :

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w(x,t) = 0 & \forall\,(x,t) \in Q \\ w(x,0) = \varphi(x) & \forall x \in (0,1) \\ \frac{\partial w}{\partial x}(0,t) = 0 & \forall t \in (0,T) \\ w(1,t) = \int_0^1 w(x,t)dx & \forall t \in (0,T) \end{cases}, \tag{P_4}$$

if u is a solution to the problem (P_1) and w is a solution to the problem (P_4), then $y = u - w$ satisfied

$$\mathcal{L}y = \frac{\partial y}{\partial t} - \Delta y(x,t) = G(x,t,y), \tag{4.1}$$

$$y(x,0) = 0, \quad \forall x \in (0,1), \tag{4.2}$$

$$\frac{\partial y}{\partial x}(0,t) = 0 \quad \forall t \in (0,t), \tag{4.3}$$

$$y(1,t)dx = 0 \quad \forall t \in (0,t), \tag{4.4}$$

or $G(x,t,y) = f(x,t,y+w)$. like the function f, the function G is also lipschitzian, that is, there is a positive constant k as

$$\|G(x,t,y_1) - G(x,t,y_2)\|_{L^2(Q)} \leq k\left(\|y_1 - y_2\|_{L^2(0,T,H^1(0,1))}\right). \tag{4.5}$$

First, we propose the concept of the studied solution.

Let $v = v(x,t)$ any function of V, as

$$V = \left\{ v \in C^1(Q),\ \frac{\partial}{\partial x}v(0,t) = v(1,t) = 0,\ t \in [0,T] \right\}.$$

multiply (4.1) by v and integrate it on Q_τ, we find

$$\int_{Q_\tau} \frac{\partial y}{\partial t}(x,t) \cdot v(x,t)dxdt - \int_{Q_\tau} \Delta y(x,t) \cdot v(x,t)dxdt$$
$$= \int_{Q_\tau} G(x,t,y) \cdot v(x,t)dxdt, \tag{4.6}$$

and use a integration by parts and the conditions on y, v we obtain

$$\int_{Q_\tau} \frac{\partial y}{\partial t}(x,t) \cdot v(x,t)dxdt + \int_{Q_\tau} \frac{\partial y}{\partial x}(x,t) \cdot \frac{\partial v}{\partial x}(x,t)dxdt$$
$$= \int_{Q_\tau} G(x,t,y) \cdot v(x,t)dxdt, \tag{4.7}$$

it then results from (4.7) that

$$A(y,v) = \int_{Q_\tau} G(x,t,y) \cdot v(x,t)dxdt, \tag{4.8}$$

or

$$A(y,v) = \int_{Q_\tau} \frac{\partial y}{\partial t}(x,t) \cdot v(x,t)dxdt + \int_{Q_\tau} \frac{\partial y}{\partial x}(x,t) \cdot \frac{\partial v}{\partial x}(x,t)dxdt.$$

Definition 1. the function $y \in L^2\left(0,T; H^1(0,1)\right)$ is said a weak solution of the problem (4.1)–(4.4) if (4.8), (4.3) and (4.4) are achieved.

Building a recurring sequence starting with $y^{(0)} = 0$. the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows : given the element $y^{(n-1)}$, then for $n =$

$1, 2, 3, \ldots$ We will solve the following problem

$$\left\{\begin{array}{rcl}\frac{\partial y^{(n)}}{\partial t}-\Delta y^{(n)} & = & G\left(x,t,y^{(n-1)}\right)\\ y^{(n)}\left(x,0\right) & = & 0\\ \frac{\partial y^{(n)}}{\partial x}(0,t) & = & 0\\ y^{(n)}(1,t)dx & = & 0\end{array}\right. , \qquad (P_5)$$

according to the study of the previous linear problem each time we fix the n, the problem (P_5) admits a unique solution $y^{(n)}\left(x,t\right)$ which is given explicitly by the variable separation method.

Now suppose $z^{(n)}(x,t)=y^{(n+1)}\left(x,t\right)–y^{(n)}(x,t)$, so we get a new problem:

$$\left\{\begin{array}{c}\frac{\partial z^{(n)}}{\partial t}-\Delta z^{(n)}=p^{(n-1)}(x,t)\\ z^{(n)}\left(x.0\right)=0\\ \frac{\partial z^{(n)}}{\partial x}\left(0,t\right)=0\\ z^{(n)}\left(1,t\right)dx=0\end{array}\right. , \qquad (P_6)$$

or

$$p^{(n-1)}(x,t)=G\left(x,t,y^{(n)}\right)-G\left(x,t,y^{(n-1)}\right).$$

Lemma 1. *Suppose the condition* (4.5) *be satisfied. So for the problem* (P_6), *we have the following a priori estimate :*

$$\|z^{(n)}\|_{L^2(0,T,H^1(0,1))}\leq c\|z^{(n-1)}\|_{L^2(0,T,H^1(0,1))},$$

or

$$c=\sqrt{\frac{k^2T\exp\left(\frac{\varepsilon}{2}T\right)}{2\varepsilon\min\left(\frac{1}{2},T\right)}}.$$

Proof. multiply

$$\frac{\partial z^{(n)}}{\partial t}-\Delta z^{(n)}=p^{(n-1)}(x,t),$$

by $z^{(n)}$, and integrate it on Q_τ, we obtain :

$$\int_{Q_\tau}\frac{\partial z^{(n)}}{\partial t}\left(x,t\right)\cdot z^{(n)}(x,,t)dxdt-\int_{Q_\tau}\Delta z^{(n)}(x,t)\cdot z^{(n)}\left(x,t\right)dxdt$$
$$=\int_{Q_\tau}p^{(n-1)}(x,t)\cdot z^{(n)}\left(x,t\right)dxdt.$$

Let us use an integration by parts for each term by taking account of the initial condition and the boundary conditions, we find :

$$\frac{1}{2}\int_0^1 (z^{(n)}(x,\tau))^2 dx + \int_{Q_\tau}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$= \int_{Q_\tau} p^{(n-1)}(x,t)\cdot z^{(n)}(x,t)\,dxdt.$$

Using Cauchy with $\varepsilon-$inequality, we obtain :

$$\frac{1}{2}\int_0^1 (z^{(n)}(x,\tau))^2 dx + \int_{Q_\tau}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leq \frac{1}{2\varepsilon}\int_{Q_\tau}(p^{(n-1)})^2 dxdt + \frac{\varepsilon}{2}\int_{Q_\tau}(z^{(n)}(x,t))^2 dxdt.$$

On the other hand we have

$$\left|p^{n-1}(x,t)\right|^2 = \left|G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right)\right|^2,$$

as G est lipschitz we find that

$$\begin{aligned}\left|p^{n-1}(x,t)\right|^2 &\leq k^2\left(\left|y^{(n)} - y^{(n-1)}\right|\right)^2\\ &= k^2\left|z^{(n-1)}\right|^2\\ &\leq k^2\left(\left|z^{(n-1)}\right|^2 + \left|z_x^{(n-1)}\right|^2\right),\end{aligned}$$

then we integrate it on Q_τ

$$\begin{aligned}\int_{Q_\tau}\left|p^{n-1}(x,t)\right|^2 dxdt &\leq k^2\int_{Q_\tau}\left(\left|z^{(n-1)}\right|^2 + \left|z_x^{(n-1)}\right|^2\right)dxdt\\ &\leq k^2\|z^{(n-1)}(x,t)\|^2_{L^2(0,T\,;\,H^1(0,1))}.\end{aligned}$$

Using Gronwall's lemma, we get :

$$\frac{1}{2}\int_0^1 (z^{(n)}(x,\tau))^2 dx + \int_{Q_\tau}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leq \frac{1}{2\varepsilon}\int_{Q_\tau}(p^{(n-1)})^2 dxdt\cdot\exp\left(\frac{\varepsilon}{2}T\right),$$

so we find

$$\frac{1}{2}\|z^{(n)}\|^2_{L^2(0,1)} + \|z^{(n)}_x\|^2_{L^2(Q_T)} \leq \frac{k^2}{2\varepsilon}\|z^{(n-1)}(x,t)\|^2_{L^2(0,T\ ;\ H^1(0,1))} \cdot \exp\left(\frac{\varepsilon}{2}T\right),$$

then integrating it on $(0,T)$, we obtain :

$$\|z^{(n)}(x,t)\|^2_{L^2(0,T\ ;\ H^1(0,1))} \leq \frac{k^2 T \exp\left(\frac{\varepsilon}{2}T\right)}{2\varepsilon \min\left\{\frac{1}{2},T\right\}}\|z^{(n-1)}(x,t)\|^2_{L^2(0,T\ ;\ H^1(0,1))}. \quad \square$$

According to the criterion of series convergence, the series $\sum_{n=1}^{\infty} z^{(n)}$ is said convergent if

$$\sqrt{\frac{k^2 T \exp\left(\frac{\varepsilon}{2}T\right)}{2\varepsilon \min\left(\frac{1}{2},T\right)}} < 1.$$

Which give

$$k < \sqrt{\frac{2\varepsilon \min\left(\frac{1}{2},T\right)}{T\exp\left(\frac{\varepsilon}{2}T\right)}},$$

as $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, and $y^{(0)}(x,t) = 0$ we have

$$\begin{aligned}\sum_{i=0}^{n-1} z^{(i)} &= \sum_{i=0}^{n-1}\left(y^{(i+1)}(x,t) - y^{(i)}(x,t)\right)\\ &= y^{(1)} - y^{(0)} + y^{(2)} - y^{(1)} + \cdots + y^{(n)} - y^{(n-1)}\\ &= y^{(n)},\end{aligned}$$

then the sequance $\left(y^{(n)}\right)_n$ defined by

$$y^{(n)}(x,t) = \sum_{i=0}^{n-1} z^{(i)},$$

is convergent towards an element $y \in L^2\left(0,T,H^1(0,1)\right)$.

Now we will demonstrate that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of the problem (4.1)−(4.4) showing that y checked :

$$A(y,v) = \int_{Q_\tau} G(x,t,y)\cdot v(x,t)\,dxdt.$$

Therefore we consider the weak formulation of the problem (P_5) following :

$$A\left(y^{(n)},v\right)=\int_{Q_\tau}\frac{\partial y^{(n)}}{\partial t}(x,t)\cdot v(x,t)dxdt+\int_{Q_\tau}\frac{\partial y^{(n)}}{\partial x}(x,t)\cdot\frac{\partial v}{\partial x}(x,t)dxdt.$$

As A is linear, we have

$$\begin{aligned}A\left(y^{(n)},v\right)&=A\left(y^{(n)}-y,v\right)+A\left(y,v\right)\\&=\int_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial t}(x,t)\cdot v(x,t)dxdt\\&\quad+\int_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial x}(x,t)\cdot\frac{\partial v}{\partial x}(x,t)dxdt\\&\quad+\int_{Q_\tau}\frac{\partial y}{\partial t}(x,t)\cdot v(x,t)dxdt+\int_{Q_\tau}\frac{\partial y}{\partial x}(x,t)\cdot\frac{\partial v}{\partial x}(x,t)dxdt,\end{aligned}\tag{4.9}$$

we apply the Cauchy Schwartz inequality on $A\left(y^{(n)}-y,v\right)$, we obtain

$$\begin{aligned}&\int_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial t}(x,t)\cdot v(x,t)dxdt+\int_{Q_\tau}\frac{\partial(y^{(n)}-y)}{\partial x}(x,t)\cdot\frac{\partial v}{\partial x}(x,t)dxdt\\&\quad\leq\|v_x\|_{L^2(Q)}\left[\|\left(y^{(n)}-y\right)_t\|_{L^2(0,T,H^1(0,1))}+\|\left(y^{(n)}-y\right)_x\|_{L^2(0,T,H^1(0,1))}\right].\end{aligned}$$

On the other hand, as

$$y^{(n)}\longrightarrow y\quad\text{dans } L^2\left(0,T,H^1\left(0,1\right)\right)\approxeq H^1\left(Q\right),$$

so

$$\begin{aligned}y^{(n)}&\longrightarrow y\quad\text{dans } L^2\left(Q\right),\\y_t^{(n)}&\longrightarrow y_t\quad\text{dans } L^2\left(Q\right),\\y_x^{(n)}&\longrightarrow y_x\quad\text{dans } L^2\left(Q\right),\end{aligned}$$

Let's go to the limit when $n\longrightarrow+\infty$, we find

$$\lim_{n\longrightarrow+\infty}A\left(y^{(n)}-y,v\right)=0.\tag{4.10}$$

From (4.10) and going to the limit in (4.9) we obtain

$$\lim_{n\longrightarrow+\infty}A\left(y^{(n)},v\right)=A\left(y,v\right).$$

Theorem 1. *If the condition* (4.5) *is satisfied and*

$$k < \sqrt{\frac{2\varepsilon \min\left(\frac{1}{2}, T\right)}{T \exp\left(\frac{\varepsilon}{2} T\right)}},$$

So the problem (4.1)−(4.4) *admits a weak solution belonging to* $L^2(0,T;$ $H^1(0,1))$.

It remains to be proven that the problem (4.1)−(4.4) admits a unique solution.

Theorem 2. *Under the condition* (4.5), *the solution of the problem* (4.1)−(4.4) *is unique.*

Proof. suppose that y_1 and y_2 in $L^2\left(0,T;\ H^1(0,1)\right)$ are two solutions of (4.1)−(4.4), then $Z = y_1 - y_2$ satisfied $Z \in L^2\left(0,T;\ H^1(0,1)\right)$ and

$$\mathcal{L}y = \frac{\partial Z}{\partial t} - \Delta Z(x,t) = \psi(x,t),$$

$$\begin{aligned} Z(x,0) &= 0, \quad \forall x \in (0,1), \\ \frac{\partial Z}{\partial x}(0,t) &= 0 \quad \forall t \in (0,t), \\ Z(1,t)dx &= 0 \quad \forall t \in (0,t), \end{aligned}$$

or

$$\psi(x,t) = G(x,t,y_1) - G(x,t,y_2).$$

Following the same method as that used for the proof of the Lemma 1, we obtain

$$\|Z\|_{L^2(0,T;\ H^1(0,1))} \leq c \|Z\|_{L^2(0,T;\ H^1(0,1))}, \tag{4.11}$$

or c is the same constant of Lemma 1.

As $c < 1$, so according to (4.11) it comes that

$$(1-c)\|Z\|_{L^2(0,T;\ H^1(0,1))} \leq 0.$$

We conclude that $y_1 = y_2$ in $L^2\left(0,T;\ H^1(0,1)\right)$. □

5. Blow-up of Solution in Finite Time for a Super-Linear Problem

We are going to study the explosion in finite time of solution for the semi-linear problem (P_1) in the particular case by taking $f(x,t,u) = u^p$, $p > 1$

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u(x,t) = u^p & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,1) \\ \frac{\partial u}{\partial x}(0,t) = 0 & \forall t \in (0,t) \\ u(1,t) = \int_0^1 k(x,t)u(x,t)dx & \forall t \in (0,t) \end{cases},$$

or k is a positive and bounded function.

Let the following Sturm Liouville's problem :

$$\begin{cases} -\Delta\xi = \lambda\xi \\ \xi'(0) = 0 \\ \xi(1) = 0 \end{cases}.$$

Where the solution is given by

$$\xi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Using the Dirichlet-Neumann conditions, we obtain the following eigenfunctions

$$\xi(x) = A\cos(2k+1)\frac{\pi}{2}x.$$

for $k = 0$, we have

$$\begin{aligned} \xi(x) &= A\cos\frac{\pi}{2}x \\ \lambda_1 &= \left(\frac{\pi}{2}\right)^2. \end{aligned}$$

Let us pose the function $\Pi(t)$ defined by

$$\Pi(t) = \int_0^1 u(x,t)\xi(x)dx.$$

multiply

$$\frac{\partial u}{\partial t} - \Delta u(x,t) = u^p,$$

by $\xi(x)$ and integrating it on $(0,1)$, we find

$$\int_0^1 \frac{\partial u}{\partial t}(x,t)\cdot\xi(x)dxdt - \int_0^1 \Delta u(x,t)\cdot\xi(x)dxdt = \int_0^1 u^p(x,t)\cdot\xi(x)dxdt,$$

by integration by parts, we get

$$\begin{aligned}\Pi'(t)+\lambda_1\Pi(t) &= -\left(\int_0^1 k(x,t)\cdot u(x,t)dx\right)\xi'(1)+\int_0^1 \xi(x)u^p(x,t)dxdt\\ &\geq \frac{\pi}{2}\left(\int_0^1 k(x,t)\cdot u(x,t)dx\right)+\int_0^1 \xi(x)u^p(x,t)dxdt\\ &\geq \frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t))\int_0^1 \xi(x)u(x,t)dx+\int_0^1 \xi(x)\cdot u^p(x,t)dxdt\\ &\geq \frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t))\,\Pi(t)+\int_0^1 \xi(x)u^p(x,t)dxdt. \end{aligned} \tag{4.12}$$

Applying Jensen's inequality, we find

$$\begin{aligned}\int_0^1 \frac{\pi}{2}u^p(x,t)\xi(x)dx &\geq \left(\frac{\pi}{2}\int_0^1 u(x,t)\xi(x)dx\right)^p \geq \left(\frac{\pi}{2}\right)^{p-1}\left(\int_0^1 u(x,t)\xi(x)dx\right)^p\\ &= \left(\frac{\pi}{2}\right)^{p-1}(\Pi(t))^p. \end{aligned} \tag{4.13}$$

Replacing (4.13) in (4.12), we obtain the following inequality :

$$\Pi'(t)+(\lambda_1-\frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t)))\Pi(t)\geq\left(\frac{\pi}{2}\right)^{p-1}(\Pi(t))^p.$$

To find the finite time of explosion which verifies the previous inequation, we need to solve the following Bernoulli equation :

$$\Pi'(t)+(\lambda_1-\frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t)))\cdot\Pi(t)-\left(\frac{\pi}{2}\right)^{p-1}(\Pi(t))^p=0. \tag{4.14}$$

To solve this equation, we use the following variable change

$$v=\Pi^{1-p}, \tag{4.15}$$

and replacing (4.15) in (4.14), we find

$$\frac{1}{1-p}v'v^{\frac{p}{1-p}}+(\lambda_1-\frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t)))\cdot v^{\frac{1}{1-p}}-\left(\frac{\pi}{2}\right)^{p-1}v^{\frac{p}{1-p}}=0.$$

So, we just solve the following Bernoulli equation :

$$v'(t) + (1-p)\cdot(\lambda_1 - \frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t)))\cdot v(t) = (1-p)\left(\frac{\pi}{2}\right)^{p-1}. \quad (4.16)$$

First we will solve the following homogeneous equation :

$$v'(t) + (1-p)\cdot(\lambda_1 - \frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t)))\cdot v(t) = 0,$$

by simple integration we get

$$v_1(t) = c_1 e^{k_0 t},$$

with

$$k_0 = \left(\lambda_1 - \frac{\pi}{2}\min_{(x,t)\in Q}(k(x,t))\right).$$

Now we move on to solving the non-homogeneous equation (4.16) by the constant variation method, where we put

$$v_2(t) = c_1(t)e^{k_0 t},$$

we find

$$c_1(t) = (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(-\frac{1}{k_0}\right)e^{-k_0 t},$$

from where

$$v_2(t) = (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(-\frac{1}{k_0}\right).$$

So the solution of (4.16) is

$$\begin{aligned} v(t) =& v_1(t) + v_2(t) \\ =& c_1 e^{k_0 t} - (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right). \end{aligned}$$

Which give

$$\Pi(t) = \left(c_1 e^{k_0 t} - (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)^{\frac{1}{1-p}},$$

on the other hand we have

$$\Pi(0) = \left(c_1 - (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)^{\frac{1}{1-p}},$$

which implies

$$c_1 = (\Pi(0))^{1-p} + (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right).$$

Then

$$\Pi(t) = \left(\left((\Pi(0))^{1-p} + (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)e^{k_0 t} - (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)^{\frac{1}{1-p}}$$
$$= \left(\frac{1}{\left((\Pi(0))^{1-p} + (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)e^{k_0 t} - (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)}\right)^{\frac{1}{p-1}}.$$

as $\frac{1}{p-1} > 0$, then

$$\Pi \longrightarrow +\infty \text{ si } \left((\Pi(0))^{1-p} + (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)\right)e^{k_0 t}$$
$$- (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right) \longrightarrow 0,$$

so, we obtain

$$T = \frac{1}{k_0}\ln\frac{(1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)}{(\Pi(0))^{1-p} + (1-p)\left(\frac{\pi}{2}\right)^{p-1}\left(\frac{1}{k_0}\right)}.$$

Hence

$$\Pi(0) = \int_0^1 \varphi(x)\cos\left(\frac{\pi}{2}x\right)dx.$$

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