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ON A CERTAIN CATEGORY OF $\mathfrak{gl}_\infty$-MODULES

CUIPO JIANG[1,a,*] AND HAISHENG LI[2,b]

[1]School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, China

[a]E-mail: cpjiang@sjtu.edu.cn

[2]Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102, USA.

[b]E-mail: hli@camden.rutgers.edu

Abstract

This is a continuation of a previous study [10] on Lie algebra $\mathfrak{gl}_\infty$ in the context of quantum vertex algebras. In this paper, we study a particular category $\mathcal{C}$ of $\mathfrak{gl}_\infty$-modules and a subcategory $\mathcal{C}_{int}$ of integrable $\mathfrak{gl}_\infty$-modules. As the main results, we classify the irreducible modules in these two categories and we show that every module in category $\mathcal{C}_{int}$ is semi-simple. Furthermore, we determine the decomposition of the tensor products of irreducible modules in category $\mathcal{C}_{int}$.

1. Introduction

In the representation theory of Kac-Moody algebras, of great importance is the category of integrable modules, where classifying irreducible integrable modules has been an open problem (see [12]). For (finite rank) affine Kac-Moody algebras, irreducible integrable modules with finite-dimensional weight spaces were classified by Chari (see [1]). Integrable modules for affine Kac-Moody algebras were studied further in [2, 3, 4] (cf. [16], [14]).

Integrable representations for infinite rank affine Kac-Moody algebras, including $\mathfrak{gl}_\infty$ (the Lie algebra of doubly infinite matrices with only finitely many nonzero entries), are also of great importance in many areas, especially in mathematical physics. Among the most important and interesting results is the remarkable relation of highest weight integrable representations with soliton equations, which was discovered and developed by Kyoto school (see

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[7], [5, 6], [11]). Lie algebra $\mathfrak{gl}_\infty$ has also been used effectively to study $\mathcal{W}$-algebras (see [8], [13]).

In a previous study [10], we exhibited a natural association of quantum vertex algebras (see [15]) to $\mathfrak{gl}_\infty$. In that study we came across a category of $\mathfrak{gl}_\infty$-modules W satisfying the condition that for any $m \in \mathbb{Z}$, $w \in W$, $E_{m,n}w = 0$ for all but finitely many integers n. (A canonical base of $\mathfrak{gl}_\infty$ consists of $E_{m,n}$ $(m, n \in \mathbb{Z})$, where $E_{m,n}$ denotes the matrix whose only nonzero entry is the (m, n)-entry which is 1.) This category contains the natural module $\mathbb{C}^\infty$ and its tensor products, whereas it excludes nontrivial highest weight modules and lowest weight modules. It is our hope to classify the irreducible objects in this category. This is the main motivation for this current paper.

In this paper, we focus on a smaller category of $\mathfrak{gl}_\infty$-modules, for which we are able to determine and classify all the irreducible objects. Specifically, we study $\mathfrak{gl}_\infty$-modules W satisfying the condition that for every $w \in W$, there exists a finite subset S of $\mathbb{Z}$ such that

$$E_{m,n}w = 0 \quad \text{for } m, n \in \mathbb{Z},\ n \notin S.$$

Denote by $\mathcal{C}$ the category of such $\mathfrak{gl}_\infty$-modules and by $\mathcal{C}_{int}$ the category of those integrable $\mathfrak{gl}_\infty$-modules. The category $\mathcal{C}$ still contains the natural module $\mathbb{C}^\infty$, and it is closed under tensor product.

Note that for a finite-dimensional simple Lie algebra, or more generally for a Kac-Moody algebra (see [12]), one has the well known category o and its subcategory o_{int} of integrable modules. To a certain extent, this category $\mathcal{C}$ of $\mathfrak{gl}_\infty$-modules is analogous to category o. Especially, it is proved that every module in category $\mathcal{C}_{int}$ is completely reducible. By using certain generalized Verma modules, we can classify irreducible modules in categories $\mathcal{C}$ and $\mathcal{C}_{int}$. On the other hand, we can also determine the decomposition of tensor products of irreducible modules in category $\mathcal{C}_{int}$ in terms of the decomposition of tensor products of irreducible modules for $\mathfrak{gl}_n$ with sufficiently large n.

We now give a more detailed description of the main results. Let S be a finite and nonempty subset of $\mathbb{Z}$. To S we associate a triangular decomposition

$$\mathfrak{gl}_\infty = \mathfrak{gl}_\infty^{(S,+)} \oplus \mathfrak{gl}_\infty^{(S,0)} \oplus \mathfrak{gl}_\infty^{(S,-)},$$

where

$$\begin{aligned}\mathfrak{gl}_\infty^{(S,+)} &= \operatorname{span}\{E_{m,p} \mid m \in \mathbb{Z},\ p \notin S\},\\ \mathfrak{gl}_\infty^{(S,-)} &= \operatorname{span}\{E_{p,n} \mid p \notin S,\ n \in S\},\\ \mathfrak{gl}_\infty^{(S,0)} &= \operatorname{span}\{E_{m,n} \mid m, n \in S\}.\end{aligned}$$

Alternatively, set

$$\mathfrak{gl}_S = \mathfrak{gl}_\infty^{(S,0)} = \operatorname{span}\{E_{m,n} \mid m, n \in S\}.$$

Given a $\mathfrak{gl}_S$-module U, using this particular triangular decomposition we define a generalized Verma module $M(S,U)$, which is a $\mathfrak{gl}_\infty$-module induced from $\mathfrak{gl}_S$-module U. When U is irreducible, $M(S,U)$ has a unique irreducible quotient module, denoted by $L(S,U)$. We show that $M(S,U)$ and $L(S,U)$ belong to category $\mathcal{C}$ and that every irreducible $\mathfrak{gl}_\infty$-module in $\mathcal{C}$ is isomorphic to $L(S,U)$ for some finite subset S of $\mathbb{Z}$ and for some irreducible $\mathfrak{gl}_S$-module U. We also give a necessary and sufficient condition that $L(S_1,U_1) \simeq L(S_2,U_2)$, where S_1, S_2 are finite subsets of $\mathbb{Z}$ and U_1, U_2 are irreducible modules for $\mathfrak{gl}_{S_1}$ and $\mathfrak{gl}_{S_2}$, respectively.

Set

$$H = \operatorname{span}\{E_{n,n} \mid n \in \mathbb{Z}\},$$

a Cartan subalgebra of $\mathfrak{gl}_\infty$. Let S be a finite nonempty subset of $\mathbb{Z}$. Set

$$H_S = \operatorname{span}\{E_{n,n} \mid n \in S\},$$

which is a Cartan subalgebra of $\mathfrak{gl}_S$. Let $\lambda \in H^*$ such that $\lambda(E_{n,n}) = 0$ for $n \notin S$. Define $M(S,\lambda)$ to be the generalized Verma module $M(S,U)$ with $U = M(\lambda_S)$, where $M(\lambda_S)$ denotes the Verma $\mathfrak{gl}_S$-module with highest weight $\lambda_S = \lambda_{H_S}$. A fact is that the weight-λ subspace of $M(S,\lambda)$ is one-dimensional. We then define an irreducible module $L(S,\lambda)$ as the quotient module of $M(S,\lambda)$ by the maximal submodule. We show that $L(S,\lambda)$ is integrable if and only if $\lambda(E_{n,n}) \in \mathbb{N}$ for $n \in \mathbb{Z}$ and $\lambda(E_{m,m}) \geq \lambda(E_{n,n})$ for $m, n \in S$ with $m < n$. Furthermore, we show that every such irreducible integrable module belongs to category $\mathcal{C}_{int}$ and every irreducible $\mathfrak{gl}_\infty$-module in $\mathcal{C}_{int}$ is isomorphic to such an irreducible module.

We furthermore study the tensor products of irreducible modules in $\mathcal{C}_{int}$. It is shown that the tensor product of any two irreducible modules in $\mathcal{C}_{int}$ is always cyclic. Let W be any $\mathfrak{gl}_\infty$-module in $\mathcal{C}$ and let S be a finite subset of $\mathbb{Z}$. Set

$$\Omega_S(W) = \{w \in W \mid E_{p,q}w = 0 \quad \text{for any } p, q \in \mathbb{Z} \text{ with } q \notin S\},$$

which is a $\mathfrak{gl}_S$-submodule of W. It is proved that if W is irreducible, there exists a finite subset S' of $\mathbb{Z}$ such that $\Omega_S(W)$ is an irreducible $\mathfrak{gl}_S$-module for any finite subset S of $\mathbb{Z}$, containing S'. Now, let W_1, W_2 be irreducible $\mathfrak{gl}_\infty$-modules in $\mathcal{C}_{int}$. It is proved that for any sufficiently large S, the decomposition of $W_1 \otimes W_2$ into irreducible $\mathfrak{gl}_\infty$-modules is determined by the decomposition of $\Omega_S(W_1) \otimes \Omega_S(W_2)$ into irreducible $\mathfrak{gl}_S$-modules. Consequently, the tensor product of any two irreducible modules in category $\mathcal{C}_{int}$ always decomposes into a finite sum of irreducible modules, unlike the case with highest weight modules.

After this paper was completed, we found a very interesting paper [18], in which Penkov and Serganova had studied the category of integrable modules with finite dimensional weight subspaces for $sl(\infty)$, $o(\infty)$, and $sp(\infty)$. Among other results they proved that the category of integrable modules with finite dimensional weight subspaces is semisimple and they also identified each irreducible module in this category. Their method is somewhat different from that of this present paper.

This paper is organized as follows: In Section 2, we define categories $\mathcal{C}$ and $\mathcal{C}_{int}$ of $\mathfrak{gl}_\infty$-modules and we establish the complete reducibility. In Section 3, we classify irreducible modules in categories $\mathcal{C}$ and $\mathcal{C}_{int}$. In Section 4, we study the decomposition of the tensor product modules.

2. Categories $\mathcal{C}$ and $\mathcal{C}_{int}$ of $\mathfrak{gl}_\infty$-modules

In this section, we introduce a category $\mathcal{C}$ of $\mathfrak{gl}_\infty$-modules and a subcategory $\mathcal{C}_{int}$ of integrable modules. As the main result of this section, we prove that every $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ is completely reducible.

We begin with Lie algebra $\mathfrak{gl}_\infty$, which is the Lie algebra of doubly infinite matrices with only finitely many nonzero entries, under the commutator bracket. A canonical base consists of $E_{m,n}$ $(m, n \in \mathbb{Z})$, where $E_{m,n}$ denotes the matrix whose only nonzero entry is the (m, n)-entry which is 1, and we have

$$[E_{m,n}, E_{r,s}] = \delta_{n,r}E_{m,s} - \delta_{m,s}E_{r,n} \tag{2.1}$$

for $m, n, r, s \in \mathbb{Z}$. Let $\mathbb{C}^\infty$ denote the vector space of doubly infinite column vectors with only finitely many nonzero entries. Denote the standard unit base vectors by v_n for $n \in \mathbb{Z}$. The natural action of $\mathfrak{gl}_\infty$ on $\mathbb{C}^\infty$ is given by

$$E_{i,j}v_k = \delta_{j,k}v_i \quad \text{for } i, j, k \in \mathbb{Z}.$$

Define $\deg E_{i,j} = j - i$ for $i, j \in \mathbb{Z}$ to make $\mathfrak{gl}_\infty$ a $\mathbb{Z}$-graded Lie algebra, where the degree-n homogeneous subspace $(\mathfrak{gl}_\infty)_{(n)}$ for $n \in \mathbb{Z}$ is linearly spanned by $E_{m,m+n}$ for $m \in \mathbb{Z}$. We have the standard triangular decomposition

$$\mathfrak{gl}_\infty = \mathfrak{gl}_\infty^+ \oplus \mathfrak{gl}_\infty^0 \oplus \mathfrak{gl}_\infty^-,$$

where $\mathfrak{gl}_\infty^\pm = \sum_{\pm(j-i)>0} \mathbb{C}E_{i,j}$ and $\mathfrak{gl}_\infty^0 = \sum_{n\in\mathbb{Z}} \mathbb{C}E_{n,n}$. Alternatively, set

$$H = \mathfrak{gl}_\infty^0 = \operatorname{span}\{E_{n,n} \mid n \in \mathbb{Z}\}, \tag{2.2}$$

a Cartan subalgebra of $\mathfrak{gl}_\infty$.

The formal completion $\overline{\mathfrak{gl}_\infty}$ of $\mathfrak{gl}_\infty$ is also a $\mathbb{Z}$-graded Lie algebra, where

$$\overline{(\mathfrak{gl}_\infty)_{(n)}} = \left\{ \sum_{m\in\mathbb{Z}} a_m E_{m,m+n} \mid a_m \in \mathbb{C} \right\}.$$

If W is a $\mathfrak{gl}_\infty$-module such that for every $w \in W$ and for every $n \in \mathbb{Z}$, $E_{m,m+n}w = 0$ for all but finitely many integers m, then W is naturally a $\overline{\mathfrak{gl}_\infty}$-module.

Definition 2.1. A $\mathfrak{gl}_\infty$-module W is said to be *integrable* if for every $n \in \mathbb{Z}$, $E_{n,n}$ is semi-simple on W and if for any $p, q \in \mathbb{Z}$ with $p \neq q$, $E_{p,q}$ is locally nilpotent on W.

A notion of integrable module for a general Lie algebra including Kac-Moody Lie algebras was introduced in [12]. This definition of an integrable module for $\mathfrak{gl}_\infty$ is just a version of that. From [12] (Proposition 3.8), a $\mathfrak{gl}_\infty$-module W is integrable if $E_{n,n}$ is semi-simple on W for *some* $n \in \mathbb{Z}$ and if $E_{j,j+1}$ and $E_{j+1,j}$ for all $j \in \mathbb{Z}$ are locally nilpotent on W.

Remark 2.2. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with Chevalley generators e_i, f_i, h_i $(1 \le i \le l)$. A $\mathfrak{g}$-module W is *integrable* if e_i, f_i $(1 \le i \le l)$ are locally nilpotent on W. From [12] (Proposition 3.8), every integrable $\mathfrak{g}$-module is $\mathfrak{g}$-locally finite, hence a direct sum of finite-dimensional irreducible modules. Consequently, on an integrable $\mathfrak{g}$-module, every root vector of $\mathfrak{g}$ is locally nilpotent and the Cartan algebra is semi-simple.

Remark 2.3. Consider the three dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the standard Chevalley generators e, f, h. Suppose that V is an integrable $\mathfrak{sl}(2, \mathbb{C})$-module with a nonzero vector v satisfying the condition that $ev = 0$ and $hv = kv$ for some $k \in \mathbb{C}$. Then $k \in \mathbb{N}$, $f^{k+1}v = 0$, and $e^k f^k v = (k!)^2 v$.

Recall that a highest weight $\mathfrak{gl}_\infty$-module with highest weight $\lambda \in H^*$ is a module W with a vector w such that

$$\begin{aligned} &E_{n,n}w = \lambda_n w, \quad E_{n,n+1}w = 0 \qquad \text{for } n \in \mathbb{Z}, \\ &W = U(\mathfrak{gl}_\infty)w, \end{aligned}$$

where $\lambda_n = \lambda(E_{n,n})$. From [12], a highest weight integrable $\mathfrak{gl}_\infty$-module is irreducible.

Definition 2.4. Denote by $\mathcal{C}$ the category of $\mathfrak{gl}_\infty$-modules W such that for any $w \in W$, there exists a finite subset S of $\mathbb{Z}$ such that $E_{m,n}w = 0$ for all $m, n \in \mathbb{Z}$ with $n \notin S$. Furthermore, define $\mathcal{C}_{int}$ to be the subcategory consisting of integrable $\mathfrak{gl}_\infty$-modules in $\mathcal{C}$.

It can be readily seen that every submodule of a $\mathfrak{gl}_\infty$-module from $\mathcal{C}$ is in $\mathcal{C}$ and the tensor product of any (finitely many) $\mathfrak{gl}_\infty$-modules from $\mathcal{C}$ is in $\mathcal{C}$. The same can be said for category $\mathcal{C}_{int}$.

Set

$$I_\infty = \sum_{n \in \mathbb{Z}} E_{n,n}, \tag{2.3}$$

which lies in the completion $\overline{\mathfrak{gl}_\infty}$ of $\mathfrak{gl}_\infty$. Notice that every $\mathfrak{gl}_\infty$-module in category $\mathcal{C}$ is naturally a $\overline{\mathfrak{gl}_\infty}$-module. Then for any $\mathfrak{gl}_\infty$-module W in category $\mathcal{C}$, I_∞ is a well defined operator on W and $[I_\infty, \mathfrak{gl}_\infty] = 0$.

Let S be a finite and nonempty subset of $\mathbb{Z}$. Set

$$\mathfrak{gl}_S = \text{span}\{E_{m,n} \mid m, n \in S\}, \tag{2.4}$$

which is a subalgebra of $\mathfrak{gl}_\infty$, and set

$$I_S = \sum_{n\in S} E_{n,n} \in \mathfrak{gl}_S. \tag{2.5}$$

The Lie algebra $\mathfrak{gl}_S$ is reductive with one-dimensional center $\mathbb{C}I_S$.

Lemma 2.5. *Let W be any $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$. Then for every $n \in \mathbb{Z}$, $E_{n,n}$ is semi-simple on W with only nonnegative integer eigenvalues. Furthermore, I_∞ is semi-simple on W with only nonnegative integer eigenvalues and H is semi-simple.*

Proof. For $p, q \in \mathbb{Z}$ with $p \neq q$, let $\mathfrak{g}_{p,q}$ denote the linear span of $E_{p,q}$, $E_{q,p}$, $E_{p,p} - E_{q,q}$, which is a subalgebra isomorphic to $\mathfrak{sl}_2$ with $E_{p,q}, E_{q,p}, E_{p,p} - E_{q,q}$ corresponding to e, f, h, respectively. With W an integrable $\mathfrak{gl}_\infty$-module, we see that W is an integrable $\mathfrak{g}_{p,q}$-module. Therefore, $E_{p,p} - E_{q,q}$ is semi-simple on W (by Remark 2.2).

Let k be any fixed integer. We now show that $E_{k,k}$ is semi-simple on W. Let w be any vector of W. Then there exists a finite subset S of $\mathbb{Z}$ such that $k \in S$ and $E_{p,q}w = 0$ for all $p, q \in \mathbb{Z}$ with $q \notin S$. Consider the $\mathfrak{gl}_S$-submodule $U(\mathfrak{gl}_S)w$ generated by w. Suppose $u \in W$ satisfies that $E_{p,q}u = 0$ for all $p, q \in \mathbb{Z}$ with $q \notin S$. Then for any $m, n \in S$ and $p, q \in \mathbb{Z}$ with $q \notin S$, we have

$$E_{p,q}E_{m,n}u = E_{m,n}E_{p,q}u + \delta_{q,m}E_{p,n}u - \delta_{n,p}E_{m,q}u = E_{m,n}E_{p,q}u - \delta_{n,p}E_{m,q}u = 0.$$

It follows from this and induction that $E_{p,q}U(\mathfrak{gl}_S)w = 0$ for all $p, q \in \mathbb{Z}$ with $q \notin S$. In particular, $E_{q,q}U(\mathfrak{gl}_S)w = 0$ for all $q \notin S$. Let $n \in S$. Pick an integer q with $q \notin S$. We have $E_{q,q} = 0$ on $U(\mathfrak{gl}_S)w$, so that $E_{n,n} - E_{q,q}$ $(= E_{n,n})$ preserves $U(\mathfrak{gl}_S)w$. Since $E_{n,n} - E_{q,q}$ is semi-simple on W from the first paragraph, $E_{n,n} - E_{q,q}$ is semi-simple on $U(\mathfrak{gl}_S)w$, which implies that $E_{n,n}$ is semi-simple on $U(\mathfrak{gl}_S)w$. We have already proved that

$E_{m,m} = 0$ on $U(\mathfrak{gl}_S)w$ for $m \notin S$. Thus H preserves $U(\mathfrak{gl}_S)w$ and is semi-simple. It then follows that H is semi-simple on W. In particular, $E_{k,k}$ is semi-simple on W.

Let $n \in \mathbb{Z}$ and let $u \in W$ be an eigenvector of $E_{n,n}$ with eigenvalue $\lambda \in \mathbb{C}$. There exists an integer m such that $m \neq n$ and $E_{p,m}u = 0$ for all $p \in \mathbb{Z}$. Then

$$(E_{n,n} - E_{m,m})u = E_{n,n}u = \lambda u \quad \text{and} \quad E_{n,m}u = 0.$$

As W is an integrable $\mathfrak{g}_{m,n}$-module, in view of Remark 2.3 we have $\lambda \in \mathbb{N}$ and $(E_{m,n})^{\lambda+1}u = 0$, as desired. □

Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$. From Lemma 2.5 we have

$$W = \oplus_{\ell \in \mathbb{N}} W[\ell],$$

a direct sum of $\mathfrak{gl}_\infty$-modules, where $W[\ell] = \{w \in W \mid I_\infty \cdot w = \ell w\}$ for $\ell \in \mathbb{N}$. In view of this, it suffices to determine $\mathfrak{gl}_\infty$-modules in category $\mathcal{C}_{int}$, on which I_∞ acts as a nonnegative integer scalar.

As a refinement of Lemma 2.5 we have:

Lemma 2.6. *Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ such that I_∞ acts as a nonnegative integer scalar ℓ. Then for every $n \in \mathbb{Z}$, $E_{n,n}$ is semi-simple with nonnegative integer eigenvalues not exceeding ℓ and $(E_{m,n})^{\ell+1} = 0$ for all $m, n \in \mathbb{Z}$ with $m \neq n$.*

Proof. Let $m, n \in \mathbb{Z}$ with $m \neq n$. Recall that $\mathfrak{g}_{m,n}$ denotes the three-dimensional simple subalgebra spanned by $E_{m,n}, E_{n,m}, E_{m,m} - E_{n,n}$. By Lemma 2.5, the eigenvalues of $E_{m,m}$ and $E_{n,n}$ are nonnegative integers which are bounded by ℓ as $I_\infty = \ell$. It follows that the eigenvalues of $E_{m,m} - E_{n,n}$ are bounded by $-\ell$ and ℓ. As W is an integrable $\mathfrak{g}_{m,n}$-module, it is a direct sum of irreducible $\mathfrak{g}_{m,n}$-modules of dimension bounded by $\ell + 1$. Consequently, $(E_{m,n})^{\ell+1} = 0$. □

The following is an immediate consequence of Lemma 2.6:

Corollary 2.7. *Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ such that $I_\infty = 0$. Then $\mathfrak{gl}_\infty$ acts trivially on W.*

Next, we have:

Proposition 2.8. *Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ such that $I_\infty = 1$ on W. Then W is a direct sum of submodules isomorphic to the natural module $\mathbb{C}^\infty$.*

Proof. Let v be any nonzero H-weight vector. As $I_\infty = 1$, there is $r \in \mathbb{Z}$ such that $E_{r,r}v = v$ and $E_{n,n}v = 0$ for all $n \in \mathbb{Z}$ with $n \neq r$. For any $m \neq r$, we have

$$E_{m,m}E_{r,m}v = E_{r,m}E_{m,m}v + \delta_{m,r}E_{m,m}v - E_{r,m}v = -E_{r,m}v.$$

Since the only eigenvalues of $E_{r,r}$ are either 1 or 0 (by Lemma 2.5), we must have $E_{r,m}v = 0$. Note that $E_{m,m}v = 0$ for $m \neq r$. Let $m, n \in \mathbb{Z}$ with $m \neq n$ and $m, n \neq r$. We have $E_{r,r}E_{n,m}v = E_{n,m}v$ and $E_{n,n}E_{n,m}v = E_{n,m}v$. As $I_\infty = 1$, we must have $E_{n,m}v = 0$. Thus $E_{p,q}v = 0$ for all $p, q \in \mathbb{Z}$ with $q \neq r$.

For $n \in \mathbb{Z}$, set $v_n = E_{n,r}v$. We have $v = v_r$. Assume $n \neq r$. Then

$$E_{n,n}v_n = E_{n,n}E_{n,r}v = E_{n,r}E_{n,n}v + E_{n,r}v = E_{n,r}v = v_n. \tag{2.6}$$

This forces $E_{m,m}v_n = 0$ for all $m \neq n$ (no matter whether $v_n \neq 0$). Next, we show that

$$E_{p,q}v_n = 0 \quad \text{for } p, q \in \mathbb{Z} \text{ with } q \neq n. \tag{2.7}$$

If $p = q \neq n$, we already have $E_{p,q}v_n = 0$. Assume $p \neq q,\ p \neq n,\ q \neq n$. We have

$$E_{q,q}(E_{p,q}v_n) = E_{p,q}E_{q,q}v_n - E_{p,q}v_n = -E_{p,q}v_n.$$

Again, since the only possible eigenvalues of $E_{q,q}$ are 0 and 1, we have $E_{p,q}v_n = 0$. Furthermore, we have $E_{m,r}v_r = E_{m,r}v = v_m$ and

$$E_{m,n}v_n = E_{m,n}E_{n,r}v = E_{n,r}E_{m,n}v + E_{m,r}v - \delta_{m,r}E_{n,n}v = E_{m,r}v = v_m \tag{2.8}$$

for $n \neq r$. It follows from (2.7) and (2.8) that $U(\mathfrak{gl}_\infty)v$ is a homomorphism image of $\mathbb{C}^\infty$. Consequently, $U(\mathfrak{gl}_\infty)v$ is isomorphic to $\mathbb{C}^\infty$. Then it follows that W is a sum of submodules isomorphic to $\mathbb{C}^\infty$. As $\mathbb{C}^\infty$ is irreducible, W is a direct sum of submodules isomorphic to $\mathbb{C}^\infty$. □

Next, we shall prove that every $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ is completely reducible. First, we prove a technical result.

Lemma 2.9. *Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$. Suppose that v is a nonzero H-eigenvector of weight λ, satisfying the condition that*

$$\begin{aligned} &E_{p,q}v = 0 \quad \text{for all } p,q \in \mathbb{Z} \text{ with } q \notin \{r, r+1, \ldots, n\}, \\ &E_{r,j+1}v = 0 \quad \text{for } r \le j \le n, \end{aligned} \tag{2.9}$$

where r and n are some fixed integers with $r < n$. Set

$$v' = (E_{r-1,r})^{\lambda_r - \lambda_{r+1}} \cdots (E_{r-1,n-1})^{\lambda_{n-1}-\lambda_n} (E_{r-1,n})^{\lambda_n} v \in W.$$

Then

$$\begin{aligned} &E_{p,q}v' = 0 \quad \text{for all } p,q \in \mathbb{Z} \text{ with } q \notin \{r-1, r, \ldots, n\}, \\ &E_{r-1,j+1}v' = 0 \quad \text{for } r-1 \le j \le n. \end{aligned} \tag{2.10}$$

Furthermore, we have

$$(E_{n,r-1})^{\lambda_n} (E_{n-1,r-1})^{\lambda_{n-1}-\lambda_n} \cdots (E_{r,r-1})^{\lambda_r - \lambda_{r+1}} v' = \alpha v, \tag{2.11}$$

where $\alpha = (\lambda_n!)^2((\lambda_{n-1}-\lambda_n)!)^2 \cdots ((\lambda_r - \lambda_{r+1})!)^2$, a nonzero integer.

Proof. First of all, by Lemma 2.5 we have $\lambda_m \in \mathbb{N}$ for all $m \in \mathbb{Z}$. From assumption (2.9), v is a singular vector in W viewed as a $\mathfrak{gl}_S$-module with $S = \{r, r+1, \ldots, n\}$. Since W is an integrable $\mathfrak{gl}_S$-module from assumption, we must have

$$\lambda_j - \lambda_{j+1} \in \mathbb{N} \quad \text{for } r \le j \le n-1.$$

Note that $\lambda_m = 0$ for $m \notin S$ from (2.9). In particular, $\lambda_{r-1} = 0 = \lambda_{n+1}$.

Let $p, q \in \mathbb{Z}$ with $q \notin \{r-1, r, \ldots, n\}$. For $r \le j \le n$, we have

$$E_{p,q}E_{r-1,j} = E_{r-1,j}E_{p,q} - \delta_{p,j}E_{r-1,q}.$$

Then it follows from induction that

$$E_{p,q}v' = 0 \quad \text{for all } p,q \in \mathbb{Z} \text{ with } q \notin \{r-1, r, \ldots, n\},$$

proving the first part of (2.10).

For $r \le m \le n$, set

$$v_m = (E_{r-1,m})^{\lambda_m - \lambda_{m+1}} \cdots (E_{r-1,n-1})^{\lambda_{n-1}-\lambda_n} (E_{r-1,n})^{\lambda_n} v.$$

Note that $v' = v_r$. Next, in several steps we prove that for every $m > r$,

$$E_{r-1,j} v_m = 0 \quad \text{for } j = m, m+1, \ldots, n,$$

from which we obtain the second part of (2.10) by taking $m = r$.

First, we have

$$E_{j,m}(E_{r-1,m_1} \cdots E_{r-1,m_s} v) = 0 \tag{2.12}$$

for any $j, m, m_1, \ldots, m_s \in \mathbb{Z}$ with $r \leq j < m, m_1, \ldots, m_s$. This is because $E_{j,m}$ commutes with E_{r-1,m_i} for $i = 1, \ldots, s$ and $E_{j,m} v = 0$ from assumption (2.9).

Second, we have

$$E_{j,r-1}(E_{r-1,m_1} \cdots E_{r-1,m_s} v) = 0 \tag{2.13}$$

for any $j, m_1, \ldots, m_s \in \mathbb{Z}$ with $r \leq j < m_1, \ldots, m_s$. This follows from induction and (2.12), as for $1 \leq i \leq s$,

$$E_{j,r-1} E_{r-1,m_i} = E_{r-1,m_i} E_{j,r-1} + E_{j,m_i}$$

and $E_{j,r-1} v = 0$ by assumption (2.9).

Third, for any $j, m_1, \ldots, m_s \in \mathbb{Z}$ with $r \leq j < m_1, \ldots, m_s$, we have

$$(E_{j,j} - E_{r-1,r-1})(E_{r-1,m_1} \cdots E_{r-1,m_s} v) = (\lambda_j - s)(E_{r-1,m_1} \cdots E_{r-1,m_s} v), \tag{2.14}$$

noticing that $E_{j,j} v = \lambda_j v$ and $E_{r-1,r-1} v = 0$ by assumption (2.9).

Fourth, for any $j, m_1, \ldots, m_s \in \mathbb{Z}$ with $r \leq j < m_1, \ldots, m_s$, we have

$$\left(E_{r-1,j}\right)^{(\lambda_j - s)+1} (E_{r-1,m_1} \cdots E_{r-1,m_s} v) = 0 \tag{2.15}$$

and

$$\begin{aligned} &\left(E_{j,r-1}\right)^{\lambda_j - s} \left(E_{r-1,j}\right)^{\lambda_j - s} (E_{r-1,m_1} \cdots E_{r-1,m_s} v) \\ &\quad = ((\lambda_j - s)!)^2 (E_{r-1,m_1} \cdots E_{r-1,m_s} v). \end{aligned} \tag{2.16}$$

Recall that for a fixed j with the above condition, $E_{j,r-1}$, $E_{r-1,j}$, $E_{j,j} - E_{r-1,r-1}$ span a Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Then the two assertions follow from (2.13) and (2.14) as W is an integrable $\mathfrak{sl}(2,\mathbb{C})$-module from assumption.

Fifth, we apply this to v_m for $r \le m \le n$. In view of (2.15) we have

$$E_{r-1,m}v_m = (E_{r-1,m})^{(\lambda_m-\lambda_{m+1})+1}v_{m+1} = 0, \tag{2.17}$$

where we set $v_{n+1} = v$. As $E_{r-1,j}$ with $j \ge r$ commute each other, by (2.17) we obtain

$$E_{r-1,j}v_r = 0 \quad \text{for } r \le j \le n. \tag{2.18}$$

This also holds for $j > n$ since $E_{r-1,j}v = 0$ from assumption. Now we have proved the second part of (2.10). The last assertion follows immediately from repeatedly applying (2.16). □

On the basis of Lemma 2.9 we have:

Proposition 2.10. *Let W be an integrable $\mathfrak{gl}_\infty$-module and let $v \in W$ be a nonzero H-weight vector satisfying the condition that*

$$\begin{aligned} &E_{p,q}v = 0 \quad \text{for } p, q \in \mathbb{Z} \ \text{ with } \ q \notin \{r, r+1, \ldots, n\}, \\ &E_{j,j+1}v = 0 \quad \text{for } r \le j \le n-1, \end{aligned} \tag{2.19}$$

where r and n are some integers with $r < n$. Then the submodule $U(\mathfrak{gl}_\infty)v$ is irreducible.

Proof. Let u be any nonzero vector in $U(\mathfrak{gl}_\infty)v$. We now prove $v \in U(\mathfrak{gl}_\infty)u$, so that $U(\mathfrak{gl}_\infty)u = U(\mathfrak{gl}_\infty)v$. As $u \in U(\mathfrak{gl}_\infty)v$, there exist two integers r' and n' with $r' \le r < n \le n'$ such that $u \in U(\mathfrak{gl}_{S'})v$, where

$$S' = \{r', r'+1, \ldots, n'\}.$$

Note that if $U(\mathfrak{gl}_{S'})v$ is an irreducible $\mathfrak{gl}_{S'}$-module, then we have $v \in U(\mathfrak{gl}_{S'})v = U(\mathfrak{gl}_{S'})u \subset U(\mathfrak{gl}_\infty)u$, as desired.

As W is an integrable $\mathfrak{gl}_{S'}$-module, any $\mathfrak{gl}_{S'}$-submodule of W generated by a singular vector is irreducible. In view of this, it suffices to prove that there is a singular vector v' such that $U(\mathfrak{gl}_{S'})v = U(\mathfrak{gl}_{S'})v'$. Note that from assumption we have $E_{r,j}v = 0$ for $r+1 \le j \le n$ and $E_{r,q}v = 0$ for $q \ge n+1$. Thus $E_{r,j}v = 0$ for all $j \ge r+1$. To summarize we have

$$\begin{aligned} &E_{p,q}v = 0 \quad \text{for all } p, q \in \mathbb{Z} \ \text{ with } q \notin \{r, r+1, \ldots, n'\}, \\ &E_{r,j}v = 0 \quad \text{for } j \ge r+1. \end{aligned}$$

By repeatedly applying Lemma 2.9, we obtain a vector v' such that

$$E_{p,q}v = 0 \quad \text{for all } p,q \in \mathbb{Z} \ \text{ with } q \notin \{r', r'+1, \dots, n'\},$$
$$E_{r',j}v = 0 \quad \text{for } j \geq r'+1$$

and such that $U(\mathfrak{gl}_{S'})v = U(\mathfrak{gl}_{S'})v'$, where the second condition implies that v' is a singular vector of the $\mathfrak{gl}_{S'}$-module $U(\mathfrak{gl}_{S'})v$. This proves that $U(\mathfrak{gl}_{S'})v$ is an irreducible $\mathfrak{gl}_{S'}$-module, $U(\mathfrak{gl}_\infty)u = U(\mathfrak{gl}_\infty)v$ for any nonzero vector u in $U(\mathfrak{gl}_\infty)v$, and $U(\mathfrak{gl}_\infty)v$ is an irreducible $\mathfrak{gl}_\infty$-module. □

Now, we are in a position to present our main result.

Theorem 2.11. *Every $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ is completely reducible.*

Proof. Let W be any $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$. By Lemma 2.5, H is semi-simple on W. To prove W is completely reducible, it suffices to show that the submodule generated by each vector is a sum of irreducible submodules.

Let $w \in W$. We shall prove that $U(\mathfrak{gl}_\infty)w$ is a sum of irreducible submodules. From assumption, there are integers r and n with $r < n$ such that

$$E_{p,q}w = 0 \quad \text{for all } p,q \in \mathbb{Z} \text{ with } q \notin \{r, r+1, \dots, n\}.$$

Set $S = \{r, r+1, \dots, n\}$. As a $\mathfrak{gl}_S$-submodule of W, $U(\mathfrak{gl}_S)w$ is integrable, so that $U(\mathfrak{gl}_S)w$ is a direct sum of finite-dimensional irreducible $\mathfrak{gl}_S$-modules. Now, it suffices to show that for every finite-dimensional irreducible $\mathfrak{gl}_S$-submodule U of $U(\mathfrak{gl}_S)w$, $U(\mathfrak{gl}_\infty)U$ is irreducible. Let $v \in U$ be a highest weight vector of weight λ. Then

$$U(\mathfrak{gl}_\infty)U = U(\mathfrak{gl}_\infty)v.$$

Now we prove that $U(\mathfrak{gl}_\infty)v$ is irreducible. Note that for any $p,q \in \mathbb{Z}$ with $q \notin S$ and for any $m,k \in S$,

$$E_{p,q}E_{m,k} = E_{m,k}E_{p,q} - \delta_{p,k}E_{m,q}.$$

By induction we get

$$E_{p,q}U(\mathfrak{gl}_S)w = 0 \quad \text{for all } p,q \in \mathbb{Z} \text{ with } q \notin S.$$

In particular, we have

$$E_{p,q}v = 0 \quad \text{for all } p, q \in \mathbb{Z} \text{ with } q \notin S = \{r, r+1, \ldots, n\}.$$

As v is a highest weight vector, we also have

$$E_{j,j+1}v = 0 \quad \text{for} \quad r \leq j \leq n-1.$$

By Proposition 2.10, $U(\mathfrak{gl}_\infty)v$ is an irreducible $\mathfrak{gl}_\infty$-module. Therefore, W is completely reducible. □

3. Classification of Irreducible $\mathfrak{gl}_\infty$-modules in $\mathcal{C}$ and $\mathcal{C}_{int}$

In this section, we classify irreducible $\mathfrak{gl}_\infty$-modules in categories $\mathcal{C}$ and $\mathcal{C}_{int}$. To achieve this goal, for any finite subset S of $\mathbb{Z}$ and for any irreducible $\mathfrak{gl}_S$-module U, through a generalized Verma module construction we construct an irreducible $\mathfrak{gl}_\infty$-module $L(S,U)$ in category $\mathcal{C}$ and we show that any irreducible module in $\mathcal{C}$ is isomorphic to a module of this form. Furthermore, for a linear functional λ on H compatible with S in a certain sense we define a generalized Verma $\mathfrak{gl}_\infty$-module $M(S,\lambda)$ and construct an irreducible module $L(S,\lambda)$. We then determine when $L(S,\lambda)$ is integrable and we show that every irreducible module in $\mathcal{C}_{int}$ is isomorphic to such an integrable module $L(S,\lambda)$.

Let S be a *finite* and *nonempty* subset of $\mathbb{Z}$, which is fixed temporarily. Recall

$$\mathfrak{gl}_S = \text{span}\{E_{m,n} \mid m, n \in S\} \subset \mathfrak{gl}_\infty.$$

To S, we associate a triangular decomposition

$$\mathfrak{gl}_\infty = \mathfrak{gl}_\infty^{(S,+)} \oplus \mathfrak{gl}_\infty^{(S,0)} \oplus \mathfrak{gl}_\infty^{(S,-)}, \tag{3.1}$$

where $\mathfrak{gl}_\infty^{(S,0)} = \mathfrak{gl}_S$,

$$\begin{aligned} \mathfrak{gl}_\infty^{(S,+)} &= \text{span}\{E_{m,n} \mid m, n \in \mathbb{Z},\ n \notin S\}, \\ \mathfrak{gl}_\infty^{(S,-)} &= \text{span}\{E_{m,n} \mid m, n \in \mathbb{Z},\ m \notin S,\ n \in S\}. \end{aligned} \tag{3.2}$$

Notice that $\mathfrak{gl}_\infty^{(S,-)}$ is an abelian subalgebra and that $\mathfrak{gl}_\infty^{(S,+)} + \mathfrak{gl}_S$ is a semi-product.

Recall that

$$I_S = \sum_{n\in S} E_{n,n} \in \mathfrak{gl}_S.$$

We have

$$\begin{aligned} &[I_S, E_{m,n}] = 0 \ \text{ for either } m,n \in S, \text{ or } m,n \notin S,\\ &[I_S, E_{m,n}] = E_{m,n} \ \text{ for } m \in S,\ n \notin S,\\ &[I_S, E_{m,n}] = -E_{m,n} \ \text{ for } m \notin S,\ n \in S. \end{aligned} \tag{3.3}$$

Using $\mathrm{ad}(I_S)$, we make $\mathfrak{gl}_\infty$ a $\mathbb{Z}$-graded Lie algebra for which

$$\begin{aligned} &\mathfrak{gl}_\infty^{(n)} = 0 \quad \text{if } |n| \geq 2,\\ &\mathfrak{gl}_\infty^{(0)} = \mathrm{span}\{E_{m,n} \mid \text{ either } m,n \in S, \text{ or } m,n \notin S\},\\ &\mathfrak{gl}_\infty^{(1)} = \mathrm{span}\{E_{m,n} \mid m \in S,\ n \notin S\},\\ &\mathfrak{gl}_\infty^{(-1)} = \mathrm{span}\{E_{m,n} \mid m \notin S,\ n \in S\}. \end{aligned} \tag{3.4}$$

Set

$$\mathfrak{gl}_{S^o} = \mathrm{span}\{E_{m,n} \mid m,n \notin S\}. \tag{3.5}$$

Then

$$\mathfrak{gl}_\infty^{(0)} = \mathfrak{gl}_S \oplus \mathfrak{gl}_{S^o}, \tag{3.6}$$

a direct product. Notice that $\mathfrak{gl}_\infty^{(\pm 1)}$ both are abelian subalgebras. We see that

$$\mathfrak{gl}_\infty^{(S,+)} = \mathfrak{gl}_\infty^{(1)} \oplus \mathfrak{gl}_{S^o}, \qquad \mathfrak{gl}_\infty^{(S,-)} = \mathfrak{gl}_\infty^{(-1)}. \tag{3.7}$$

Let U be a $\mathfrak{gl}_S$-module. Let $\mathfrak{gl}_\infty^{(S,+)}$ act trivially on U, to make U a $(\mathfrak{gl}_\infty^{(S,+)} + \mathfrak{gl}_S)$-module. Then form a generalized Verma $\mathfrak{gl}_\infty$-module

$$M(S,U) = U(\mathfrak{gl}_\infty) \otimes_{U\left(\mathfrak{gl}_\infty^{(S,+)} + \mathfrak{gl}_S\right)} U. \tag{3.8}$$

In view of the P-B-W theorem we have

$$M(S,U) = U(\mathfrak{gl}_\infty^{(-1)}) \otimes U = S(\mathfrak{gl}_\infty^{(-1)}) \otimes U.$$

By endowing U with degree 0, we make $M(S,U)$ a $\mathbb{Z}$-graded $\mathfrak{gl}_\infty$-module. (The homogeneous subspaces are infinite-dimensional in general.)

Lemma 3.1. *The $\mathfrak{gl}_\infty$-module $M(S,U)$ belongs to the category $\mathcal{C}$.*

Proof. Let W consist of every $w \in M(S,U)$, satisfying the condition that there exists a finite subset T of $\mathbb{Z}$ such that $E_{m,n}w = 0$ for all $m,n \in \mathbb{Z}$ with $n \notin T$. By the construction we have $U \subset W$. Then it suffices to prove that W is a submodule. Assume that $w \in W$ with a finite subset T of $\mathbb{Z}$ such that $E_{m,n}w = 0$ for all $m,n \in \mathbb{Z}$ with $n \notin T$. Let $p,q \in \mathbb{Z}$ be arbitrarily fixed. For $m,n \in \mathbb{Z}$, we have

$$E_{m,n}(E_{p,q}w) = E_{p,q}E_{m,n}w + \delta_{n,p}E_{m,q}w - \delta_{q,m}E_{p,n}w.$$

One sees that $E_{m,n}(E_{p,q}w) = 0$ for all $m,n \in \mathbb{Z}$ with $n \notin T \cup \{p\}$. This proves $E_{p,q}w \in W$. Then the lemma follows. □

Definition 3.2. Let U be a $\mathfrak{gl}_S$-module as before. Denote by $L(S,U)$ the quotient of the $\mathbb{Z}$-graded $\mathfrak{gl}_\infty$-module $M(S,U)$ by the maximal graded submodule with trivial degree-0 homogeneous subspace.

Remark 3.3. Assume that U is a $\mathfrak{gl}_S$-module on which I_S acts as a scalar $\alpha \in \mathbb{C}$. Then $M(S,U)$ is a canonically graded $\mathfrak{gl}_\infty$-module

$$M(S,U) = \bigoplus_{n\in\mathbb{N}} M(S,U)_{\alpha-n}, \tag{3.9}$$

where $M(S,U)_{\alpha-n} = \{w \in M(S,U) \mid I_S \cdot w = (\alpha-n)w\}$ for $n \in \mathbb{N}$.

Let W be a $\mathfrak{gl}_\infty$-module and let S be a finite subset of $\mathbb{Z}$ as before. Set

$$\Omega_S(W) = \{w \in W \mid E_{p,q}w = 0 \quad \text{for all } p,q \in \mathbb{Z} \ \text{ with } q \notin S\}. \tag{3.10}$$

It can be readily seen that $\Omega_S(W)$ is a $\mathfrak{gl}_S$-submodule of W.

Proposition 3.4. *Let S be a finite subset of $\mathbb{Z}$ and let U be an irreducible $\mathfrak{gl}_S$-module. Then $L(S,U)$ is an irreducible $\mathfrak{gl}_\infty$-module belong to category $\mathcal{C}$. On the other hand, every irreducible $\mathfrak{gl}_\infty$-module in category $\mathcal{C}$ is isomorphic to $L(S,U)$ for some finite subset S of $\mathbb{Z}$ and for some irreducible $\mathfrak{gl}_S$-module U.*

Proof. Since U is an irreducible $\mathfrak{gl}_S$-module, U is necessarily countable-dimensional. Then I_S acts on U as a scalar, say $\alpha \in \mathbb{C}$. From Remark 3.3, $M(S,U)$ is canonically $\mathbb{C}$-graded by the eigenspaces of I_S, where $M(S,U) =$

$\oplus_{n\in\mathbb{N}}M(S,U)_{\alpha-n}$ with $M(S,U)_\alpha = U$. We see that every submodule of $M(S,U)$ is graded. It follows that $M(S,U)$ has a unique maximal submodule. Consequently, $L(S,U)$ is an irreducible $\mathfrak{gl}_\infty$-module. From Lemma 3.1, $L(S,U)$ belongs to $\mathcal{C}$.

Now, let W be an irreducible $\mathfrak{gl}_\infty$-module in category $\mathcal{C}$. Pick a nonzero vector w in W. Then there exists a finite and nonempty subset S of $\mathbb{Z}$ such that $w \in \Omega_S(W)$. Set $U = U(\mathfrak{gl}_S)w$, a $\mathfrak{gl}_S$-submodule of $\Omega_S(W)$. Using the P-B-W theorem we get

$$W = U(\mathfrak{gl}_\infty)w = U(\mathfrak{gl}_\infty^{(-1)})U.$$

As W belongs to category $\mathcal{C}$, I_∞ acts on W and commutes with the action of $\mathfrak{gl}_\infty$. With W an irreducible $\mathfrak{gl}_\infty$-module, W must be countable dimensional (over $\mathbb{C}$). It then follows that I_∞ acts as a scalar on W, say α. Notice that $I_S \cdot w = I_\infty \cdot w = \alpha w$. Consequently, I_S acts on U as scalar α. As $W = U(\mathfrak{gl}_\infty^{(-1)})U$, I_S is semisimple on W with eigenvalues contained in $\{\alpha - n \mid n \in \mathbb{N}\}$ and U is the eigenspace of eigenvalue α. It follows that U is an irreducible $\mathfrak{gl}_S$-module. By the construction of $M(S,U)$, there exists an epimorphism from $M(S,U)$ to W, which reduces to an isomorphism from $L(S,U)$ to W. This completes the proof. □

From the second part of the proof of Proposition 3.4 we immediately have:

Lemma 3.5. *Let W be an irreducible $\mathfrak{gl}_\infty$-module in $\mathcal{C}$. Then there exists a finite subset S of $\mathbb{Z}$ such that $\Omega_S(W) \neq 0$. Furthermore, for any such finite subset S of $\mathbb{Z}$, $\Omega_S(W)$ is an irreducible $\mathfrak{gl}_S$-module and $W \simeq L(S, \Omega_S(W))$.*

The following is also immediate:

Lemma 3.6. *Let S be a finite subset of $\mathbb{Z}$ and let U be an irreducible $\mathfrak{gl}_S$-module. Then $\Omega_S(L(S,U)) = U$.*

We next determine the isomorphism classes of irreducible $\mathfrak{gl}_\infty$-modules $L(S,U)$. Let S be a finite subset of $\mathbb{Z}$ and S_1 a subset of S. Assume that U_1 is an irreducible $\mathfrak{gl}_{S_1}$-module on which I_{S_1} acts as a scalar $\alpha \in \mathbb{C}$. In the following we associate an irreducible $\mathfrak{gl}_S$-module to U_1. Set

$$N = \mathrm{span}\{E_{p,q} \mid p,q \in S,\ q \notin S_1\} \quad \text{and} \quad B = N + \mathfrak{gl}_{S_1}.$$

We see that both B and N are subalgebras of $\mathfrak{gl}_S$ and B contains N as an ideal. Letting N act on U_1 trivially, we make U_1 a B-module. Then form an induced module

$$\mathrm{ind}_{\mathfrak{gl}_{S_1}}^{\mathfrak{gl}_S}(U_1) = U(\mathfrak{gl}_S) \otimes_{U(B)} U_1. \tag{3.11}$$

As we have seen before, I_{S_1} gives rise to an $(\alpha+\mathbb{Z})$-grading on $\mathrm{ind}_{\mathfrak{gl}_{S_1}}^{\mathfrak{gl}_S}(U_1)$ with U_1 as the degree-α subspace. It follows that $\mathrm{ind}_{\mathfrak{gl}_{S_1}}^{\mathfrak{gl}_S}(U_1)$ has a unique maximal submodule. Then we define U_1^S to be the (unique) irreducible quotient $\mathfrak{gl}_S$-module of $\mathrm{ind}_{\mathfrak{gl}_{S_1}}^{\mathfrak{gl}_S}(U_1)$.

Lemma 3.7. *Let S be a finite subset of $\mathbb{Z}$ and S_1 a subset of S. Assume that U_1 is an irreducible $\mathfrak{gl}_{S_1}$-module. Then $L(S_1,U_1) \simeq L(S,U_1^S)$.*

Proof. Set $U = U(\mathfrak{gl}_S)U_1 \subset L(S_1,U_1)$. Note that $E_{p,q}U_1 = 0$ for $p,q \in \mathbb{Z}$ with $q \notin S$. It follows that

$$E_{p,q} \cdot U(\mathfrak{gl}_S)U_1 = 0 \quad \text{for } p,q \in \mathbb{Z} \text{ with } q \notin S.$$

Then

$$L(S_1,U_1) = U(\mathfrak{gl}_\infty) \cdot U$$

and I_S acts on $U = U(\mathfrak{gl}_S)U_1$ also as scalar α since $I_S = I_{S_1}$ on U_1. Just as with $M(S,U)$, we see that $L(S_1,U_1)$ is naturally an $(\alpha+\mathbb{Z})$-graded $\mathfrak{gl}_\infty$-module by I_S, with U as the degree-α subspace. Since $L(S_1,U_1)$ is irreducible, it follows that U is an irreducible $\mathfrak{gl}_S$-module and $L(S_1,U_1) \simeq L(S,U)$. From the construction of U_1^S, we have $U \simeq U_1^S$. Thus $L(S_1,U_1) \simeq L(S,U_1^S)$. □

As an immediate consequence of Lemmas 3.6 and 3.7 we have:

Corollary 3.8. *Let S_1 and S_2 be finite subsets of $\mathbb{Z}$ and let U_1 and U_2 be irreducible modules for $\mathfrak{gl}_{S_1}$ and $\mathfrak{gl}_{S_2}$, respectively. Set $S = S_1 \cup S_2$. Then $L(S_1,U_1) \simeq L(S_2,U_2)$ if and only if $U_1^S \simeq U_2^S$.*

Next, we study an analog of Verma module. For $i \in \mathbb{Z}$, let ε_i denote the linear functional on H defined by $\varepsilon_i(E_{j,j}) = \delta_{i,j}$ for $j \in \mathbb{Z}$. The root system of $\mathfrak{gl}_\infty$ with respect to Cartan subalgebra H is given by

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i,j \in \mathbb{Z},\ i \neq j\}. \tag{3.12}$$

Note that the usual polarization is given by $\Delta_\pm = \{\pm(\varepsilon_i - \varepsilon_j) \mid i, j \in \mathbb{Z},\ i < j\}$.

Let S be a subset of $\mathbb{Z}$. Recall $\mathfrak{gl}_S = \mathrm{span}\{E_{m,n} \mid m, n \in S\}$. Set

$$\begin{aligned} \mathfrak{gl}_S^\pm &= \mathrm{span}\{E_{m,n} \mid m, n \in S,\ \pm(n-m) > 0\}, \\ \mathfrak{gl}_S^0 &= H_S = \mathrm{span}\{E_{n,n} \mid n \in S\}. \end{aligned} \tag{3.13}$$

For $\lambda \in H^*$, set

$$\mathrm{supp}(\lambda) = \{m \in \mathbb{Z} \mid \lambda_m\ (= \lambda(E_{m,m})) \neq 0\}.$$

Let $\lambda \in H^*$ with $\mathrm{supp}(\lambda) \subset S$. Denote by λ_S the restriction of λ on H_S. Let $M(\lambda_S)$ and $L(\lambda_S)$ denote the Verma module and the irreducible quotient module for Lie algebra $\mathfrak{gl}_S$, respectively.

Define $M(S, \lambda)$ to be the generalized Verma $\mathfrak{gl}_\infty$-module $M(S, U)$ with $U = M(\lambda_S)$. On the other hand, we have a generalized Verma $\mathfrak{gl}_\infty$-module $M(S, L(\lambda_S))$, which is a quotient module of $M(S, \lambda)$. Furthermore, denote by $L(S, \lambda)$ the irreducible quotient module of $M(S, L(\lambda_S))$.

Set

$$\Delta_-(S) = \{(\varepsilon_i - \varepsilon_j) \mid i, j \in S,\ i > j\} \cup \{(\varepsilon_p - \varepsilon_i) \mid i \in S,\ p \notin S\}. \tag{3.14}$$

Furthermore, set

$$Q_-(S) = \mathbb{N} \cdot \Delta_-(S) \subset H^*. \tag{3.15}$$

We have

$$M(S, \lambda) = \oplus_{\alpha \in Q_-(S)} M(S, \lambda)_{\lambda+\alpha}, \tag{3.16}$$

where $M(S, \lambda)_\lambda$ is 1-dimensional. It is straightforward to show that every weight subspace is finite-dimensional.

Definition 3.9. Let W be a $\mathfrak{gl}_\infty$-module and let S be a subset of $\mathbb{Z}$. A nonzero vector $v \in W$ is called an *S-singular vector* if v is an H-eigenvector such that

$$\begin{aligned} E_{p,q} v = 0 &\quad \text{for } p, q \in \mathbb{Z} \ \text{ with } \ q \notin S, \\ E_{m,n} v = 0 &\quad \text{for } m, n \in S \ \text{ with } m < n. \end{aligned} \tag{3.17}$$

The following universal property of $M(S,\lambda)$ is straightforward to prove:

Lemma 3.10. *Let W be a $\mathfrak{gl}_\infty$-module and let w be an S-singular vector of weight λ in W. Then there exists a $\mathfrak{gl}_\infty$-module homomorphism θ from $M(S,\lambda)$ to W, uniquely determined by $\theta(v)=w$, where v is an S-singular vector in $M(S,\lambda)$ of weight λ. Furthermore, if W is irreducible, we have $W\simeq L(S,\lambda)$.*

We also have the following result:

Lemma 3.11. *Let S be a finite subset of $\mathbb{Z}$ and let $\lambda\in H^*$ with $\mathrm{supp}(\lambda)\subset S$. Then S-singular vectors in $L(S,\lambda)$ are unique up to scalar multiples.*

Proof. Set $\ell=\sum_{m\in\mathbb{Z}}\lambda_m=\sum_{m\in S}\lambda_m$. Then $I_S\cdot w=\ell w$ for $w\in L(\lambda_S)$ (the irreducible highest weight $\mathfrak{gl}_S$-module of highest weight λ_S). We see that $L(S,\lambda)$ is an $(\ell+\mathbb{Z})$-graded $\mathfrak{gl}_\infty$-module with

$$L(S,\lambda)=\bigoplus_{n\in\mathbb{N}}L(S,\lambda)_{\ell-n},$$

where $L(S,\lambda)_{\ell-n}=\{w\in L(S,\lambda)\mid I_S\cdot w=(\ell-n)w\}$ and $L(S,\lambda)_\ell=L(\lambda_S)$. It is straightforward to see that the subspace spanned by all S-singular vectors in $L(S,\lambda)$ is I_S-stable, so that it is a graded subspace. Let u be any homogeneous S-singular vector of degree $\ell-k$ with $k\in\mathbb{N}$. We have

$$U(\mathfrak{gl}_\infty)u=U(\mathfrak{gl}_\infty^{(-1)})U(\mathfrak{gl}_S)u\subset\bigoplus_{n\geq k}L(S,\lambda)_{\ell-n}.$$

Since $L(S,\lambda)$ is irreducible, k must be zero, so that every S-singular vector is contained in $L(S,\lambda)_\ell$. Then each S-singular vector is a singular vector in $L(\lambda_S)$ viewed as a $\mathfrak{gl}_S$-module, which is known to be unique up to scalar multiples. Consequently, S-singular vectors in $L(S,\lambda)$ are unique up to scalar multiples. □

As an immediate consequence we have:

Corollary 3.12. *Let S be a finite subset of $\mathbb{Z}$ and let $\lambda,\mu\in H^*$ such that $\mathrm{supp}(\lambda)$, $\mathrm{supp}(\mu)\subset S$. Then $L(S,\lambda)\simeq L(S,\mu)$ if and only if $\lambda=\mu$.*

Lemma 3.13. *Let W be an irreducible $\mathfrak{gl}_\infty$-module in category $\mathcal{C}$, satisfying the condition that for any vector $w\in W$ and for any finite subset S of $\mathbb{Z}$, $U(\mathfrak{gl}_S)w$ is a $\mathfrak{gl}_S$-module in category o. Then $W\simeq L(S,\lambda)$ where $S=$*

$\{r, r+1, \dots, n\}$ for some integers r and n with $r < n$ and for some $\lambda \in H^*$ with $\operatorname{supp}(\lambda) \subset S$.

Proof. Let w be a nonzero vector in W. As W belongs to category $\mathcal{C}$, there exist integers r and n with $r < n$ such that

$$E_{p,q}w = 0 \quad \text{for all } p, q \in \mathbb{Z} \ \text{ with } q \notin \{r, r+1, \dots, n\}.$$

Set $S = \{r, r+1, \dots, n\}$. We have $w \in \Omega_S(W)$ and hence $U(\mathfrak{gl}_S)w \subset \Omega_S(W)$. From our assumption, in the $\mathfrak{gl}_S$-submodule $U(\mathfrak{gl}_S)w$ there exists a highest weight vector v. Then we have

$$E_{p,q}v = 0 \quad \text{for all } p, q \in \mathbb{Z} \text{ with } q \notin S,$$
$$E_{m,n}v = 0 \quad \text{for } m, n \in S \text{ with } m < n.$$

On the other hand, as $E_{q,q}v = 0$ for $q \in \mathbb{Z} \backslash S$, v is an H-eigenvector of some weight $\lambda \in H^*$ with $\operatorname{supp}(\lambda) \subset S$. Then v is an S-singular vector of weight λ. It follows from Lemma 3.10 that $W \simeq L(S, \lambda)$. □

By Corollary 3.8 we have:

Corollary 3.14. *Let S_1, S_2 be finite subsets of $\mathbb{Z}$ and let $\lambda, \mu \in H^*$ be such that* $\operatorname{supp}(\lambda) \subset S_1$ *and* $\operatorname{supp}(\mu) \subset S_2$. *Then $L(S_1, \lambda) \simeq L(S_2, \mu)$ if and only if $L(\lambda_{S_1})^S \simeq L(\mu_{S_2})^S$, where $S = S_1 \cup S_2$, and $L(\lambda_{S_1})$, $L(\mu_{S_2})$ are the irreducible highest weight modules for $\mathfrak{gl}_{S_1}$ and $\mathfrak{gl}_{S_2}$, respectively.*

Remark 3.15. Let σ be a permutation on $\mathbb{Z}$. It can be readily seen that σ becomes an automorphism of the Lie algebra $\mathfrak{gl}_\infty$ by defining

$$\sigma(E_{m,n}) = E_{\sigma(m),\sigma(n)} \quad \text{for } m, n \in \mathbb{Z}. \tag{3.18}$$

For any $\mathfrak{gl}_\infty$-module W, we denote by $W^{[\sigma]}$ the $\mathfrak{gl}_\infty$-module with W as the underlying space and with the action given by

$$a \cdot w = \sigma(a)w \quad \text{for } a \in \mathfrak{gl}_\infty, \ w \in W.$$

We see that if W is an integrable $\mathfrak{gl}_\infty$-module, then $W^{[\sigma]}$ is still an integrable module. Furthermore, if W is in category $\mathcal{C}_{int}$, $W^{[\sigma]}$ is still in category $\mathcal{C}_{int}$.

The following is straightforward to prove:

Lemma 3.16. *Let $\lambda \in H^*$ be such that* $\mathrm{supp}(\lambda) \subset S$*, and let σ be a permutation on $\mathbb{Z}$ such that $\sigma(m) < \sigma(n)$ for any $m, n \in S$ with $m < n$. Then $L(S,\lambda)^{[\sigma]} \simeq L(\sigma^{-1}(S), \lambda \circ \sigma)$.*

Next, we determine when $L(S,\lambda)$ is an integrable module.

Definition 3.17. Denote by $P_+(S)$ the set of $\lambda \in H^*$ such that

$$\begin{aligned} &\lambda(E_{i,i}) \in \mathbb{N} \quad \text{for } i \in \mathbb{Z}, \\ &\lambda(E_{i,i}) = 0 \quad \text{whenever } i \notin S, \\ &\lambda(E_{i,i}) \geq \lambda(E_{j,j}) \quad \text{for } i,j \in S \ \text{ with } i < j. \end{aligned} \tag{3.19}$$

Note that if $\lambda \in P_+(S)$, then $\mathrm{supp}(\lambda) \subset S$.

Remark 3.18. We here mention a fact which we need in the proof of the next proposition. Let λ be a (dominant integral) weight for Lie algebra $\mathfrak{gl}_{n+1}$ such that

$$\lambda_j \in \mathbb{N} \quad \text{for } 1 \leq j \leq n+1, \text{ and } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}.$$

Then the irreducible highest weight $\mathfrak{gl}_{n+1}$-module $L(\lambda)$ is finite-dimensional. Let μ be the lowest weight of $L(\lambda)$. We claim that $\mu_j \in \mathbb{N}$ for $1 \leq j \leq n+1$. It was known (cf. [9]) that $\mu = \sigma(\lambda)$ where σ is the longest Weyl group element. Suppose γ is any weight such that $\gamma_j \in \mathbb{N}$ for $1 \leq j \leq n+1$. For $1 \leq i \leq n$, with the reflection r_i, we have

$$r_i(\gamma) = \gamma - \langle \gamma, \alpha_i^\vee \rangle \alpha_i = \gamma - (\gamma_i - \gamma_{i+1})(\varepsilon_i - \varepsilon_{i+1}).$$

Then

$$r_i(\gamma)(E_{j,j}) = \begin{cases} \gamma_j & \text{if } j \neq i, i+1, \\ \gamma_{i+1} & \text{if } j = i, \\ \gamma_i & \text{if } j = i+1, \end{cases}$$

which implies $r_i(\gamma)_j \in \mathbb{N}$ for $1 \leq j \leq n+1$. Then the claim follows from induction.

We have:

Proposition 3.19. *Let S be a finite subset of $\mathbb{Z}$ and let $\lambda \in H^*$ be such that* $\mathrm{supp}(\lambda) \subset S$. *Then $L(S, \lambda)$ is an integrable $\mathfrak{gl}_\infty$-module if and only if $\lambda \in P_+(S)$.*

Proof. In view of Remark 3.14 and Lemma 3.16, it suffices to prove the proposition for $S = \{r, r+1, \ldots, n\} \subset \mathbb{Z}$, where r and n are fixed integers with $r < n$.

Assume $L(S, \lambda)$ is an integrable $\mathfrak{gl}_\infty$-module. Then $L(S, \lambda)$ is in category C_{int}. By Lemma 2.5, $\lambda_m \in \mathbb{N}$ for all $m \in \mathbb{Z}$. As an integrable $\mathfrak{gl}_\infty$-module, $L(S, \lambda)$ is necessarily an integrable $\mathfrak{gl}_S$-module, containing $L(\lambda_S)$ as a submodule. Then it follows that $\lambda_r \geq \lambda_{r+1} \geq \cdots \geq \lambda_n$. Thus $\lambda \in P_+(S)$.

Conversely, assume $\lambda \in P_+(S)$. By definition, $\lambda_m \in \mathbb{N}$ for all $m \in \mathbb{Z}$ and $\lambda_r \geq \lambda_{r+1} \geq \cdots \geq \lambda_n$. Notice that $E_{m,m}$ for $m \in \mathbb{Z}$ are semi-simple on $L(S, \lambda)$. We now show that $L(S, \lambda)$ is integrable, by proving that $E_{j,j+1}, E_{j+1,j}$ for $j \in \mathbb{Z}$ are locally nilpotent. We shall freely use the following observation: Suppose that W is an irreducible $\mathfrak{gl}_\infty$-module. For any $p, q \in \mathbb{Z}$ with $p \neq q$, if $(E_{p,q})^k w = 0$ for some nonzero vector $w \in W$ and for some positive integer k, then $E_{p,q}$ is locally nilpotent on the whole space W. This simply follows from the fact that $\mathrm{ad}E_{p,q}$ is locally nilpotent on $\mathfrak{gl}_\infty$.

Let v be a highest weight vector in $\mathfrak{gl}_S$-module $L(\lambda_S) \subset L(S, \lambda)$.

(1) For $r \leq j \leq n-1$, we claim that $E_{j,j+1}$ and $E_{j+1,j}$ are locally nilpotent on $L(S, \lambda)$. With the assumption on λ, we know that $L(\lambda_S)$ is an (irreducible) integrable $\mathfrak{gl}_S$-module, so that $E_{j,j+1}$ and $E_{j+1,j}$ are locally nilpotent on $L(\lambda_S)$. Then it follows from the simple observation.

(2) Let $p, q \in \mathbb{Z}$ with $p \neq q,\ q \notin S$. As $E_{p,q}v = 0$, it follows that $E_{p,q}$ is locally nilpotent on $L(S, \lambda)$.

(3) We claim that $E_{n+1,n}$ is locally nilpotent on $L(S, \lambda)$.

Recall that

$$E_{p,q}v = 0 \quad \text{for all } p, q \in \mathbb{Z} \ \text{ with } q \notin S = \{r, r+1, \ldots, n\},$$

$$E_{r,r+1}v = E_{r+1,r+2}v = \cdots = E_{n-1,n}v = 0.$$

With $n+1 \notin S$ we also have $E_{n,n+1}v = 0$. Thus v is also an $\bar{S}$-singular vector with $\bar{S} = S \cup \{n+1\}$. By Lemma 3.10, we have $L(S, \lambda) \simeq L(\bar{S}, \lambda)$

which contains $L(\lambda_{\bar{S}})$ as a $\mathfrak{gl}_{\bar{S}}$-submodule. Notice that

$$\lambda_i \in \mathbb{N} \;\text{ for } i \in \mathbb{Z} \;\text{ and }\; \lambda_r \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} = 0.$$

Then $L(\lambda_{\bar{S}})$ is an integrable $\mathfrak{gl}_{\bar{S}}$-module. In particular, $E_{n+1,n}$ is locally nilpotent on $L(\lambda_{\bar{S}})$. Then it follows that $E_{n+1,n}$ is locally nilpotent on $L(S,\lambda)$.

(4) We claim that $E_{r-1,r}$ is locally nilpotent on $L(S,\lambda)$.

Recall that $L(\lambda_S)$ is an integrable $\mathfrak{gl}_S$-module. Consequently, $L(\lambda_S)$ is finite-dimensional. Let v_* be a lowest weight vector in $\mathfrak{gl}_S$-module $L(\lambda_S)$, so that

$$E_{j+1,j}v_* = 0 \quad \text{ for } r \leq j \leq n-1.$$

We also have

$$E_{p,q}v_* = 0 \quad \text{ for } p,q \in \mathbb{Z} \;\text{ with }\; q \notin S.$$

In particular, we have $E_{r,r-1}v_* = 0$. Set $\tilde{S} = S \cup \{r-1\}$. Then v_* is a lowest-weight singular vector in the $\mathfrak{gl}_{\tilde{S}}$-module $U(\mathfrak{gl}_{\tilde{S}})v_*$. We see that $L(S,\lambda) = U(\mathfrak{gl}_\infty)v_*$ can be naturally $\mathbb{Z}$-graded by $I_{\tilde{S}}$ with $U(\mathfrak{gl}_{\tilde{S}})v_*$ as the highest degree subspace. As $L(S,\lambda)$ is irreducible, it follows that $U(\mathfrak{gl}_{\tilde{S}})v_*$ is an irreducible $\mathfrak{gl}_{\tilde{S}}$-module.

Let μ be the H-weight of v_*. By Remark 3.18, we have $\mu_r \in \mathbb{N}$. Now consider the vector $(E_{r-1,r})^{\mu_r+1}v_*$. As

$$E_{r,r-1}v_* = 0 \quad \text{and} \quad (E_{r,r} - E_{r-1,r-1})v_* = E_{r,r}v_* = \mu_r v_*,$$

we have

$$E_{r,r-1} \cdot (E_{r-1,r})^{\mu_r+1}v_* = 0.$$

For $r \leq j \leq n-1$, since $[E_{j+1,j}, E_{r-1,r}] = 0$ and $E_{j+1,j}v_* = 0$, we have

$$E_{j+1,j} \cdot (E_{r-1,r})^{\mu_r+1}v_* = 0.$$

Furthermore, for $p,q \in \mathbb{Z}$ with $q \notin S$ and $q \neq r-1$, we have

$$E_{p,q}E_{r-1,r} = E_{r-1,r}E_{p,q} - \delta_{r,p}E_{r-1,q}.$$

By induction we get $E_{p,q} \cdot (E_{r-1,r})^{\mu_r+1} v_* = 0$. Thus, $(E_{r-1,r})^{\mu_r+1} v_*$, if not zero, is another lowest-weight singular vector in the $\mathfrak{gl}_{\tilde{S}}$-module $U(\mathfrak{gl}_{\tilde{S}})v_*$. As v_* and $(E_{r-1,r})^{\lambda_r+1} v_*$ have different H-weights, $(E_{r-1,r})^{\lambda_r+1} v_*$ must be zero. It then follows that $E_{r-1,r}$ is locally nilpotent on $L(S,\lambda)$.

To summarize, we have proved that $E_{i,i+1}$ and $E_{i+1,i}$ for $i \in \mathbb{Z}$ are locally nilpotent on $L(S,\lambda)$. It was known that H is semi-simple on $L(S,\lambda)$. Therefore, $L(S,\lambda)$ is an integrable $\mathfrak{gl}_\infty$-module. □

With Proposition 3.19 and Theorem 2.11, using a standard argument (see [12]) we obtain:

Corollary 3.20. *Let $S = \{r, r+1, \ldots, n\}$ with $r < n$ and let $\lambda \in P_+(S)$. Then the maximal submodule of $M(S,\lambda)$ is generated by $E_{n+1,n}^{\lambda_n+1} v$, $E_{r-1,r}^{\lambda_r+1} v$, and $E_{i+1,i}^{\lambda_i-\lambda_{i+1}+1} v$ for $r \le i \le n-1$, where v is a highest weight vector of weight λ.*

To summarize we have:

Theorem 3.21. *Let $S = \{r, r+1, \ldots, n\}$ where r and s are integers with $r < n$ and let $\lambda \in H^*$ be such that* $\mathrm{supp}(\lambda) \subset S$ *and $\lambda \in P_+(S)$. Then $L(S,\lambda)$ belongs to category $\mathcal{C}_{int}$. On the other hand, every irreducible $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ is isomorphic to a module of this form.*

Proof. The first assertion follows from Lemma 3.1 and Proposition 3.19. Now, let W be an irreducible $\mathfrak{gl}_\infty$-module in $\mathcal{C}_{int}$. By Proposition 3.4, $W \simeq L(S,U)$ for some finite subset S of $\mathbb{Z}$ and for some irreducible $\mathfrak{gl}_S$-module U. As W is an integrable $\mathfrak{gl}_\infty$-module, W is an integrable $\mathfrak{gl}_S$-module containing U as a submodule. Then U is finite-dimensional. Let v be a highest weight vector in U viewed as a $\mathfrak{gl}_S$-module. Noticing that $E_{q,q}v = 0$ for $q \notin S$, we see that v is an H-weight vector of a weight $\lambda \in H^*$ with $\mathrm{supp}(\lambda) \subset S$. It follows that $W = U(\mathfrak{gl}_\infty)v \simeq L(S,\lambda)$. Furthermore, by Proposition 3.19 we have $\lambda \in P_+(S)$. □

4. Decomposition of Tensor Product Modules In Category $\mathcal{C}_{int}$

In this section, we construct the irreducible integrable $\mathfrak{gl}_\infty$-modules $L(S,\lambda)$ by using the natural module $\mathbb{C}^\infty$, and we also determine the decomposition of tensor product modules in category $\mathcal{C}_{int}$.

Let A denote the polynomial algebra $\mathbb{C}[x_m \mid m \in \mathbb{Z}]$. Define $\deg x_m = 1$ for $m \in \mathbb{Z}$ to make A a $\mathbb{Z}$-graded algebra

$$A = \oplus_{r\geq 0} A_r.$$

It was well known that Lie algebra $\mathfrak{gl}_\infty$ naturally acts on A with

$$E_{m,n} = x_m \frac{\partial}{\partial x_n} \quad \text{for } m, n \in \mathbb{Z}.$$

It can be readily seen that A is a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$. We see that A_r for $r \geq 0$ are submodules with $A_0 = \mathbb{C}$ and $A_1 \simeq \mathbb{C}^\infty$. In fact, A is isomorphic to the symmetric algebra $S(\mathbb{C}^\infty)$ with v_m identified with x_m for $m \in \mathbb{Z}$.

One can show that for each $r \geq 0$, A_r is an irreducible $\mathfrak{gl}_\infty$-module. For each $n \in \mathbb{Z}$, let ε_n be the linear functional on H defined by

$$\varepsilon_n(E_{m,m}) = \delta_{n,m} \quad \text{for } m \in \mathbb{Z}.$$

Clearly, ε_n $(n \in \mathbb{Z})$ are linearly independent. For $i_1, \ldots, i_r \in \mathbb{Z}$, the H-weight of monomial $x_{i_1} \cdots x_{i_r}$ is $\varepsilon_{i_1} + \cdots + \varepsilon_{i_r}$. We see that every H-weight space of A_r is 1-dimensional. Then any nonzero submodule of A_r must contain a monomial of degree r. For $j_1, \ldots, j_k \in \mathbb{Z}$, $n_1, \ldots, n_k \in \mathbb{N}$ with $j_1 < j_2 < \cdots < j_k$ and $n_1 + \cdots + n_k = r$, we have

$$(E_{j_1,j_2})^{n_2} \cdots (E_{j_1,j_k})^{n_k} \left(x_{j_1}^{n_1} \cdots x_{j_k}^{n_k} \right) = n_2! \cdots n_k! x_{j_1}^r.$$

For any $p, q \in \mathbb{Z}$ with $p \neq q$, we have

$$(E_{p,q})^r x_q^r = r! x_p^r.$$

We also have

$$\frac{1}{m_1!} \cdots \frac{1}{m_s!} (E_{i_1,t})^{m_1} \cdots (E_{i_s,t})^{m_s} \cdot x_t^r = \binom{r}{m_1} \binom{r-m_1}{m_2} \cdots \binom{m_s}{m_s} x_{i_1}^{m_1} \cdots x_{i_s}^{m_s}$$

for $i_1, \ldots, i_s, t \in \mathbb{Z}$, $m_1, \ldots, m_s \in \mathbb{N}$ with $i_1 < i_2 < \cdots < i_s$, $m_1 + \cdots + m_s = r$, and $t \neq i_1, \ldots, i_s$. It then follows that A_r is an irreducible $\mathfrak{gl}_\infty$-module. We see that $A_r \simeq L(S, r\varepsilon_1)$ with $S = \{1\}$.

On the other hand, $\mathfrak{gl}_\infty$ naturally acts on the exterior algebra $\Lambda(\mathbb{C}^\infty)$, which is also an $\mathbb{N}$-graded module

$$\Lambda(\mathbb{C}^\infty) = \bigoplus_{n\in\mathbb{N}} \Lambda^n.$$

We have $\Lambda^0 = \mathbb{C}$ and $\Lambda^1 = \mathbb{C}^\infty$. For $n \geq 2$, the submodule Λ^n has a basis consisting of vectors

$$v_{i_1} \wedge \cdots \wedge v_{i_n}$$

for $i_1, \ldots, i_n \in \mathbb{Z}$ with $i_1 < i_2 < \cdots < i_n$. One sees that the H-weight of vector $v_{i_1} \wedge \cdots \wedge v_{i_n}$ is $\varepsilon_{i_1} + \cdots + \varepsilon_{i_n}$ of multiplicity one. Similarly, one can show that for every $n \geq 0$, Λ^n is an irreducible $\mathfrak{gl}_\infty$-module. For $n \geq 1$, we have $\Lambda^n \simeq L(S, \varepsilon_1 + \cdots + \varepsilon_n)$ with $S = \{1, 2, \ldots, n\}$.

Note that for any permutation σ on $\mathbb{Z}$, A_m and Λ^n are σ-invariant.

Remark 4.1. Let n be a positive integer and let $\lambda \in H^*$ be such that $\lambda_m = 0$ for $m \notin \{1, 2, \ldots, n\}$, $\lambda_i \in \mathbb{N}$ for $1 \leq i \leq n$, and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Note that

$$\begin{aligned}\lambda \;=\; & (\lambda_1 - \lambda_2)\varepsilon_1 + (\lambda_2 - \lambda_3)(\varepsilon_1 + \varepsilon_2) + \cdots + (\lambda_{n-1} - \lambda_n)(\varepsilon_1 + \cdots + \varepsilon_{n-1}) \\ & + \lambda_n(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n).\end{aligned}$$

For $1 \leq k \leq n$, set

$$w_k = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k \in \Lambda^k.$$

Furthermore, set

$$w_\lambda = w_1^{\otimes(\lambda_1-\lambda_2)} \otimes (w_2)^{\otimes(\lambda_2-\lambda_3)} \otimes \cdots \otimes (w_{n-1})^{\otimes(\lambda_{n-1}-\lambda_n)} \otimes (w_n)^{\otimes\lambda_n},$$

which lies in the $\mathfrak{gl}_\infty$-module

$$(\Lambda^1)^{\otimes(\lambda_1-\lambda_2)} \otimes (\Lambda^2)^{\otimes(\lambda_2-\lambda_3)} \otimes \cdots \otimes (\Lambda^{n-1})^{\otimes(\lambda_{n-1}-\lambda_n)} \otimes (\Lambda^n)^{\otimes\lambda_n}.$$

Set $S = \{1, 2, \ldots, n\}$. It can be readily seen that w_λ is an S-singular vector of weight λ. Then it follows that $U(\mathfrak{gl}_\infty)w_\lambda \simeq L(S, \lambda)$.

For the rest of this section, we discuss the decomposition of tensor product modules in category $\mathcal{C}_{int}$. In view of Theorem 2.11, the tensor product of any two irreducible modules in category $\mathcal{C}_{int}$ is completely reducible. For example, we have

$$A_1 \otimes A_1 = \mathbb{C}^\infty \otimes \mathbb{C}^\infty = S^2(\mathbb{C}^\infty) \oplus \Lambda^2(\mathbb{C}^\infty) = A_2 \oplus \Lambda^2.$$

This typical example indicates that the tensor product of two irreducible modules in category $\mathcal{C}_{int}$ can be a finite sum of irreducible submodules. Next, we show that indeed this is the case. First, we establish a technical result.

Lemma 4.2. *Let S be a finite subset of $\mathbb{Z}$ and let $\lambda \in P_+(S)$. Let $v \in L(S,\lambda)_\lambda$ nonzero and set $K_S^+ = \mathrm{span}\{E_{p,i} \mid p > \max(S),\ i \in S\}$, an abelian subalgebra of $\mathfrak{gl}_\infty$. Then there exists a nonzero H-weight vector $v' \in U(K_S^+)v \subset L(S,\lambda)$ such that*

$$E_{p,i}v' = 0 \quad \text{for all } p \in \mathbb{Z},\ i \in S.$$

Proof. Let $i \in S$. Assume $E_{i,i}v = kv$ where k is a nonnegative integer (by Lemma 2.5). Then

$$E_{i,i}\left(E_{p_1,i} \cdots E_{p_r,i}v\right) = (k-r)E_{p_1,i} \cdots E_{p_r,i}v$$

for any integers $p_1, \ldots, p_r$ outside S. As $E_{i,i}$ has only nonnegative integer eigenvalues, we have

$$E_{p_1,i} \cdots E_{p_r,i}v = 0 \quad \text{whenever } r \geq k+1.$$

Since K_S^+ is abelian and S is finite, it follows that there exists a nonzero H-weight vector $v' \in U(K_S^+)v$ such that

$$E_{p,i}v' = 0 \quad \text{for } p > \max(S),\ i \in S. \tag{4.1}$$

Let p be a fixed integer such that $p > \max(S)$ and $E_{q,p}v' = 0$ for all $q \in \mathbb{Z}$. For $i, j \in S$, we have $E_{p,j}v' = 0$, $E_{i,p}v' = 0$, and $E_{p,p}v' = 0$, so that

$$E_{i,j}v' = E_{i,p}E_{p,j}v' - E_{p,j}E_{i,p}v' + \delta_{i,j}E_{p,p}v' = 0. \tag{4.2}$$

Furthermore, let $q \notin S,\ i \in S$. As $E_{i,i}v' = 0$, we have

$$E_{i,i}(E_{q,i}v') = E_{q,i}E_{i,i}v' - E_{q,i}v' = -(E_{q,i}v').$$

Because $E_{i,i}$ has only nonnegative integer eigenvalues, we must have $E_{q,i}v' = 0$. Therefore, we have $E_{p,i}v' = 0$ for all $p \in \mathbb{Z},\ i \in S$, as desired. $\square$

We shall also need the following simple fact:

Lemma 4.3. *Let W be a $\mathfrak{gl}_\infty$-module in category $\mathcal{C}_{int}$ and let S be a finite subset of $\mathbb{Z}$. Suppose that U is a $\mathfrak{gl}_S$-submodule of $\Omega_S(W)$ such that U generates W as a $\mathfrak{gl}_\infty$-module. If $U = \coprod_{\alpha\in I} L(\lambda^\alpha_S)$ is a $\mathfrak{gl}_S$-module with $\lambda^\alpha \in H^*$ such that* $\mathrm{supp}(\lambda^\alpha) \subset S$ *for $\alpha \in I$, then*

$$W \simeq \coprod_{\alpha\in I} L(S, \lambda^\alpha).$$

Furthermore, $\Omega_S(W) = U$.

Proof. Since W is an integrable $\mathfrak{gl}_S$-module, $\Omega_S(W)$ as a submodule is a direct sum of finite-dimensional irreducible $\mathfrak{gl}_S$-modules. Noticing that for any $v \in \Omega_S(W)$, $E_{q,q}v = 0$ for $q \notin S$, we see that any singular vector in $\Omega_S(W)$ viewed as a $\mathfrak{gl}_S$-module is an S-singular vector. Suppose that v is a singular vector in $\Omega_S(W)$ viewed as a $\mathfrak{gl}_S$-module of H-weight λ^α with $\alpha \in I$. By Proposition 2.10, $U(\mathfrak{gl}_\infty)v$ is irreducible, so that $U(\mathfrak{gl}_\infty)v \simeq L(S, \lambda^\alpha)$. It then follows that there are singular vectors v_β $(\beta \in J)$ in U $(\subset \Omega_S(W))$ with H-weights λ^β such that

$$W = \oplus_{\beta\in J} U(\mathfrak{gl}_\infty)v_\beta \simeq \oplus_{\beta\in J} L(S, \lambda^\beta).$$

From this, using Lemma 3.6 we get

$$\Omega_S(W) = \oplus_{\beta\in J}\Omega_S(U(\mathfrak{gl}_\infty)v_\beta) = \oplus_{\beta\in J} L(\lambda^\beta_S) \subset U.$$

Therefore $\Omega_S(W) = U$. $\square$

Now, we give a decomposition into irreducible submodules of the tensor product of any two irreducible modules in category $\mathcal{C}_{int}$.

Theorem 4.4. *Let $S = \{r, r+1, \ldots, n\}$ be a finite subset of $\mathbb{Z}$ with $r \le n$, and let $\lambda, \mu \in P_+(S)$. Then $L(S, \lambda) \otimes L(S, \mu)$ is a cyclic $\mathfrak{gl}_\infty$-module.*

Furthermore, there exists an integer k with $k \geq n$ such that for any integer $\bar{n} \geq k$, the decomposition of $L(S,\lambda) \otimes L(S,\mu)$ into irreducible $\mathfrak{gl}_\infty$-submodules agrees with the decomposition of $L(\lambda_{\bar{S}}) \otimes L(\mu_{\bar{S}})$ into irreducible $\mathfrak{gl}_{\bar{S}}$-irreducible submodules where $\bar{S} = \{r, r+1, \ldots, \bar{n}\}$.

Proof. Let $u \in L(S,\lambda)_\lambda$ and $v \in L(S,\mu)_\mu$, both nonzero. Then $u \in \Omega_S(L(S,\lambda))$, $v \in \Omega_S(L(S,\mu))$. By Lemma 4.2, there exists a nonzero H-weight vector $v' \in U(K_S^+)v$ such that $E_{p,q}v' = 0$ for all $p \in \mathbb{Z}$, $q \in S$. We now prove that $u \otimes v'$ generates $L(S,\lambda) \otimes L(S,\mu)$ as a $\mathfrak{gl}_\infty$-module. Set

$$K = \mathrm{span}\{E_{p,i} \mid p \in \mathbb{Z},\ i \in S\}.$$

We have $U(K)u = U(\mathfrak{gl}_\infty)u = L(S,\lambda)$ (using the P-B-W theorem) and $K \cdot v' = 0$. Then

$$U(K)(u \otimes v') = U(K)u \otimes v' = L(S,\lambda) \otimes v'.$$

As $L(S,\mu) = U(\mathfrak{gl}_\infty)v'$, it follows that $U(\mathfrak{gl}_\infty)(u \otimes v') = L(S,\lambda) \otimes L(S,\mu)$. This proves that $L(S,\lambda) \otimes L(S,\mu)$ is cyclic on $u \otimes v'$.

Furthermore, let k be an integer larger than n such that

$$v' \in \langle E_{p,i} \mid n < p \leq k,\ i \in S\rangle \cdot v,$$

where $\langle \cdot \rangle$ denotes the generated subalgebra of $U(\mathfrak{gl}_\infty)$. Let $\bar{n}$ be any integer larger than k and set $\bar{S} = \{r, r+1, \ldots, \bar{n}\}$. Noticing that

$$u \in \Omega_{\bar{S}}(L(S,\lambda)), \quad v' \in \Omega_{\bar{S}}(L(S,\mu)),$$

we have

$$L(S,\lambda) \otimes L(S,\mu) = U(\mathfrak{gl}_\infty)(u \otimes v') = U(\mathfrak{gl}_\infty^{(\bar{S},-)})U(\mathfrak{gl}_{\bar{S}})(u \otimes v').$$

As u and v are singular vectors in the integrable $\mathfrak{gl}_{\bar{S}}$-modules $L(S,\lambda)$ and $L(S,\mu)$, respectively, we get

$$U(\mathfrak{gl}_{\bar{S}})u = L(\lambda_{\bar{S}}) \text{ and } U(\mathfrak{gl}_{\bar{S}})v = L(\mu_{\bar{S}}).$$

Noticing that $U(\mathfrak{gl}_{\bar{S}})v' \subset U(\mathfrak{gl}_{\bar{S}})v = L(\mu_{\bar{S}})$, we have

$$U(\mathfrak{gl}_{\bar{S}})(u \otimes v') \subset L(\lambda_{\bar{S}}) \otimes L(\mu_{\bar{S}}) \subset \Omega_{\bar{S}}\left(L(S,\lambda) \otimes L(S,\mu)\right).$$

By Lemma 4.3 we get

$$U(\mathfrak{gl}_{\bar{S}})(u\otimes v') = L(\lambda_{\bar{S}})\otimes L(\mu_{\bar{S}}) = \Omega_{\bar{S}}\left(L(S,\lambda)\otimes L(S,\mu)\right) \tag{4.3}$$

and

$$L(S,\lambda)\otimes L(S,\mu)\simeq L\left(\bar{S}, L(\lambda_{\bar{S}})\otimes L(\mu_{\bar{S}})\right).$$

This proves the second assertion. □

Example 4.5. For an illustration, consider $A_m\otimes A_n$ with m,n positive integers, where

$$A_m = L(S, m\varepsilon_1) \quad \text{and} \quad A_n = L(S, n\varepsilon_1)$$

with $S=\{1\}$. In this case, we can show that $A_m\otimes A_n$ is cyclic on $x_1^m\otimes x_2^n$. Set $K=\operatorname{span}\{E_{p,1}\mid p\neq 1\}$. Then

$$A_m = U(\mathfrak{gl}_\infty)x_1^m = U(K)x_1^m \quad \text{and} \quad K\cdot x_2^n = 0.$$

We have

$$U(K)(x_1^m\otimes x_2^n) = U(K)x_1^m\otimes x_2^n = A_m\otimes x_2^n,$$

from which it follows that $U(\mathfrak{gl}_\infty)(x_1^m\otimes x_2^n) = A_m\otimes A_n$. Set $\bar{S}=\{1,2\}$. We have $\mathfrak{gl}_{\bar{S}} = \mathfrak{gl}_2$, linearly spanned by $E_{i,j}$ for $1\leq i,j\leq 2$, and

$$U(\mathfrak{gl}_{\bar{S}})x_1^m = L(m\varepsilon_1), \quad U(\mathfrak{gl}_{\bar{S}})x_2^n = L(n\varepsilon_1).$$

For $\mathfrak{gl}_2$-modules, we have

$$L(m\varepsilon_1)\otimes L(n\varepsilon_1) = \bigoplus_{j=0}^{|m-n|} L\left((m+n-j)\varepsilon_1 + j\varepsilon_2\right).$$

This gives rise to a decomposition of $A_m\otimes A_n$ as a $\mathfrak{gl}_\infty$-module

$$A_m\otimes A_n = \bigoplus_{j=0}^{|m-n|} L\left(\bar{S}, (m+n-j)\varepsilon_1 + j\varepsilon_2\right).$$

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