

ON FUSION PROCEDURE FOR THE TWO-PARAMETER QUANTUM ALGEBRA IN TYPE A

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Dedicated to Robert Griess, Jr. in honor of his 71st birthday

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Abstract

Finite dimensional irreducible modules of the two-parameter quantum enveloping algebra $U_{r,s}(\mathfrak{sl}_n)$ are explicitly constructed using the fusion procedure when rs^{-1} is generic. This provides an alternative and combinatorial description of the Schur-Weyl duality for the two-parameter quantum linear algebras of type A.

1. Introduction

Schur-Weyl duality is one of the main methods to construct irreducible modules of the classical simple Lie groups out of the fundamental representations [22]. The quantum version for the quantum enveloping algebra $U_q(\mathfrak{sl}_n)$ and the Hecke algebra $H_q(\mathfrak{S}_m)$ has played an important role in the fervent development of quantum groups. They provide one of the first examples [13] to show the similarity between the classical and quantum theories.

Two-parameter general and special linear quantum groups [21, 8, 4] are further generalization of the corresponding one-parameter Drinfeld-Jimbo quantum groups [7, 12]. The two-parameter quantum groups also had their origin in the quantum inverse scattering method [20] as well as other approaches [14, 6]. In particular, the Schur-Weyl duality was also generalized

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to the two-parameter case [2]. As in the classical situation, let V be the natural n -dimensional module of $U_{r,s}(\mathfrak{sl}_n)$, then arbitrary irreducible modules can be constructed using the (r, s) -symmetric tensor $S_{r,s}^2(V)$ and the R-matrix $R = R_{VV}$ for two-parameter quantum group $U_{r,s}(\mathfrak{sl}_n)$. Alternatively any finite dimensional irreducible $U_{r,s}(\mathfrak{sl}_n)$ -module can be built from the tensor product $V^{\otimes m}$ using the symmetry of the Hecke algebra $H_q(\mathfrak{S}_m)$, where $q = \sqrt{s/r}$.

In this work we give an alternative description of all irreducible representations (rs^{-1} is not a root of unity) using the developments [11] of the fusion procedure [3] in the quantum inverse scattering method. It seems that the two-parameter case can be treated quite similarly using the fusion procedure, thus one can more or less apply the known results from the one-parameter case to get corresponding formulas. As there is an abstract argument available to construct the irreducible modules in the two-parameter case, we nevertheless give a detailed description of all irreducible $U_{r,s}(\mathfrak{sl}_n)$ -modules using the fusion procedure to supplement the existing theory.

Much of the results in the paper are expected for the experts, and we hope the current presentation can further show the similarity and connection to the one-parameter case. The one-parameter case is adapted into the two parameter situation in a self-contained manner and we try to be complete as much as possible for pedagogical purpose.

2. Two-parameter Quantum Group $U_{r,s}(\mathfrak{sl}_n)$ and R-matrix

We start with the basic definition of the two-parameter quantum groups $U_{r,s}(\mathfrak{gl}_n)$, $U_{r,s}(\mathfrak{sl}_n)$ following the notations in [2]. Let $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} | j = 1, 2, \dots, n-1\}$ be the set of the simple roots of type A_{n-1} , where $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ are an orthonormal basis of a Euclidean space with inner product $\langle \cdot, \cdot \rangle$. The root system is then $\Phi = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$.

Fix two distinct nonzero complex numbers r, s , and assume they are in general position.

Definition 2.1. The two-parameter quantum enveloping algebra $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ is the unital associative algebra over \mathbb{C} generated by e_j, f_j , ($1 \leq j < n$) and $a_i^{\pm 1}, b_i^{\pm 1}$, ($1 \leq i \leq n$) with the following relations:

$$(R1) \quad a_i^{\pm 1}, b_i^{\pm 1} \text{ commute with each other, and } a_i a_i^{-1} = b_i b_i^{-1} = 1,$$

$$(R2) \quad a_i e_j a_i^{-1} = r^{\langle \epsilon_i, \alpha_j \rangle} e_j \text{ and } a_i f_j a_i^{-1} = r^{-\langle \epsilon_i, \alpha_j \rangle} f_j,$$

$$(R3) \quad b_i e_j b_i^{-1} = s^{\langle \epsilon_i, \alpha_j \rangle} e_j \text{ and } b_i f_j b_i^{-1} = s^{-\langle \epsilon_i, \alpha_j \rangle} f_j,$$

$$(R4) \quad [e_i, f_j] = \frac{\delta_{ij}}{r-s} (a_i b_{i+1} - a_{i+1} b_i),$$

$$(R5) \quad [e_i, e_j] = [f_i, f_j] = 0, \text{ if } |i - j| > 1,$$

$$(R6) \quad e_i^2 e_{i+1} - (r + s) e_i e_{i+1} e_i + r s e_{i+1} e_i^2 = 0 \text{ and}$$

$$e_i e_{i+1}^2 - (r + s) e_{i+1} e_i e_{i+1} + r s e_{i+1}^2 e_i = 0,$$

$$(R7) \quad f_i^2 f_{i+1} - (r^{-1} + s^{-1}) f_i f_{i+1} f_i + r^{-1} s^{-1} f_{i+1} f_i^2 = 0 \text{ and}$$

$$f_i f_{i+1}^2 - (r^{-1} + s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1} s^{-1} f_{i+1}^2 f_i = 0,$$

where $[x, y] = xy - yx$ is the commutator.

The algebra $U = U_{r,s}(\mathfrak{sl}_n)$ is the subalgebra of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements e_j, f_j, ω_j and ω'_j ($1 \leq j < n$), where $\omega_j = a_j b_{j+1}$, and $\omega'_j = a_{j+1} b_j$. These elements satisfy the relations (R5)-(R7) along with

$$(R1') \quad \text{The } \omega_i, \omega'_i \text{ are invertible and they all commute with each other,}$$

$$(R2') \quad \omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i \text{ and } \omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i,$$

$$(R3') \quad \omega'_i e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega'_i \text{ and } \omega'_i f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega'_i,$$

$$(R4') \quad [e_i, f_j] = \frac{\delta_{ij}}{r-s} (\omega_i - \omega'_i).$$

Clearly when $r = q, s = q^{-1}$, the algebra U modulo the ideal generated by the elements $\omega_j^{-1} - \omega'_j, 1 \leq j < n$, is isomorphic to $U_q(\mathfrak{sl}_n)$.

The algebra $U_{r,s}(\mathfrak{sl}_n)$ is a Hopf algebra under the comultiplication Δ such that ω_i, ω'_i are group-like elements and the other nontrivial comultiplications, counits and antipodes are given by:

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i, \\ \epsilon(e_i) &= \epsilon(f_i) = 0, S(e_i) = -\omega_i^{-1} e_i, S(f_i) = -f_i \omega_i'^{-1}. \end{aligned}$$

The representation theory of $U_{r,s}(\mathfrak{sl}_n)$ is quite similar to that of one-parameter case. We recall some of the basic notations for later discussion. Let $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ be the weight lattice of \mathfrak{gl}_n , $Q = \mathbb{Z}\Phi$ the root lattice, and $Q_+ = \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \alpha_i$, where ϵ_i are the orthonormal vectors as before. Recall that Λ is equipped with the partial order given by $\nu \leq \lambda$ if and only if $\lambda - \nu \in Q_+$.

For each $\lambda \in \Lambda$ one defines the algebra homomorphism $\widehat{\lambda} : \widetilde{U}^0 \rightarrow \mathbb{C}$ by:

$$\widehat{\lambda}(a_i) = r^{\langle \epsilon_i, \lambda \rangle}, \quad \widehat{\lambda}(b_i) = s^{\langle \epsilon_i, \lambda \rangle}, \quad (2.1)$$

where \widetilde{U}^0 is the subalgebra of $U_{r,s}(\mathfrak{gl}_n)$ generated by $a_i^{\pm 1}, b_i^{\pm 1}$ ($1 \leq i \leq n$). Then the restriction of $\widehat{\lambda} : U^0 \rightarrow \mathbb{C}$ of $\widehat{\lambda}$ to the subalgebra U^0 of U generated by ω_j, ω'_j ($1 \leq j < n$) satisfies:

$$\widehat{\lambda}(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle}, \quad \widehat{\lambda}(\omega'_j) = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}. \quad (2.2)$$

It was shown in [1] when rs^{-1} is not a root of unity, the homomorphisms $\widehat{\lambda} = \widehat{\mu}$ if and only if the corresponding weights $\lambda = \mu$. These homomorphisms are called generalized weights. For an algebra homomorphism $\chi : U^0 \mapsto \mathbb{C}$ one defines the generalized weight subspace of $U_{r,s}$ -module M by

$$M_\chi = \{v \in M \mid (\omega_i - \chi(\omega_i))^m v = (\omega'_i - \chi(\omega'_i))^m v = 0, \text{ for all } i \text{ and for some } m\}.$$

If $m = 1$, the subspace M_χ becomes a weight subspace associated with the homomorphism χ . Since U^0 is commutative, it is easy to see that any finite dimensional $U_{r,s}$ -module M can be decomposed into a sum of generalized weight subspaces:

$$M = \bigoplus_{\chi} M_\chi. \quad (2.3)$$

When all generalized weights in M are of the form $\chi \cdot (-\widehat{\alpha})$ for a fixed χ and α varying in Q_+ (here $\chi \cdot (-\widehat{\alpha})(\omega_i) = \chi(\omega_i)(-\widehat{\alpha})(\omega_i)$ and $\chi \cdot (-\widehat{\alpha})(\omega'_i) = \chi(\omega'_i)(-\widehat{\alpha})(\omega'_i)$), we say M is a *highest weight module* of weight χ and write $M = M(\chi)$. Benkart and Witherspoon [2] have shown that when M is simple, all generalized weight subspaces are actually weight subspaces. Moreover, if all generalized weights are of the form $\widehat{\lambda}$ for $\lambda \in \Lambda$, we will simply write M_λ for $M_{\widehat{\lambda}}$, and similarly the highest weight module $M(\widehat{\lambda})$ will be denoted as $M(\lambda)$.

One can also define the notion of Verma modules $M(\lambda)$ as in the classical situation. It is known [2] that all finite dimensional simple $U_{r,s}(\mathfrak{sl}_n)$ -modules are realized as simple quotients of Verma modules. We will write by $V(\lambda)$ the simple quotient of the Verma module $M(\lambda)$.

Now let us look at the simplest irreducible module $V(\bar{\omega}_1)$. Let V be the n -dimensional vector space over \mathbb{C} with basis $\{v_j | 1 \leq j \leq n\}$, and $E_{ij} \in \text{End}(V)$ be defined by $E_{ij}v_k = \delta_{jk}v_i$. Define the $U_{r,s}(\mathfrak{sl}_n)$ -action on V by

$$\begin{aligned} e_j &= E_{j,j+1}, \quad f_j = E_{j+1,j}, \\ \omega_j &= rE_{jj} + sE_{j+1,j+1} + \sum_{k \neq j, j+1} E_{kk}, \\ \omega'_j &= sE_{jj} + rE_{j+1,j+1} + \sum_{k \neq j, j+1} E_{kk}, \end{aligned}$$

where $1 \leq j \leq n-1$. It is clear that $V = \bigoplus_{j=1}^n V_{\epsilon_j}$ and it is the simple $U_{r,s}(\mathfrak{sl}_n)$ -module $V(\bar{\omega}_1)$.

Let $\check{R} = \check{R}_V : V \otimes V \longrightarrow V \otimes V$ be the R -matrix defined by

$$\check{R} = \sum_{i=1}^n E_{ii} \otimes E_{ii} + r \sum_{i < j} E_{ji} \otimes E_{ij} + s^{-1} \sum_{i < j} E_{ij} \otimes E_{ji} + (1 - rs^{-1}) \sum_{i < j} E_{jj} \otimes E_{ii}, \quad (2.4)$$

which is essentially determined by the simple module V and the comultiplication Δ (see Prop. 2.2). For each $1 \leq i < k$ let \check{R}_i be the isomorphism on $V^{\otimes k}$ defined by

$$\check{R}_i(w_1 \otimes w_2 \otimes \cdots \otimes w_k) = w_1 \otimes \cdots \otimes \check{R}(w_i \otimes w_{i+1}) \otimes w_{i+2} \otimes \cdots \otimes w_k.$$

Then we have the braid relations:

$$\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1} \quad \text{for } 1 \leq i < k-1, \quad (2.5)$$

The construction also implies that for $|i-j| \geq 2$,

$$\check{R}_i \check{R}_j = \check{R}_j \check{R}_i.$$

Moreover one can directly check that

$$\check{R}_i^2 = \left(1 - \frac{r}{s}\right) \check{R}_i + \frac{r}{s} Id \quad (2.6)$$

for all $1 \leq i < k$. In particular, the minimum polynomial of \check{R} on $V \otimes V$ is $(t-1)(t + \frac{r}{s})$ if $s \neq -r$ [2].

Proposition 2.2 ([2]). *The endomorphisms $\check{R}_i \in \text{End}(V^{\otimes k})$ commute with the action of $U_{r,s}(\mathfrak{sl}_n)$ on $V^{\otimes k}$.*

Proof. By the Hopf algebra structure of $U_{r,s}(\mathfrak{sl}_n)$ it is enough to check this for $k = 2$. Then it is a direct verification to confirm that \check{R} commutes with $\Delta(e_i), \Delta(f_i), \Delta(\omega_i), \Delta(\omega'_i)$ for $i = 1, \dots, n-1$ on the fundamental representation $V(\bar{\omega}_1)$. \square

3. Yang-Baxterization and the Wedge Modules of $U_{r,s}(\mathfrak{sl}_n)$

The fusion procedure relies on the spectral parameter dependent R-matrix $\check{R}(z)$, which satisfies the so-called quantum Yang-Baxter equation. One form of the Yang-Baxter equation (YBE) is the following matrix equation on $V^{\otimes 3}$:

$$\check{R}_1(z)\check{R}_2(zw)\check{R}_1(w) = \check{R}_2(w)\check{R}_1(zw)\check{R}_2(z). \quad (3.1)$$

where $\check{R}(0) = \check{R}$. The Yang-Baxterization method recovers the spectral parameter dependent R-matrix $\check{R}(z)$ from the initial condition (2.5) at $z = 0$.

The Yang-Baxterization process was carried out for the two-parameter R-matrix in [16] using the method of [9].

Proposition 3.1 ([16]). *For the braid group representation $\check{R} = \check{R}_{VV}$, the R-matrix $\check{R}(z)$ is given by*

$$\begin{aligned} \check{R}(z) = & (1 - zrs^{-1}) \sum_{i=1}^n E_{ii} \otimes E_{ii} + (1 - z)(r \sum_{i>j} + s^{-1} \sum_{i<j}) E_{ij} \otimes E_{ji} \\ & + z(1 - rs^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj} + (1 - rs^{-1}) \sum_{i>j} E_{ii} \otimes E_{jj}. \end{aligned} \quad (3.2)$$

Remark 3.2. Clearly $\check{R}(0) = \check{R}$. Moreover, when $r = q$ and $s = q^{-1}$, the R-matrix $\check{R}(z)$ turns into

$$\begin{aligned} \check{R}_q(z) = & (1 - zq^2) \sum_{i=1}^n E_{ii} \otimes E_{ii} + (1 - z)q \sum_{i \neq j} E_{ij} \otimes E_{ji} \\ & + (1 - q^2) \left(\sum_{i>j} + z \sum_{i<j} \right) E_{ii} \otimes E_{jj}, \end{aligned}$$

which is exactly the Jimbo R-matrix for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ [12]. In this regard we can view $\check{R}(z)$ as an (r, s) -analogue of the R-matrix $\check{R}_q(z)$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ [10]. The one-parameter R-matrix can also be used to treat the quantum algebra $U_q(\mathfrak{gl}(m|n))$ [19].

For convenience, we introduce a normalized R-matrix with two spectral parameters

$$\begin{aligned} \check{R}(x, y) &= \frac{sy\check{R}(x/y)}{y-x} \\ &= \frac{sy-rx}{y-x} \sum_{i=1}^n E_{ii} \otimes E_{ii} + (sr \sum_{i>j} + \sum_{i<j}) E_{ij} \otimes E_{ji} \\ &\quad + \frac{(s-r)x}{y-x} \sum_{i<j} E_{ii} \otimes E_{jj} + \frac{(s-r)y}{y-x} \sum_{i>j} E_{ii} \otimes E_{jj}. \end{aligned} \quad (3.3)$$

The original YBE immediately implies the following Yang-Baxter equation:

$$\check{R}_i(x, y)\check{R}_{i+1}(x, z)\check{R}_i(y, z) = \check{R}_{i+1}(y, z)\check{R}_i(x, z)\check{R}_{i+1}(x, y). \quad (3.4)$$

Let $S_{r,s}^2(V)$ be the subspace of $V \otimes V$ spanned by $\{v_i \otimes v_i | 1 \leq i \leq n\} \cup \{v_i \otimes v_j + sv_j \otimes v_i | 1 \leq i < j \leq n\}$ and $\Lambda_{r,s}^2(V)$ be the subspace of $V \otimes V$ spanned by $\{v_i \otimes v_j - rv_j \otimes v_i | 1 \leq i < j \leq n\}$.

Proposition 3.3 ([16]). *The subspace $S_{r,s}^2(V)$ is equal to the image of $\check{R}(1, r^{-1}s)$ on $V \otimes V$, and $\Lambda_{r,s}^2(V)$ is equal to the image of $\check{R}(1, rs^{-1})$ on $V \otimes V$.*

Remark 3.4. The above result is equivalent to $S_{r,s}^2(V) = \text{Ker } \check{R}(1, sr^{-1})$ and $\Lambda_{r,s}^2(V) = \text{Ker } \check{R}(1, s^{-1}r)$. This suggests that special values of the Yang-Baxter matrix can lead to irreducible modules, which we will show in general in Section 6.

Proposition 3.5 ([16]). *The k th fundamental representation of $U_{r,s}(\mathfrak{sl}_n)$ can be realized as the following quotient of the k -fold tensor product*

$$V^{\otimes k} / \sum_{i=0}^{k-2} V^{\otimes i} \otimes S_{r,s}^2(V) \otimes V^{\otimes(k-i-2)} \cong V(\bar{\omega}_k). \quad (3.5)$$

In Section 6 we will give an alternative way to construct all irreducible modules using the Yang-Baxter equation.

4. The Hecke Algebra and the Schur-Weyl Duality for $U_{r,s}(\mathfrak{gl}_n)$

For any $r, s \in \mathbb{C} \setminus \{0\}$, we introduce the Hecke algebra $H_m(r, s)$ as follows.

Definition 4.1. The Hecke algebra $H_m(r, s)$ is the unital associative algebra over \mathbb{C} with generators T_i , $1 \leq i < m$, subject to the following relations:

$$(H1): T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i < m - 1,$$

$$(H2): T_i T_j = T_j T_i, \quad |i - j| > 1,$$

$$(H3): (T_i - 1)(T_i + \frac{r}{s}) = 0.$$

Remark 4.2. When $r \neq 0$, the elements $t_i = \sqrt{\frac{s}{r}} T_i$ satisfy

$$(H3'): (t_i - \sqrt{\frac{s}{r}})(t_i + \sqrt{\frac{r}{s}}) = 0.$$

If we set $q = \sqrt{\frac{s}{r}}$, the two parameter Hecke algebra $H_m(r, s)$ is isomorphic to $H_m(q)$, the Hecke algebra associated to the symmetric group \mathfrak{S}_m . By a well-known result of Hecke algebras $H_m(r, s)$ is semisimple whenever $\sqrt{\frac{s}{r}}$ is not a root of unity.

From Section 2, it is easy to verify that the $U_{r,s}(\mathfrak{gl}_n)$ -module $V^{\otimes m}$ affords a representation of Hecke algebra $H_m(r, s)$:

$$H_m(r, s) \rightarrow \text{End}_{U_{r,s}(\mathfrak{gl}_n)}(V^{\otimes m}) \quad (4.1)$$

$$T_i \mapsto \check{R}_i \quad (1 \leq i < m).$$

Benkart and Witherspoon [2] gave a two parameter analogue of the Schur-Weyl duality for $U_{r,s}(\mathfrak{sl}_n)$ and the Hecke algebra $H_{r,s}(\mathfrak{S}_k)$ associated with the symmetric group \mathfrak{S}_k , which we recall as follows.

Proposition 4.3 ([2]). *Assume rs^{-1} is not a root of unity. Then:*

(i) $H_m(r, s)$ maps surjectively onto $\text{End}_{U_{r,s}(\mathfrak{gl}_n)}(V^{\otimes m})$.

(ii) if $n \geq m$, $H_m(r, s)$ is isomorphic to $\text{End}_{U_{r,s}(\mathfrak{gl}_n)}(V^{\otimes m})$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition, where $l = l(\lambda)$ is the length.

Corollary 4.4. *When rs^{-1} is not a root of unity, as an $(U_{r,s}(\mathfrak{gl}_n) \otimes H_m(r, s))$ -module, the space $V^{\otimes m}$ has the following decomposition:*

$$V^{\otimes m} \cong \bigoplus_{\lambda} V_{\lambda} \otimes V^{\lambda},$$

where the partition λ of m runs over the set of partition such that $l(\lambda) \leq n$, V_{λ} is the $U_{r,s}(\mathfrak{gl}_n)$ -module associated to λ , and V^{λ} is the $H_m(r, s)$ -module corresponding to λ .

Remark 4.5. The above theorem is a two-parameter version of the well-known result of Jimbo [13].

In section 6 we will give detailed information on the idempotents, which will then give a realization of all irreducible modules.

5. The Orthogonal Primitive Idempotents of $H_m(r, s)$

For any index $i = 1, \dots, m-1$, let $s_i = (i, i+1)$ be the adjacent transposition in the symmetric group \mathfrak{S}_m . Take any element $\sigma \in \mathfrak{S}_m$ and choose a reduced decomposition $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$. Denote $T_{\sigma} = T_{i_1} T_{i_2} \cdots T_{i_k}$, this element in $H_m(r, s)$ does not depend on the reduced decompositions of σ [11].

The Jucys-Murphy elements of $H_m(r, s)$ are defined inductively by

$$y_1 = 1, \quad y_{k+1} = \frac{s}{r} T_k y_k T_k, \quad (5.1)$$

where $k = 1, \dots, m-1$. These elements satisfy

$$y_k T_l = T_l y_k, \quad l \neq k, k-1.$$

In particular, y_1, y_2, \dots, y_m generate a commutative subalgebra of $H_m(r, s)$. For any $k = 1, \dots, m$, we let w_k denote the unique longest element of symmetric group \mathfrak{S}_k , which is regarded as a natural subgroup of \mathfrak{S}_m . The corresponding elements $T_{w_k} \in H_m(r, s)$ are given by $T_{w_1} = 1$ and

$$T_{w_k} = T_1(T_2 T_1) \cdots (T_{k-2} T_{k-3} \cdots T_1)(T_{k-1} T_{k-2} \cdots T_1)$$

$$= (T_1 \cdots T_{k-2} T_{k-1})(T_1 \cdots T_{k-3} T_{k-2}) \cdots (T_1 T_2) T_1, \quad k = 2, \dots, m. \quad (5.2)$$

It can be verified easily that

$$T_{w_k} T_j = T_{k-j} T_{w_k}, \quad 1 \leq j < k \leq m, \quad (5.3)$$

$$T_{w_k}^2 = \left(\frac{r}{s}\right)^{\frac{k(k-1)}{2}} y_1 y_2 \cdots y_k, \quad k = 1, \dots, m. \quad (5.4)$$

For each $i = 1, \dots, m-1$, we define the elements [18]:

$$T_i(x, y) = sT_i + \frac{s-r}{\frac{y}{x} - 1}, \quad (5.5)$$

where x, y are complex variables. We will regard $T_i(x, y)$ as a rational functions in x, y with values in $H_m(r, s)$. These functions satisfy the braid relations:

$$T_i(x, y) T_{i+1}(x, z) T_i(y, z) = T_{i+1}(y, z) T_i(x, z) T_{i+1}(x, y). \quad (5.6)$$

We will identify a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of m with its Young diagram. The Young diagram is a left-justified array of rows of cells such that the first row contains λ_1 cells, the second row contains λ_2 cells, etc. A cell outside λ is called addable to λ if the union of λ and the cell is a (proper) Young diagram. A λ -tableau is obtained by filling in the cells of the Young diagram bijectively with the numbers $1, \dots, m$. A tableau T is called standard if the entries of the tableau increase along the rows and down the columns. If a cell occurs in the (i, j) -th position, its (r, s) -content will be defined as $(\frac{s}{r})^{j-i}$. Let σ_k denote the (r, s) -content of the cell occupied by k in T .

Example 5.1. For $\lambda = (2, 1)$, the corresponding Young diagram is



The cells at the positions $(1, 3)$, $(2, 2)$, $(3, 1)$ are the addable cells. For the λ -tableau $T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, its (r, s) -contents are $\sigma_1 = 1$, $\sigma_2 = \frac{r}{s}$, $\sigma_3 = \frac{s}{r}$.

A set of primitive idempotents of $H_m(r, s)$ parameterized by partitions λ of m and the standard λ -tableaux T can now be defined inductively by

the following rule [5]. Set $E_T^\lambda = 1$ if $m = 1$, whereas for $m \geq 2$, one defines inductively

$$E_T^\lambda = E_U^\mu \frac{(y_m - \rho_1) \cdots (y_m - \rho_k)}{(\sigma - \rho_1) \cdots (\sigma - \rho_k)}, \quad (5.7)$$

where U is the tableau obtained from T by removing the cell α occupied by m , μ is the shape of U , and ρ_1, \dots, ρ_k are the (r,s) -contents of all the addable cells of μ except for α , while σ is the (r,s) -content of the latter.

These elements satisfy the characteristic property that if λ and λ' are partitions of m

$$E_T^\lambda E_{T'}^{\lambda'} = \delta_{\lambda, \lambda'} \delta_{T, T'} E_T^\lambda \quad (5.8)$$

for arbitrary standard tableaux T and T' of shapes λ and λ' respectively. Moreover,

$$\sum_{\lambda} \sum_T E_T^\lambda = 1, \quad (5.9)$$

summed over all partitions λ of m and all the standard λ -tableaux T .

6. Fusion Formulas for the Orthogonal Primitive Idempotents of

$$H_m(r, s)$$

We now apply the fusion formulas [11] to the situation of the two-parameter quantum algebra to derive a corresponding formula for the idempotents of $H_m(r, s)$, which can then be used to construct all the irreducible $U_{r,s}(\mathfrak{sl}_n)$ -modules.

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of m , the conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_l)$ of λ the partition of m whose diagram obtained by reflection in the main diagonal. Hence λ'_i is the number of nodes in the i th column of λ . Define

$$b(\lambda) = \sum_{i \geq 1} \binom{\lambda'_i}{2}.$$

If $\alpha = (i, j)$ is a cell of λ , then the corresponding hook is defined as $h_\alpha = \lambda_i + \lambda'_j - i - j + 1$.

Now we introduce the rational function in complex variables u_1, \dots, u_m with values in $H_m(r, s)$,

$$\Psi(u_1, \dots, u_m) = \prod_{k=1, \dots, m-1} (T_k(u_1, u_{k+1})T_{k-1}(u_2, u_{k+1}) \cdots T_1(u_k, u_{k+1})) \cdot T_{w_m}^{-1},$$

where the product is carried out in the order of $k = 1, \dots, m - 1$. The following theorem is obtained by a similar argument as in the one-parameter case [11].

Theorem 6.1. *For the partition λ of m and a standard λ -tableau T , the idempotents E_T^λ can be obtained by the consecutive evaluations*

$$E_T^\lambda = f(\lambda)\Psi(u_1, \dots, u_m)|_{u_1=\sigma_1}|_{u_2=\sigma_2} \cdots |_{u_m=\sigma_m}, \quad (6.1)$$

where

$$f(\lambda) = \left(\frac{s}{r}\right)^{b(\lambda')} s^{-\binom{m}{2}} \left(1 - \frac{s}{r}\right)^m \prod_{\alpha \in \lambda} \left(1 - \left(\frac{s}{r}\right)^{h_\alpha}\right)^{-1}.$$

Example 6.2. For $m = 2$ and $\lambda = (2)$ we get

$$E_T^\lambda = f(\lambda)(sT_1 + r),$$

where $f(\lambda) = \frac{1}{r+s}$, for the standard tableau $T = \boxed{12}$. In particular, $\sigma_1 = 1$, $\sigma_2 = \frac{s}{r}$.

Example 6.3. For $m = 2$ and $\lambda = (1, 1)$, we get

$$E_T^\lambda = f(\lambda) \frac{s^2}{r} (1 - T_1),$$

where $f(\lambda) = \frac{r}{s(r+s)}$, for the standard tableau $T = \boxed{1 \atop 2}$. In particular, $\sigma_1 = 1$, $\sigma_2 = \frac{r}{s}$.

Example 6.4. For $m = 3$, $\lambda = (1, 1, 1)$, $T = \boxed{1 \atop 2 \atop 3}$, we have

$$\Psi(u_1, u_2, u_3)|_{u_1=\sigma_1}|_{u_2=\sigma_2}|_{u_3=\sigma_3} = \frac{s^6}{r^3} (1 - T_1 - T_2 + T_1T_2 + T_2T_1 - T_1T_2T_1),$$

and $f(\lambda) = \frac{r^3}{(s+r)(s^2+rs+r^2)s^3}$. Thus we get

$$E_T^\lambda = \frac{s^3}{(s+r)(s^2+rs+r^2)}(1 - T_1 - T_2 + T_1T_2 + T_2T_1 - T_1T_2T_1),$$

which is the same as that in formula (5.7).

From Corollary 4.4, we have

$$E_T^\lambda(V^{\otimes m}) \cong E_T^\lambda\left(\bigoplus_{\mu} V_{\mu} \otimes V^{\mu}\right).$$

Since E_T^λ annihilates the irreducible $H_m(r, s)$ -module V^λ , we have

$$E_T^\lambda(V^{\otimes m}) \cong E_T^\lambda\left(\bigoplus_{\mu} V_{\mu} \otimes V^{\mu}\right) \cong V_{\lambda} \otimes E_T^\lambda V^{\lambda}.$$

Furthermore, since E_T^λ acts on the irreducible module V^λ of $H_m(r, s)$ as a projector on a 1-dimensional subspace, we can get the following explicit description of the irreducible modules of $U_{r,s}(\mathfrak{sl}_n)$.

Theorem 6.5. *For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of m with length $l \leq n$ and T a standard λ -tableau,*

$$V(\lambda) = E_T^\lambda(V^{\otimes m})$$

is the finite dimensional irreducible representation of $U_{r,s}(\mathfrak{sl}_n)$ with the highest weight $\sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\bar{\omega}_i$.

Remark 6.6. We can see that Proposition 3.3 is a special case of Theorem 6.5. Actually, it is easy to check that $\check{R}(1, r^{-1}s) = s(s\check{R} + r)$. While from Example 6.2 for $\lambda = (2)$, we have

$$E_T^\lambda = \frac{1}{r-s}(sT_1 + r).$$

Thus we have $S_{r,s}^2(V) = \check{R}(1, r^{-1}s)V^{\otimes 2} = E_T^{(2)}(V^{\otimes 2})$. Similarly, using Example 6.3 we have $\Lambda_{r,s}^2(V) = \check{R}(1, rs^{-1})V^{\otimes 2} = E_T^{(1,1)}(V^{\otimes 2})$.

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