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CONJUGACY OF EMBEDDINGS OF ALTERNATING GROUPS IN EXCEPTIONAL LIE GROUPS

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Abstract

We discuss conjugacy classes of embeddings of Alternating groups in Exceptional Lie groups. We settle the count of classes of embeddings in E_8 of a subgroup Alt_{10} and its double cover. This involves computation and the reduction of the problems to relative eigenvector problems. We update previously published tables of embeddings. We comment on the improvements present in our table and on the remaining unsettled conjugacy questions.

1. Introduction

In their seminal 1987 paper [1], Arjeh Cohen and Bob Griess set out a program to classify the finite simple subgroups that embed in exceptional Lie groups. They reduced the problem to an investigation of the status of a small set of candidate subgroups. Over the next 15 years, these cases were resolved by a number of group theorists in [2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22, 23, 32]. Eventually, [16] gave a table with a complete list of subgroups. This table also contains partial information about the important matter of conjugacy classes of embeddings. Relatively little has been done to settle remaining conjugacy questions since the publication of [16]. The only new results [14, 25] concern conjugacy of embeddings of alternating groups.

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In this paper, we summarize this progress and deal with the new cases of embeddings of Alt_{10} and its double cover into E_8.

Table 1 updates the part of Table QE of [16] that relates to alternating subgroups. The notation matches that of the old table, and is explained in Section 2. Those entries marked with a † represent improvements to the old table, while those marked ‡ are improvements due to new results in this paper. Any entry that contains an inequality represents a conjugacy question that remains open.

Table 1: Projective Embeddings of Alternating groups in Exceptional Lie groups.

z	Finite Simple Group	$G_2(\mathbb{C})$	$F_4(\mathbb{C})$	$3E_6(\mathbb{C})$	$2E_7(\mathbb{C})$	$E_8(\mathbb{C})$
1	Alt_5	4	13(8)	15(10)	19(12)	31(19)†
2		4	12(21)	18(32)	96(51)†	103(58)†
1	Alt_6	0	$\geq 5(\geq 3)$ † Z	$\geq 10(\geq 6)$ † Z	$\geq 12(\geq 7)$ † S	$\geq 18(\geq 11)$†
2		0	2(1)†	3(2)†	3(2)†	17(9)†
3		$2(1); 3A_2$	8(4)†	22(9)†	22(12)†	$\geq 34(\geq 17)$ † Z
6		0	0	4(1) † $6A_5$	12(4)†	12(4)†
1	Alt_7	0	0	1(1) † $6A_5$	1(1) † 2^2D_6	$\geq 2(\geq 2)$ † Z
2		0	1(1) † $4A_3$	1(1)†	1(1)†	4(4)†
3		0	0	2(1) † $6A_5$	2(2)†	1(1)†
6		0	0	4(1) † $6A_5$	4(2)†	2(1)†
1	Alt_8	0	0	0	1(1) † $4A_7$	1(1)†
2		0	1(1)†	1(1)†	1(1)†	3(3)†
1	Alt_9	0	0	0	3(2) † $4A_7$	4(3)†
2		0	1(1)†	1(1)†	1(1)†	$\geq 4(\geq 3)$†
1	Alt_{10}	0	0	0	0	1(1) ‡ $3A_8$
2		0	1(1)†	1(1)†	1(1)†	2(2) ‡ $2D_8$
	Alt_n					
2	$n=11,\ldots,17$	0	0	1(1) † $n \leq 11$	1(1) † $n \leq 13$	1(1) † $n \leq 17; 2D_8$

An embedding $G \leq E$ of a quasisimple group G in a Lie group E is called *Lie imprimitive* if it factors through a smaller Lie group. Other embeddings are *Lie primitive*. We recall that there is a chain of natural embeddings $G_2 < F_4 < 3E_6 < 2E_7 < E_8$. Hence, a projective embedding into one of them is a projective embedding into E_8. This means that although [1] stressed the problem of embeddings into E_8, the possibility of imprimitive embeddings that factor through a smaller exceptional group forced consideration of embeddings into all of the exceptional Lie groups. Moreover, the central extensions that appear in the chain of exceptional groups naturally leads us to consider projective embeddings rather than just embeddings. Most of the difficult questions that remained after [1] concerned the status of finite groups that could only embed Lie primitively into one of the exceptional groups. Other imprimitive cases reduce to a question of an embedding into a classical Lie group.

The majority of questions about Lie primitive embeddings were resolved by exhaustive computer search. (Indeed, for cases such as the embeddings of 2-dimensional linear groups given in [32], where a machine free argument exists, alternative computational solutions are also available.) Various computational strategies have been applied, but all ended with a classification up to conjugacy. In contrast, knowledge of just one imprimitive embedding of a group, settled its status for the project of [1], but did so without necessarily resolving conjugacy. In fact, in many cases where imprimitive embeddings are known there is still an interesting open question of whether there might also be primitive embeddings.

In the case of embeddings of Alt_{10} into E_8 that we treat in this paper, the possibility of a new Lie primitive embedding was left open in [16], but was ruled out in 4.4.4 of Litterick's thesis [24]. Our new analysis also rules out a Lie primitive embedding and resolves conjugacy of imprimitive embeddings, which was not treated in [24]. The computational methods developed for the Lie primitive cases seem appropriate for our present work. We chose to use the method introduced in [17], which in hindsight is by far the most convenient of the strategies applied. We discuss this method in Section 3.

We should also mention [9], the first in a series of papers by David Craven that attempt to severely narrow the possibilities for Lie primitive quasisimple finite subgroups of groups of exceptional Lie type in all characteristics. Litterick's thesis [24] is part of that larger effort.

2. Alternating Subgroups of Exceptional Lie Groups

Since the publication of [16], progress has been made on the conjugacy problem in the case of the alternating groups and their nonsplit covers. The most definitive conjugacy results for alternating groups in [16, Table QE] concern embeddings of Alt_5 and its double cover $SL(2,5)$. This summarizes the work of [10], [11], [12] and [13]. The rows in [16, Table QE] that cover these cases contain the following entries:

z	G	$G_2(\mathbb{C})$	$F_4(\mathbb{C})$	$3E_6(\mathbb{C})$	$2E_7(\mathbb{C})$	$E_8(\mathbb{C})$
1	Alt_5	4	13(8)	15(10)	19(12)	$\geq 31(\geq 19)Z$
2		4	12(21)	18(32)	$\geq 96(\geq 51)$	$\geq 103(\geq 58)S$

The first row (where $z = 1$) deals with the simple group Alt_5 and the second row its double cover. An entry such as the 13(8) in the F_4 column for Alt_5 means that there are 13 classes of injective homomorphisms $Alt_5 \to F_4(\mathbb{C})$ and that the images of these homomorphisms give 8 classes of subgroups in $F_4(\mathbb{C})$. In general, the number of classes of homomorphisms is at least as large as the number of classes of subgroups. In the case of embeddings of Alt_5 into G_2, the table indicates that the two counts agree and are both 4.

In the E_8-column of the first row of the table in [16] there was an inequality, which has been replaced by a definitive entry in our Table 1. The inequality represented an uncompleted conjugacy problem. The issue was the possibility of Lie primitive embeddings of Alt_5 with one particular fusion pattern (see Definition 1 below). Similarly, in $2E_7$, there was the possibility of a Lie primitive class of $2Alt_5$ subgroups which caused uncertainty in the entries in both the E_7 and E_8 columns. The letters Z and S appended to these uncertain entries summarize information about centralizers of possible extra embeddings — Z meaning zero-dimensional, and S small dimensional. The only additional unexplained notation that appears in our Table 1 is that certain entries are followed by the name of another Lie group. For example, there is an entry $2(1); 3A_2$ in the G_2 column for the row $3Alt_6$. This means that the two embeddings of $3Alt_6$ into $G_2(\mathbb{C})$ factor through an intermediate subgroup $3A_2$.

The open questions about embeddings of Alt_5 and its double cover were settled in 2003 by Lusztig in [25] using a method suggested by J.P. Serre. We now know there are no Lie primitive Alt_5 or $2Alt_5$ subgroups in any of the exceptional complex Lie groups.

In [14], the case of Alt_n and its nonsplit covers for $n \geq 6$ was addressed, leading to several improvements in Table QE. The methods used in [14] were similar to those used in [10], [11], [12] and [13]. But as with the cases of Alt_5 and $2.Alt_5$ in E_8, there were several unresolved cases left over from [14],

$\underline{E_8}$:

Alt_6: There are several unresolved cases here. In the case of Alt_6 Fusion Patterns 11 and 18, there are no known embeddings of groups with these fusion patterns, but since the centralizer is 0-dimensional, there could be Lie primitive embeddings. In the cases of Fusion Patterns 131 and 243 (see [14, Table 25]), we have embeddings in smaller Lie subgroups (in fact, by an argument due to Borovik, two nonconjugate embeddings in the case of fusion pattern 243), but because the centralizers are 0-dimensional, there could also be Lie primitive instances of groups with these fusion patterns.

Alt_7: There is a class with Fusion Pattern 1 which can be constructed in the $2D_8$ subgroup but has 0-dimensional centralizer. So there could also be Lie primitive classes. The character is $10_a + 10_b + 14_a^2 + 15^4 + 35^4$.

$2Alt_9$: Since the central involution has type 2B, we can assume this class is in the $2D_8$ subgroup (sharing the central involution of the $2D_8$ subgroup), and an instance of $2Alt_9$ with fusion pattern 1 can be constructed. However, we cannot say that this embedding is unique in $2D_8$, which is necessary to determine whether it is unique in E_8. The adjoint character for this embedding is $8 + 28^2 + 56 + 8_{a2} + 8_{b2} + 56_{a2}^2$. (The characters with a "2" in their subscript are the faithful ones. The others are Alt_9 characters.)

$3Alt_6$: Since there is a central element of order 3, there will be no Lie primitive embeddings, and in fact, all instances of this group in E_8 have a 3B central element. The centralizer of a 3B element has type $3A_2E_6$, so one is reduced to constructing $3Alt_6$ subgroups in this group. Each fusion pattern has one possible construction except there are four that might have more than one because they involve Alt_6 subgroups of $3E_6$ with Fusion Patterns 252 and 260 which may have Lie primitive classes. So this problem reduces to those two problems.

3. Computer Construction of Embeddings

In this section, we give an overview of a method to classify embeddings from a finite group G into a Lie group $E(\mathbb{C})$. This method was first applied computationally in [17], and was based on a computer free approach for constructing embeddings introduced in [29]. It was also used in [30], which contains full details of the procedure, where it appears as Algorithm 2.3. We begin with three observations, each of which changes the underlying field to a form slightly more suitable for machine computation.

In general, to compute embeddings into Lie groups, it is convenient to work over a field of finite characteristic rather than over the complex numbers. The change of field allows for exact arithmetic and gives us access to the standard computational tools of modular representation theory such as the Meataxe [26] and condensation [27, 28]. Larsen's powerful $0-p$ correspondence [18] gives the option to change field in this way. The correspondence is a bijection between conjugacy classes of embeddings of a finite group G into $E(\mathbb{C})$ and its embeddings into a Chevalley group $E(\overline{k})$, whenever $\overline{k}$ is an algebraically closed field with characteristic co-prime to $|G|$.

In order to count embeddings $G \leq E(\overline{k})$, we first count actions of G on the Lie algebra $\mathcal{E}$ of $E(\overline{k})$. (Our proof of Theorem 1 in Section 4 is an example of the sort of argument that can relate the two counts.) We begin with a natural action of G on some larger Lie algebra $\overline{\mathcal{L}}$ and search for all invariant subalgebras of type $\mathcal{E}$. Of course, we should try to ensure that $\overline{\mathcal{L}}$ is as small as possible, and Theorem 2.2 of [30] shows that a classical (i.e. linear, orthogonal or symplectic) Lie algebra defined on a minimal module for $\mathcal{E}$ serves as a good choice for $\overline{\mathcal{L}}$.

Our computational procedure to locate particular G-invariant subalgebras of $\overline{\mathcal{L}}$ begins from input that specifies the action of G on a natural module for $\overline{\mathcal{L}}$ and structure constants for the (Lie) multiplication of basis vectors in the algebra. This input data is a finite set of matrices, whose entries therefore lie in a finite subfield $k \leq \overline{k}$. In particular, the structure constants for the algebra define a G-invariant k-Lie algebra $\mathcal{L}$ with $\mathcal{L} \otimes \overline{k} = \overline{\mathcal{L}}$. In this way, we end up with a problem suitable for machine computation: we input a G-invariant Lie algebra $\mathcal{L}$ defined over a finite field k and determine G-invariant subalgebras of $\overline{\mathcal{L}}$ that meet appropriate other conditions.

The conditions that we shall impose to pick out desirable subalgebras of $\overline{\mathcal{L}}$ are representation theoretic and always have the following form:

> The subalgebra $\mathcal{E}$ contains copies of one or more particular (absolutely) irreducible G-modules $U, W, \ldots$ and is contained in a particular submodule $\overline{M} \leq \overline{\mathcal{L}}$.

For example, in Section 5 where $G = Alt_{10}$ and we seek G-invariant algebras of type E_8 whose adjoint character restricts to $9 + 35 + 36 + 84^2$ on G, we set $\overline{M}$ to be the sum of all irreducible constituents of $\overline{\mathcal{L}}$ with these characters and U and W to be the irreducibles with characters 9 and 35. These choices are clearly designed to catch all E_8-subalgebras with the required character. We aim to find all images of $U, W, \ldots$ in $\overline{\mathcal{L}}$ that can lie in an algebra with these properties. The subalgebras generated by such images will give us large pieces of the desired subalgebras of $\overline{\mathcal{L}}$ (and in the Lie primitive case, give us exactly the desired subalgebras).

For computational purposes $\overline{M}$ is specified by giving a basis inside $\overline{\mathcal{L}}$. It is possible that a finite extension of the field k might be required to specify this basis, in which case we just enlarge k as needed. We write M for the kG-module spanned by the basis. A further finite extension of k might be needed to accommodate one or more of the irreducibles $U, W, \ldots$. The potential need for multiple field extensions seems to have little computational significance. In practice we have rarely needed more than a quadratic extension of a prime field.

A precise statement of our computational problem in the case where we make use of two irreducible G-modules U and W follows. (It is easy to formulate very similar problems for any number of input modules, and our later strategy for finding solutions also applies similarly. An example of such modifications, for the case where we make use of just one irreducible W, appears in Section 4.)

Problem 1. Let U and W be kG-modules, let $\mathcal{L}$ be a G-invariant k-Lie algebra and M a G-submodule of $\mathcal{L}$. Determine all pairs $(h_U, h_W) \in Hom_{kG}(U, \mathcal{L}) \otimes \overline{k} \times Hom_{kG}(W, \mathcal{L}) \otimes \overline{k}$ with $[h_U(U), h_W(W)] \subset M \otimes \overline{k}$.

Let us write H_U and H_W for $Hom_{kG}(U, \mathcal{L}) \otimes \overline{k}$ and $Hom_{kG}(W, \mathcal{L}) \otimes \overline{k}$. Observe that the requirement that $[h_U(U), h_W(W)] \subset M \otimes \overline{k}$ is a bilinear condition on the pair (h_U, h_W). Therefore, as explained in detail in [30], the solutions to Problem 1 correspond to elements $h_U \otimes h_W$ in a computable subspace $N \leq H_U \otimes H_W$. Our strategy is to determine all pure tensors in this space. (A pure tensor is the tensor product of two vectors, rather than the more typical sum of tensor products.)

Let $a_1, \ldots, a_\ell$ be a basis of the dual space of H_W; these elements map $H_U \otimes H_W$ to H_U in the usual way. Any pure tensor x can be factored as a product $h_u \otimes h_w$ and it has proportional images under the maps $a_1, \dots a_\ell$ because: $xa_i = (h_w, a_i)h_u$. In this way, we reduce Problem 1 to an instance of:

Problem 2. (Relative Eigenvectors) Given a set of linear maps $a_i : X \to Y, 1 \leq i \leq \ell$ find all vectors x for which there exist a vector y and scalars $\lambda_1, \lambda_2, \dots, \lambda_\ell$ such that $xa_1 = \lambda_1 y, xa_2 = \lambda_2 y, \dots, xa_k = \lambda_k y$.

Equivalently, rephrasing this in terms of matrices gives the relative eigenvector problem described in [17] and [30].

Problem 3. (Relative Eigenvectors) Given a set of $r \times c$ matrices $A_1, A_2, \dots A_\ell$ find all *relative eigenvectors* x for which there exist a vector y and scalars $\lambda_1, \lambda_2, \dots, \lambda_\ell$ such that $xA_1 = \lambda_1 y, xA_2 = \lambda_2 y, \dots, xA_k = \lambda_k y$.

The special case where $r = c$, $\ell = 2$ and A_1 is the identity matrix is just the ordinary eigenvector problem. As in the ordinary eigenvector problem, we expect and allow for extension of scalars when seeking relative eigenvectors. (Of course, this requirement is implicit in the extension of scalars written into the formulation of Problem 1.)

In general, there is no efficient algorithm for the relative eigenvector problem; it is NP-hard [31]. However, some special cases of the problem are easy to solve. For example, if any one of the maps, a_1 say, has a right inverse r, then clearly relative eigenvectors are just common eigenvectors of $a_2r, a_3r, \dots, a_kr$. Instances of this case of the problem were obtained in all prior applications described in [17, 30].

Moreover, if $\ell = 2$, the Relative Eigenvector Problem has a well understood solution [33]. A recursive algorithm described in [31] calculates a set of subspaces whose union contains all relative eigenvectors. The subspaces are all spanned by relative eigenvectors and only one contains any vectors that are not relative eigenvectors. This gives an approach to the general problem: find spaces containing relative eigenvectors for several pairs of the matrices and compute intersections. Unfortunately, it is possible that some subspaces involved in this procedure might not be proper, so the strategy has no guarantee of complete success. However, it gives enough of a restriction on the relative eigenvectors to classify embeddings in the cases treated later in this paper. Although we will not make use of the possibility, we remark that the Relative Eigenvector Problem can always be reduced to a Gröbner Basis calculation.

4. Embeddings of $2Alt_{10}$ in E_8

As discussed in Section 2, we give the resolution of the conjugacy problem for $2Alt_{10}$ in E_8 with Fusion Pattern 1. To begin, we determine the possibilities for the restriction of the adjoint character for E_8 to a subgroup $G \cong 2Alt_{10}$ of E_8 with this fusion pattern. We follow [7] and look at feasible characters of G for E_8.

Definition 2. Suppose $E(\mathbb{C})$ is a complex Lie group, V is an irreducible module for $E(\mathbb{C})$ of degree n over $\mathbb{C}$, and μ the corresponding character for V. A character ϕ of degree n for a finite group $G \leq E(\mathbb{C})$ is said to be **feasible for** $E(\mathbb{C})$ if, for each $g \in G$, there exists $h \in E(\mathbb{C})$ such that $\phi(g) = \mu(h)$.

As in [7], we sometimes loosen this definition and call a character ϕ of G feasible for $E(\mathbb{C})$ if $\phi(g) = \mu(h)$ for some element orders in G.

Generally speaking, if the module for $E(\mathbb{C})$ is small enough and the irreducible characters for G are large enough, we find feasible characters by an exhaustive computer search through all possible n-dimensional characters of G, eliminating a possibility whenever its value at an element of G cannot match a corresponding value on $E(\mathbb{C})$.

If G is a $2Alt_{10}$ subgroup of E_8 with Fusion Pattern 1, this method shows that the only feasible character of G is $36+84+64_a+64_b$ (using Atlas [8] notation).

As shown in [14, §5.6], there is exactly one other conjugacy class of $2Alt_{10}$ subgroups in E_8. The adjoint character restricts to the character $1^{21}+9^7+36+16^8_{a2}$ for this second class. (The faithful characters are denoted with a "2" in the subscript. The other characters are Alt_{10} characters.) Our following computer analysis shows that there is also exactly one class of embeddings with Fusion Pattern 1. Together, these two classes account for our new entry of 2 in the $2Alt_{10}$-row of Table 1.

We now apply the method of Section 3 to classify embeddings $2Alt_{10} < E_8(\mathbb{C})$ that have adjoint character $36 + 84 + 64_a + 64_b$. In this section we fix the following choices for the objects described in Section 3: k is the finite field $\mathbb{Z}_{11}$ and $\overline{k}$ is its algebraic closure, G is $2Alt_{10}$, $\mathcal{L}$ is the Lie algebra of a non-singular G-invariant symmetric bilinear form on a 248-dimensional kG-module V with character $36 + 84 + 64_a + 64_b$. We remark that as a kG-module, $\mathcal{L}$ is isomorphic to the skew square of V — see [29]. We obtain an explicit representation of G on V in the usual way by applying the Meataxe to decompose tensor products of small kG-modules.

Let W be an irreducible submodule of V with character 64_a. An application of the Meataxe and condensation shows that $Hom_{kG}(W, \Lambda^2 V)$ is 8-dimensional and gives an explicit basis for this space. (In this context, explicit means that each basis element is a matrix with 64 rows and $247 \times 248/2$ columns. The large number of columns is the reason why condensation needs to be used along with the Meataxe.) As explained in Section 3, we pick a module M that contains all copies of $36, 64_a, 64_b$ and 84 in $\mathcal{L}$. Here, M is 2188-dimensional, its decomposition into irreducibles is $36^9 + 64_a^8 + 64_b^8 + 84^{10}$.

In this case, we work with just one irreducible module W, and adapt Problem 1 to use a map that turns an element $(h_w, h_{w'}) \in H_W \times H_W$ to the space $[h_w(W), h_{w'}(W)]$. Again, there is a computable subspace $N \leq H_W \otimes H_W$ consisting of tensors with images inside $M \otimes \overline{k}$. We need to search for particular elements of N that have the form $h_w \otimes h_w$. Since this element belongs to $S^2(H_W)$, we gain by using $N \cap S^2(H_W)$ in place of N when we pass, as in Section 3, to a relative eigenvector problem.

We compute $dim(N) = 21$ and $dim(N \cap S^2(H_W)) = 13$. This means we must solve a relative eigenvector problem with 8 matrices of size 13×8. Now, since $13 > 8$, none of the matrices can have a right inverse, and there is no straight reduction to an ordinary eigenvector problem. However, the strategy of repeatedly cutting the size of the solution space by finding relative eigenvectors for pairs of matrices is completely successful. It produces a set of four relative eigenvectors. Each has entries in a field F_{11^2}, a quadratic extension of k. Moreover, each relative eigenvector corresponds to a homomorphism that maps W to an image that does generate an algebra of type E_8.

We have now shown that $\overline{\mathcal{L}}$ has exactly four G-invariant subalgebras of type E_8. We further note that if X is an irreducible constituent of V and c_X is a linear transformation that acts as -1 on X and 1 on the other constituents then c_X is an automorphism of $\overline{\mathcal{L}}$ that centralizes G. We check by computation that c_{36} and c_{64a} transitively permute the four G-invariant copies of $\mathcal{E}$ in $\overline{\mathcal{L}}$. We now have:

Theorem 1. *There is exactly one conjugacy class of embeddings of* $2Alt_{10}$ *in* $E_8(\mathbb{C})$ *that have character* $36 + 84 + 64_a + 64_b$.

Proof. Let G be the group $2Alt_{10}$, and E be the group $E_8(\overline{k})$, where $\overline{k}$ is the algebraic closure of F_{11}. Let $\mathcal{E}$ be the Lie algebra of E, and Γ be the general linear group of invertible linear transformations of $\mathcal{E}$. Let $\langle\ ,\ \rangle$ be the Killing form of $\mathcal{E}$ and let Λ be the orthogonal subgroup of Γ that preserves this form. Let $\mathcal{L}$ be the Lie algebra of Λ. Suppose that θ_1 and θ_2 are injective homomorphisms from G to E, whose images have character $36 + 84 + 64_a + 64_b$. Let C_i be the centralizer in Λ of the image of G under θ_i, for $i = 1, 2$. Write V for the representation of G given by the map $\theta_1 : G \to E \leq \Gamma$.

Observe that although there are infinitely many G-invariant bilinear forms on V, these differ only by (independent) multiplication by scalar factors on each of the four irreducible constituents of V. These bilinear forms are all equivalent under the centralizer $C_\Gamma(g^{\theta_1})$ (because Γ consists of matrices over an algebraically closed field). Accordingly, V and $\langle\ ,\ \rangle$ are equivalent to the representation and bilinear form considered in our machine analysis. In particular, $\mathcal{L}$ contains exactly four G^{θ_1}-invariant subalgebras of type E_8.

There is an element $\alpha \in \Gamma$ with $\theta_2 = \theta_1^\alpha$, since the embeddings give representations of G with the same character. Moreover, α^{-1} transforms the G^{θ_2}- invariant form $\langle\ ,\ \rangle$ to a G^{θ_1}-invariant form, which as we noted is the image of $\langle\ ,\ \rangle$ under an element $\beta \in C_\Gamma(g^{\theta_1})$. Hence, $\gamma = \alpha^{-1}\beta^{-1}$ is an element of Γ that transforms θ_2 to θ_1 and belongs to Λ since it fixes $\langle\ ,\ \rangle$.

We now observe that $\mathcal{E}^\gamma$ is another G^{θ_1}-invariant subalgebra of $\mathcal{L}$ of type E_8. (It is G^{θ_1}-invariant because $\mathcal{E}$ is G^{θ_2}-invariant and γ transforms θ_2 to θ_1 and it is contained in $\mathcal{L}$ because $\gamma \in \Lambda$.) Hence, $\mathcal{E}^\gamma$ has the form $\mathcal{E}^c$ for some $c \in C_1$ (since it is one of the four invariant subalgebras of type E_8, all of which have this form according to our computation). Now let $\delta = \gamma c^{-1}$. This element transforms θ_2 to θ_1 and belongs to E since it acts on $\mathcal{E}$. In particular, θ_1 and θ_2 are E-conjugate.

We have now shown there there is just one class of embeddings from G to E, and the theorem follows by applying Larsen's $0-p$ correspondence. □

5. Embeddings of Alt_{10} in E_8

In this section, we discuss the resolution of the conjugacy problem for Alt_{10} in E_8. If G is such an Alt_{10} subgroup of E_8, we calculate the 248-dimensional characters for G which are feasible for E_8 by doing an exhaustive computer search of all 248-dimensional characters for G, eliminating those which do not give the exact values needed for an Alt_{10} subgroup of E_8. The only character that is feasible is $9+35+36+84^2$ (using Atlas [8] notation).

Our machine analysis in this case is designed to show that any embedding into E_8 factors through an intermediate subgroup $3A_8$. We did carry the computation further to obtain and classify embeddings into E_8, but a machine free argument can supply this additional information too.

It is convenient to work over the finite field $k = \mathbb{Z}_{11}$ and its algebraic closure $\overline{k}$. Here, G is Alt_{10}, and $\mathcal{L}$ is the Lie algebra of a non-singular G-invariant symmetric bilinear form on a 248-dimensional kG-module V with character $9+35+36+84^2$. Our computation requires an explicit choice of module and bilinear form. As usual, the module V is made from constituents constructed with the Meataxe and we select a form that decomposes as a direct sum of invariant bilinear forms on five irreducible constituents of V. We may and do choose identical G-invariant forms on the two 84-dimensional constituents. (When we extend scalars to $\overline{k}$ there is just one class of G-invariant form. This means we can safely choose any invariant form. However, certain other choices might require additional extensions to the scalars in the course of computation, and this is undesirable.) We write Λ for the orthogonal group that fixes the bilinear form and C for $C_\Lambda(G)$.

In our computation, we apply the procedure of Section 3 more than once. In each application we impose the condition that the product inside $\mathcal{L}$ of two unknown G-invariant modules only involves composition factors with characters $9, 35, 36$ and 84. The largest submodule $M \leq \mathcal{L}$ with these characters has dimension $2457 = 6\times 9 + 9\times 35 + 16\times 36 + 18\times 84$. We work with images of G-modules U and W with respective characters 9 and 35. The spaces H_U and H_W have dimensions 6 and 9, respectively. As in Section 4, we compute explicit bases for them using the Meataxe and condensation. If $h \in H_W$ and $h(W)$ belongs to a subalgebra of $\mathcal{L}$ with type E_8, then $[h(W), h(W)] \subset M$. As usual, this reduces to a relative eigenvector problem. In this case a partial solution to the problem shows that h lies in a particular 4-dimensional subspace $H_W^\dagger \leq H_W$.

Now, all pairs $(h_W, h_U) \in H_W^\dagger \times H_U$ for which $h_W(W)$ and $h_U(U)$ could lie in an E_8-subalgebra of $\mathcal{L}$ satisfy $[h_W(W), h_U(U)] \subset M$. Again this is a relative eigenvector problem. In this case, the solution shows that either h_W belongs to a particular 1-dimensional subspace $H_W^\ddagger \leq H_W^\dagger$ or h_U belongs to one of two particular 2-dimensional subspaces $H_U^\dagger$ and $H_U^\ddagger$ in H_U. Further, $H_U^\dagger$ and $H_U^\ddagger$ are interchanged by an element of C — this means we need only track solutions from one of these alternatives since solutions in the other follow in lock step.

The subspace $H_W^\ddagger(W)$ turns out to be barren, but instead of an intricate computation to verify that it cannot extend to an E_8-subalgebra, we just observe by computation that $H_W^\ddagger(W)$ generates a copy of A_8 in $\mathcal{L}$. Hence, even if it could extend to a copy of E_8, this copy would contain an invariant A_8-subalgebra.

We must now consider the other case, of an E_8-subalgebra in $\overline{\mathcal{L}}$ that is C-conjugate to one that contains the image of a non-trivial element of $H_U^\dagger$. Here, we compute that the subalgebra of $\mathcal{L}$ generated by $H_U^\dagger(U)$ has type $A_8 \oplus A_8$. (As it happens, it contains the A_8-subalgebra mentioned in the last paragraph.) Working over a quadratic extension of k, we obtain elements h_1 and h_2 in $H_U^\dagger$ for which the two images $h_i(U)$ generate commuting algebras of type A_8 in $\mathcal{L}$. Moreover, exactly two elements of the form $h = h_1 + xh_2$, with $0 \neq x \in \overline{k}$, have the property that $h(U)$ generates a copy of A_8. For all other values of x we get a subalgebra with at least two 9-dimensional Alt_{10}-constituents. (This is because the intersection of $[[h(U), h(U)], h(U)]$ with $H_U(U)$ depends cubically on x. The condition that this intersection should coincide with $h(U)$ is a cubic in x with a known root $x = 0$ and two other roots, which turn out to be mutually negative.) We have now established:

Theorem 2. *Any Alt_{10}-invariant E_8-subalgebra of $\mathcal{L}$ contains an Alt_{10}-invariant subalgebra of type A_8.*

We computed further properties of the four $H_U^\dagger$-elements h_1, h_2 and $h_1 \pm xh_2$ described above. Only the images of the last two extend to G-invariant copies of E_8 in $\mathcal{L}$. However, this machine computation is not needed here.

Observe that there are known embeddings $Alt_{10} \leq 3A_8(\mathbb{C})$ and $3A_8(\mathbb{C}) \leq E_8(\mathbb{C})$. Moreover, any two Alt_{10} subgroups of the $3A_8$ subgroup are conjugate, because there is only one nontrivial 9-dimensional character for Alt_{10}. Further, since our machine calculation shows that any Alt_{10} subgroup of E_8 is conjugate to a subgroup of the $3A_8$ subgroup of E_8, we apply [11, Remark 4.14] to Alt_{10} to see that any two Alt_{10} subgroups of E_8 are conjugate in E_8.

A final computation shows that each of the A_8-subalgebras that extend to copies of E_8 contains a G-invariant subalgebra with character 36 and type B_4. This also follows from the observation that the embedding into E_8 can be realized through an intermediate subgroup $2D_8$. Accordingly, any G-invariant algebra of type E_8 must contain G-invariant subalgebras with types A_8 and D_8, with respective characters $9+35+36$ and $36+84$. The intersection of these subalgebras is a G-invariant subalgebra which must have character 36 (since 36 has multiplicity 1 in the character of the embedding of G in E_8).

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