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HOPF-COLE TRANSFORMATION

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Abstract

The Hopf-Cole transformation turning the strongly nonlinear Burgers equation into the linear heat equation plays an important role in the development of mathematical sciences. In this article the transformation is viewed from historical perspective. Some open problems concerning the application of the Hopf-Cole transformation are also raised.

1. Introduction

The Hopf -Cole transformation, named after Eberhard Hopf, [18], and Julian D. Cole, [6], is to transform the *strongly nonlinear* Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \kappa u_{xx}$$

to the *linear* heat equation

$$\phi_t = \kappa\phi_{xx}.$$

It is striking that a nonlinear equation can be transformed *exactly* into a linear equation. This realisation has motivated further research in mathematical sciences not necessarily directly related to gas dynamics, which motivated the study of the Burgers equation. Research directly related to the transformation has also been substantial. We will illustrate these with concrete examples. We conclude with some historical perspectives on the

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Hopf-Cole transformation and Burgers equation, and raise some open problems.

The Hopf-Cole transformation is performed in steps. First, form the Hamilton-Jacobi equation for the antiderivative U,

$$U_t + \frac{1}{2}(U_x)^2 = \kappa U_{xx},\ U_x = u.$$

Then introduce the *Hopf-Cole relation*

$$U(x,t) = -2\kappa \log[\phi(x,t)]$$

and the Hamilton-Jacobi equation for $U(x,t)$ becomes the heat equation for the new function $\phi(x,t)$:

$$\phi_t = \kappa\phi_{xx}.$$

The general solution to the initial value problem with initial value $\phi(x,0)$ is given as convolution of the initial value with the heat kernel $H(x,t)$:

$$\begin{cases} \phi(x,t) = \displaystyle\int_{-\infty}^{\infty} H(x-y,t)\phi(y,0)dy, \\ H(x,t) \equiv \dfrac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}. \end{cases}$$

Normalize the anti-derivative $U(x,t)$ by $U(0,t) = 0$ and the above formula for $\phi(x,t)$ yields the solution formula for the initial value problem for the Hamilton-Jacobi equation:

$$U(x,t) = -2\kappa \log\Big[\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0_-}^{y} u(z,0)dz} dy\Big].$$

Finally, the formula for the solution $u = U_x$ of the Burgers equation is:

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0-}^{y} u(z,0)dz} dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0-}^{y} u(z,0)dz} dy}.$$

The Hopf-Cole transformation therefore gives an explicit representation of the solution to the general initial value problem for the Burgers equation.

2. Linearized Hopf-Cole Transformation

The Hopf-Cole transformation has a straightforward linearized version. Consider a solution $\bar{u}(x,t)$ of the Burgers equation

$$\begin{cases} \bar{u}_t + \left(\dfrac{\bar{u}^2}{2}\right)_x = \kappa \bar{u}_{xx} \\ \bar{U}_x = \bar{u},\ \bar{U}(x,t) = -2\kappa \log[\bar{\phi}(x,t)] \end{cases}$$

and the Burgers equation linearized around $\bar{u}$:

$$v_t + (\bar{u}v)_x = \kappa v_{xx}.$$

The Hopf-Cole transformation can be applied to solve the linearized Burgers equation. This is done by following the above steps, with the original Hopf-Cole relation

$$V + \bar{U} = -2\kappa \log[\bar{\phi} + \zeta].$$

replaced by the *linearized Hopf-Cole relation*:

$$V = -2\kappa \frac{\zeta}{\bar{\phi}}.$$

The above steps lead also to the heat equation for the perturbed function ζ:

$$\zeta_t = \kappa \zeta_{xx},$$

which is solved to yield the solution of the initial value problem for the linearized Burgers equation:

$$\begin{cases} V(x,t) = \dfrac{\int_{-\infty}^{\infty} [\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y,0) V(y,0)] dy}{\bar{\phi}(x,t)}, \\ \zeta(x,t) = -\dfrac{1}{2\kappa} \displaystyle\int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y,0) V(y,0)\Big] dy, \\ v(x,t) = V_x(x,t). \end{cases}$$

3. Hopf Equation

The explicit formula for the solutions of the Burgers equation yields a formula for solving the inviscid equation, termed the Hopf equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

by letting the viscosity to tend to zero $\kappa \to 0_+$. One may attempt to solve the Hopf equation by the method of characteristics:

$$\frac{d}{dt}u(x(t),t) = 0,\ \frac{d}{dt}x(t) = u(x(t),t).$$

However, as is well-known, the *characteristics* $x = x(t)$ may coalesce and *shock waves* develop, so the initial value problem cannot be solved by the characteristic method. The Hopf-Cole transformation offers an explicit formula for the solution of the initial value problem for the Hopf equation by taking the *zero dissipation* limit $\kappa \to 0_+$ of the explicit expression for the Burgers solution. For the explicit dependence on the viscosity, write the solution formula as:

$$\begin{cases} u(x,t) = u(x,t;\kappa) = \dfrac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{1}{2\kappa}F(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\kappa}F(x,y,t)}dy}, \\ F(x,y,t) = \dfrac{(x-y)^2}{2t} + \displaystyle\int_{0_-}^{y} u(z,0)dz. \end{cases}$$

As $\kappa \to 0_+$, the method of steepest descent yields the limiting solution

$$\lim_{\kappa \to 0_+} u(x,t,\kappa) = \frac{x-\xi}{t},$$

here $y = \xi$ is where the location the function $F(x,y,t)$ assumes its minimum value. As $F_y(x,y,t) = \frac{y-x}{t} + u(y,0)$, we also have

$$\lim_{\kappa \to 0_+} u(x,t,\kappa) = u(\xi,0),\ \min_y F(x,y,t) = F(x,\xi,t).$$

This gives simple geometric construction of the solution. The initial information $u(\xi,0)$ propagates along the *characteristic line* $x = \xi + u(\xi,0)t$ to reach the location (x,t). This is consistent with the characteristics method. However, which initial information reaches the given location (x,t) requires global consideration of the minimum of $F(x,y,t)$. When the minimum is attained at more than one location, it represents a discontinuity, a shock wave in the inviscid solution. In general, a smooth initial data can give rise to infinite, even dense set of shock waves. The Hopf's solutions offer a concrete example to the notion of weak solutions.

4. Nonlinearity

One measure of the basic importance of the Burgers equation is that the equation contains the critical nonlinearity. This can be seen by considering the simplest situation, the dissipation around a constant state. Take the base state to be zero and consider a compact supported perturbation:

$$u(x,0)=0 \text{ for } |x|>M$$

for some positive constant M. Due to *dissipation term* u_{xx}, the solution tends to zero time-asymptotically,

$$u(x,t)\to 0, \text{ as } t\to\infty.$$

With the linear dissipative equation, the heat equation, the dissipation is governed by a multiple of the heat kernel

$$u(x,t)\to AH(x,t)=\frac{A}{\sqrt{4\pi\kappa t}}e^{-\frac{x^2}{4\kappa t}}, \text{ as } t\to\infty,\ A=\int_{-\infty}^{\infty}u(x,0)dx.$$

In $L_1(x)$ norm, we have

$$\int_{-\infty}^{\infty}|u(x,t)-AH(x,t)|dx=O(1)t^{-\frac{1}{2}}, \text{ as } t\to\infty.$$

We expect the Burgers solution to have similar decaying rates as heat kernel:

$$\left.\begin{array}{llll} u(x,t) &= O(1)t^{-\frac{1}{2}}, & u_t(x,t) &= O(1)t^{-\frac{3}{2}}, \\ u_x(x,t) &= O(1)t^{-1}, & u_{xx}(x,t) &= O(1)t^{-\frac{3}{2}}, \end{array}\right. \quad \text{as } t\to\infty.$$

From this, we see that the *nonlinear term* uu_x has the same dissipation rate as the other terms in the Burgers equation:

$$uu_x(x,t)=O(1)t^{\frac{1}{2}}t^{-1}=O(1)t^{-\frac{3}{2}}\cong u_t(x,t)\cong u_{xx}(x,t), \text{ as } t\to\infty.$$

In other words, the dissipation phenomenon of Burgers solutions cannot be governed by the heat equation. Instead, one needs to consider the *Burgers* *kernel* $b(x,t)=b(x,t;A)$:

$$\begin{cases} b_t+\left(\dfrac{b^2}{2}\right)_x=\kappa b_{xx}, \\ b(x,0)=A\delta(x). \end{cases}$$

The Hopf-Cole transformation yields the explicit expression of the Burgers kernel

$$b(x,t;A)=\frac{\frac{\sqrt{\kappa}}{\sqrt{t}}(e^{\frac{A}{2\kappa}}-1)e^{-\frac{x^2}{4\kappa t}}}{\sqrt{\pi}+\int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty}(e^{\frac{A}{2\kappa}}-1)e^{-y^2}dy}.$$

The heat kernel and the Burgers kernel distribute the mass differently. We compare $b(x,t;A)$ with $Ah(x,t)$ as they carry the same mass;

$$\int_{-\infty}^{\infty}(b(x,t;A)-Ah(x,t))dx=0,\ t\geq 0.$$

The Burgers kernel is non-symmetric with respect to x-axis; while the heat kernel is symmetric with respect to x-axis. For $A>0$ ($A<0$), the Burgers kernel has more mass for $x>0$ ($x<0$). From the explicit expressions we see that the two distributions of mass does not tend to zero time-asymptotically:

$$\int_{-\infty}^{\infty}|b(x,t;A)-Ah(x,t)|dx=O(1),\ \text{as } t\to\infty.$$

This is to be compared with the difference between the Burgers solution $u(x,t)$ with the associated Burgers kernel $b(x,t;A)$:

$$\int_{-\infty}^{\infty}|u(x,t)-b(x,t;A)|dx=O(1)t^{-\frac{1}{2}},\ \text{as } t\to\infty.$$

In contrast to the Burgers equation, for nonlinear equation

$$u_t+\left(\frac{u^p}{p}\right)_x=\kappa u_{xx},\ p>2,$$

the nonlinear term $u^{p-1}u_x$ decays at the higher rate than the other terms in the equation. In this case, the dissipation phenomenon can be accurately governed by the heat equation, ignoring the nonlinear term.

For general conservation law with convex flux

$$u_t + f(u)_x = \kappa u_{xx},\ \ f''(u) > 0,$$

the time-asymptotic state is governed by the Burgers equation. To see this we first perform a linear transform $x \to x - f'(0)t,\ u \to (f''(0))^{-1}u$ and then use the Taylor expansion to write the equation as

$$u_t + \left(\frac{u^2}{2}\right)_x = \kappa u_{xx} + (O(1)u^3)_x.$$

This equation can be accurately approximated, in time-asymptotic sense, by the Burgers equation as the truncation term $(O(1)u^3)_x$ decays at higher rate of t^{-2} than other terms in the equation. For instance, it can easily proved that the difference between the Burgers kernel and the solution decay at the rate of $t^{-1/2}$ in $L_1(x)$ norm.

There is another way of relating the general conservation law to the Burgers equation through the characteristic speed λ:

$$\lambda_t + \left(\frac{\lambda^2}{2}\right)_x = \kappa\lambda_{xx} - \kappa f'''(u)(u_x)^2,\ \ \lambda = f'(u).$$

The error term $-\kappa f'''(u)(u_x)^2$ decays at faster rates than the terms in the Burgers equation. For the hyperbolic conservation law

$$u_t + f(u)_x = 0,\ \ f''(u) > 0$$

this relation is exact:

$$\lambda_t + \left(\frac{\lambda^2}{2}\right)_x = 0,\ \ \lambda = f'(u).$$

However, this works only for smooth solutions. For weak solutions, the Rankine-Hugoniot conditions for the two equations are close only for shocks with small strength.

5. Metastable States

For the Burgers equation, we have seen that the large-time behavior of solutions of finite mass A is the Burgers kernel $b(x, t; A)$, the self-similar solutions. However, the solution goes through a *meta-stable state,* which depends on the initial data and the strength of the viscosity κ, before it approaches the time-asymptotic state $b(x,t;A)$. Consider the scaling $v(x,t)=u(A\alpha x,\alpha t)/A$ so that the Burgers equation with initial data of the magnitude A turns into the Burgers equation with different viscosity and with initial data of magnitude one:

$$\begin{cases} u_t+uu_x=\kappa u_{xx}\to v_t+vv_x=\mu v_{xx}, \\ v(x,t)=\dfrac{1}{A}u(A\alpha x,\alpha t),\ \mu=\dfrac{\kappa}{A^2\alpha},\ u(x,0)\cong A,\ v(x,0)\cong 1. \end{cases}$$

When the *effective Reynolds number* $R=A/\kappa$ is large, that is, the nonlinearity, measured by A, is large compared to the viscosity κ, the solution is accurately approximated by the solution of the inviscid Burgers equation, namely the Hopf equation for a long period of time. We see this by taking the parameter α to be large so that $\mu=1/(RA\alpha)$ is small and $v(x,t)$ is close to the Hopf solution for finite time t, or equivalently, the original solution $u(\cdot,\alpha t)$ is close to a Hopf solution for the time period of finite multiple of the large constant α, or for a period of the order of R.

This leads us to the consideration of the large time behavior of the solution $h(x,t)$ of the Hopf equation. Consider the scaling

$$\begin{cases} h_t+\left(\dfrac{h^2}{2}\right)_x=0, \\ h(x,0)=h_0(x); \end{cases} \Rightarrow \begin{cases} k_t+\left(\dfrac{k^2}{2}\right)_x=0,\ k(x,t)=\beta h(\beta x,\beta^2 t), \\ k(x,0)=\beta h_0(\beta x); \end{cases}$$

By taking $\beta\to\infty$, for initial data of finite mass A, the large time behavior of $h(x,t)$ is therefore dictated by the solution of

$$\begin{cases} w_t+\left(\dfrac{w^2}{2}\right)_x=0, \\ w(x,0)=A\delta(x). \end{cases} .$$

For nonlinear equation to have a distribution $A\delta(x)$ as initial data, an understanding is required. By $w(x,0)=A\delta(x)$, we mean that $w(x,t),\ t>0$, is a measurable function and is a weak solution in the sense of distribution and that

$$\lim_{t\to 0^+} w(x,t) = 0, \text{ for } x \neq 0, \int_{-\infty}^{\infty} w(x,t)dx = A, \text{ for } t > 0.$$

Strikingly, there are *infinite* many *N-wave solutions* $N(x,t;p,q)$ to this initial value problem:

$$w(x,t) = N(x,t;p,q) = \begin{cases} \dfrac{x}{t}, & \text{for } -\sqrt{-2pt} < x < \sqrt{2qt}, \\ 0, & \text{otherwise.} \end{cases}$$

The initial condition is satisfied in the above sense here:

$$\lim_{t\to 0^+} N(x,t;p,q) = 0, \text{ for } x \neq 0; \int_{-\infty}^{\infty} N(x,t;p,q)dx = p + q = A.$$

The solution depend on the constants p and q. The constants p and q have only the constraints

$$p \leq 0 \leq q, \ p + q = A;$$

otherwise are arbitrary.

The N-waves represent accurate approximation of the long time, but not time-asymptotic state of the viscous solutions. Thus, for the Burgers solutions, the N-waves represent *meta-stable* states. We therefore conclude that the two limits, time-asymptotic and zero dissipation limits, do not commute:

$$\lim_{\kappa\to 0}\lim_{t\to\infty} u(x,t;\kappa) \neq \lim_{t\to\infty}\lim_{\kappa\to 0} u(x,t;\kappa).$$

The first limit is $\lim_{\kappa\to 0} b(x,t;A)$ with one time invariant A; while the second limit is the N-wave with two time invariants p and q.

We have seen that the Burgers equation has a strong nonliinearity that is critical. There is another aspect that separates the Burgers and Hopf equation from the equations

$$\begin{aligned} u_t + (\frac{u^3}{3})_x &= \kappa u_{xx}, \\ u_t + (\frac{u^3}{3})_x &= 0. \end{aligned}$$

It turns out that the solution operator of this inviscid equation has only one time invariant. This can also be understood by studying

$$\begin{cases} u_t + \left(\dfrac{u^3}{3}\right)_x = 0, \\ u(x,0) = A\delta(x), \end{cases}$$

which, unlike the Burgers' case with two time invariants, has only one solution, expressed in terms of the characteristic speed u^2, [25]: For $A > 0$,

$$u^2(x,t) \;=\; \begin{cases} \dfrac{x}{t}, & \text{for } 0 < x < \left(\dfrac{3A}{2}\right)^{\frac{2}{3}} t^{\frac{1}{3}}, \\ 0, & \text{otherwise;} \end{cases}$$

for $A < 0$,

$$u^2(x,t) \;=\; \begin{cases} \dfrac{x}{t}, & \text{for } -\left(\dfrac{-3A}{2}\right)^{\frac{2}{3}} t^{\frac{1}{3}} < x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

6. Nonlinear Waves

In the study of gas dynamics, there are nonlinear waves, such as the shock and rarefaction waves. The Hopf-Cole transformation is crucial for the understanding of the interaction of these waves with the initial layer. This is so because these waves can be approximated by the corresponding Burgers waves, see next two sections.

To study the interaction of the initial layer with the nonlinear waves, we consider the *Burgers rarefaction wave* $b_R(x,t)$ and the *Burgers formation of shock wave* $u_S(x,t)$. These are solutions of the Burgers equation with *Riemann initial data* consisting of two constant states. Without loss of generality, take these waves to be symmetric with respect to x by choosing a positive constant $\lambda_0 > 0$ and considering $\pm\lambda_0$ as the initial values:

$$\begin{cases} (b_R)_t + b_R(b_R)_x = \kappa(b_R)_{xx}, \\ b_R(x,0) = \begin{cases} -\lambda_0, & \text{for } x < 0, \\ \lambda_0, & \text{for } x > 0; \end{cases} \end{cases}$$

$$\begin{cases} (u_S)_t + u_S(u_S)_x = \kappa(b_S)_{xx}, \\ u_S(x,0) = \begin{cases} \lambda_0, & \text{for } x < 0, \\ -\lambda_0, & \text{for } x > 0. \end{cases} \end{cases}$$

For the shock wave, it is interesting to study the process of the forming of the permanent shock profile $b_S(x)$, resulting from the *interaction of the shock and initial layers*:

$$b_S(x) = \lim_{t\to\infty} u_S(x,t).$$

The Hopf-Cole transformation yields

$$\begin{cases} u_s(x,t) = -\lambda_0 \frac{Erfc(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{\kappa}} Erfc(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}{Erfc(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{\kappa}} Erfc(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}, \\ b_S(x) = \lim_{t\to\infty} u_S(x,t) = -\lambda_0 \tanh\left(\frac{\lambda_0 x}{2\kappa}\right). \end{cases}$$

It follows from the above identities that the thickness of the initial layer is attained when the error functions $Erfc$ approach $\sqrt{\pi}$ for a fixed location x. This is so if $\lambda_0 t/\sqrt{4\kappa t}$ is larger than a given constant, or equivalently, the thickness T_0 of initial layer is given by

$$\frac{\lambda_0 T_0}{\sqrt{4\kappa T_0}} = O(1), \quad \text{or} \quad T_0 = O(1)\frac{\kappa}{(\lambda_0)^2}.$$

The thickness of the initial layer T_0 is large if the strength of the viscosity κ is large, an understandable fact as the viscosity delays the formation of the shock. T_0 is small when the shock strength λ_0 is large. This is because, for larger λ_0, the effect of nonlinearity is also larger and thereby accelerates the formation of the shock profile.

Next we study the rarefaction waves $b_R(x,t)$ for the Burgers equation. The rarefaction wave $h_R(x,t)$ for the Hopf equation:

$$\begin{cases} (h_R)_t + \Big(\dfrac{(h_R)^2}{2}\Big)_x = 0, \\ h_R(x,0) = \begin{cases} -\lambda_0, & \text{for } x < 0, \\ \lambda_0, & \text{for } x > 0; \end{cases} \end{cases}$$

is a centered rarefaction wave, found easily by characteristic method:

$$h_R(x,t) = \begin{cases} \dfrac{x}{t}, & \text{for } -\lambda_0 t < x < \lambda_0 t, \\ -\lambda_0, & \text{for } x < -\lambda_0 t, \\ \lambda_0, & \text{for } x > \lambda_0 t. \end{cases}$$

The rate of expansion in the wave region $-\lambda_0 t < x < \lambda_0 t$ is linear in time:

$$\frac{\partial}{\partial x} h_R(x,t) = \frac{1}{t}$$

The viscous rarefaction wave $b_R(x,t)$, found by Hopf-Cole transformation, is

$$b_R(x,t) = \lambda_0 \frac{e^{\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}{e^{\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}.$$

This expression indicates a rich wave structure. We only point out that in the region well within the inviscid rarefaction wave, the Burgers rarefaction wave is close to the inviscid wave x/t:

$$\begin{aligned} b_R(x,t) - \frac{x}{t} &= O(1)\Big[\frac{1}{|x-\lambda_0 t|} + \frac{1}{|x+\lambda_0 t|}\Big] \\ &\text{for } x \in (-\lambda_0 t + M\sqrt{4\kappa t}, \lambda_0 t - M\sqrt{4\kappa t}),\ t > O(1)\frac{\kappa}{(\lambda_0)^2}. \end{aligned}$$

Near the boundary of the inviscid rarefaction wave, the rate of expansion degenerates from $(b_R)_x = O(1)t^{-1}$ to the usual dissipation rate of $t^{-1/2}$. Dissipation dominates around the edges $x = \pm\lambda_0 t$ of the inviscid wave region as well as during the initial time period, $t < O(1)\dfrac{\kappa}{(\lambda_0)^2}$.

7. Green's Functions

To study the behavior of a Burgers solution around a nonlinear wave, the basic information is provided by the Green's function for the Burgers equation linearized around the nonlinear wave. For this, we use the linearized Hopf-Cole transformation mentioned before.

Consider the Green's function $G_S(x,t;x_0,t_0) = G_S(x,t;x_0,t-t_0)$ for the Burgers equation linearized around the shock profile $b_S(x)$:

$$\begin{cases} (G_S)_t + b_S(G_S)_x = \kappa(G_S)_{xx}, \\ G_S(x,0) = \delta(x-x_0); \end{cases}$$

Note that we have written above the Green's function for the anti-derivative of the Burgers equation linearized around the shock profile $b_S(x)$. This is done in order to eliminate the freedom of the phase shift of the shock profile. The linearized Hopf-Cole transformation yields

$$G_S(x,t;x_0) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-x_0)^2}{4\kappa t}} \frac{e^{\frac{\lambda_0 x_0}{2\kappa}} + e^{-\frac{\lambda_0 x_0}{2\kappa}}}{e^{-\frac{\lambda_0 x}{2\kappa}} + e^{-\frac{\lambda_0 x}{2\kappa}}} e^{\frac{(\lambda_0)^2 t}{4\kappa}}.$$

The Green's function can be rewritten as weighted combination of the heat kernel with speeds $\pm\lambda_0$:

$$G_S(x,t;x_0) = \frac{1+e^{-\frac{\lambda_0|x_0|}{\kappa}}}{1+e^{-\frac{\lambda_0|x|}{\kappa}}} \begin{cases} H(x+\lambda_0 t,t), & \text{for } x>0,\ x_0>0; \\ e^{-\frac{\lambda_0|x|}{\kappa}} H(x+\lambda_0 t,t), & \text{for } x<0,\ x_0>0; \\ H(x-\lambda_0 t,t), & \text{for } x<0,\ x_0<0; \\ e^{-\frac{\lambda_0|x|}{\kappa}} H(x-\lambda_0 t,t), & \text{for } x>0,\ x_0<0. \end{cases}$$

When both the source x_0 and the target x are positive,

$$G_S(x,t;x_0) \cong H(x+\lambda_0 t,t),\ x>0,\ x_0>0,$$

and so the propagation of the source is basically according to the heat kernel with negative speed $-\lambda_0$. Similarly, when both the source x_0 and the target x are negative,

$$G_S(x,t;x_0) \cong H(x-\lambda_0 t,t),\ x<0,\ ,x_0<0,$$

The above are consistent with propagation of information around the inviscid shock, along the characteristic lines with speed $-\lambda_0$ for $x > 0$ and λ_0 for $x < 0$:

$$h_S(x,t) = \begin{cases} -\lambda_0, & \text{for } x > 0, \\ \lambda_0, & \text{for } x < 0, \end{cases}$$

and so when the source and target are of the same sign, the propagation is governed by the dissipative version of the transport equation $u_t \pm \lambda_0 u_x = 0$. On the other hand, when the source x_0 and the target x are of different signs, the propagation speed is determined by the source, but, as the inviscid speed at the target is different, there is an exponentially decaying term in x:

$$G_S(x,t;x_0) \cong e^{-\frac{\lambda_0|x|}{\kappa}} H(x \pm \lambda_0 t), \text{ for } \pm x_0 > 0, \; xx_0 < 0.$$

The above exact analysis of the Green's function is the consequence of the application of the linearized Hopf-Cole transformation. The Hopf-Cole transformation is more crucially needed for the more subtle analysis of the wave propagation over a rarefaction wave. Consider the Green's function $G_R(x,t;x_0,t_0)$ for wave propagation over the rarefaction wave $b_R(x,t)$:

$$\begin{cases} (G_R)_t + b_R(G_R)_x = \kappa(G_R)_{xx}, \\ G_R(x,t_0;x_0,t_0) = \delta(x - x_0). \end{cases}$$

The linear Hopf-Cole transformation gives

$$\begin{aligned} G_R(x,t;x_0,t_0) \;=\; & e^{-\frac{(x-x_0)^2}{4\kappa(t-t_0)}} e^{-\frac{(\lambda_0)^2(t-t_0)}{4\kappa}} \\ & \times \frac{e^{-\frac{\lambda_0 x_0}{2\kappa}} Erfc(\frac{-x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}}) + e^{-\frac{\lambda_0 x_0}{2\kappa}} Erfc(\frac{x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}})}{e^{-\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) + e^{\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}. \end{aligned}$$

Unlike the shock profile $b_S(x)$, which has a permanent shape and is invariant in time, the rarefaction wave $b_R(x,t)$ evolves in time and so there is no time invariant, $G_R(x,t;x_0,t_0)$ and not $G_R(x,t-t_0;x_0)$. As we have seen, $b_R(x,t)$ incorporates both the linear hyperbolic expansion rate and the sub-linear dissipation rate. It is therefore interesting to understand the propagation over such a wave. The above exact form gives several scales for the behavior of the propagation. Take the case of propagation when both the target (x,t)

and the source (x_0, t_0) are both well inside the inviscid rarefaction wave and after the initial layer time $O(1)\kappa/(\lambda_0)^2$:

$$\begin{aligned} x &\in (-\lambda_0 t + M\sqrt{4\kappa t}, \lambda_0 t - M\sqrt{4\kappa t}), \\ x_0 &\in (-\lambda_0 t_0 + M\sqrt{4\kappa t_0}, \lambda_0 t - M\sqrt{4\kappa t_0}), \end{aligned} \qquad t > t_0 > O(1)\frac{\kappa}{(\lambda_0)^2}.$$

By direct calculations using the above exact form, we have the quantitative estimate of the Green's function $G_R(x, t; x_0, t_0)$:

$$G_R(x,t;x_0,t_0) \cong \frac{\frac{\sqrt{\kappa t_0}}{\lambda_0 t_0 + x_0} + \frac{\sqrt{\kappa t_0}}{\lambda_0 t_0 - x_0}}{\frac{\sqrt{\kappa t}}{\lambda_0 t + x} + \frac{\sqrt{\kappa t}}{\lambda_0 t - x}} \frac{e^{-\frac{t(x_0 - t_0 x/t)^2}{4\kappa t_0 (t-t_0)}}}{\sqrt{4\kappa t_0 (t - t_0)}}$$

We now analyze this expression. First, the propagation of waves is around the zero line of the exponential:

$$\frac{t(x_0 - t_0 x/t)^2}{4\kappa t_0 (t - t_0)} = 0, \text{ or, } x = \frac{t}{t_0} x_0,$$

that is, along the characteristic line through the source at (x_0, t_0) of the rarefaction wave according to the inviscid Hopf solution $h_R(x, t)$. The essential support of the information is in the region given by

$$\frac{t(x_0 - t_0 x/t)^2}{4\kappa t_0 (t - t_0)} = O(1), \text{ or } \left| x - \frac{t}{t_0} x_0 \right| = O(1)\sqrt{\kappa(t - t_0)\frac{t}{t_0}}.$$

Thus, for short and intermediate time propagation, $0 < t - t_0 < t/2$, the width of the support is

$$O(1)\sqrt{\kappa(t - t_0)\frac{t}{t_0}} = O(1)\sqrt{\kappa(t - t_0)},$$

for short and intermediate time, $0 < t - t_0 < t/2$.

For short and intermediate time, that is, the source time is close to the target time, the effect of the inviscid expansion of the rarefaction wave is not obvious and the width $O(1)\sqrt{\kappa(t - t_0)}$ is of the same order as the linear dissipation. On the other hand, for large time propagation,

$$O(1)\sqrt{\kappa(t - t_0)\frac{t}{t_0}} = O(1)\sqrt{\frac{\kappa}{t_0}}t, \text{ for large time, } \frac{t}{2} < t - t_0 < t,$$

the width is linear in time t, the same rate of expansion of the inviscid rarefaction wave. Notice that the strength of the source

$$\frac{\frac{\sqrt{\kappa t_0}}{\lambda_0 t_0+x_0}+\frac{\sqrt{\kappa t_0}}{\lambda_0 t_0-x_0}}{\frac{\sqrt{\kappa t}}{\lambda_0 t+x}+\frac{\sqrt{\kappa t}}{\lambda_0 t-x}}$$

varies significantly also. This has to be so as the $L_1(x)$ norm of the Green's function is kept as the constant one. As the source and target approach the boundary of the inviscid rarefaction wave

$$\frac{\lambda_0 t_0+x_0}{\sqrt{\kappa t_0}} \ll 1,\ \frac{\lambda_0 t_0-x_0}{\sqrt{\kappa t_0}} \ll 1,\ \frac{\lambda_0 t+x}{\sqrt{\kappa t}} \ll 1,\ \frac{\lambda_0 t-x}{\sqrt{\kappa t}} \ll 1,$$

the effect of diffusion becomes as important as the inviscid nonlinear expansion. The study of rarefaction waves is interesting even for scalar equation, [16], and richer wave phenomenon occurs for system, [26]. We emphasis that these rich wave phenomena can be studied only by resorting to the explicit expression obtained by Hopf-Cole transformation.

8. Hyperbolic Conservation Laws

We turn now to the system of equations in this and next sections and show that the theory for the Burgers and Hopf equations plays an essential role in the study of these systems. This section is concerned with systems of hyperbolic conservation laws. First we recall that a scalar convex conservation law

$$u_t + f(u)_x = 0,\ u \in \mathbb{R},\ f''(u) > 0,$$

is turned into Hopf equation by considering the *characteristic speed* $\lambda(u) = f'(u)$:

$$\lambda_t + \lambda\lambda_x = 0,\ \lambda = \lambda(u) = f'(u).$$

This is valid for smooth solutions, but as conservation laws, the two equations

$$u_t + f(u)_x = 0,\ \lambda_t + (\frac{\lambda^2}{2})_x = 0,$$

are not equivalent in the sense of weak solutions. The jump conditions are, respectively,

$$\begin{cases} s = \frac{f(u_+)-f(u_-)}{u_+-u_-} = \frac{f'(u_+)+f'(u_-)}{2} + O(1)|u_+ - u_-|^2; \\ s = \frac{\lambda_+ + \lambda_-}{2} = \frac{f'(u_+)+f'(u_-)}{2}. \end{cases}$$

Thus the approximation by the Hopf equation is of second order accuracy.

Consider system of hyperbolic conservation laws

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,\ \boldsymbol{u} \in \mathbb{R}^n, \\ \boldsymbol{l}_j(\boldsymbol{u})\boldsymbol{f}'(\boldsymbol{u}) = \lambda_j(\boldsymbol{u})\boldsymbol{l}_j(\boldsymbol{u}),\ \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{r}_k = \lambda_k(\boldsymbol{u})\boldsymbol{r}_k(\boldsymbol{u}), \\ \boldsymbol{l}_j(\boldsymbol{u}) \cdot \boldsymbol{r}_k(\boldsymbol{u}) = \delta_{jk}. \end{cases}$$

An important example that motivates the general study of system of hyperbolic conservation laws is the Euler equations in gas dynamics

$$\begin{cases} \rho_t + (\rho v)_x = 0, & \text{conservation of mass,} \\ (\rho v)_t + (\rho v^2 + p)_x = 0, & \text{conservation of momentum,} \\ (\rho E)_t + (\rho E v + p v)_x = 0, & \text{conservation of energy.} \end{cases}$$

The approximation by the Hopf equation can be generalized to system of hyperbolic conservation laws for *i-simple waves*:

$$\boldsymbol{u}(x,t) \in \boldsymbol{R}_i(\boldsymbol{u}_0),\ \boldsymbol{R}_i(\boldsymbol{u}_0) \text{ integral curve of } \boldsymbol{r}_i(\boldsymbol{u}) \text{ through a state } \boldsymbol{u}_0.$$

By definition, $\boldsymbol{u}_x(x,t)$ and $\boldsymbol{u}_t(x,t)$ are parallel to $\boldsymbol{r}_i(\boldsymbol{u}(x,t))$:

$$\begin{matrix} \boldsymbol{u}_x(x,t) = \alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u}(x,t)), \\ \boldsymbol{u}_t(x,t) = \beta(x,t)\boldsymbol{r}_i(\boldsymbol{u}(x,t)), \end{matrix} \quad \text{for some scalar functions } \alpha(x,t), \beta(x,t).$$

Plug this into the hyperbolic conservation laws to obtain

$$\beta(x,t) + \alpha(x,t)\lambda_i(\boldsymbol{u}(x,t)) = 0.$$

The convexity condition for scalar equations is that the characteristic speed $\lambda(u) = f'(u)$ is a strictly monotone function of the state u. The generalization to the system is that this is so for the simple waves, that is, the characteristic speed $\lambda_i(\boldsymbol{u})$ is strictly monotone in the characteristic direction

$\boldsymbol{r}_i(\boldsymbol{u})$:

$$\nabla\lambda_i(\boldsymbol{u})\cdot\boldsymbol{r}_i(\boldsymbol{u})\neq 0,\ \text{for genuine nonlinearity.}$$

For the Euler equations in gas dynamics, the characteristic families for the acoustic waves are genuinely nonlinear. Besides $\boldsymbol{l}_j(\boldsymbol{u})\cdot\boldsymbol{r}_k(\boldsymbol{u})=\delta_{jk}$, we may make the additional normalization of

$$\nabla\lambda_i(\boldsymbol{u})\cdot\boldsymbol{r}_i(\boldsymbol{u})=1,\ \text{for genuine nonlinearity.}$$

For i-simple waves,

$$\begin{aligned}&\frac{\partial}{\partial x}\lambda_i(\boldsymbol{u}(x,t))=\nabla\lambda_i(\boldsymbol{u}(x,t))\cdot\boldsymbol{u}_x(x,t)=\alpha(x,t),\\ \text{and similarly}\quad&\frac{\partial}{\partial t}\lambda_i(\boldsymbol{u}(x,t))=\beta(x,t).\end{aligned}$$

We thus obtain the Hopf equation for the simple wave $\boldsymbol{u}(x,t)$, [22]:

$$\lambda_t+\lambda\lambda_x=0,\ \lambda=\lambda_i(\boldsymbol{u}(x,t)).$$

Thus we construct i-simple waves $\boldsymbol{u}(x,t)$ for the system by requiring the wave to take values along characteristic curve and the movement along the curve according to a Hopf solution $\lambda(x,t)$:

$$\boldsymbol{u}(x,t)\in\boldsymbol{R}_i(\boldsymbol{u}_0),\ \lambda_i(\boldsymbol{u}(x,t))=\lambda(x,t),\ i\text{-simple wave.}$$

Again, the construction of i-shock waves for the system according to the Hopf equation is not exact and is of second order accuracy.

The theory of N-waves for the Hopf equation can be generalized to systems. For scalar laws, the convergence to the N-waves can be expressed in pointwise sense. Two relatively strong shock waves emerge in the solution and in between the solution is dominated by rarefaction wave. This is done for general convex scalar conservation laws. In particular the convergence is at the rate of $t^{-1/2}$ in $L_1(x)$. The time invariants are directly computed from the initial data

$$p=\min_x\int_{-\infty}^{x}u(y,0)dy,\ q=\max_x\int_{x}^{\infty}u(y,0)dy.$$

For systems, there are *nonlinear interactions* between waves pertaining to distinct characteristic fields. Consider

$$C_{ijk} = \boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{f}''(\boldsymbol{u})\boldsymbol{r}_j(\boldsymbol{u})\boldsymbol{r}_k(\boldsymbol{u}).$$

The coefficients C_{ijk} for $i = l = k$ measure the nonlinearity of the give characteristic field, for instance $C_{iii} \neq 0$ means that the i-th characteristic field is genuinely nonlinear. The other coefficients measure the *coupling* of distinct characteristic fields. Due to the interaction of waves pertaining to distinct characteristic fields, the convergence to N-waves for system is slower. Two relatively strong shock waves emerge for each genuinely nonlinear field. Each N-waves carries two time invariants, which can only be computed time-asymptotically. There are waves propagating between the primary waves and cause the convergence to N-waves to be of the slower rate. In $L_1(x)$, the rate is $t^{-1/4}$, [23]. This is obtained by careful wave tracing, [21], and strong estimates on the secondary waves between the primary waves, using the Glimm scheme, [14]. The analysis for the primary waves is motivated and uses the theory of N waves for the Hopf equation.

The study of N-waves for the systems was initiated for the 2×2 systems, [9]. This is natural, as there exist the Riemann invariants coordinates for such systems and the scalar equations by the earlier studies can be viewed as measuring the variation for one of the Riemann invariants. Moreover, there is the difficult paper of Glimm-Lax, [15], on the decay of solutions for the 2×2 systems. As seen above, it is more natural to consider the characteristic values in forming the Hopf equation.

9. Viscous Conservation Laws

Consider a general system of viscous conservation laws:

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})\boldsymbol{u}_x)_x, \ \ \boldsymbol{u} \in \mathbb{R}^n.$$

The viscosity matrix $\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})$ contains dissipative parameters, which in general depend on the solution, $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{u})$.

A prime example is the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + (\rho v)_x = 0, & \text{conservation of mass,} \\ (\rho v)_t + (\rho v^2 + p)_x = (\nu u_x)_x, & \text{conservation of momentum,} \\ (\rho E)_t + (\rho E v + p v)_x = (\nu u u_x + \kappa T_x)_x, & \text{conservation of energy.} \end{cases}$$

There are two dissipation parameters $\boldsymbol{\mu} = (\nu, \kappa)$, the viscosity and heat conductivity coefficients. From kinetic theory, these coefficients are functions of the temperature T.

The viscosity matrix $\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})$ tends to zero as $\boldsymbol{\mu}$ vanishes and the system becomes hyperbolic conservation laws. Similar to the theory for hyperbolic conservation laws, the Burgers equation, with its Hopf-Cole transformation, is essential for the study of viscous conservation laws. To see the Burgers equation approximation, we again start with the i-simple waves $\boldsymbol{u}(x,t) \in \boldsymbol{R}_i(\boldsymbol{u}_0)$, for a genuinely nonlinear field $\nabla\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) = 1$. Due to the viscosity term, the simple waves are only approximate solutions. There is also the question of determining the viscosity coefficient κ for the Burgers equation. Following the reasoning for the hyperbolic conservation laws, we have

$$\beta(x,t) + \alpha(x,t)\lambda_i(\boldsymbol{u}(x,t)) = \boldsymbol{l}_i(\boldsymbol{u})[\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})\alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u})]_x.$$

The viscosity term is rewritten as

$$\begin{aligned} \boldsymbol{l}_i(\boldsymbol{u})[\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})\alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u})]_x &= [\boldsymbol{l}_i(\boldsymbol{u})\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})\alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u})]_x \\ &\quad -\boldsymbol{l}_i(\boldsymbol{u})_x[\mathbb{B}(\boldsymbol{u}, \boldsymbol{\mu})\alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u})]. \end{aligned}$$

For the consideration of solutions around a given state $\boldsymbol{u}_0$, the viscosity coefficient κ for the associated Burgers equation is

$$\kappa = \boldsymbol{l}_i(\boldsymbol{u}_0)\mathbb{B}(\boldsymbol{u}_0, \boldsymbol{\mu})\boldsymbol{r}_i(\boldsymbol{u}_0).$$

Same as in the hyperbolic case, $\lambda_i(\boldsymbol{u}(x,t))_x = \alpha(x,t)$, $\lambda_i(\boldsymbol{u}(x,t))_t = \beta(x,t)$, and we obtain the Burgers equation

$$\lambda_t + \lambda\lambda_x = \kappa\lambda_{xx}, \ \lambda = \lambda_i(\boldsymbol{u}).$$

Here we have ignored the term

$$\boldsymbol{l}_i(\boldsymbol{u})_x[\mathbb{B}(\boldsymbol{u},\boldsymbol{\mu})\alpha(x,t)\boldsymbol{r}_i(\boldsymbol{u})] = \lambda_x \boldsymbol{l}_i(\boldsymbol{u})_x \mathbb{B}(\boldsymbol{u},\boldsymbol{\mu})\boldsymbol{r}_i(\boldsymbol{u}).$$

This term is of the order of $(\lambda_x)^2$, a higher decaying term when one considers the dissipation around a constant state $\boldsymbol{u}_0$.

As noted before, the Burgers equation is a good approximation also for nonlinear waves such as the shock and rarefaction waves, and the Hopf-Cole transformation is useful for the study of initial layer and these nonlinear waves for large time, [17]. The analysis carries over to the Boltzmann equation in the kinetic theory, [30]. There is a connection of the Boltzmann equation to the compressible Navier-Stokes equations through the Chapman-Enskog expansion.

In addition to the coupling induced by the hyperbolic nonlinearity through the coefficients C_{ijk}, there is additional coupling due to viscosity. For the above approximate solution $\boldsymbol{u}(x,t)$ based on the Burgers equation, we have the truncation error for the system:

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\mathbb{B}(\boldsymbol{u},\boldsymbol{\mu})\boldsymbol{u}_x)_x + \text{truncation error}, \\ \text{truncation error} = \sum_{j\neq i}[\lambda_x(\boldsymbol{l}_j\mathbb{B}\boldsymbol{r}_i\boldsymbol{r}_i]_x, \end{cases}$$

where the aforementioned higher order term is again ignored. The truncation error induces the interaction of wave of distinct characteristic families, $j \neq i$, due to the presence of the viscosity matrix $\mathbb{B}$. The two coupling mechanisms, one due to the nonlinearity of the flux $\boldsymbol{f}(\boldsymbol{u})$ and one due to the coupling of the flux and the viscosity matrix $\mathbb{B}(\boldsymbol{u},\boldsymbol{\mu})$, yield a rich wave structure, [24], [27]. For a finite mass perturbation of a constant state $\boldsymbol{u}_0$, the main waves are Burgers kernels $b(x,t;A)$ for each genuinely nonlinear field and heat kernels for fields of weaker nonlinearity. Due to the coupling, the convergence to the combination of Burgers and heat kernels is slower than for the scalar equations. Interestingly, the convergence rate in $L_1(x)$ is $t^{-1/4}$, same as for the convergence to N-waves in the hyperbolic case.

The study of nonlinear wave interactions is based on the explicit construction. For the study of shock waves, the Burgers Green's function is used in the construction of the Green's function, [28].

10. Concluding Remarks and Open Problems

10.1. Burgers

The main motivation of J. M. Burgers for the study of the equation $u_t + uu_x = \mu u_{xx}$ was to understand certain key elements of turbulence for compressible flows, [3, 4, 5]. It is the simplest model with strong nonlinearity and dissipation. It is remarkable that the simple nonlinear equation is capable of exhibiting some of the essential features of turbulence in the transfer of energy between low frequency and high frequency waves. Based on earlier works on laminar and turbulent solutions, Burgers first considered the 2×2 system

$$\begin{cases} \frac{\partial v}{\partial t} = U(v-w) + \nu\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial v}{\partial y} + 2w\frac{\partial w}{\partial y}, \\ \frac{\partial w}{\partial t} = U(v+w) + \nu\frac{\partial^2 w}{\partial y^2} + 2w\frac{\partial v}{\partial y} + 2v\frac{\partial w}{\partial y}. \end{cases}$$

He then proposed the model equation for $U = 0$

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial v}{\partial y}.$$

Stability criteria are obtained for these equations. The model equation is the simplest model with strong nonlinearity and dissipation. Burgers' initial success in his series of papers on the difficult subject of turbulence drew attention to the equation that now bears his name. The Hopf-Cole transformation allows for further studies of the statistical properties by Burgers himself, [5].

It is interesting to mention that the nonlinear term uu_x in the Burgers equation is often compared to the convection terms $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in the fluid dynamics equations. Note, however, that to study the stationary shock waves, the fluid speed normal to the shock is around the sonic speed. This would also indicate that it is natural to consider one of the characteristic speeds $\lambda_1 = v - c,\ \lambda_3 = v + c$ for the acoustic field in gas flow to be near zero. This agrees with the study of simple waves as mentioned before.

The Burgers equation was already proposed in the 1915 paper by H. Bateman [1]:

"The possibility of the solution of the equations of motion of a viscous fluid becoming discontinuous when the viscosity approaches the value zero, may perhaps be illustrated by a consideration of the equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}."$$

Bateman then considered the viscous profiles and demonstrated his point. As Cole put it: Bateman considered the equation as "worthy of study and gave a special solution." The richness of the subject was envisioned by Burgers much later.

10.2. Friedrichs

In [13], K. O. Friedrichs studied the N-waves to approximate the stationary flow around a supersonic airfoil. There is the leading shock in front of the airfoil induced by the compression of the flow due to the presence of the airfoil. This is followed by expansion wave as the airfoil becomes thinner. Then the flows on the two sides of the airfoil meet toward the end of the airfoil and the resulting compression creates another shock wave. Thus the general flow can be approximated by a rarefaction wave sandwiched by two shock waves. Friedrichs derived a scalar equation and carried the analysis for it.[1] Although the idea of N-waves was mentioned earlier, [10], Friedrichs computed the decay rate of $t^{-1/2}$ of the shock and the width $t^{1/2}$ of the expansion wave. Here t represents the distance from the airfoil. The similarity in rates of this *inviscid dissipation* with the usual viscous dissipation is striking. This is an important consideration for Hopf's investigation of Burgers equation later.

10.3. Hopf and Cole

The Hopf-Cole transformation was found independently by E. Hopf, [18] and J. Cole, [6].

[1] During 1976-1977, as a junior visitor at NYU, the author explained to Friedrichs his work on the N-waves for general system of hyperbolic conservation laws in Friedrichs office. Friedrichs said, in his usual modest manner, that "I visited Cal. Tech. for a summer. People there said to me that you are a good applied mathematician and you can solve problems for us. But for the whole summer I only did the study of N-waves." Friedrichs asked the author to describe the N-waves for a system.

Cole was aware that Bateman proposed the Burgers equation as a model problem for shock waves. Cole actually derived the Burgers equation from the full gas dynamics equations to govern the perturbation of gas velocity around a sonic velocity:

$$\frac{\partial w}{\partial t}+\beta\frac{\partial w}{\partial x}=\frac{4}{3}\nu^*\frac{\partial^2 w}{\partial x^2}.$$

"for w = excess of flow velocity over a sonic velocity, where $\beta=(\gamma+1)/2$, ν^* = the kinematic viscosity at sonic condition."

Hopf claimed in his paper that

"The reduction of (1) to the heat equation was known to me since the end of 1946. However, it was not until 1949 that I became sufficiently acquainted with the recent development of fluid dynamics to be convinced that a theory of (1) could serve as an instructive introduction into some of the mathematical problems involved."

Hopf 'became sufficiently acquainted' with Burgers equation, equation (1), in two aspects. He learned that Burgers equation was used for the study of fluid dynamics. He also learned the behavior of shock waves, particularly the N-waves, for the inviscid Burgers equation, the Hopf equation. Hopf learned these through his contacts with IPST at University of Maryland in the 40's[2]. These prompted Hopf to use his transformation to derive the formula for the inviscid solutions and the time invariants for the N-waves. Hopf then concluded that the N-waves are the meta-stable states for the Burgers solutions and that the time-asymptotic and zero dissipation limits do not commute.

10.4. Forsyth

While the ingenuity of the Hopf-Cole transform is clear, it is interesting to note that, already in the early 20th century, in one volume of the classical books he wrote, A. R. Forsyth put the transform as an exercise, [12]. Before the era of functional analytic approach to establish the existence of

[2]The authors had conversations with Avron Douglis in the 70's and 80's when both were faculty at University of Maryland. There were opportunities talking to Burgers at IPST, where Burgers spent final years of his career.

solutions without having to obtain explicit formulas, much efforts focused on transformations to simplify the equation which, in some cases, allows one to solve the equation and obtain explicit solution formulas. For the parabolic equation

$$\frac{\partial^2 z}{\partial x^2} + 2\alpha\frac{\partial z}{\partial x} + 2\beta\frac{\partial z}{\partial y} + \gamma z = 0,$$

it is to consider linear transformations $z \to z'$, which transforms the equation to another of the same form

$$\frac{\partial^2 z'}{\partial x^2} + 2\alpha'\frac{\partial z'}{\partial x} + 2\beta'\frac{\partial z'}{\partial y} + \gamma' z' = 0,$$

Depending on the transformation, there are *invariants*, combination of the coefficients that are unchanged under the transformation. In the case $z' = \lambda z$, for a known function $\lambda = \lambda(x, y)$, there are two invariants $I = I',\ J = J'$:

$$I = \beta,\ J = \frac{\partial}{\partial x}\Big[\frac{1}{\beta}\Big(\gamma - \frac{\partial \alpha}{\partial x} - \alpha^2\Big)\Big] - 2\frac{\partial \alpha}{\partial y}.$$

For more complicated transformations, it is not clear that the form of the equation is preserved. In Volume VI of Forsyth's books, on Page 102, Ex. 3, a transform is considered:

$$z_1 = \frac{\partial z}{\partial x} + (\alpha + u)z.$$

Straightforward calculations show that the form of the equation is kept under the transformation

$$\frac{\partial^2 z_1}{\partial x^2} + 2\alpha_1\frac{\partial z_1}{\partial x} + 2\beta_1\frac{\partial z_1}{\partial y} + \gamma_1 z_1 = 0,$$

when the function $u = u(x, y)$ satisfies

$$\frac{\partial}{\partial x}\Big[\frac{1}{\beta}\Big(\frac{\partial u}{\partial x} - u^2\Big)\Big] + 2\frac{\partial u}{\partial y} = J,$$

and the new coefficients are given as

$$\begin{cases} \alpha_1 = \alpha - \dfrac{1}{2\beta}\dfrac{\partial \beta}{\partial x}, \\ \beta_1 = \beta, \\ \gamma_1 = \gamma - \dfrac{\alpha}{\beta}\dfrac{\partial \beta}{\partial x} + \dfrac{u}{\beta}\dfrac{\partial \beta}{\partial x} - 2\dfrac{\partial u}{\partial x}. \end{cases}$$

J is no longer an invariant and further calculations, as also requested in the exercise, yield

$$J_1 - J = \frac{\partial}{\partial y}\Big(\frac{1}{\beta}\frac{\partial \beta}{\partial x}\Big) + \frac{1}{2}\frac{\partial}{\partial x}\Big[\frac{1}{\beta}\frac{\partial}{\partial x}\Big(\frac{1}{\beta}\frac{\partial \beta}{\partial x}\Big)\Big] - \frac{\partial}{\partial x}\Big[\frac{1}{u}\frac{\partial}{\partial x}\Big(\frac{u^2}{\beta}\Big)\Big].$$

This does not lead to ways for finding the solution formulas. Nevertheless, the form of the equation for $z_1(x, y)$ is shown to be the same as that for the original function $z(x, y)$ and the role of the function $u(x, y)$ is made definite. In particular, $z_1 = 0$ is a solution and the process yields the Hopf-Cole transformation by setting $z_1 = \frac{\partial z}{\partial x} + (\alpha + u)z = 0$.

The spirit of going back to the era of finding explicit ways for solving specific equations comes back in recent decades and is essential for many studies, such as in the theory of solitons.

10.5. Modern theory of shock waves

Before Hopf-Cole transformation, the shock wave theory was focusing on the study of gas dynamics, as documented in the classical 1948 book of Courant-Friedrichs, [7]. This is natural as the shock wave theory, as inaugurated by Riemann and Stokes, was concerned exclusively with the Euler and Naveir-Stokes equations in gas dynamics. The Hopf-Cole transformation, particularly Hopf's analysis, indicated that it is fruitful to consider general scalar hyperbolic conservation laws. Hopf's explicit formula for solving the Hopf equation was generalized to general convex conservation laws by Lax and by Oleinik. The characterisation of the two time invariants p, and q for N-waves are also generalised to general convex laws, [20]. For the theory of general system of hyperbolic conservation laws, the first step is the solution of the Riemann problem by Lax, [20]. This leads to the fundamental breakthrough by James Glimm, [14] for general initial value problem. There is now a sophisticated theory for one spatial dimensional hyperbolic conservation laws, c.f. [8]. It can be argued that the Hopf-Cole transformation helped to inaugurate the modern theory of shock waves. Instead of considering particular physical systems, such as the Euler equations in gas dynamics, one considers conservation laws of general form

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0.$$

Several decades of the general study provides people with solid understanding of shock wave theory. With the confidence, in recent years, people increasingly go back to the classical concern of specific physical models.

10.6. Nonlinear transformations

The Hopf-Cole transformation is a strikingly, truly nonlinear transformation. Its importance goes beyond solving the Burgers equation and takes on certain philosophical meaning: It is possible that strongly nonlinear equations can be *linearized* through a nonlinear transformation. This has inspired researchers working on quite different nonlinear phenomena. An important example is the soliton theory. It is based on linearisation, in a very nontrivial sense. There is the Miura transformation, [29], between two strongly nonlinear equations, that inaugurates the soliton theory:

$$\begin{cases} V_t - 6VV_x + V_{xxx} = 0, & \text{KdV} \\ \phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0, & \text{Modified KdV,} \\ V = \phi^2 \pm \phi_x, & \text{Miura transformation.} \end{cases}$$

As Miura put it, [29]:

“It is rare and surprising to find a transformation between two simple nonlinear partial differential equations of independent interest. One is reminded of the Hopf-Cole transformation of quadratically nonlinear Burgers equation into the heat conduction (diffusion) equation. A number of investigators (including us) have attempted unsuccessfully to find a similar simple linearizing transformation for the KdV equation, but a complicated one will be given in VI.”

10.7. Statistical studies

The Hopf-Cole transformation allows for further studies of the statistical properties by Burgers himself, [5]. Other researchers have followed Burgers in the statistical study, for instance, [11] studies the Burgers equation with stochastic forcing

$$u_t + uu_x = \mu u_{xx} + \frac{\partial F}{\partial x}(x,t).$$

The Burgers equation, with its explicit solution solver, provides the basic framework for the study of complex physical phenomena. The seminal paper [19] initiates the study of the evolution of the profile of a growing interface. Let $h(\boldsymbol{x},t)$ be the height and consider the local growth of the profile given by anti-derivative of the Burgers equaiton, a Hamilton-Jacobi equation, plus a noise η:

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\boldsymbol{x},t).$$

The Hopf-Cole transform provides the first step by turning this into linear equation with a source:

$$\begin{cases} \dfrac{\partial W}{\partial t} = \nu\nabla^2 W + \dfrac{\lambda}{2\nu}\eta(\boldsymbol{x},t)W, \\ W(\boldsymbol{x},t) = e^{\frac{\lambda}{2\nu}h(\boldsymbol{x},t)}. \end{cases}$$

Strikingly, a new scaling, distinct from deterministic dissipation equations such as the Burgers and heat equations, comes up due to the noise. The authors marvel that they are able to relate the Burgers equation to their efforts, [19]:

"We thus have an intriguing connection between evolutions of a hydrodynamic and a growth pattern!"

This fruitful research direction is being earnestly pursued.

10.8. Open problems

In fluid dynamics and kinetic theory, there are initial, shock and boundary layers. The Hopf-Cole transformation gives definite information on the interaction of initial and shock layers for the Burgers equation. This has been generalised, through approximation by simple waves, to the compressible Navier-Stokes equations, [17] and to the Boltzmann equation, [30]. The transformation also gives definite information on the interaction of initial layer and rarefaction wave. It would be desirable to generalise this to general viscous conservation laws and to the Boltzmann equation.

More generally, one would like to consider the interaction of initial layer with combination of distinct types of nonlinear waves. This occurs naturally in the Riemann problem. By simple scaling, large-time behavior of the Riemann solutions is related to the zero dissipation limit. The first step to resolve the Riemann problem for system with physical viscosity would be to understand the initial layer by the Green's function approach. The starting point of the Green's function approach for systems would be to use Hopf-Cole transformation for understanding the nonlinear waves for systems. How to use the transformation for different wave types contained in the Riemann solution is a worthy problem. It would be interesting to study the Riemann problem for the Boltzmann equation in the kinetic theory. The initial data consist of two Maxwellian states. By scaling, the study of the Riemann problem is directly related to zero dissipation limit for viscous conservation laws and the zero mean free path limit for the Boltzmann equation. The zero dissipation limit for systems with artificial viscosity

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \mu \boldsymbol{u}_{xx}$$

was solved by [2].

As mentioned before, for both system of hyperbolic conservation laws and viscous conservation laws, the convergence rates to their respective time-asymptotic states are the same, $t^{-1/4}$ in $L_1(x)$. In light of this, it should be possible to understand the sense in which the N-waves for hyperbolic system represent the meta-stable states for the viscous system.

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