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CONSTRUCTION OF HOLOMORPHIC VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24 USING THE LEECH LATTICE AND LEVEL p LATTICES

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Abstract

In this article, we discuss a more uniform construction of all 71 holomorphic vertex operator algebras in Schellekens' list using an idea proposed by G. Höhn. The main idea is to try to construct holomorphic vertex operator algebras of central charge 24 using some sublattices of the Leech lattice Λ and level p lattices. We study his approach and try to elucidate his ideas. As our main result, we prove that for an even unimodular lattice L and a prime order isometry g, the orbifold vertex operator algebra $V_{L_g}^{\hat{g}}$ has group-like fusion. We also realize the construction proposed by Höhn for some special isometry of the Leech lattice of prime order.

1. Introduction

The theory of vertex operator algebra (VOA) has its origin in mathematical physics and has been found to be a useful tool for studying 2-dimensional conformal field theory. The notion of vertex operator algebras is also well-motivated by Frenkel-Lepowsky-Meurman's construction of the

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moonshine VOA $V^\natural$ whose full automorphism group is the Monster simple group [23]. The construction of the moonshine VOA $V^\natural$ does not only prove a conjecture by McKay-Thompson but it also plays a fundamental role in shaping the theory of VOA. In addition, the theory of VOA provides a very effective platform for studying certain mysterious properties of the Monster simple group and some other sporadic simple groups. In the introduction of their book [23], Frenkel-Lepowsky-Meurman conjectured that the moonshine VOA $V^\natural$ can be characterized by the following three conditions:

(a) the VOA $V^\natural$ is the only irreducible (ordinary) module for itself;

(b) the central charge of $V^\natural$ is 24;

(c) the weight one space of $V^\natural$ is trivial, i.e., $V_1^\natural = 0$.

These are natural analogues of conditions which characterize the binary Golay code and the Leech lattice. This conjecture is still open and is considered as one of the most difficult questions in the theory of VOA.

A simple rational VOA V is said to be *holomorphic* if it itself is the only irreducible (ordinary) module. In particular, FLM's moonshine VOA $V^\natural$ is a very special example of a holomorphic VOA of central charge 24. In 1993, Schellekens [43] studied the structures of holomorphic VOAs of central charge 24 and obtained a partial classification by determining possible Lie algebra structures for the weight one subspaces of holomorphic VOAs of central charge 24. There are 71 cases in his list but only 39 of the 71 cases were known explicitly at that time. It is also an open question if the Lie algebra structure of the weight one subspace will determine the VOA structure uniquely when the central charge is 24.

In the recent years, many new holomorphic VOAs of central charge 24 have been constructed. The main technique is often referred to as "Orbifold construction". In [31, 33], 17 holomorphic VOAs were constructed using the theory of framed VOAs. Moreover, three holomorphic VOAs have been constructed in [39, 42] using $\mathbb{Z}_3$-orbifold constructions associated with lattice VOAs. Inspired by the work of Miyamoto [39], van Ekeren, Möller and Scheithauer established in [21] the general $\mathbb{Z}_n$-orbifold construction for automorphisms of arbitrary orders. In particular, constructions of five holomorphic VOAs were discussed. In [34], five other holomorphic VOAs have been obtained using an orbifold construction associated with inner automorphisms. In addition, a construction of a holomorphic VOA with one of two remaining Lie algebras has been obtained in [35] using a $\mathbb{Z}_7$-orbifold construction on the Leech lattice VOA. The final case is also studied in [32] and a construction of such a VOA is obtained. As a consequence, all 71 Lie algebras in Schellekens' list can be realized as the weight one Lie algebras of some holomorphic VOAs of central charge 24. For the list of the 71 Lie algebras and the references for holomorphic VOAs, see Appendix. Nevertheless, the constructions are based on case by case analysis and a uniform approach is missing.

Very recently, G. Höhn [28] has proposed a very provocative idea for studying the list of Schellekens using automorphisms of Niemeier lattices and Leech lattice. In particular, he suggested a more uniform construction of all 71 holomorphic VOAs in Schellekens' list using the Leech lattice. His idea is to use certain simple current extensions of lattice VOAs and some orbifold subVOAs in the Leech lattice VOA. In this article, we will study his approach. The main purpose is to try to elucidate his idea and try to realize his proposed construction for some special cases. As our main result, we prove that the orbifold VOA $V_{L_g}^{\hat{g}}$ has group-like fusion for an even unimodular lattice L and an isometry g of prime order. We also realize the construction proposed by Höhn for some special isometry of the Leech lattice of prime order.

The organization of the article is as follows. In Section 2, we review some basic notions about integral lattices and VOAs. In Section 3, we discuss a more uniform construction for all 71 cases in Schellekens' list proposed by Höhn. In Section 4, we study the orbifold VOA $V_{L_g}^{\hat{g}}$, where L is an even unimodular lattice and g is an isometry of L. We prove that the orbifold VOA $V_{L_g}^{\hat{g}}$ has group-like fusion (i.e., all irreducible modules are simple currents) if g has prime order (cf. Theorem 4.12) . In Section 5, we realize the construction proposed by Höhn for some isometry of the Leech lattice of prime order.

2. Preliminary

We first review some basic terminology and notation for integral lattices and VOAs.

2.1. Integral lattices

By *lattice*, we mean a free abelian group of finite rank with a rational valued, positive definite symmetric bilinear form $\langle\ ,\ \rangle$. A lattice L is *integral* if $\langle L, L\rangle < \mathbb{Z}$ and it is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$. Note that an even lattice is integral. We use L^* to denote the dual lattice of L,

$$L^* = \{v \in \mathbb{Q} \otimes_{\mathbb{Z}} L \mid \langle v, L\rangle < \mathbb{Z}\},$$

and denote the discriminant group L^*/L by $\mathcal{D}(L)$. Note that if a lattice L is integral, then $L \subset L^*$. Let $\{x_1, \dots, x_n\}$ be a basis of L. The Gram matrix of L is defined to be the matrix $G = (\langle x_i, x_j\rangle)_{1\le i,j\le n}$. The determinant of L, denoted by $\det(L)$, is the determinant of G. Note that $\det(L) = |\mathcal{D}(L)|$.

Let L be an integral lattice. For any positive integer m, let $L_m = \{x \in L \mid \langle x, x\rangle = m\}$ be the set of all norm m elements in L. *The summand of* L *determined by the subset* S of L is the intersection of L with the $\mathbb{Q}$-span of S. An *isometry* g of L is a linear isomorphism $g \in GL(\mathbb{Q} \otimes_{\mathbb{Z}} L)$ such that $g(L) \subset L$ and $\langle gx, gy\rangle = \langle x, y\rangle$ for all $x, y \in L$. We denote the group of all isometries of L by $O(L)$.

Definition 2.1. Let L be an even lattice. The *theta series* of L is defined to be the series $\Theta_L(q) := \sum_{x\in L} q^{\langle x,x\rangle/2}$, where q is a formal variable.

Definition 2.2. Let p be a prime. An integral lattice L is said to be *p-elementary* if $pL^* < L$. A 1-elementary lattice is also called *unimodular*.

For $\ell \in \mathbb{Z}_{>0}$, we use ${}^{\ell}L$ to denote the scaling: ${}^{\ell}L$ equals L as a lattice but the bilinear form is multiplied by ℓ. Note that ${}^{\ell}L \cong \sqrt{\ell}L = \{\sqrt{\ell}x \mid x \in L\}$.

Definition 2.3. Let L be an even lattice. The *level* of L is defined to be the smallest positive integer ℓ such that ${}^{\ell}(L^*)$ is again even.

Remark 2.4. Let p be a prime. If $p \ne 2$, then L has level p if and only if L is p-elementary and $L^* \ne L$. If $p = 2$, then L has level 2 if and only if L is 2-elementary, $L^* \ne L$ and ${}^{2}(L^*)$ is even. Note also that if L has level p, then so is ${}^{p}(L^*)$.

Remark 2.5. Let L be an even lattice of level ℓ (not necessary prime). Set $q = e^{2\pi\sqrt{-1}\tau}$, where τ is in the upper half plane of $\mathbb{C}$. Then the theta series $\Theta_L(q)$ is a modular form for the congruence subgroup $\Gamma_0(\ell)$.

Definition 2.6. Let L be an integral lattice and $g \in O(L)$. We denote the fixed point sublattice of g by $L^g = \{x \in L \mid gx = x\}$. The *coinvariant lattice* of g is defined to be

$$L_g = Ann_L(L^g) = \{x \in L \mid \langle x, y\rangle = 0 \text{ for all } y \in L^g\}.$$

Lemma 2.7. *Let L be an even unimodular lattice. Let $g \in O(L)$ be an isometry of order $\ell > 1$ such that $L^g \neq 0$. Then $\ell(L^g)^* < L^g$ and $\mathcal{D}(L^g) \cong \mathcal{D}(L_g)$.*

Proof. Since L is unimodular, the natural projection $p_0 : L \to (L^g)^*$ is a surjection. Moreover, we have

$$p_0(x) = \frac{1}{\ell}(x + gx + \cdots + g^{\ell-1}x) \quad \text{for any } x \in L.$$

Hence, $\ell p_0(L) < L^g$ and we have $\ell(L^g)^* < L^g$. That $\mathcal{D}(L^g) \cong \mathcal{D}(L_g)$ also follows from the fact that L is unimodular. □

Next we will recall the definition for the root system of an even lattice L (cf. [41]). A vector $v \in L$ is *primitive* if the sublattice spanned by v is a direct summand of L. A primitive vector v is called a *root* if $2\langle v, L\rangle/\langle v, v\rangle < \mathbb{Z}$. The set of roots

$$R(L) = \{v \in L \mid v \text{ is primitive, } 2\langle v, L\rangle/\langle v, v\rangle < \mathbb{Z}\}$$

is called the *root system* of L. Notice that the root system of a lattice is also a root system (up to a scaling) in the usual sense in Lie algebra theory.

Remark 2.8 (cf. [41])**.** If L has a prime level p, then

$$R(L) = \{v \in L \mid \langle v, v\rangle = 2\} \cup \{v \in p(L^*) < L \mid \langle v, v\rangle = 2p\}.$$

2.2. Leech lattice

Next we recall the definition and some basic properties about the Leech lattice.

Let $\Omega = \{1, 2, 3, \ldots, 24\}$ be a set of 24 element and let $\mathcal{G} \subset P(\Omega)$ be the extended binary Golay code of length 24 indexed by Ω. A subset $S \subset \Omega$ is called a $\mathcal{G}$-set if $S = \operatorname{supp} \alpha$ for some codeword $\alpha \in \mathcal{G}$. We will identify a

$\mathcal{G}$-set with the corresponding codeword in $\mathcal{G}$. The automorphism group of the Golay code $\mathcal{G}$ is the Mathieu group M_{24}.

The following is a standard construction of the Leech lattice.

Definition 2.9 (Standard Leech lattice [6, 24]). Let $e_i := \frac{1}{\sqrt{8}}(0,\dots,4,\dots,0)$ for $i \in \Omega$. Then $(e_i, e_j) = 2\delta_{i,j}$. Denote $e_X := \sum_{i\in X} e_i$ for $X \in \mathcal{G}$. The *standard Leech lattice* Λ is a lattice of rank 24 generated by the vectors:

$$\frac{1}{2}e_X\,, \quad \text{where } X \text{ is a generator of the Golay code } \mathcal{G};$$
$$\frac{1}{4}e_\Omega - e_1\,;$$
$$e_i \pm e_j\,,\ i,j \in \Omega.$$

By the definition, it is clear that the Mathieu group M_{24} acts on Λ as a permutation subgroup on $\{e_1,\dots,e_{24}\}$.

2.3. Vertex operator algebras

Throughout this article, all vertex operator algebras are defined over the field $\mathbb{C}$ of complex numbers. First, we recall the notion of vertex operator algebras from [3, 23, 22]. A *vertex operator algebra* (VOA) $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{m\in\mathbb{Z}} V_m$ equipped with a linear map

$$Y(a,z) = \sum_{i\in\mathbb{Z}} a_{(i)} z^{-i-1} \in (\mathrm{End}(V))[[z, z^{-1}]], \quad a \in V$$

and the *vacuum vector* $\mathbf{1}$ and the *conformal vector* ω satisfying certain axioms ([3, 23]). For $a \in V$ and $n \in \mathbb{Z}$, we call the operator $a_{(n)}$ the *n-th mode* of a. We also note that the operators $L(n) = \omega_{(n+1)}, n \in \mathbb{Z}$ satisfy the Virasoro relation:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\, c\ \mathrm{id}_V,$$

where c is a complex number and is called the *central charge* of V.

A linear automorphism g of V is called *an automorphism* of V if

$$g\omega = \omega \quad \text{and} \quad gY(v,z) = Y(gv,z)g \quad \text{for all } v \in V.$$

A *vertex operator subalgebra* (or a *subVOA*) is a graded subspace of V which has a structure of a VOA such that the operations and its grading agree with the restriction of those of V and that they share the vacuum vector. A subVOA is called a *full subVOA* if it has the same conformal vector as V. For an automorphism g of a VOA V, we use V^g to denote the set of all fixed-points of g, which is a full subVOA of V.

A VOA is *rational* if any module is completely reducible. A VOA V is said to be *of CFT-type* if $V_0 = \mathbb{C}\mathbf{1}$ and is said to be *C_2-cofinite* if $\dim(V/C_2(V)) < \infty$, where $C_2(V) = Span\{u_{(-2)}v \mid u, v \in V\}$. A module is *self-dual* if its contragredient module is isomorphic to itself. A VOA is said to be *strongly regular* if it is rational, C_2-cofinite, self-dual and of CFT-type. A rational VOA is said to be *holomorphic* if it itself is the only irreducible module, up to equivalence.

2.4. Fusion products and simple current extensions

Let V^0 be a simple rational C_2-cofinite VOA of CFT-type and let W^1 and W^2 be V^0-modules. It was shown in [30] that the V^0-module $W^1 \boxtimes_{V^0} W^2$, called the *fusion product*, exists. This product is commutative and associative. A V^0-module M is called a *simple current* if for any irreducible V^0-module X, the fusion product $M \boxtimes_{V^0} X$ is also irreducible.

Let $R(V^0)$ denote the set of (the isomorphism classes of) all irreducible V^0-modules. By the rationality (or C_2-cofiniteness) of V^0, $R(V^0)$ is a finite set. Note that if all elements in $R(V^0)$ are simple currents, then $R(V^0)$ forms an abelian group under the fusion product. In this case, we say that V^0 *has group-like fusion.*

Let $\{V^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible V^0-modules indexed by an abelian group D. A simple VOA $V_D = \bigoplus_{\alpha\in D} V^\alpha$ is called a *simple current extension* of V^0 if it carries a D-grading and every V^α is a simple current. Note that $V^\alpha \boxtimes_{V^0} V^\beta \cong V^{\alpha+\beta}$.

Proposition 2.10. ([16, Proposition 5.3]) *Let V^0 be a simple rational C_2-cofinite VOA of CFT type and let $V_D = \bigoplus_{\alpha\in D} V^\alpha$ and $\tilde{V}_D = \bigoplus_{\alpha\in D} \tilde{V}^\alpha$ be simple current extensions of V^0. If $V^\alpha \cong \tilde{V}^\alpha$ as V^0-modules for all $\alpha \in D$, then V_D and $\tilde{V}_D$ are isomorphic VOAs.*

2.5. Modular invariance and holomorphic VOAs

Now, let us recall the modular invariance property of VOAs from [46]. Let V be a rational VOA of central charge c and let $M^0(=V), M^1, \dots, M^p$ be all the irreducible V-modules. For $0 \le i \le p$, there is a number λ_i such that $M^i_{\lambda_i} \neq 0$ and M^i decomposes as

$$M^i = \bigoplus_{n=0}^{\infty} M^i_{\lambda_i+n}.$$

The number λ_i is called *the (lowest) conformal weight* of M^i.

Let $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the upper half plane in $\mathbb{C}$. The trace function associated with M^i is defined as follows: For any homogeneous element $v \in V$ and $\tau \in \mathfrak{H}$,

$$Z_{M^i}(v,\tau) := \operatorname*{tr}_{M^i} o(v) q^{L(0)-c/24} = q^{\lambda_i - c/24} \sum_{n=0}^{\infty} \operatorname*{tr}_{M^i_{\lambda_i+n}} o(v) q^n,$$

where $o(v) = v_{(\mathrm{wt} v-1)}$ and $q = e^{2\pi i\tau}$. Assume further that V is C_2-cofinite. Then $Z_{M^i}(v,\tau)$ converges to a holomorphic function on the domain $|q| < 1$ [13], [46].

Recall that the full modular group $SL(2,\mathbb{Z})$ has generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It acts on $\mathfrak{H}$ as follows:

$$\gamma : \tau \longmapsto \frac{a\tau + b}{c\tau + d}, \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

The modular group also has an action on the trace functions. More precisely, we have the following result.

Theorem 2.11 ([46] (also see [13])). *Let V be a rational and C_2-cofinite vertex operator algebra with the irreducible V-modules $M^0, \dots, M^p$. Then the vector space spanned by $Z_{M^0}(v,\tau), \dots, Z_{M^p}(v,\tau)$ is invariant under the action of $SL(2,\mathbb{Z})$ defined above, i.e., there is a representation ρ of $SL(2,\mathbb{Z})$*

on this vector space. Moreover, the transformation matrices are independent of the choice of $v \in V$*.*

Now let V be a strongly regular holomorphic VOA. Then V has only one irreducible module, up to equivalence. Let $Z_V(q) = Z_V(\mathbf{1}, \tau) = \sum_{n=0}^{\infty} \dim V_n \, q^{n-c/24}$ be the character of V. By the modular invariance,

$$Z_V(\mathbf{1}, \gamma\tau) = \rho(\gamma) Z_V(\mathbf{1}, \gamma\tau) \quad \text{for all } \gamma \in SL(2, \mathbb{Z}),$$

where $\rho : SL(2, \mathbb{Z}) \to \mathbb{C}^{\times}$ is a character. Recall that $\rho(S^2) = \rho((ST)^3) = 1$. Since $S(i) = i$ on $\mathfrak{H}$, we have $\rho(S) = 1$ and $\rho(T) = e^{-2\pi\sqrt{-1}(c/24)}$. This implies that the central charge c of V is a multiple of 8 by $\rho((ST)^3) = 1$. Remark that $Z_V(q) \in \mathbb{C}[j^{1/3}]$ (cf. [27]), where j is the famous elliptic j-function.

For $c = 8$ or 16, it was proved in [16] that a holomorphic VOAs of $c = 8$ or 16 is isomorphic to lattice VOAs, V_{E_8}, $V_{E_8 \oplus E_8}$ or $V_{D_{16}^+}$.

For $c = 24$, the classification is much more complicated; nevertheless, it is commonly believed that the VOA structure of a strongly regular holomorphic VOA of central charge 24 is determined by its weight one subspace V_1.

Proposition 2.12 (cf. [3, 23])**.** *Let V be a VOA of CFT-type. Then, the weight one subspace V_1 forms a Lie algebra with the bracket defined by $[a, b] = a_{(0)}b$ for any $a, b \in V_1$. There is also an symmetric invariant bilinear form defined by $(a, b) \cdot \mathbf{1} = a_{(1)}b$ for $a, b \in V_1$. Moreover, the n-th modes $v_{(n)}$, $v \in V_1$, $n \in \mathbb{Z}$, define an affine representation of the Lie algebra V_1 on V.*

Notation 2.13. For a simple Lie subalgebra $\mathfrak{a}$ of V_1, the *level* of $\mathfrak{a}$ is defined to be the scalar by which the canonical central element acts on V as the affine representation. When the type of the root system of $\mathfrak{a}$ is X_n and the level of $\mathfrak{a}$ is k, we denote the type of $\mathfrak{a}$ by $X_{n,k}$.

For the study of the weight one Lie algebras of holomorphic VOAs of central charge 24, the following two propositions proved by Dong and Mason are very useful.

Proposition 2.14 ([17, Theorem 1.1, Corollary 4.3])**.** *Let V be a strongly regular VOA. Then V_1 is reductive. Let $\mathfrak{s}$ be a simple Lie subalgebra of V_1. Then V is an integrable module for the affine representation of $\mathfrak{s}$ on V, and* *the subVOA generated by $\mathfrak{s}$ is isomorphic to the simple affine VOA associated with $\mathfrak{s}$ at positive integral level.*

Proposition 2.15 ([15, (1.1), Theorem 3 and Proposition 4.1]). *Let V be a strongly regular, holomorphic VOA of central charge 24. If the Lie algebra V_1 is neither $\{0\}$ nor abelian, then V_1 is semisimple, and the conformal vectors of V and the subVOA generated by V_1 are the same. In addition, for any simple ideal of V_1 at level k, the identity*

$$\frac{h^\vee}{k} = \frac{\dim V_1 - 24}{24}$$

holds, where $h^\vee$ is the dual Coxeter number.

In 1993, Schellekens [43] studied the Lie algebra structures for weight one subspaces of holomorphic VOAs of central charge 24. He also determined a list of 71 possible weight one Lie algebra structures of holomorphic VOAs of $c = 24$; see Appendix for the list. This list was recently verified and reproved mathematically in [21].

As we mentioned in the introduction, it was proved that all 71 Lie algebras in Schellekens' list can be realized as the weight one Lie algebra of some holomorphic VOAs of central charge 24. Although all of them are achieved by applying orbifold construction, this approach still heavily depends on case by case analysis and there are usually several different methods to construct the same VOAs.

3. Reconstruction of Holomorphic VOAs

Very recently, Höhn [28] proposed a more uniform construction of all 71 holomorphic VOAs of central charge 24 using subVOAs of the Leech lattice VOA and other lattice VOAs. In this section, we will discuss and try to elucidate his ideas.

3.1. Lattice vertex operator algebras

We first recall the notion of lattice VOAs and review some of their properties. We use the standard notation for the lattice vertex operator algebra

$$V_L = M(1) \otimes \mathbb{C}\{L\}$$

associated with a positive definite even lattice L [23]. In particular, $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$ is an abelian Lie algebra and the bilinear form is extended to $\mathfrak{h}$ by $\mathbb{C}$-linearity. Moreover, $\hat{\mathfrak{h}} = \mathfrak{h}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}k$ is the corresponding affine algebra and $\mathbb{C}k$ is the 1-dimensional center of $\hat{\mathfrak{h}}$. The subspace $M(1) = \mathbb{C}[\alpha(n)|1 \leq i \leq d, n<0]$ for a basis $\{\alpha_1,\ldots,\alpha_d\}$ of $\mathfrak{h}$, where $\alpha(n) = \alpha\otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)\cdot 1 = 0$ for all $\alpha\in\mathfrak{h}$ and n nonnegative, and k acts as the scalar 1. Also, $\mathbb{C}\{L\} = span\{e^\beta \mid \beta\in L\}$ is the twisted group algebra of the additive group L such that $e^\beta e^\alpha = (-1)^{\langle\alpha,\beta\rangle}e^\alpha e^\beta$ for any $\alpha,\beta\in L$. The vacuum vector $\mathbf{1}$ of V_L is $1\otimes e^0$ and the Virasoro element ω is $\frac{1}{2}\sum_{i=1}^{d}\beta_i(-1)^2\cdot\mathbf{1}$, where $\{\beta_1,\ldots,\beta_d\}$ is an orthonormal basis of $\mathfrak{h}$. For the explicit definition of the corresponding vertex operators, we refer to [23] for details. We also note that V_L is strongly regular and the central charge of V_L is equal to the rank of L.

Let L^* be the dual lattice of L. For $\alpha+L\in L^*/L$, denote $V_{\alpha+L} = M(1)\otimes\mathbb{C}\{\alpha+L\}$, where $\mathbb{C}\{\alpha+L\} = Span\{e^\beta \mid \beta\in\alpha+L\}\subset\mathbb{C}\{L^*\}$. Then $V_{\alpha+L}$ is an irreducible V_L-module [23]. It was also proved in [7] that any irreducible V_L-module is isomorphic to $V_{\alpha+L}$ for some $\alpha+L\in L^*/L$. In particular, we have the following result.

Theorem 3.1. *Let L be an even unimodular, i.e., $L^* = L$. Then V_L is a strongly regular holomorphic VOA.*

3.2. Parafermion VOAs

Next we review the definition of parafermion VOAs and some of the basic properties [10, 19].

Definition 3.2. Let V be a VOA of CFT-type and W a subVOA of V. Let ω and ω' be the conformal vectors of V and W, respectively. Assume that $\omega'\in V_2$ and $L(1)\omega' = 0$. The *commutant* of W in V is defined to be the subVOA

$$W^c = \mathrm{Com}_V(W) = \{v\in V \mid w_{(n)}v = 0 \text{ for all } w\in W, n\geq 0\}$$

with conformal vector $\omega-\omega'$.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and $\widehat{\mathfrak{g}}$ the affine Kac-Moody Lie algebra associated with $\mathfrak{g}$. For a positive integer k, let $L_{\widehat{\mathfrak{g}}}(k,0)$ be the irreducible vacuum module for $\widehat{\mathfrak{g}}$ with level k. Then $L_{\widehat{\mathfrak{g}}}(k,0)$ is a simple VOA and it contains a Heisenberg VOA $M_{\mathfrak{h}}(k)$ corresponding to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The commutant $K(\mathfrak{g},k)$ of the Heisenberg VOA in $L_{\widehat{\mathfrak{g}}}(k,0)$ is called a *parafermion VOA*. In other words, $K(\mathfrak{g},k) = \mathrm{Com}_{L_{\widehat{\mathfrak{g}}}(k,0)}(M_{\mathfrak{h}}(k))$.

Let Q be the root lattice of $\mathfrak{g}$ and let Q_L be the sublattice spanned by the long roots. It is well-known that the affine VOA $L_{\widehat{\mathfrak{g}}}(k,0)$ also contains the lattice VOA $V_{\sqrt{k}Q_L}$ and that it is the commutant of the parafermion VOA $K(\mathfrak{g},k)$ in $L_{\widehat{\mathfrak{g}}}(k,0)$. Let λ be a dominant integral weight of level k of $\mathfrak{g}$ and let $L_{\widehat{\mathfrak{g}}}(k,\lambda)$ be the irreducible $L_{\widehat{\mathfrak{g}}}(k,0)$-module with highest weight λ. Then, we have the decomposition:

$$L_{\widehat{\mathfrak{g}}}(k,\lambda) = \bigoplus_{\beta_i \in Q/kQ_L} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\lambda+\beta_i)} \otimes M^{\lambda,\lambda+\beta_i}$$

as modules for $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$, where

$$M^{\lambda,\mu} = \{v \in L_{\widehat{\mathfrak{g}}}(k,\lambda) \mid h(m)v = \mu(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}.$$

It is clear that $M^{\lambda,\mu} \cong M^{\lambda,\mu'}$ if $\mu - \mu' \in kQ_L$.

3.3. Reconstruction based on simple current extensions

Next we will discuss an idea of Höhn [28], in which a more uniform construction of all 71 holomorphic VOAs of central charge 24 has been proposed. The main idea is to use certain simple current extensions of lattice VOAs and some orbifold subVOAs in the Leech lattice VOA.

Let $\mathfrak{g}$ be a Lie algebra in Schellekens' list. Suppose that $\mathfrak{g}$ is semisimple and $\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{r,k_r}$, where $\mathfrak{g}_{i,k_i}$'s are simple ideals of $\mathfrak{g}$ at level k_i. Let V be a strongly regular holomorphic VOA of central charge 24 such that $V_1 \cong \mathfrak{g}$. Then by Proposition 2.14, the subVOA U generated by V_1 is isomorphic to the tensor of simple affine VOAs as follows:

$$U \cong L_{\widehat{\mathfrak{g}_1}}(k_1,0) \otimes \cdots \otimes L_{\widehat{\mathfrak{g}_r}}(k_r,0).$$

Moreover, by Proposition 2.15, U is a full subVOA of V. For each $1 \leq i \leq r$, the affine VOA $L_{\mathfrak{g}_i}(k_i,0)$ contains the lattice VOA $V_{\sqrt{k_i}Q_L^i}$, where Q_L^i is the lattice spanned by the long roots of $\mathfrak{g}_i$. Let $Q_{\mathfrak{g}} = \sqrt{k_1}Q_L^1 \oplus \cdots \oplus \sqrt{k_r}Q_L^r$. Then, we have

$$\mathrm{Com}_U(V_{Q_{\mathfrak{g}}}) \cong K(\mathfrak{g}_1,k_1) \otimes \cdots \otimes K(\mathfrak{g}_r,k_r).$$

Set $W = \mathrm{Com}_V(V_{Q_\mathfrak{g}})$ and $X = \mathrm{Com}_V(K(\mathfrak{g}_1, k_1) \otimes \cdots \otimes K(\mathfrak{g}_r, k_r))$. Then it is clear that

$$X \supset V_{Q_\mathfrak{g}} \quad \text{and} \quad W \supset K(\mathfrak{g}_1, k_1) \otimes \cdots \otimes K(\mathfrak{g}_r, k_r).$$

Moreover, $\mathrm{Com}_V(X) = W$ and $\mathrm{Com}_V(W) = X$.

Since an extension of a lattice VOA is again a lattice VOA, there exists an even lattice $L_\mathfrak{g} > Q_\mathfrak{g}$ such that $X \cong V_{L_\mathfrak{g}}$. By [11, Corollary 12.10], $R(V_{L_\mathfrak{g}})$ has group-like fusion. In addition, $R(V_{L_\mathfrak{g}})$ is isomorphic to $\mathcal{D}(L_\mathfrak{g}) = L_\mathfrak{g}^*/L_\mathfrak{g}$ as quadratic spaces. Indeed, $R(V_{L_\mathfrak{g}})$ has the quadratic form $q : R(V_{L_\mathfrak{g}}) \to \mathbb{Z}/n\mathbb{Z}$ defined by

$$q(V_{\alpha+L_\mathfrak{g}}) = n\mathrm{wt}(V_{\alpha+L_\mathfrak{g}}) = \frac{n\langle \alpha, \alpha \rangle}{2} \bmod n,$$

where $\mathrm{wt}(\cdot)$ denotes the conformal weight of the module and n is the exponent of the discriminant group $\mathcal{D}(L_\mathfrak{g})$.

Now, assuming that $R(W)$ also forms a quadratic space isomorphic to $R(V_{L_\mathfrak{g}})$, we will analyse the structure of V. The VOA V defines a maximal totally singular subspace of $R(V_{L_\mathfrak{g}}) \times R(W)$ and hence it induces an isomorphism of quadratic spaces $\varphi : (R(V_{L_\mathfrak{g}}), q) \to (R(W), q')$ such that $q(M) + q'(\varphi(M)) = 0$ for all $M \in R(V_{L_\mathfrak{g}})$.

Conversely, let $\varphi : (R(V_{L_\mathfrak{g}}), q) \to (R(W), -q')$ be an isomorphism of quadratic spaces. Then the set $\{(M, \varphi(M)) \mid M \in R(V_{L_\mathfrak{g}})\}$ is a maximal totally singular subspace of $R(V_{L_\mathfrak{g}}) \times R(W)$ and hence $U = \bigoplus_{M \in R(V_{L_\mathfrak{g}})} M \otimes \varphi(M)$ has a structure of a holomorphic VOA. Notice that $(R(W), q') \cong (R(W), -q')$ as quadratic spaces. The key idea of Höhn is to try to describe the VOA W using a certain coinvariant sublattice of the Leech lattice Λ. The following is his main conjecture.

Conjecture 3.3. There exists an isometry $g \in O(\Lambda)$ such that $R(V_{\Lambda_g}^{\hat{g}}) \cong R(V_{L_\mathfrak{g}})$ as quadratic spaces.

Remark 3.4. In [28], Höhn also gives some description of the isometry g. He conjectured that all of them can be chosen from the standard M_{24} in $O(\Lambda)$.

If this conjecture is true, then we can reconstruct the holomorphic VOA V with $V_1 \cong \mathfrak{g}$ as a simple current extension of $V_{L_\mathfrak{g}} \otimes V_{\Lambda_g}^{\hat{g}}$. We will discuss such extensions later for some conjugacy classes of $O(\Lambda)$.

4. Orbifold VOA $V_{L_g}^{\hat{g}}$

Let L be an even unimodular lattice and g an isometry of L. In this section, we study the orbifold VOA $V_{L_g}^{\hat{g}}$. In particular, we show that the orbifold VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, i.e., $R(V_{L_g}^{\hat{g}})$ forms an abelian group with respect to the fusion product, if the order of g is prime. Recall that this is equivalent to the condition that any element in $R(V_{L_g}^{\hat{g}})$ is a simple current.

4.1. Coinvariant lattices and group-like fusion

Let V be a VOA and $g \in \mathrm{Aut}(V)$. For any irreducible module M of V, we denote the g-conjugate of M by $M \circ g$, i.e., $M \circ g = M$ as a vector space and the vertex operator $Y_{M\circ g}(u, z) = Y_M(gu, z)$ for $u \in V$.

If $V = V_L$ is a lattice VOA and $\hat{g}$ is a lift of an isometry $g \in O(L)$, then $V_{\alpha+L} \circ \hat{g} \cong V_{g\alpha+L}$ for $\alpha + L \in L^*/L$.

Proposition 4.1 ([14, Theorem 6.1]). *Let V be a simple VOA and $g \in \mathrm{Aut}(V)$. Let M be an irreducible module of V. Suppose $M \not\cong M \circ g$. Then M is also irreducible as a V^g-module and $M \cong M \circ g$ as V^g-modules.*

Therefore, if $V_L^{\hat{g}}$ has group-like fusion, then $\alpha + L = g\alpha + L$ for all $\alpha + L \in L^*/L$; otherwise, $\alpha + L \neq g\alpha + L$ for some α and $V_{\alpha+L} \boxtimes_{V_L^{\hat{g}}} V_{\alpha+L} \supset V_{2\alpha+L} + V_{\alpha+g\alpha+L}$. In our setting, i.e., L is even unimodular, this condition always holds by the following lemma.

Lemma 4.2. *Let L be an even unimodular lattice and $g \in O(L)$. Suppose $g \neq 1$. Then $(1 - g)L_g^* < L_g$ and hence $\alpha + L_g = g\alpha + L_g$ for all $\alpha + L_g \in L_g^*/L_g$.*

Proof. By the definition, it is clear that $(1-g)L$ is orthogonal to L^g. Hence, $(1 - g)L < L_g$. Since L is unimodular, the natural projection from L to L_g^* is a surjection. For any $x \in L_g^*$, let $y \in (L^g)^*$ such that $x + y \in L$. Then $(1 - g)x = (1 - g)(x + y) \in L_g$ and $(1 - g)L_g^* = (1 - g)L < L_g$. □

The following lemma will be used in the computation of quantum dimensions.

Lemma 4.3 ([26, Lemma A.1]). *Let L be a lattice. Let $g \in O(L)$ be fixed point free isometry of prime order p. Then $L/(1 - g)L$ is an elementary abelian group of order p^m, where $m(p - 1) = \mathrm{rank}(L)$.*

We will only consider a prime order isometry g of the Leech lattice Λ in the later section. To deal with a general g, one needs to answer the following question.

Question 1. Let $g \in M_{24} < O(\Lambda)$. Suppose g has the cycle shape $n_1^{k_1} \cdots n_\ell^{k_\ell}$. Is it true that

$$\frac{\Lambda_g}{(1-g)\Lambda_g} \cong \prod_{i=1}^{\ell} \mathbb{Z}_{n_i}^{k_i}$$

as an abelian group?

4.2. $\hat{g}$-twisted modules for V_L

In this subsection, we review the construction of some twisted modules for lattice VOA. The details are essentially given in [12, 36].

Let L be a positive definite even lattice with a $\mathbb{Z}$-bilinear form $\langle\ ,\ \rangle$. Define $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend $\langle\ ,\ \rangle$ $\mathbb{C}$-linearly to $\mathfrak{h}$. Let g be an isometry of L of order p. Then $\langle ga, gb\rangle = \langle a, b\rangle$ for any $a, b \in L$. Denote $\mathfrak{h}_{(n)} = \mathfrak{h}_{(n;g)} = \{\alpha \in \mathfrak{h} \mid g\alpha = \nu^n \alpha\}$ for $n \in \mathbb{Z}$, where $\nu = \nu_p = \exp(2\pi\sqrt{-1}/p)$ is a primitive p-th root of unity.

The g-twisted affine Lie algebra $\hat{\mathfrak{h}}[g]$ is the algebra $\hat{\mathfrak{h}}[g] = \bigoplus_{n\in\mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^{n/p} \oplus \mathbb{C}c$ equipped with the bracket $[x \otimes t^{m/p}, y \otimes t^{n/p}] = \langle x, y\rangle (m/p)\delta_{m+n,0}c$, for $x \in \mathfrak{h}_{(m)}, y \in \mathfrak{h}_{(n)}, m, n \in \mathbb{Z}$, and $[c, \hat{\mathfrak{h}}[g]] = 0$. Denote

$$\hat{\mathfrak{h}}[g]^+ = \coprod_{n>0} \mathfrak{h}_{(n)} \otimes t^{n/p}, \quad \hat{\mathfrak{h}}[g]^- = \coprod_{n<0} \mathfrak{h}_{(n)} \otimes t^{n/p}, \quad \text{and} \quad \hat{\mathfrak{h}}[g]^0 = \mathfrak{h}_{(0)} \otimes t^0 \oplus \mathbb{C}c,$$

and form an induced $\hat{\mathfrak{h}}[g]$-module

$$M(1)[g] = U(\hat{\mathfrak{h}}[g]) \otimes_{U(\hat{\mathfrak{h}}[g]^+ \oplus \hat{\mathfrak{h}}[g]^0)} \mathbb{C} \cong S(\hat{\mathfrak{h}}[g]^-) \quad \text{(linearly)},$$

where $\hat{\mathfrak{h}}[g]^+$ acts trivially on $\mathbb{C}$ and c acts as 1 on $\mathbb{C}$, and $U(\cdot)$ and $S(\cdot)$ denote the universal enveloping algebra and symmetric algebra, respectively.

When $g = 1$, we will denote $M(1)[g] = M(1)$. Note that $M(1)$ has a vertex operator algebra structure known as "free bosonic vertex operator algebra" (cf. [23]). Moreover, for any $g \in O(L)$, $M(1)[g]$ is an irreducible g-twisted $M(1)$-module (cf. [12, 23]).

Suppose that g is a fixed point free isometry of L of order p. As in [12, 36], set $q = p$ if p is even and $q = 2p$ if p is odd. Let $\langle \kappa_q \rangle$ be a cyclic group of order q and denote $\kappa_n = \kappa_q^{\frac{q}{n}}$ if n divides q. Define two commutator maps $c : L \times L \to \langle \kappa_q \rangle$ and $c^g : L \times L \to \langle \kappa_q \rangle$ by

$$c(\alpha, \beta) = \kappa_q^{q\langle \alpha,\beta\rangle/2} \qquad \text{and} \qquad c^g(\alpha, \beta) = \kappa_q^{\sum_{r=0}^{p-1}(q/2+xr)\langle g^r\alpha,\beta\rangle}, \tag{4.1}$$

where $x = 1$ if p is even and $x = 2$ if p is odd. Then c and c^g define two central extensions

$$1 \to \langle \kappa_q \rangle \to \hat{L} \bar{\to} L \to 1 \quad \text{and} \quad 1 \to \langle \kappa_q \rangle \to \hat{L}_g \bar{\to} L \to 1$$

such that $aba^{-1}b^{-1} = c(\bar{a}, \bar{b})$ (or $aba^{-1}b^{-1} = c^g(\bar{a}, \bar{b})$) for $a, b \in \hat{L}$ (or $a, b \in \hat{L}_g$). Note that $c(\alpha, \beta) = \kappa_2^{\langle \alpha,\beta\rangle}$ and $c^g(\alpha, \beta) = \kappa_p^{\sum_{r=0}^{p-1} r\langle g^r\alpha,\beta\rangle}$ as L is even and $\sum_{r=0}^{p-1} g^r = 0$ and one can lift g to an automorphism $\hat{g}$ of $\hat{L}$ ([23, Proposition 5.4.1]).

Remark 4.4. If p is odd, then we can take a lift $\hat{g}$ of g as the same order p.

For $\alpha, \beta \in L$, define $\varepsilon_0(\alpha, \beta) = \kappa_q^{\sum_{0<r<p/2}(q/2+xr)\langle g^r\alpha,\beta\rangle}$, where x is defined as in (4.1). Then there is a set theoretic identification between $\hat{L}$ and $\hat{L}_g$ such that the respective group multiplication $\times$ and $\times_g$ are related by $a \times b = \varepsilon_0(\alpha, \beta) a \times_g b$. Note that ε_0 is g-invariant and g can be lifted to an automorphism $\hat{g}$ of $\hat{L}_g$ (cf. [12, Remark 2.2]).

Let T be an $\hat{L}_g$-module on which $K = \{a^{-1}\hat{g}(a) | a \in \hat{L}_g\}$ acts trivially and κ_q acts as multiplication by ν_q. Then the vector space

$$V_L^T = M(1)[g] \otimes T$$

has a structure of $\hat{g}$-twisted V_L-module. Moreover, V_L^T is irreducible if and only if T is irreducible as an $\hat{L}_g$-module. The $L(0)$-weight of $1 \otimes t$, $t \in T$ is given by

$$\frac{1}{4p^2} \sum_{k=1}^{p-1} k(p-k) \dim \mathfrak{h}_{(k)}. \tag{4.2}$$

We refer to [23, 12] for the details of the corresponding vertex operators.

All the irreducible $\hat{L}_g$-modules such that $K = \{a^{-1}\hat{g}(a)|a \in \hat{L}_g\}$ acts trivially and κ_q acts as multiplication by ν_q has been classified by Lepowsky [36].

Proposition 4.5 ([36, Proposition 6.2]). *Let $R = \{a \in L\,|\,c^g(a,b) = 1$ for all $b \in L\}$, $M = (1-g)L$ and denote the pull back of R in $\hat{L}_g$ by $\hat{R}$. Then, there are exactly $|R/M|$ central character $\chi : \hat{R}/K \to \mathbb{C}^*$ such that $\chi(\kappa_q) = \nu_q$. For each such χ, there is a unique (up to equivalence) irreducible $\hat{L}_g$-module T_χ on which $\hat{R}$ acts according to χ and every irreducible $\hat{L}_g$-module on which κ_q acts as multiplication by ν_q and $K = \{a^{-1}\hat{g}(a)|a \in \hat{L}_g\}$ acts trivially is equivalent to one of these. Each of such module has dimension $|L/R|^{\frac{1}{2}}$.*

4.3. Quantum dimensions of twisted modules of V_L

In this subsection, we will compute the quantum dimensions for irreducible g-twisted modules for V_L. The calculation is a generalization of that in [23, Chapter 10] and it is essentially based on a manuscript of Abe [2].

We first recall some facts about quantum dimensions of irreducible modules of vertex operator algebras from [9]. Let V be a strongly regular VOA and let $M^0 = V, M^1, \ldots, M^p$ be all the irreducible V-modules. The *quantum dimension* of M^i is defined to be

$$\mathrm{qdim}_V M^i = \lim_{y\to 0} \frac{Z_{M^i}(iy)}{Z_V(iy)},$$

where $Z_M(\tau) = Z_M(\mathbf{1},\tau)$ is the character of M and y is real and positive.

The following result was proved in [9].

Theorem 4.6. *Let V be a strongly regular vertex operator algebra, $M^0 = V, M^1, \ldots, M^p$ be all the irreducible V-modules. Assume further that the conformal weights of $M^1, \ldots, M^p$ are greater than 0. Then*

(1) $\mathrm{qdim}_V M^i \geq 1$ *for any* $0 \leq i \leq p$;

(2) M^i *is a simple current if and only if* $\mathrm{qdim} M^i = 1$.

Next we determine the character for g-twisted V_L-modules. Let g be a fixed point free isometry of L of order p. Since the characteristic polynomial

$\det(x-g)$ of g on L has integral coefficients, eigenspaces of g whose eigenvalues are primitive k-th roots of unity have the same dimension, say, n_k. Then

$$\det(x-g)=\prod_{k|p}\Phi_k(x)^{n_k}, \tag{4.3}$$

where $\Phi_k(x)$ is the k-th cyclotomic polynomial with leading coefficient 1.

Remark 4.7. Since $\mathfrak{h}_{(j;g)}$ is the eigenspace of g of eigenvalue ν_p^j on $\mathfrak{h}$, we have $\dim\mathfrak{h}_{(j;g)}=n_k$ if ν_p^j is a primitive k-th root of unity.

Note also that for $k\in\mathbb{Z}_{>0}$,

$$x^k-1=\prod_{d|k}\Phi_d(x).$$

By the Möbius inversion formula, we also have

$$\Phi_k(x)=\prod_{d|k}(x^d-1)^{\mu(k/d)}$$

where $\mu(k)\in\mathbb{Z}$ is the Möbius function. In particular,

$$\det(x-g)=\prod_{k\in\mathbb{N},k|p}\Phi_k(x)^{n_k}=\prod_{d|p}(x^d-1)^{m_d}, \tag{4.4}$$

where $m_d=\sum_{k|p}\sum_{d|k}n_k\mu(k/d)$. By comparing the degrees of polynomials in the both hand sides, we have $\sum_{d|p}dm_d=\mathrm{rank}(L)$.

Let $r_i=\dim\mathfrak{h}_{(i;g)}$ for $i=0,1,\cdots,p-1$. Recall that g is fixed point free, i.e., $r_0=0$. Then $\det(x-g)=\prod_{i=1}^{p-1}(x-\nu_p^i)^{r_i}$. Since $x^d-1=\prod_{j=0}^{d-1}(x-\nu_p^{jp/d})$, we have

$$\prod_{i=1}^{p-1}(x-\nu_p^i)^{r_i}=\det(x-g)=\prod_{d|p}\prod_{j=0}^{d-1}(x-\nu_p^{jp/d})^{m_d}.$$

In particular, $r_i=\sum_{d|p,p|di}m_d$ for $i=1,\cdots,p-1$ and $\sum_{d|p}m_d=0$.

For $c\in\mathbb{Q}_{>0}$, let

$$a_c(\tau)=\prod_{n=0}^{\infty}(1-q^{c+n}),\quad q=e^{2\pi\sqrt{-1}\tau}$$

on the upper half plane. For $c=1$, we have

$$a_1(\tau)=\prod_{n=1}^{\infty}(1-q^n)=q^{-\frac{1}{24}}\eta(\tau),$$

where $\eta(\tau)$ is the Dedekind Eta function.

By the discussions above, we have

$$\begin{aligned}\prod_{i=1}^{p-1}a_{i/p}(\tau)^{r_i}&=\prod_{i=1}^{p-1}\prod_{d|p,p|di}a_{i/p}(\tau)^{m_d}\\&=\prod_{d|p}\prod_{j=1}^{d-1}a_{j/d}(\tau)^{m_d}\\&=\prod_{d|p}\prod_{j=1}^{d-1}\prod_{n=0}^{\infty}(1-q^{j/d+n})^{m_d}\\&=\prod_{d|p}\prod_{n=1}^{\infty}\frac{(1-q^{n/d})^{m_d}}{(1-q^n)^{m_d}}\\&=\prod_{d|p}q^{-\frac{m_d}{24d}}\eta\left(\tau/d\right)^{m_d}.\end{aligned}$$

Note that the last equation follows from $\sum_{d|p}m_d=0$.

Recall that $V_L^T=M(1)[g]\otimes T$. As a consequence, we have the following result.

Lemma 4.8. *Let $g\in O(L)$ be fixed point free. Then*

$$\begin{aligned}Z_{V_L^T}(q)&=\frac{(\dim T)q^{\left(\sum_{i=1}^{p-1}\frac{i(p-i)r_i}{4p^2}\right)-\frac{\ell}{24}}}{\prod_{j=1}^{p-1}\prod_{n=0}^{\infty}(1-q^{j/p+n})^{r_j}}\\&=(\dim T)q^{\left(\sum_{i=1}^{p-1}\frac{i(p-i)r_i}{4p^2}\right)-\frac{\ell}{24}}\prod_{d|p}\frac{q^{\frac{m_d}{24d}}}{\eta(\tau/d)^{m_d}},\end{aligned}$$

where $\ell=\operatorname{rank}(L)$.

Recall that the character of V_L is given by

$$Z_{V_L}(q)=\frac{\Theta_L(\tau)}{\eta(\tau)^{\ell}},$$

where $\Theta_L(\tau)$ is the theta function of L. Then we have

$$\frac{Z_{V_L^T}(iy)}{Z_{V_L}(iy)} = (\dim T) q^{\left(\sum_{i=1}^{p-1} \frac{i(p-i)r_i}{4p^2}\right) - \frac{\ell}{24} + \sum_{d|p} \frac{m_d}{24d}} \times \frac{\eta(iy)^\ell}{\prod_{d|p} \eta(iy/d)^{m_d}} \frac{1}{\Theta_L(iy)}. \tag{4.5}$$

Proposition 4.9. *Let* $v = \sqrt{|\mathcal{D}(L)|}$ *(i.e., v is the volume of L in $\mathbb{R}^\ell$). Then we have*

$$\lim_{y\to 0^+} \frac{\eta(iy)^\ell}{\prod_{d|p} \eta(iy/d)^{m_d}} \frac{1}{\Theta_L(iy)} = \frac{v}{\sqrt{\prod_{d|p} d^{m_d}}}.$$

Proof. Recall that

$$\eta(-\tau^{-1}) = \sqrt{-i\tau}\eta(\tau).$$

This implies

$$\eta(iy) = y^{-1/2}\eta(i/y) = y^{-1/2} e^{-2\pi/y} \prod_{n=1}^{\infty} (1 - e^{-2\pi n/y})$$

and

$$\eta(iy/d) = y^{-1/2} d^{1/2} \eta(id/y) = y^{-1/2} d^{1/2} e^{-2\pi d/y} \prod_{n=1}^{\infty} (1 - e^{-2\pi n d/y}).$$

It is also well-known [20, Proposition 2.1] that

$$\Theta_L(iy) = y^{-\ell/2} v^{-1} \Theta_{L^*}(iy^{-1}).$$

Therefore, we have

$$\frac{\eta(iy)^\ell}{\prod_{d|p} \eta(iy/d)^{m_d}} \frac{1}{\Theta(iy)} = \frac{y^{-\ell/2} e^{-2\pi\ell/y} y^{\ell/2} v \prod_{n=1}^{\infty}(1 - e^{-2\pi n/y})}{\prod_{d|p} y^{-m_d/2} d^{m_d/2} e^{-2\pi d m_d/y} \Theta_{L^*}(i/y) \prod_{n=1}^{\infty}(1 - e^{-2\pi n d/y})}.$$

Since $\sum_{d|p} m_d = 0$ and $\sum_{d|p} d m_d = \ell$, we have

$$\frac{\eta(iy)^\ell}{\prod_{d|p} \eta(\tau/d)^{m_d}} \frac{1}{\Theta_L(iy)} = \frac{v \prod_{n=1}^{\infty}(1 - e^{-2\pi n/y})}{\prod_{d|p} d^{m_d/2} \Theta_{L^*}(i/y) \prod_{n=1}^{\infty}(1 - e^{-2\pi n d/y})}$$

$$\longrightarrow \frac{v}{\sqrt{\prod_{d|p} d^{m_d}}}$$

as $y \to 0^+$, that is $e^{-c/y} \to 0$ for $c \in \mathbb{R}_{>0}$. □

Since $q = e^{-2\pi y} \to 1^-$ as $y \to 0^+$, the term $q^{\left(\sum_{i=1}^{p-1} \frac{i(p-i)r_i}{4p^2}\right) - \frac{\ell}{24} + \sum_{d|p} \frac{p_d}{24d}}$ tends to 1. Combining (4.5) and Proposition 4.9, we have the following theorem.

Theorem 4.10. *Let L be positive definite even lattice of rank ℓ. Let g be a fixed point free isometry of L of order p. Let $\hat{g}$ be a lift of g. For any $\widehat{L}_g$-module T, the quantum dimension of the $\hat{g}$-twisted module V_L^T exists and*

$$\mathrm{qdim}_{V_L} V_L^T := \lim_{q \to 1^-} \frac{Z_{V_L^T}(q)}{Z_{V_L}(q)} = \frac{v \dim T}{\prod_{d|p} d^{m_d/2}},$$

where $v = \sqrt{|\mathcal{D}(L)|}$ and m_d are integers given by $\det(x-g) = \prod_{d|p}(x^d-1)^{m_d}$.

As a corollary, we have

Corollary 4.11. *Let L be an even unimodular lattice and let $g \in O(L)$ be of prime order. Then*

$$\mathrm{qdim}_{V_{L_g}} V_{L_g}^T = 1$$

for any irreducible $(\widehat{L_g})_g$-module T.

Proof. First we note that $(1-g)L = (1-g)L_g^*$ since L is unimodular. By definition, the action of g on L_g is fixed point free. It follows from Proposition 4.5 that $(\dim T)^2 = |L/R|$. Note that $(1-g)L < R$. Moreover, there are exactly $|R/(1-g)L_g|$ irreducible $\hat{g}$-twisted modules for V_{L_g} by Proposition 4.5. On the other hand, by [13, Theorem 10.2], there are at most L_g^*/L_g irreducible $\hat{g}$-twisted modules for V_{L_g} since all irreducible V_{L_g}-modules are $\hat{g}$-invariant. Hence we must have $R = (1-g)L$ and $R/(1-g)L_g \cong \mathcal{D}(L_g)$. Since g is fixed point free on L_g, there is an $m \in \mathbb{Z}_+$ such that $(p-1)m = \mathrm{rank}(L_g)$. Then $\det(x-g)|_{L_g} = (x^p-1)^m/(x-1)^m$ and $|L_g/(1-g)L_g| = p^m$ by Lemma

4.3. Thus, by Theorem 4.10, we have

$$\begin{aligned}
(\mathrm{qdim}_{V_{L_g}} V_{L_g}^T)^2 &= \frac{1}{p^m}\left(\left|\frac{L_g^*}{L_g}\right| \cdot \dim(T)^2\right) \\
&= \frac{1}{p^m}\left(\left|\frac{(1-g)L_g^*}{(1-g)L_g}\right| \cdot \left|\frac{L_g}{(1-g)L}\right|\right) \\
&= \frac{1}{p^m} \cdot \left|\frac{L_g}{(1-g)L_g}\right| \\
&= \frac{1}{p^m} \cdot p^m = 1
\end{aligned}$$

as desired. □

Theorem 4.12. *Let L be an even unimodular lattice. Let g be an order p element in $O(L)$ and $\hat{g}$ a lift of g of order p. Then the VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, namely, all irreducible modules of $V_{L_g}^{\hat{g}}$ are simple currents.*

Proof. Since $V_{L_g}^{\hat{g}}$ is C_2-cofinite and rational [40, 4], it was proved in [18] that any irreducible $V_{L_g}^{\hat{g}}$-module is a submodule of an irreducible $\hat{g}^i$-twisted V_{L_g}-module for some $0 \le i \le p-1$. In addition, since the lowest conformal weights of irreducible irreducible $\hat{g}^i$-twisted V_{L_g}-modules are positive for $1 \le i \le p-1$, so are irreducible $V_{L_g}^{\hat{g}}$-modules except for $V_{L_g}^{\hat{g}}$ itself.

Let M be an irreducible g^i-twisted V_{L_g}-module for some $0 \le i \le p-1$. It follows from [18, Corollary 4.5] and Corollary 4.11 that

$$\mathrm{qdim}_{V_{L_g}^{\hat{g}}} M = p.$$

For $i = 0$, it follows from Lemma 4.2 that all irreducible $V_{L_\mathfrak{g}}$-modules are $\hat{g}$-stable. For $1 \le i \le p-1$, it is also known [12, 23] that all irreducible g^i-twisted $V_{L_\mathfrak{g}}$-modules are $\hat{g}$-stable. Hence, the eigenspace decomposition of $\hat{g}$ on any g^i-twisted V_{L_g}-module M gives a direct sum of p irreducible $V_{L_g}^{\hat{g}}$-submodules of M. Hence, by Theorem 4.6, every irreducible $V_{L_g}^{\hat{g}}$-submodule of M has quantum dimension 1, and thus it is a simple current. Hence all irreducible $V_{L_g}^{\hat{g}}$-modules are simple currents. □

Finally, we end this section with the following question about the Leech lattice Λ.

Question 2. For which $g \in O(\Lambda)$, the VOA $V_{\Lambda_g}^{\hat{g}}$ has group-like fusion?

5. Construction of holomorphic VOAs using Leech lattice

In this section, we will realize the construction proposed by Höhn using some $g \in O(\Lambda)$ of prime order.

5.1. g has the cycle shape $1^8 2^8$

Let $g \in O(\Lambda)$ be an element in the conjugacy class $2A$. Then g is conjugate to an element in the standard M_{24} of cycle shape $1^8 2^8$. The fixed point sublattice $\Lambda^g \cong BW_{16}$, the Barnes-Wall lattice of rank 16 and the coinvariant sublattice $\Lambda_g \cong \sqrt{2}E_8$. Moreover, g acts on Λ_g as the (-1)-isometry.

The following theorem is proved in [44] (see also [25]).

Theorem 5.1 ([44]). *We have* $R(V^+_{\sqrt{2}E_8}) \cong 2^{10}_+$ *and* $\mathrm{Aut}(V^+_{\sqrt{2}E_8}) \cong O^+(10,2)$.

Theorem 5.2 ([41]). *Up to isometry, there are exactly* 17 *even lattices of level* 2 *which has rank* 16 *and determinant* 2^6. *They are uniquely determined by their root systems, and the corresponding root systems are as follows:*

$$16A_1, \quad 4A_3\, 4\,{}^2A_1, \quad 2D_4\, 4C_2, \quad 2A_5\, 2\,{}^2A_2C_2, \quad 2D_5\, 2\,{}^2A_3, \quad A_7\,{}^2A_3\, 2C_3,$$
$$D_6C_4\, 2\,{}^2B_3, \quad A_9\,{}^2A_4\,{}^2B_3, \quad 4C_4, \quad E_6\,{}^2A_5C_5, \quad 2C_6\,{}^2B_4, \quad D_8\, 2\,{}^2B_4,$$
$$D_9\,{}^2A_7, \quad E_7\,{}^2B_5\, F_4, \quad C_8\, 2F_4, \quad C_{10}\,{}^2B_6, \quad E_8\,{}^2B_8,$$

where mX_n *denote the union of mutually disjoint orthogonal* m *root systems of type* X_n.

Theorem 5.3. *Let* X *be one of the 17 lattices listed in Theorem* 5.2 *and let* $L = {}^2(X^*)$. *Then* L *is an even lattice of level* 2 *and* $\mathcal{D}(L) \cong 2^{10}$ *is a non-singular quadratic space.*

Proof. Since X has level 2, $L = \sqrt{2}X^*$ is also even (see Remark 2.4). Moreover, $L^* = \frac{1}{\sqrt{2}}X$ and hence $\sqrt{2}(L^*) = X$ is again even and L has level 2. In this case, we also have

$$\frac{L^*}{L} = \frac{\frac{1}{\sqrt{2}}X}{\sqrt{2}X^*} \cong \frac{X}{2X^*} \cong 2^{10}.$$

Let $x \in X$ such that $\langle x, y\rangle \in 2\mathbb{Z}$ for all $y \in X$. Then $x \in 2X^*$. Hence, $X/2X^*$ is a non-singular quadratic space with respect to the quadratic form $q(x) = \langle x, x\rangle/2 \bmod 2$. □

Remark 5.4. It turns out that all lattices in Theorem 5.3 are in the same genus [41] and $\mathcal{D}(L) \cong 2_+^{10}$.

Recall from [45] that the (lowest) conformal weight $\mathrm{wt}(N)$ for an irreducible $V_{\sqrt{2}E_8}^+$-module N is given as follows in terms of quadratic spaces:

$$\mathrm{wt}(N) = \begin{cases} 0 & \text{if } N = V_{\sqrt{2}E_8}^+, \\ 1 & \text{if } N \text{ is singular and } N \neq V_{\sqrt{2}E_8}^+, \\ 1/2 & \text{if } N \text{ is non-singular.} \end{cases}$$

Moreover, the dimension of the top weight space is 1 if N is non-singular and is 8 if N is singular but $N \neq V_{\sqrt{2}E_8}^+$. Now let $\varphi : (R(V_L), q) \to (R(V_{\sqrt{2}E_8}^+), -q)$ be an isomorphism of quadratic spaces. Then $U = \bigoplus_{M \in R(V_L)} M \otimes \varphi(M)$ is a holomorphic VOA of central charge 24. Moreover,

$$\dim U_1 = \sum_{M \in R(V_L)} \dim(M \otimes \varphi(M))_1.$$

Since $(V_{\sqrt{2}E_8}^+)_1 = 0$ and the lowest conformal weight of $(M \otimes \varphi(M)) > 1$ for any singular $M \neq V_L$, we have

$$\dim U_1 = \dim(V_L)_1 + \sum_{M \in R(V_L), \mathrm{wt}(M)=1/2} \dim M_{1/2}.$$

Since $L^* = \frac{1}{\sqrt{2}} X$ and $\dim(V_{\lambda+L})_{1/2} = |\{\alpha \in \lambda + L \mid \langle \alpha, \alpha \rangle = 1\}|$, we also have

$$\sum_{M \in R(V_L), \mathrm{wt}(M)=1/2} \dim M_{1/2} = |X_2|.$$

Hence, we have $\dim U_1 = \dim(V_L)_1 + |X_2| = \mathrm{rank}(L) + |L_2| + |X_2|$. Recall from Remark 2.8 that the set

$$R(L) = \{v \in L \mid \langle v, v \rangle = 2\} \cup \{v \in \sqrt{2}X < L \mid \langle v, v \rangle = 4\}$$

defines the root system for the lattice L and $|R(L)| = |L_2| + |X_2|$.

In Schellekens' list [43], there are exactly 17 holomorphic VOAs of central charge 24 such that V_1 has rank 16 and the level of the irreducible components are powers of 2. The corresponding root systems match with the root systems of the lattices given in Theorem 5.2. The detail is given in Table 1.

Table 1: Rank 16 Lie algebras with level 2

$V_1 = \mathfrak{g}$	$Q_{\mathfrak{g}}$	Root system of $\sqrt{2}L_{\mathfrak{g}}^*$
${A_{1,2}}^{16}$	$\sqrt{2}A_1^{16}$	$16A_1$
${A_{1,1}}^4{A_{3,2}}^4$	$A_1^4+\sqrt{2}A_3^4$	$4A_3\ 4^2A_1$
${C_{2,1}}^4{D_{4,2}}^2$	$(\sqrt{2}\mathbb{Z}^2)^4+\sqrt{2}D_4^2$	$2D_4\ 4C_2$
${A_{2,1}}^2{A_{5,2}}^2C_{2,1}$	$A_2^2+\sqrt{2}A_5^2+\sqrt{2}\mathbb{Z}^2$	$2A_5\ 2^2A_2\ C_2$
$A_{3,1}A_{7,2}{C_{3,1}}^2$	$A_3+\sqrt{2}A_7+(\sqrt{2}\mathbb{Z}^3)^2$	$A_7\ {}^2A_3\ 2C_3$
${A_{3,1}}^2{D_{5,2}}^2$	$A_3^2+\sqrt{2}D_5^2$	$2D_5\ 2^2A_3$
${C_{4,1}}^4$	$(\sqrt{2}\mathbb{Z}^4)^4$	$4C_4$
${B_{3,1}}^2C_{4,1}D_{6,2}$	$D_3^2+\sqrt{2}\mathbb{Z}^4+\sqrt{2}D_6$	$D_6\ C_4\ 2^2B_3$
$A_{4,1}A_{9,2}B_{3,1}$	$A_4+\sqrt{2}A_9+D_3$	$A_9\ {}^2A_4\ {}^2B_3$
$A_{5,1}C_{5,1}E_{6,2}$	$A_5+\sqrt{2}\mathbb{Z}^5+\sqrt{2}E_6$	$E_6\ {}^2A_5C_5$
$B_{4,1}{C_{6,1}}^2$	$D_4+(\sqrt{2}\mathbb{Z}^6)^2$	$2C_6{}^2B_4$
${B_{4,1}}^2D_{8,2}$	$D_4^2+\sqrt{2}D_8$	$D_8\ 2\,{}^2B_4$
$A_{7,1}D_{9,2}$	$A_7+\sqrt{2}D_9$	$D_9\ {}^2A_7$
$C_{8,1}{F_{4,1}}^2$	$\sqrt{2}\mathbb{Z}^8+D_4^2$	$C_8\ 2F_4$
$B_{5,1}E_{7,2}F_{4,1}$	$D_5+\sqrt{2}E_7+D_4$	$E_7\ {}^2B_5\ F_4$
$B_{6,1}C_{10,1}$	$D_6+\sqrt{2}\mathbb{Z}^{10}$	$C_{10}{}^2B_6$
$B_{8,1}E_{8,2}$	$D_8+\sqrt{2}E_8$	$E_8\ {}^2B_8$

Remark 5.5. The construction of the 17 holomorphic VOAs in Schellekens' list with Lie rank 16 using $V_{\sqrt{2}E_8}^+$ and rank 16 lattices with determinant 2^{10} has also been discussed in [29].

5.2. g has the cycle shape 1^63^6

Let $g \in O(\Lambda)$ be an element in the conjugacy class $3B$. Then g is conjugate to an element in the standard M_{24} of cycle shape 1^63^6. In this case, the fixed point lattice and the coinvariant lattice are both isometric to the Coxeter-Todd lattice K_{12} of rank 12.

Theorem 5.6 ([5]). *Let g be a fixed point free isometry of order 3 in $Z(O(K_{12}))$ and $\hat{g}$ a lift of g in $\mathrm{Aut}(V_{K_{12}})$. Then $R(V_{K_{12}}^{\hat{g}}) \cong 3_-^8$ and $\mathrm{Aut}(V_{K_{12}}^{\hat{g}}) \cong \Omega^-(8,3).2$.*

Recall also from [5] that the fusion group $R(V_{K_{12}}^{\hat{g}})$ forms a quadratic space with the quadratic form $q(N) = 3\mathrm{wt}(N) \bmod 3$, where $\mathrm{wt}(N)$ denotes the lowest conformal weight of an irreducible module N. Moreover,

$$\mathrm{wt}(N) = \begin{cases} 1 & \text{if } q(N) = 0 \text{ but} N \neq V_{K_{12}}^{\hat{g}}, \\ 4/3 & \text{if } q(N) = 1, \\ 2/3 & \text{if } q(N) = 2, \end{cases}$$

and the dimension of the top weight space is 1 if $q(N) = 2$.

Theorem 5.7 ([41])**.** *Up to isometry, there are exactly* 6 *even lattices of level* 3 *which has rank* 12 *and determinant* 3^4*. They are uniquely determined by their root systems, and the corresponding root systems are as follows:*

$$6A_2, \quad A_5D_4\ 3\,{}^3\!A_1, \quad A_8\ 2\,{}^3\!A_2, \quad D_7\ {}^3\!A_3G_2, \quad E_6\ 3G_2, \quad E_7\ {}^3\!A_5.$$

By the same argument as in Theorem 5.3, we have the theorem.

Theorem 5.8. *Let X be one of the* 6 *lattices listed in Theorem 5.7 and let $L = {}^3(X^*)$. Then L is an even lattice of level* 3 *and $\mathcal{D}(L) \cong 3^8$ is a non-singular quadratic space.*

Remark 5.9. Notice that all lattices in Theorem 5.8 are in the same genus and $\mathcal{D}(L) \cong 3^{8}_{-}$ (cf. [41]).

Now let $\varphi : (R(V_L), q) \to (R(V_{K_{12}}^{\hat{g}}), -q)$ be an isomorphism of quadratic spaces. Then $U = \bigoplus_{M \in R(V_L)} M \otimes \varphi(M)$ is a holomorphic VOA of central charge 24. By the similar argument as in Section 5.1, we have

$$\dim U_1 = \mathrm{rank}(L) + |L_2| + |X_2|.$$

Notice that $R(L) = \{v \in L \mid \langle v, v\rangle = 2\} \cup \{\sqrt{3}x \in L \mid x \in X, \langle x, x\rangle = 2\}$ is the root system of L (cf. Remark 2.8).

In Schellekens' list [43], there are exactly 6 holomorphic VOAs of central charge 24 such that V_1 has rank 12 and the level of the irreducible component is 1 or 3. The corresponding root systems also match with the root systems of the lattices given in Theorem 5.7. The detail is given in Table 2.

Table 2: Rank 12 Lie algebras with level 3

$V_1 = \mathfrak{g}$	$Q_{\mathfrak{g}}$	Root system of $\sqrt{3}L_{\mathfrak{g}}^*$
${A_{2,3}}^6$	$\sqrt{3}A_2^6$	$6A_2$
${A_{1,1}}^3 A_{5,3} D_{4,3}$	$A_1^3 + \sqrt{3}A_5 + \sqrt{3}D_4$	$A_5 D_4\ 3\,{}^3\!A_1$
$A_{8,3} {A_{2,1}}^2$	$\sqrt{3}A_8 + A_2^2$	$A_8\ 2\,{}^3\!A_2$
$A_{3,1} D_{7,3} G_{2,1}$	$A_3 + \sqrt{3}D_7 + A_2$	$D_7\ {}^3\!A_3 G_2$
$E_{6,3} {G_{2,1}}^3$	$\sqrt{3}E_6 + A_2^3$	$E_6\ 3G_2$
$A_{5,1} E_{7,3}$	$A_5 + \sqrt{3}E_7$	$E_7\ {}^3\!A_5$

5.3. g has the cycle shape $1^4 5^4$

Let g be an element in the conjugacy class $5B$. The fixed point sublattice and coinvariant sublattice of g on Λ has been studied in [26]. The fixed point sublattice Λ^g has rank 8 and the coinvariant sublattice Λ_g has rank 16. Both lattices has the discriminant group isomorphic to 5_+^4. Moreover, $O(\Lambda^g) \cong O^+(4,5)$ and $O(\Lambda_g)$ is isomorphic to an index 2 subgroup of $\mathrm{Frob}(20) \times O^+(4,5)$ which contains neither direct factor. By Theorem 4.12, $V_{\Lambda_g}^{\hat{g}}$ has group-like fusion.

Since Λ_g has level 5, the lattice $Y = \sqrt{5}\Lambda_g^*$ is even and $\Lambda^* = \frac{1}{\sqrt{5}}Y$. Thus, all irreducible modules for V_{Λ_g} have conformal weights in $\frac{1}{5}\mathbb{Z}$. By (4.2), the (lowest) conformal weight for $V_{\Lambda_g}^T$ is given by

$$\frac{1}{4\cdot 5^2}\sum_{i=1}^{4} i(5-i)\dim \mathfrak{h}_{(i)} = \frac{1}{100}(4+6+6+4)\cdot 4 = \frac{4}{5}.$$

Therefore, all irreducible $V_{\Lambda_g}^{\hat{g}}$-modules have conformal weights in $\frac{1}{5}\mathbb{Z}$.

Recall from [21, Proposition 3.5] that the map $q_\Delta : R(V_{\Lambda_g}^{\hat{g}}) \to \mathbb{Q}/\mathbb{Z}$ defined by the conformal weight modulo $\mathbb{Z}$ is a quadratic form with respect to the fusion product. Moreover, the corresponding bilinear form $b_\Delta(M,N) := q_\Delta(M \boxtimes N) - q_\Delta(M) - q_\Delta(N)$ is non-degenerate. Since all irreducible $V_{\Lambda_g}^{\hat{g}}$-modules have conformal weights in $\frac{1}{5}\mathbb{Z}$, we have

$$b_\Delta(M^{\boxtimes 5}, N) = 5b_\Delta(M,N) \equiv 0 \bmod \mathbb{Z}$$

for all $M, N \in R(V_{\Lambda_g}^{\hat{g}})$. Thus, $M^{\boxtimes 5} = V_{\Lambda_g}^{\hat{g}}$ and $R(V_{\Lambda_g}^{\hat{g}})$ forms an elementary abelian 5-group and the order is 5^6.

Notice that there are at least two level 5 lattices of rank 8 with determinant 5^2. They are A_4^2 and an over-lattice X of $R = D_6 + \sqrt{5}A_1^2$ such that $[X : R] = 2^2$. In Schellekens' list, there are also 2 holomorphic VOAs of central charge 24 such that the level of the irreducible components are 1 and 5. The weight one Lie algebras have the type $A_{4,5}^2$ and $D_{6,5}A_{1,1}^2$. The corresponding lattice $L_{\mathfrak{g}}$ are $\sqrt{5}(A_4^*)^2$ and an overlattice of $\sqrt{5}D_6 + A_1^2$, which is equal to $\sqrt{5}X^*$.

5.4. g has the cycle shape $1^3 7^3$

Let g be an element in the conjugacy class $7B$. Then g is conjugate to an element in the standard M_{24} of cycle shape $1^3 7^3$. The fixed point sublattice has rank 6. By Lemma 4.3, we have $\mathcal{D}(\Lambda^g) \cong \mathcal{D}(\Lambda_g) \cong 7^3$. Moreover, $R(V_{\Lambda_g}^{\hat{g}})$ has group-like fusion by Theorem 4.12. By (4.2), the lowest conformal weight of any irreducible $\hat{g}$-twisted module $V_{\Lambda_g}^T$ is $6/7 \in (1/7)\mathbb{Z}$. By a similar argument as in Section 5.3, we have $R(V_{\Lambda_g}^{\hat{g}}) \cong 7^5$. Recall that an even lattice of rank 6 with determinant 7 is unique up to isometry, and it is isometric to the root lattice of A_6. Hence, $\sqrt{7}A_6^*$ is the only even lattice of level 7 with determinant 7^5.

In Schellekens' list, there is only one holomorphic VOA of central charge 24 such that the level of the irreducible component is 7. The weight one Lie algebra $V_1 = \mathfrak{g}$ has type $A_{6,7}$. The corresponding lattice $L_{\mathfrak{g}}$ is isometric to $\sqrt{7}A_6^*$ and $\sqrt{7}L_{\mathfrak{g}}^* \cong A_6$.

Remark 5.10. In [29], the case when $g \in O(\Lambda)$ has the cycle shape 2^{12} was studied. In particular, the construction of holomorphic VOAs of central 24 and weight one Lie algebras of the type $B_{12,2}$, $B_{6,2}^2$, $B_{4,2}^3$, $B_{3,2}^4$, $B_{2,2}^6$, $A_{1,4}^{12}$, $A_{8,2}F_{4,2}$, $C_{4,2}A_{4,2}^2$ and $D_{4,4}A_{2,2}^4$ was considered. Recall that $\Lambda^g \cong \Lambda_g \cong \sqrt{2}D_{12}^+$ in this case and $R(V_{\sqrt{2}D_{12}^+}^+) \cong 2^{10}4^2$ as an abelian group ([1]).

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Appendix A. Table of 71 holomorphic VOAs of central charge 24

In this appendix, we give a list of 71 holomorphic VOAs of central charge 24.

Table 3: 71 holomorphic VOAs of central charge 24.

$\dim(V_1)$	Lie algebra	rank	Existence	$\dim(V_1)$	Lie algebra	rank	Existence
0	$\emptyset$	0	[23]	24	$U(1)^{24}$	24	[23]
36	$C_{4,10}$	4	[21]	36	$A_{2,6}D_{4,12}$	6	[21]
36	${A_{1,4}}^{12}$	12	[8]	48	$A_{6,7}$	6	[35]
48	${A_{4,5}}^{2}$	8	[21]	48	${A_{2,3}}^{6}$	12	[42]
48	$A_{1,2}D_{5,8}$	6	[31]	48	$A_{1,2}A_{5,6}C_{2,3}$	8	[34]
48	$A_{1,2}{A_{3,4}}^{3}$	10	[31]	48	${A_{1,2}}^{16}$	16	[8]
60	${C_{2,2}}^{6}$	12	[8]	60	$A_{2,2}F_{4,6}$	6	[32]
60	${A_{2,2}}^{4}D_{4,4}$	12	[33]	72	$A_{1,1}C_{5,3}G_{2,2}$	8	[21]
72	${A_{1,1}}^{2}D_{6,5}$	8	[34]	72	${A_{1,1}}^{2}C_{3,2}D_{5,4}$	10	[31]
72	${A_{1,1}}^{3}A_{7,4}$	10	[31]	72	${A_{1,1}}^{3}A_{5,3}D_{4,3}$	12	[42]
72	${A_{1,1}}^{4}{A_{3,2}}^{4}$	16	[8]	72	${A_{1,1}}^{24}$	24	[23]
84	${B_{3,2}}^{4}$	12	[8]	84	${A_{4,2}}^{2}C_{4,2}$	12	[33]
96	${C_{2,1}}^{4}{D_{4,2}}^{2}$	16	[8]	96	$A_{2,1}C_{2,1}E_{6,4}$	10	[21]
96	${A_{2,1}}^{2}A_{8,3}$	12	[34]	96	${A_{2,1}}^{2}{A_{5,2}}^{2}C_{2,1}$	16	[31]
96	${A_{2,1}}^{12}$	24	[23]	108	${B_{4,2}}^{3}$	12	[8]
120	$E_{6,3}{G_{2,1}}^{3}$	12	[39]	120	$A_{3,1}D_{7,3}G_{2,1}$	12	[34]
120	$A_{3,1}C_{7,2}$	10	[31]	120	$A_{3,1}A_{7,2}{C_{3,1}}^{2}$	16	[31]
120	${A_{3,1}}^{2}{D_{5,2}}^{2}$	16	[8]	120	${A_{3,1}}^{8}$	24	[23]
132	$A_{8,2}F_{4,2}$	12	[33]	144	${C_{4,1}}^{4}$	16	[8]
144	${B_{3,1}}^{2}C_{4,1}D_{6,2}$	16	[8]	144	$A_{4,1}A_{9,2}B_{3,1}$	16	[33]
144	${A_{4,1}}^{6}$	24	[23]	156	${B_{6,2}}^{2}$	12	[8]
168	${D_{4,1}}^{6}$	24	[23]	168	$A_{5,1}E_{7,3}$	12	[34]
168	$A_{5,1}C_{5,1}E_{6,2}$	16	[33]	168	${A_{5,1}}^{4}D_{4,1}$	24	[23]
192	$B_{4,1}{C_{6,1}}^{2}$	16	[31]	192	${B_{4,1}}^{2}D_{8,2}$	16	[8]
192	${A_{6,1}}^{4}$	24	[23]	216	$A_{7,1}D_{9,2}$	16	[8]
216	${A_{7,1}}^{2}{D_{5,1}}^{2}$	24	[23]	240	$C_{8,1}{F_{4,1}}^{2}$	16	[33]
240	$B_{5,1}E_{7,2}F_{4,1}$	16	[33]	240	${A_{8,1}}^{3}$	24	[23]
264	${D_{6,1}}^{4}$	24	[23]	264	${A_{9,1}}^{2}D_{6,1}$	24	[23]
288	$B_{6,1}C_{10,1}$	16	[33]	300	$B_{12,2}$	12	[8]
312	${E_{6,1}}^{4}$	24	[23]	312	$A_{11,1}D_{7,1}E_{6,1}$	24	[23]
336	${A_{12,1}}^{2}$	24	[23]	360	${D_{8,1}}^{3}$	24	[23]
384	$B_{8,1}E_{8,2}$	16	[33]	408	$A_{15,1}D_{9,1}$	24	[23]
456	$D_{10,1}{E_{7,1}}^{2}$	24	[23]	456	$A_{17,1}E_{7,1}$	24	[23]
552	${D_{12,1}}^{2}$	24	[23]	624	$A_{24,1}$	24	[23]
744	${E_{8,1}}^{3}$	24	[23]	744	$D_{16,1}E_{8,1}$	24	[23]
1128	$D_{24,1}$	24	[23]				

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