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CLOSED RANGE PROPERTY FOR $\overline{\partial}$ ON THE POINCÁRE DISK

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Abstract

Making use of the Poincáre inequality with respect to a complete metric on the real line, we will give an elementary proof of the closed range property for $\overline{\partial}$-operator on the unit disk endowed with Poincáre metric.

1. Introduction

$\overline{\partial}$-equation plays a central role in complex analysis and geometry. On bounded domains in $\mathbb{C}^n$, there are two pioneer work related to the existence and regularity of the $\overline{\partial}$-equation see ([7], [8], [6], [10]).

Theorem 1.1 (Hörmander). *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $f \in L^2_{(p,q)}(\Omega)$ with $\overline{\partial} f = 0$ in the sense of distribution, where $0 \leq p \leq n, 1 \leq q \leq n$. Then there exists $u \in L^2_{(p,q-1)}(\Omega)$ such that $\overline{\partial} u = f$. Moreover, $\|u\| \leq C\|f\|$, where C is a constant only depending on the diameter of Ω and q.*

Theorem 1.1 tells us that on bounded pseudoconvex domains the $\overline{\partial}$-equation always have solutions. This is equivalent to say that the cohomology $H^{p,q}_{L^2,\overline{\partial}}(\Omega) := \frac{\mathrm{Ker}\overline{\partial}}{\mathrm{Im}\overline{\partial}}$ associated to the $\overline{\partial}$-operator vanishes for any $q \geq 1$. Thus, the range of $\overline{\partial}$-operator denoted by $\mathrm{Rang}(\overline{\partial})$ is a closed subspace of $L^2_{(p,q)}(\Omega)$.

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When the boundary $\partial\Omega$ is smooth, we have the boundary regularity for $\overline{\partial}$-equation.

Theorem 1.2 (Kohn). *Let $\Omega \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^∞ smooth boundary. For any $f \in C^\infty_{(p,q)}(\overline{\Omega})$ with $\overline{\partial} f = 0$ for $0 \le p \le n, 1 \le q \le n$ there exists $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ such that $\overline{\partial} u = f$.*

When the domain is not pseudoconvex, there are also plentiful results (see [12], [13], [14]) related to the existence and regularity for $\overline{\partial}$-equation. Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbb{C}^n, n \ge 3$ with $\Omega_2 \Subset \Omega_1$. Put $\Omega = \Omega_1 \setminus \overline{\Omega_2}$. Then the subelliptic estimate does not hold on Ω in general. Making use of the growing weights $e^{-t|z|^2}$ when t large, for $1 \le q \le n-2$, the author in [12] established a weaker estimate than the one obtained by Hörmander [7] on pseudoconvex domains which is sufficient to prove that $H^{p,q}_{L^2,\overline{\partial}}(\Omega)$ is a finite dimensional space. This also implies that the range of $\overline{\partial}$-operator from $L^2_{p,q-1}(\Omega)$ to $L^2_{p,q}(\Omega)$ is a closed subspace. In a recent work [14], Shaw completely solved the $\overline{\partial}$-problems on annulus with smooth boundaries in $\mathbb{C}^n$.

Theorem 1.3 (Shaw). *Let Ω be the annulus between two bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries. If we denote by $H^{p,q}_{L^2,\overline{\partial}}(\Omega)$ the cohomology associated to the $\overline{\partial}$-operator, then $H^{p,q}_{L^2,\overline{\partial}}(\Omega) = 0$ for any $0 \le p \le n, 1 \le q \le n-2$. In the critical case, for $q = n-1$, $H^{p,n-1}_{L^2,\overline{\partial}}(\Omega) = \infty$.*

When studying the extension of CR functions from the boundary of a complex manifold or the extension of CR structures to complex structures, it is useful to consider the $\overline{\partial}$-problems on domains with mixed boundary conditions. For this subject, we refer the readers to [1, 2, 3, 9, 11].

Related to the $\overline{\partial}$-problems, there are also generous results related to the closed range property for the $\overline{\partial}$-operator. In the view of functional analysis, if we denote by $\mathrm{Rang}(\overline{\partial})$ the range of $\overline{\partial}$ in the L^2-setting which is closed, then it will give us probability to solve the $\overline{\partial}$-equation. In [14], Shaw proved that the $\overline{\partial}$-operator has closed range property in the critical case when $q = n-1$ on annulus between two bounded pseudoconvex domains with smooth boundaries although the cohomology group is of infinity dimension. Recently, Shaw and Thiébaut in [15] show that if $\Omega \Subset \mathbb{C}^2$ is a domain with Lipschitz boundary such $\mathbb{C}^2 \setminus \Omega$ is connected, then the $\overline{\partial}$-operator will not have closed range from $L^2(\Omega)$ to $L^2_{(0,1)}(\Omega)$ if Ω is not pseudoconvex.

Let Ω be a bounded domain in $\mathbb{C}^n$ or in a complex manifold. Usually, the Hermitian metric we choose on Ω is induced from the ambient Hermitian manifold. However, if we choose a Hermitian metric on Ω which is a complete Riemann metric or in particular we choose the Bergman metric on Ω, Donnelly and Fefferman [5] proved

Theorem 1.4. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ endowed with its Bergman metric. If $p+q=n$, then $H^{p,q}_{L^2,\overline{\partial}}(\Omega)$ has infinity dimension and the complex Laplacian associated to $\overline{\partial}$-operator has closed range.*

In this note, we will consider $\overline{\partial}$-operator on the unit disk endowed with Poincáre metric which is the Bergman metric on the unit disk. Making use of the Poincáre inequality with respect to a complete metric on the real line, we will give an elementary proof of the closed range property for $\overline{\partial}$-operator in the L^2-setting.

2. Closed Range Property for $\overline{\partial}$-operator on Pincáre Disk

Let $D=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk in complex plane $\mathbb{C}$ with coordinates denoted by $z=x+iy$ and let $h=\frac{1}{(1-|z|^2)^2}dz\otimes d\overline{z}$ be the Pincáre metric which is a complete metric on D. For $q=0,1$, let $L^2_{(0,q)}(D,h)$ be the completion of smooth $(0,q)$-forms which have compact support in D under the inner product induced by the Poincáre metric. When $q=0$, we write $L^2(D,h)=L^2_{(0,0)}(D,h)$ for convenience. Let $\overline{\partial}:L^2(D,h)\to L^2_{(0,1)}(D,h)$ be the $\overline{\partial}$-operator defined in the sense of distribution. Let $\text{Rang}(\overline{\partial})$ denote the range of $\overline{\partial}$-operator in $L^2_{(0,1)}(D,h)$. Set $H^{0,1}_{L^2,\overline{\partial}}(D)=\frac{L^2_{(0,1)}(D,h)}{\text{Rang}(\overline{\partial})}$. Then

Theorem 2.1. *$H^{0,1}_{L^2,\overline{\partial}}(D)$ is an infinite dimensional space and* $\text{Rang}(\overline{\partial})$ *is closed in $L^2_{(0,1)}(D,h)$.*

Proof. First, let $f=f(z)d\overline{z}$ be any smooth $(0,1)$-form with f smooth up to the boundary ∂D. The L^2-norm of f with respect to the Pincáre metric on D is given by

$$\|f\|^2=\int_D\langle f(z)d\overline{z}|f(z)d\overline{z}\rangle_h dv,$$

where $dv=\frac{1}{(1-|z|^2)^2}dx\wedge dy$ is the volume form with respect to the Poincáre metric on the unit disk. Obviously, $f\in L^2_{(0,1)}(D,h)$. In particular, for any

$m \in \mathbb{N}$, set $f_m = z^m d\overline{z}$. We will show that the equation $\overline{\partial} u = f_m$ will not have a solution $u \in L^2(D, h)$. We prove this by seeking a contradiction. It is obvious that $\overline{\partial}(z^m \overline{z}) = f_m$. Suppose we have a solution $u_m \in L^2(D, h)$ such that $\overline{\partial} u_m = f_m$. Then $\overline{\partial}(u_m - z^m \overline{z}) = 0$ in the sense of distribution. Thus, $u_m - z^m \overline{z}$ is a holomorphic function on D. By Taylor's expansion

$$u_m = z^m \overline{z} + \sum_{k=0}^{\infty} a_k z^k.$$

By the assumption $u_m \in L^2(D, h)$ we have

$$\int_D \left(\sum_{k=0}^{\infty} a_k z^k + z^m \overline{z} \right) \overline{\left(\sum_{k=0}^{\infty} a_k z^k + z^m \overline{z} \right)} \frac{1}{(1 - |z|^2)^2} dx \wedge dy < \infty. \quad (2.1)$$

Taking polar coordinates, for any $0 < \tau < 1$,

$$\sum_{k=0}^{\infty} |a_k|^2 \int_0^{\tau} \frac{r^{2k+1}}{(1-r^2)^2} dr + (a_{m-1} + \overline{a_{m-1}}) \int_0^{\tau} \frac{r^{2m+1}}{(1-r^2)^2} dr + \int_0^{\tau} \frac{r^{2m+3}}{(1-r^2)^2} dr < \infty. \quad (2.2)$$

By (2.2), for any $0 < \tau < 1$, we have

$$(a_{m-1} + \overline{a_{m-1}}) \int_0^{\tau} \frac{r^{2m+1}}{(1 - r^2)^2} dr < \infty. \quad (2.3)$$

Taking $\tau \to 1$ and since the integral on the left hand side of (2.3) is divergent, thus we have $a_{m-1} + \overline{a_{m-1}} = 0$. Substituting it to (2.2) and taking $\tau \to 1$, we have $\int_0^1 \frac{r^{2m+3}}{(1-r^2)^2} dr < \infty$. Contradiction. Thus the equation $\overline{\partial} u = f_m$ does not have any solution $u \in L^2(D, h)$. This implies that $\dim H^{0,1}_{L^2, \overline{\partial}}(D) = \infty$.

For the second part of Theorem 2.1, we need to show that exists a constant $c > 0$ such that

$$\|\overline{\partial} g\|^2 \geq c \|g\|^2, \forall g \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Ker}(\overline{\partial})^{\perp}. \quad (2.4)$$

First, we show that $\mathrm{Ker}(\overline{\partial}) = \{0\}$. For any $u \in \mathrm{Ker}(\overline{\partial})$, we have $\overline{\partial} u = 0$ and

$$\int_D |u|^2 \frac{1}{(1 - |z|^2)^2} dx \wedge dy < \infty. \quad (2.5)$$

By Taylor's expansion, $u = \sum_{k=0}^{\infty} a_k z^k$. Substituting it to (2.5) and using the polar coordinates we have

$$\sum_{k=0}^{\infty}|a_k|^2\int_0^1\frac{r^{2k+1}}{(1-|r|^2)^2}<\infty. \tag{2.6}$$

Since the integral on the left hand side of (2.6) is divergent for every k, thus $a_k=0,\quad\forall k$, that is, $u=0$.

We only need to prove (2.4) when $g\in\mathrm{Dom}(\overline{\partial})$. Since the Poincáre metric on D is complete, then $C_0^\infty(D)$ is dense in $\mathrm{Dom}(\overline{\partial})\subset L^2(D,h)$. Thus we only need to prove (2.4) when $g\in C_0^\infty(D)$.

Set $z=re^{i\theta}$. Since

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial z}e^{i\theta}+\frac{\partial g}{\partial \overline{z}}e^{-i\theta}$$

we have

$$\left|\frac{\partial g}{\partial r}\right|^2 \le 2\left(\left|\frac{\partial g}{\partial z}\right|^2+\left|\frac{\partial g}{\partial \overline{z}}\right|^2\right). \tag{2.7}$$

Since

$$\begin{aligned}\|\overline{\partial}g\|^2 &= \int_D|\overline{\partial}g|_h^2\frac{1}{(1-|t|^2)^2}idz\wedge d\overline{z}\\ &= \int_D\left|\frac{\partial g}{\partial \overline{z}}\right|^2 idz\wedge d\overline{z}\\ &= \frac{1}{2}\int_D\left(\left|\frac{\partial g}{\partial \overline{z}}\right|^2+\left|\frac{\partial g}{\partial z}\right|^2\right)idz\wedge d\overline{z}. \end{aligned}\tag{2.8}$$

The last equality in (2.8) comes from the assumption that $g\in C_0^\infty(D)$. Substituting (2.7) to (2.8), we have

$$\|\overline{\partial}g\|^2\ge c_1\int_0^{2\pi}d\theta\int_0^1\left|\frac{\partial g}{\partial r}\right|^2 rdr. \tag{2.9}$$

Before the computing of the norm $\|g\|$ with respect to the Poincáre metric, we first give the following Poincáre type inequality on the real line.

Lemma 2.1. *Let f be a smooth function over $[0,1]$ and $f(1)=0$, then*

$$\int_0^1|f(x)|^2\frac{x}{(1-x)^2}dx\le c_0\int_0^1|f'(x)|^2xdx \tag{2.10}$$

where c_0 is a constant which does not depend on f.

Proof.

$$\begin{aligned}
&\int_0^1 |f(x)|^2 \frac{x}{(1-x)^2}dx \\
&= \int_0^1 |f(x)|^2 x \left(\frac{1}{1-x}\right)' dx \\
&= \frac{|f(x)|^2 x}{1-x}\Big|_0^1 - \int_0^1 \frac{1}{1-x}(|f(x)|^2 + xf'(x)\overline{f(x)} + xf(x)\overline{f'(x)})dx \\
&= -\int_0^1 \frac{1}{1-x}(|f(x)|^2 + xf'(x)\overline{f(x)} + xf(x)\overline{f'(x)})dx \\
&\leq 2\int_0^1 \frac{x}{1-x}|f(x)| \cdot |f'(x)|dx \\
&= 2\int_0^1 \frac{\sqrt{x}}{(1-x)}|f(x)| \cdot \sqrt{x}|f'(x)|dx \\
&\leq \varepsilon \int_0^1 |f(x)|^2 \frac{x}{(1-x)^2} + \frac{1}{\varepsilon}\int_0^1 |f'(x)|^2 x dx \qquad (2.11)
\end{aligned}$$

That is,

$$\int_0^1 |f(x)|^2 \frac{x}{(1-x)^2}dx \leq \frac{1}{\varepsilon(1-\varepsilon)}\int_0^1 |f'(x)|^2 x dx \qquad (2.12)$$

□

Now, we turn to the proof of the main theorem. Since g has compact support in D, we use the estimate (2.10) in Lemma 2.1 and we have

$$\begin{aligned}
\int_0^1 \left|\frac{\partial g}{\partial r}\right|^2 r dr &\geq c_0 \int_0^1 |g(r,\theta)|^2 \frac{r}{(1-r)^2}dr \\
&\geq c_0 \int_0^1 |g(r,\theta)|^2 \frac{r}{(1-r^2)^2}dr \qquad (2.13)
\end{aligned}$$

Substituting (2.13) to (2.9) we have

$$\|\overline{\partial} g\|^2 \geq c_1 c_0 \|g\|^2, \quad \forall g \in C_0^\infty(D). \qquad (2.14)$$

We get the conclusion of (2.4) and thus the $\mathrm{Rang}(\overline{\partial})$ is closed in $L^2_{(0,1)}(D,h)$. □

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