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CURVES CONTAINED IN A SMOOTH HYPERPLANE SECTION OF A VERY GENERAL QUINTIC 3-FOLD

EDOARDO BALLICO

Dipartimento di Matematica, Università di Trento, 38123 Povo (TN), Italy.
E-mail: edoardo.ballico@unitn.it

Abstract

Let $W \subset \mathbb{P}^4$ a very general quintic hypersurface. We study the existence/non-existence of non-complete intersection curves $T \subset W$ with T spanning a hyperplane H and $H \cap W$ smooth (non-existence if the hyperplanes vary in a family not containing a line or a conic of W).

1. Introduction

Let T be an integral algebraic variety over $\mathbb{C}$. We say that a property α is true for a general (resp. a very general) point of T if there is a finite (resp. countable) union Δ of proper subvarieties of T such each $o \in T \setminus \Delta$ satisfies α. Let $W \subset \mathbb{P}^4$ be a very general complex projective hypersurface of degree 5, i.e. any $W \in |\mathcal{O}_{\mathbb{P}^4}(5)|$ outside a countable union Δ of proper subvarieties of $|\mathcal{O}_{\mathbb{P}^4}(5)|$. These hypersurfaces are the target of Clemens' conjecture, which states that for each positive integer d the hypersurface W has only finitely many degree d rational curves, all of them smooth, except degree 5 plane sections of W (of course of degree 5) with geometric genus 0 ([3], [4], [5], [6], [11], [12], [13], [15]). It is expected that one can say more about curves contained in the intersection of W with a hyperplane (see [16, Corollaire at page 610] for general hypersurfaces of $\mathbb{P}^4$ of degree ≥ 7), e.g. each smooth rational curve of degree ≥ 4 should span $\mathbb{P}^4$. In this note we

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look at curves (with arbitrary geometric genus), which are contained in a hyperplane H, but they are not the complete intersection of $W \cap H$ with another hypersurface. Let $\mathbb{P}^{4\vee}$ denote the set of all hyperplanes of $\mathbb{P}^4$. Fix any integral and quasi-projective family $\mathbb{I}$ of integral curves $T \subset W$. We assume that for a general $T \in \mathbb{I}$ there is a unique hyperplane H containing T. Restricting if necessary $\mathbb{I}$ we assume that this is true for all $T \in \mathbb{I}$. We call $\pi(\mathbb{I})$ the set of all hyperplanes spanned by some $T \in \mathbb{I}$. For any $T \in \mathbb{I}$ let $\langle T \rangle$ denote the hyperplane spanned by T. We make the following restrictive assumptions:

1. the hyperplanes move, i.e. $\dim(\pi(\mathbb{I})) > 0$;
2. a general $H \in \pi(\mathbb{I})$ is not tangent to W;
3. a general $H \in \pi(\mathbb{I})$ contains no line of W.

In this note we prove the following result.

Theorem 1. *Let $W \subset \mathbb{P}^4$ be a very general quintic hypersurface. Assume the existence of an integral positive dimensional quasi-projective variety $\mathbb{I} \subset$* Chow(W) *such that any $T \in \mathbb{I}$ spans a hyperplane $\langle T \rangle$, $W \cap \langle T \rangle$ is smooth, T is not the complete intersection of $W \cap \langle T \rangle$ with another hypersurface and* $\dim(\pi(\mathbb{I})) > 0$. *Assume the non-existence of a line $L \subset W$ such that $L \subset H$ for all $H \in \pi(\mathbb{I})$. Then there is a smooth conic $D \subset W$ such that $\pi(\mathbb{I})$ is an open subset of the pencil of all hyperplanes containing D and for a very general $T \in \mathbb{I}$ we have $\mathcal{O}_{W\cap\langle T\rangle}(T) \cong \mathcal{O}_{W\cap\langle T\rangle}(x)(-yD)$ for some $x, y \in \mathbb{Z}$.*

Fix a hyperplane $H \subset \mathbb{P}^4$. The set of all smooth quintic surfaces $S \subset H$ with $\mathrm{Pic}(S) \neq \mathbb{Z}\mathcal{O}_S(1)$ is a countable union of subvarieties of codimension 4, plus the set of all $S \subset H$ containing either a line or a smooth conic ([17, Th. 0.2]; [18] shows that this is not true for surfaces with large degree) and their union is dense in the Zariski topology and in the euclidean topology of $|\mathcal{O}_{\mathbb{P}^3}(5)|$ ([2]). Since $\dim(\mathbb{P}^{4\vee}) = 4$, a dimensional count suggests that a general quintic 3-fold $W \subset \mathbb{P}^4$ contains at most countably many curves T spanning a hyperplane H containing no line and no conic of W and with T not a complete intersection of $W \cap H$ and another hypersurface. Call $\mathcal{H}_d$ the set of all hyperplanes $H = \langle T \rangle$ for some T as above with $\deg(T) = d$. For any $H \in \mathcal{H}_d$ the surface $H \cap W$ has families of non-complete intersection subcurve with arbitrarily large dimension (use $\mathcal{O}_{W\cap H}(x)(yT)$) with $y \in \mathbb{Z} \setminus \{0\}$ and $x \gg |y|$). So the question is not about the non-existence of large families of non-complete intersection degenerate subcurve of W, but that the associated hyperplanes do not move, i.e. if for each d the set $\mathcal{H}_d$ is finite. See Remark 4 for the finiteness of $\mathcal{H}_d$, $d \le 5$.

Question 1. *Is $\mathcal{H}_d$ finite for all $d \ge 6$? Is $\bigcup_{d\ge 6} \mathcal{H}_d$ dense in $\mathbb{P}^{4\vee}$ (in the Zariski and/or the euclidean topology)?*

Question 2. *Let $W \subset \mathbb{P}^4$ be a very general quintic hypersurface. Is there a finite upper bound for the rank of the Picard scheme (resp. class group) for all smooth (resp. all) hyperplane sections of W? Is this upper bound equal to 3?*

See Remarks 1, 2 and 3 for smooth hyperplane sections of a general quintic 3-fold and with Picard group of rank ≤ 3.

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2. Proof of Theorem 1

Let $\mathcal{W}$ denote the set of all smooth quintic hypersurfaces $W \subset \mathbb{P}^4$ satisfying the thesis of [5]. In particular for each $W \in \mathcal{W}$ we assume that for each integer $x \le 11$ the smooth 3-fold W contains finitely many curves of degree x and geometric genus 0, all of them smooth and pairwise disjoint, except rational plane quintics, and all of them with normal bundle isomorphic to a product of two line bundles of degree -1. For instance W contains no reducible conic. For any positive integers d let $\mathbb{I}_d$ be the set of all (T, W) with $W \in \mathcal{W}$, $T \subset W$ and T a degree d integral, rational curve. It is known that $\mathbb{I}_d$ is irreducible if and only if $d \le 11$ ([5, Theorem 1.1], [12]). We only need the irreducibility of $\mathbb{I}_d$ for very low d to check in the following remarks that certain natural hyperplane sections of a general $W \in \mathcal{W}$ have a Picard group with the expected rank.

Remark 1. Fix a general $W \in \mathcal{W}$. W has 2875 lines and any two of them are disjoint ([10], [13, page 158]). Take lines $L, R \subset W$ such that $L \ne R$. Since $L \cap R = \emptyset$, $L \cup R$ spans a hyperplane $H \subset \mathbb{P}^4$. Since $h^1(\mathbb{P}^4, \mathcal{I}_{L\cup R}(5)) = 0$ for any 2 disjoint lines L, R of $\mathbb{P}^4$, the Galois group of the covering $\mathbb{I}_1 \to \mathcal{W}$ is 2-transitive (or see the case $n = 4$ of [10]). Set $S := H \cap W$. We claim that S is smooth. Since the Galois group G of the covering $\mathbb{I}_1 \to \mathcal{W}$ is 2-transitive, this is true for one pair (L, R) if and only if it is true for all pairs of different lines of W. Fix two disjoint lines $D, T \subset H$ and let $Y \subset H$ be a general degree 5 surface containing $D \cup T$. Since a general $W \in \mathcal{W}$ contains a pair of disjoint lines and G is 2-transitive, to prove that S is smooth it is sufficient to prove that Y is smooth. Since $D \cup T$ is the base locus of $|\mathcal{I}_{D\cup T,H}(5)|$, Y is smooth outside $D \cup T$ by Bertini's theorem. Since $D \cup T$ is a smooth curve, Y is smooth by [7, Theorem 2.1] (in the set-up of [7, Theorem 2.1] either $\mathrm{Sing}(Y) = \emptyset$ or $\mathrm{Sing}(Y)$ has codimension 2 in $D \cup T$). We claim that for a very general S the group $\mathrm{Pic}(S)$ has rank 3, generated by L, R and $\mathcal{O}_S(1)$. It is sufficient to prove that for a very general Y $\mathrm{Pic}(Y)$ has rank 3, generated by D, T and $\mathcal{O}_Y(1)$. We have $h^1(H, \mathcal{I}_{D\cup T}(t)) = 0$ for all $t \geq 1$ and so for each $t \geq 2$ a very general surface $Y \subset H$ containing $D \cup T$ is normal with class group freely generated by $\mathcal{O}_Y(1)$, D and T ([1, Theorem 1.1]). Let $J \subset H$ be any line with $J \neq T$ and $J \neq D$. Since $5 > \deg(D \cup T \cup J)$, it is easy to check that $h^1(H, \mathcal{I}_{D\cup T\cup J}(5)) = 0$, i.e. $h^0(H, \mathcal{I}_{D\cup T\cup J}(5)) = h^0(H, \mathcal{I}_{D\cup T\cup J}(5)) - 6 + \sharp(J \cap (D \cup T))$. Since H has ∞^4 lines, only ∞^3 of them meeting $D \cup T$, only ∞^1 intersecting both D and T, and Y is general in $|\mathcal{I}_{D\cup T,H}(5)|$, D and T are the only lines contained in Y. Hence L and R are the only lines of S and hence (by the irreducibility of $\mathbb{I}_1$) for a general $W \in \mathcal{W}$ no 3 of the lines of W are contained in a hyperplane and there are $\binom{2875}{2}$ hyperplanes of $\mathbb{P}^4$ containing 2 lines of W and none of them is tangent to W.

Remark 2. Fix a general $W \in \mathcal{W}$ and take any line $L \subset W$ and any smooth conic $D \subset W$. We know that $D \cap L = \emptyset$. Here we check that $D \cup L$ spans $\mathbb{P}^4$ and hence we cannot get a hyperplane section with Picard group of rank at least 3 taking the linear span of $D \cup L$. Take any hyperplane $H \subset \mathbb{P}^4$, any smooth conic $T \subset H$ and any line $R \subset H$ such that $R \cap T = \emptyset$. The set of all such triples (H, T, R) has dimension 16. Since $h^1(\mathbb{P}^4, \mathcal{I}_{R\cup T}(5)) = h^1(H, \mathcal{I}_{R\cup T,H}(5)) = 0$, we have $h^0(\mathbb{P}^4, \mathcal{I}_{R\cup T}(5)) = \binom{9}{4} - 17$. Hence a general $W \in \mathcal{W}$ contains no $T \cup R$. The set of all hyperplanes $H \subset \mathbb{P}^4$ containing D is a pencil. Since the dual variety of a smooth hypersurface of degree > 1 is a hypersurface), there is $H \subset \mathbb{P}^4$ with $H \supset D$ and $H \cap W$ singular. We check here that a general hyperplane $H \subset \mathbb{P}^4$ with $H \supset D$ is smooth. We fix a hyperplane $H \subset \mathbb{P}^4$ and a smooth conic $D \subset \mathbb{P}^4$. Since the homogeneous ideal of D in H is generated by forms of degree ≤ 2, a general element of $S \in |\mathcal{I}_{D,H}(5)|$. Any smooth quintic hypersurface $W' \subset \mathbb{P}^4$ with $W' \cap H = S$ contains a conic D and a hyperplane $H \supset D$ with $H \cap W'$ smooth. Since $\mathbb{I}_2$ is irreducible, for a general $W \in \mathcal{W}$ this is true for all conics contained in W.

Remark 3. Let Γ be the set of all complete intersection $T \subset \mathbb{P}^4$ of one hyperplane and 2 quadric hypersurfaces. The set Γ is an irreducible variety of dimension 20. Fix any $T \in \Gamma$. Since $h^1(\mathbb{P}^4, \mathcal{I}_T(5)) = 0$, we have $h^0(\mathbb{P}^4, \mathcal{I}_T(5)) = \binom{9}{4} - 20$. Therefore a general $W \in \mathcal{W}$ contains only finitely may $T \in \Gamma$, all of them smooth elliptic curves, and the associated incidence correspondence $\mathbb{E}$ is irreducible and $\dim(\mathbb{E}) = 125$. Fix a general $W \in \mathcal{W}$ and take $T \in \Gamma$ with $T \subset W$. Call H the linear span of T. Since W has only ∞^3 tangent hyperplanes and $\dim(\mathcal{W}) = \dim(\mathbb{E})$, for a general W the surface $W \cap H$ is smooth (or you may quote [7, Theorem 2.1]). Since the homogeneous ideal of T in H is generated by two smooth quadric surfaces, a general quintic surface $S \subset H$ containing T is smooth. By [1, Theorem 1.1] $\mathrm{Pic}(S)$ is freely generated by T and $\mathcal{O}_S(1)$. Since $\mathbb{E}$ is irreducible, we get that $W \cap H$ is freely generated by T and $\mathcal{O}_{W\cap H}(1)$.

Remark 4. Fix a general $W \in \mathcal{W}$ and assume the existence of an integral curve $T \subset W$ of degree $d \le 5$ and whose linear span $\langle T \rangle$ has dimension ≤ 3. First assume that $\langle T \rangle$ is a plane. We know the cases $d = 1, 2$ since W has 2875 lines and $609,250$ conics, all of them smooth ([13, Theorem 3.1]). If $d = 5$, then T is a plane section of W. If $d = 3$, then T is linked by $\langle T \rangle$ to a plane conic contained in W (we also know by [5] that T is a smooth elliptic curve). If $d = 4$, then T is linked by $\langle T \rangle$ to a line contained in W and so we know that W has 2875 integral 3-dimensional families of such curves T. Now assume that $\langle T \rangle$ is a hyperplane. If $d = 3$, then T is a rational normal curve and we know that W has only finitely many such curves. If $d = 4$ the irreducibility of $\mathbb{I}_4$ and [5] gives that any such T is a smooth elliptic curve (see Remark 3 for a description of this case). Now assume $d = 5$. We have have $p_a(T) \le 2$ by Castelnuovo's upper bound for the arithmetic genus of non-degenerate curves. We have $h^1(\mathbb{P}^4, \mathcal{I}_T(5)) = h^1(H, \mathcal{I}_T(5)) = 0$ ([9]). Hence $h^0(\mathbb{P}^4, \mathcal{I}_T(5)) = \binom{9}{4} - 25 - 1 + p_a(T)$. Since $\dim(\mathbb{P}^{4\vee}) = 4$ and H contains only ∞^{20} non-degenerate curves with degree 5 and $p_a(T) \in \{0, 1, 2\}$, we get that a general $W \in \mathcal{W}$ contains such a curve T only if $p_a(T) = 2$. In this case the singular ones have lower dimension. Hence W only has finitely many T, each of them being smooth and of genus 2. In particular $\mathcal{H}_5$ is finite.

Proof of Theorem 1. Take $\mathbb{I}$ as in the statement of Theorem 1. Assume for the moment that $\dim(\mathbb{I}) = 1$. Fix a general $p \in \mathbb{P}^4$. We assume $p \notin W$ and that $p \notin H$ for a general $H \in \pi(\mathbb{I})$, say $p \notin \langle T \rangle$ for all $T \in \mathbb{I}$ in a dense open subset $\mathbb{J}$ of $\mathbb{I}$. Let $\ell : \mathbb{P}^4 \setminus \{p\} \to \mathbb{P}^3$ denote the linear projection from p. We get a family $\ell(T)$, $T \in \mathbb{J}$, of $\deg(T)$ integral space curves and a family $\ell(W \cap \langle T\rangle)$ of degree 5 surfaces with $\ell(T) \subset \ell(W \cap \langle T\rangle)$. Fix $T \in \mathbb{J}$. Since W is not a cone with vertex p, there are only finitely many $T_1 \in \mathbb{J}$ with $\ell(W \cap \langle T_1\rangle) = \ell(W \cap \langle T\rangle)$. Since $\dim(\mathbb{J}) = 1$ and $\dim(\mathbb{P}^{4\vee}) = 4$, taking the linear projection from varying $W \in \mathcal{W}$ we get a family Γ of smooth degree 5 surfaces of $\mathbb{P}^3$ such that each $S \in \Gamma$ contains a $\deg(T)$ integral curve and Γ has codimension ≤ 3 in $|\mathcal{O}_{\mathbb{P}^3}(5)|$. By [17, Th. 0.2] a general $S \in \Gamma$ contains either a line or a conic (see [8] and [16] for the characterization of the surfaces containing a line). Since W contains only finitely many lines and conics, all of them smooth, either there is a line $L \subset H$ for all $H \in \pi(\mathbb{J})$ or there is a smooth conic D such that $D \subset \pi(T)$ for all $T \in \mathbb{I}$. We excluded the former case. Assume the existence of the conic D. Since $h^0(\mathbb{P}^4, \mathcal{I}_D(1)) = 2$, $\mathbb{I}$ is induced by the pencil of all hyperplanes containing D. To conclude (for a general $W \in \mathcal{W}$) it is sufficient to prove that a general degree 5 surface $S \subset \mathbb{P}^3$ containing a smooth conic T is smooth and $\mathrm{Pic}(S)$ is freely generated by $\mathcal{O}_S(T)$ and $\mathcal{O}_S(1)$. S is smooth, because the homogeneous ideal of D is generated by forms of degree ≤ 2 (or you may quote [7, Theorem 2.1]). $\mathrm{Pic}(S)$ is freely generated by $\mathcal{O}_S(T)$ and $\mathcal{O}_S(1)$ by [14, II.3.8] or [1, Theorem 1.1], because $h^1(\mathcal{I}_T(t)) = 0$ and a general $A \in |\mathcal{I}_T(t)|$ is smooth for all $t > 0$.

Now assume $\dim(\mathbb{I}) > 1$. Take any integral $\mathbb{I}' \subset \mathbb{I}$ such that $\dim(\mathbb{I}') = 1$ and $\dim(\pi(\mathbb{I}')) > 0$. By part (a) either there is a conic $D \subset H$ for all $H \in \pi(\mathbb{I}')$ or there is a line $L \subset H$ for all $H \in \pi(\mathbb{I}')$. Since W has only finitely many lines or conic, the same line or the same conics works for all $\mathbb{I}'$. If there is a conic, then $\dim(\mathbb{I}) = 1$, a contradiction. We excluded the case of a line in the statement of Theorem 1, but by the irreducibility of $\mathbb{I}_1$ we also know that for a general $W \in \mathcal{W}$, any line $L \subset W$ and a general hyperplane H containing L the surface $W \cap H$ is smooth and its Picard scheme is freely generated by $\mathcal{O}_{W\cap H}(1)$ and L ([14, II.3.8] or [1, Theorem 1.1]). □

References

1. J. Brevik and S. Nollet, Noether-Lefschetz theorem with base locus, *Int. Math. Res. Not. IMRN* 2011, No. 6, 1220-1244.

2. C. Ciliberto, J. Harris and R. Miranda, General components of the Noether-Lefschetz locus and their density in the space of all surfaces, *Math. Ann.*, **282** (1988), 667-680.

3. H. Clemens, Some results about Abel-Jacobi mappings, *Topics in transcendental algebraic geometry* (Princeton, N.J., 1981/1982), 289–304, *Ann. of Math. Stud.*, 106, Princeton Univ. Press, Princeton, NJ, 1984.

4. E. Cotterill, Rational curves of degree 10 on a general quintic threefold, *Comm. Algebra*, **33** (2005), 1833-1872.

5. E. Cotterill, Rational curves of degree 11 on a general quintic 3-fold, *Quart. J. Math.*, **63** (2012), 539-568.

6. D'Almeida, Courbes rationnelles de degré 11 sur une hypersurface quintique générale de $\mathbb{P}^4$, *Bull. Sci. Math.*, **136** (2012), 899-903.

7. S. Diaz and D. Harbater, Strong Bertini theorems, *Trans. Amer. Math. Soc.* **324** (1991), No. 1, 73-86.

8. M. Green, Components of maximal dimension in the Noether-Lefschetz theorem, *J. Differential Geomety*, **27** (1988), 295-302.

9. L. Gruson, R. Lazarsfeld and Ch. Peskine, On a theorem of Castelnuovo, and the equations defining space curves, *Invent. Math.*, **72** (1983), 491-506.

10. J. Harris, Galois groups of enumerative problems, *Duke Math. J.*, **50** (1983), 1127-1135.

11. T. Johnsen and S. Kleiman, Rational curves of degree at most 9 on a general quintic threefold, *Comm. Algebra*, **24** (1996), 2721-2753.

12. T. Johnsen and S. Kleiman, Toward Clemens' Conjecture in Degrees between 10 and 24, *Serdica Math. J.*, **23** (1997), 131-142.

13. S. Katz, On the finiteness of rational curves on quintic threefolds, *Compositio Math.* **60** (1986), 151-162.

14. A. F. Lopez, Noether-Lefschetz theory and the Picard group of projective surfaces, *Mem. Amer. Math. Soc.*, **89**, (1991).

15. P. G. J. Nijsse, Clemens' conjecture for octic and nonic curves, *Indag. Math.*, **6** (1995), 213-221.

16. C. Voisin, Une précision concernant le théorème de Noether, *Math. Ann.*, **280** (1988), 605-611.

17. C. Voisin, Composantes de petite codimension du lieu de Noether-Lefschetz, *Comm. Math. Helvetici*, **64** (1989), 515-526.

18. C. Voisin, Contrexemple à une conjecture de J. Harris, *C. R. Acad. Sci. Paris Sér. I Math.*, **313** (1991), No. 10, 685–687.

19. C. Voisin, On a conjecture of Clemens on rational curves on hypersurfaces, *J. Differential Geometry*, **44** (1996), No. 1, 200–213.

20. C. Voisin, Correction to "On a conjecture of Clemens on rational curves on hypersurfaces", *J. Differential Geometry*, **49** (1998), 601-611.