Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **11** (2016), No. 2, pp. 371-392

ON SOME NEW ORLICZ SEQUENCE SPACES DERIVED BY USING RIESZ MEAN AND MULTIPLIER SEQUENCE

SERKAN DEMİRİZ

University of Gaziosmanpasa, Faculty of Arts and Science, Department of Mathematics, Tokat, Turkey.
E-mail: serkandemiriz@gmail.com

Abstract

In this paper, we introduce the Orlicz sequence spaces generated by Riesz mean associated with a fixed multiplier sequence of non-zero scalars. Furthermore, we emphasize several algebraic and topological properties relevant to these spaces. Finally, we determine the Köthe-Toeplitz dual of the spaces $\ell^{'}_{M}(R^q,\Lambda)$ and $h_M(R^q,\Lambda)$.

1. Introduction

By ω, we shall denote the space of all complex valued sequences. Any vector subspace of ω is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p ; we denote the spaces of all bounded, convergent, absolutely and p- absolutely convergent series, respectively; where $1 \leq p < \infty$. A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space* (see Chaudary and Nanda ([2, pp.272-273]).

A function $M : [0,\infty) \to [0,\infty)$ which is convex with $M(u) \geq 0$ for $u \geq 0$, and $M(u) \to \infty$ as $u \to \infty$, is called as an *Orlicz function.* An Orlicz

Received September 1, 2015 and in revised form November 10, 2015.
AMS Subject Classification: 40C05, 40A05, 46A45.
Key words and phrases: Orlicz function, Riesz mean, sequence spaces, Köthe-Toeplitz dual.

function M can always be represented in the following integral form

$$M(u) = \int_0^u p(t)dt$$

where p the kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel p associated with the Orlicz function M and let

$$q(s) = \sup\{t : p(t) \leq s\}.$$

Then, q possesses the same properties as the function p. Suppose now

$$\Phi(x) = \int_0^x q(s)ds.$$

Then, Φ is an Orlicz function. The functions M and Φ are called *mutually complementary Orlicz functions*.

Now, we give the following well-known results.

Let M and Φ be mutually complementary Orlicz functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \qquad \text{(Young's Inequality)}. \tag{1.1}$$

(ii) For all $u \geq 0$,

$$up(u) = M(u) + \Phi(p(u)). \tag{1.2}$$

(iii) For all $u \geq 0$ and $0 < \lambda < 1$,

$$M(\lambda u) < \lambda M(u). \tag{1.3}$$

An Orlicz function M is said to satisfy the Δ_2-condition for small u or at 0 if for each $k \in \mathbb{N}$, there exists $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an Orlicz function M is said to satisfy the

Δ_2-condition if and only if

$$\limsup_{u\to 0^+} \frac{M(2u)}{M(u)} < \infty.$$

Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b]. \tag{1.4}$$

Orlicz used the Orlicz function to introduce the sequence space ℓ_M (see Musielak [3]; Lindenstrauss and Tzafriri [4]), as follows

$$\ell_M = \left\{ x = (x_k) \in \omega : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some } \rho > 0 \right\}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. For relevant terminology and additional knowledge on the Orlicz sequence spaces and related topics, the reader may refer to [3-19].

Throughout the present article, we assume that $\Lambda = (\lambda_k)$ is the sequence of non-zero complex numbers. Then, for a sequence space E, the multiplier sequence space $E(\Lambda)$ associated with the multiplier sequence Λ is defined by

$$E(\Lambda) = \{x = (x_k) \in \omega : \Lambda x = (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. G. Goes and S. Goes defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences $(1/k)$ and (k) in [20], respectively. A multiplier sequence can be used to accelerate the convergence of sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus, it also covers a larger class of sequences for study.

Let $t = (t_k)$ be a sequence of non-negative real numbers with $t_0 > 0$ and write

$$T_n = \sum_{k=0}^{n} t_k \quad \text{for all} \quad n \in \mathbb{N}.$$

Then, the *Riesz means* with respect to the sequence $t = (t_k)$ is defined by the matrix $R^t = (r^t_{nk})$ which is given by

$$r^t_{nk} = \begin{cases} \dfrac{t_k}{T_n}, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$ [1].

Definition 1.1. Let M be any Orlicz function and

$$\delta(M, x) := \sum_k M(|x_k|)$$

where $x = (x_k) \in \omega$. Then, we define the sets $\widetilde{\ell}_M(R^t, \Lambda)$ and $\widetilde{\ell}_M$ by

$$\widetilde{\ell}_M(R^t, \Lambda) \ := \ \left\{ x = (x_k) \in \omega : \widehat{\delta}_{R^t}(M, x) = \sum_k M\left(\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{T_k} \right) < \infty \right\}$$

and

$$\widetilde{\ell}_M \ := \ \{ x = (x_k) \in \omega : \delta(M, x) < \infty \}.$$

Definition 1.2. Let M and Φ be mutually complementary functions. Then, we define the set $\ell_M(R^t, \Lambda)$ by

$$\ell_M(R^t, \Lambda) = \left\{ x = (x_k) \in \omega : \sum_k \left(\frac{\sum_{j=0}^{k} \lambda_j t_j x_j}{T_k} \right) y_k \text{ converges for all } y = (y_k) \in \widetilde{\ell}_\Phi \right\}$$

which is called as Orlicz sequence space associated with the multiplier sequence $\Lambda = (\lambda_k)$ and generated by Riesz matrix.

The α-dual or Köthe-Toeplitz dual X^α of a sequence space X is defined by

$$X^\alpha = \left\{ a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty \quad \text{for all } x = (x_k) \in X \right\}.$$

It is known that if $X \subset Y$, then $Y^\alpha \subset X^\alpha$. It is clear that $X \subset X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$, then X is called as an α space. In particular, an α space is called a Köthe space or a perfect sequence space.

The main purpose of this paper is to introduce the sequence spaces $\ell_M(R^t,\Lambda), \widetilde{\ell}_M(R^t,\Lambda), \ell'_M(R^t,\Lambda)$ and $h_M(R^t,\Lambda)$, and investigate their certain algebraic and topological properties. Furthermore, it is proved that the spaces $\ell'_M(R^t,\Lambda)$ and $h_M(R^t,\Lambda)$ are topologically isomorphic to the spaces $\ell_\infty(R^t,\Lambda)$ and $c_0(R^t,\Lambda)$ when $M(u) = 0$ on some interval, respectively. Finally, the α-dual of the spaces $\ell'_M(R^t,\Lambda)$ and $h_M(R^t,\Lambda)$ are determined, and therefore the non-perfectness of the space $\ell'_M(R^t,\Lambda)$ is showed when $M(u) = 0$ on some interval.

2. Main Results

In this section, we emphasize the sequence spaces $\ell_M(R^t,\Lambda)$, $\widetilde{\ell}_M(R^t,\Lambda)$, $\ell'_M(R^t,\Lambda)$ and $h_M(R^t,\Lambda)$, and give their some algebraic and topological properties.

Proposition 2.1. *For any Orlicz function M, the inclusion $\widetilde{\ell}_M(R^t,\Lambda) \subset \ell_M(R^t,\Lambda)$ holds.*

Proof. Let $x = (x_k) \in \widetilde{\ell}_M(R^t,\Lambda)$. Then, since $\sum_k M\left(\frac{|\sum_{j=0}^k \lambda_j t_j x_j|}{T_k}\right) < \infty$ we have from (1.1) that

$$\begin{aligned}\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| &\le \sum_k \left|\left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| \\ &\le \sum_k M\left(\left|\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right|\right) + \sum_k \Phi(|y_k|) < \infty\end{aligned}$$

for every $y = (y_k) \in \widetilde{\ell}_\Phi$. Thus, $x = (x_k) \in \ell_M(R^t,\Lambda)$. □

Proposition 2.2. *For each $x = (x_k) \in \ell_M(R^t,\Lambda)$,*

$$\sup\left\{\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| : \delta(\Phi, y) \le 1\right\} < \infty. \tag{2.1}$$

Proof. Suppose that (2.1) does not hold. Then, for each $n \in \mathbb{N}$, there exists y^n with $\delta(\Phi, y^n) \leq 1$ such that

$$\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k^n\right| > 2^{n+1}.$$

Without loss of generality, we can assume that $\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}, y^n \geq 0$. Now, we can define a sequence $z = (z_k)$ by

$$z_k = \sum_n \frac{1}{2^{n+1}} y_k^n$$

for all $k \in \mathbb{N}$. By the convexity of Φ, we have

$$\begin{aligned}
\Phi\left(\sum_{n=0}^{l} \frac{1}{2^{n+1}} y_k^n\right) &\leq \frac{1}{2}\left[\Phi(y_k^0) + \Phi\left(y_k^1 + \frac{y_k^2}{2} + \cdots + \frac{y_k^l}{2^{l-1}}\right)\right] \\
&\leq \sum_{n=0}^{l} \frac{1}{2^{n+1}} \Phi(y_k^n)
\end{aligned}$$

for any positive integer l. Hence, using the continuity of Φ, we have

$$\delta(\Phi, z) = \sum_k \Phi(z_k) \leq \sum_k \sum_n \frac{1}{2^{n+1}} \Phi(y_k^n) \leq \sum_n \frac{1}{2^{n+1}} = 1.$$

But for every $l \in \mathbb{N}$, it holds

$$\begin{aligned}
\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) z_k &\geq \sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) \sum_{n=0}^{l} \frac{1}{2^{n+1}} y_k^n \\
&= \sum_{n=0}^{l} \sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) \frac{1}{2^{n+1}} y_k^n \geq l.
\end{aligned}$$

Hence, $\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) z_k$ diverges and this implies that $x \notin \ell_M(R^t, \Lambda)$, a contradiction. This leads us to the required result. □

The preceding result encourages us to introduce the following norm $\|.\|_M^{R^t}$ on $\ell_M(R^t, \Lambda)$.

Proposition 2.3. *The following statements hold:*

(i) $\ell_M(R^t,\Lambda)$ *is a normed linear space under the norm* $\|.\|_M^{R^t}$ *defined by*

$$\|.\|_M^{R^t} = \sup\left\{\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| : \delta(\Phi, y) \le 1\right\}. \tag{2.2}$$

(ii) $\ell_M(R^t,\Lambda)$ *is a Banach space under the norm defined by* (2.2).

(iii) $\ell_M(R^t,\Lambda)$ *is a BK-space under the norm defined by* (2.2).

Proof. (i) It is easy to verify that $\ell_M(R^t,\Lambda)$ is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequences. Now we show that $\|.\|_M^{R^t}$ is a norm on the space $\ell_M(R^t,\Lambda)$.

If $x = 0$, then obviously $\|.\|_M^{R^t} = 0$. Conversely, assume $\|.\|_M^{R^t} = 0$. Then, using the definition of the norm given by (2.2), we have

$$\sup\left\{\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| : \delta(\Phi, y) \le 1\right\} = 0.$$

This implies that $\left|\sum_k \left(\frac{\sum_{j=0}^k \lambda_j t_j x_j}{T_k}\right) y_k\right| = 0$ for all y such that $\delta(\Phi, y) \le 1$. Now considering $y = e^k$ if $\Phi(1) \le 1$ otherwise considering $y = e^k/\Phi(1)$ so that $\lambda_k t_k x_k = 0$ for all $k \in \mathbb{N}$, where e^k is a sequence whose only non-zero terms is 1 in k^{th} place for each $k \in \mathbb{N}$. Hence, we have $x_k = 0$ for all $k \in \mathbb{N}$, since (λ_k) is a sequence of non-zero scalars and $t = (t_k)$ be a sequence of non-negative real numbers with $t_0 > 0$. Thus, $x = 0$.

It is easy to show that $\|\alpha x\|_M^{R^t} = |\alpha|\|x\|_M^{R^t}$ and $\|x+y\|_M^{R^t} \le \|x\|_M^{R^t} + \|y\|_M^{R^t}$ for all $\alpha \in \mathbb{C}$ and $x, y \in \ell_M(R^t,\Lambda)$.

(ii) Let (x^p) be any Cauchy sequence in the space $\ell_M(R^t,\Lambda)$. Then, for any $\varepsilon > 0$, there exists a positive integer n_0 such that $\|x^p - x^q\|_M^{R^t} < \varepsilon$ for all $p, q \ge n_0$. Using the definition of norm given by (2.2), we get

$$\sup\left\{\left|\sum_k \left[\frac{\sum_{j=0}^k \lambda_j t_j (x_j^p - x_j^q)}{T_k}\right] y_k\right| : \delta(\Phi, y) \le 1\right\} < \varepsilon$$

for all $p, q \ge n_0$. This implies that

$$\left|\sum_k \left[\frac{\sum_{j=0}^k \lambda_j t_j (x_j^p - x_j^q)}{T_k}\right] y_k\right| < \varepsilon$$

for all y with $\delta(\Phi, y) \leq 1$ and for all $p, q \geq n_0$. Now considering $y = e^k$ if $\Phi(1) \leq 1$, otherwise considering $y = e^k/\Phi(1)$ we have $\{\lambda_k t_k x_k^p\}_k$ is a Cauchy sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$. Hence, it is a convergent sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$.

Let

$$\lim_{p\to\infty} \lambda_k t_k x_k^p = x_k$$

for each $k \in \mathbb{N}$. Using the continuity of the modulas, we can derive for all $p \geq n_0$ as $q \to \infty$, that

$$\sup\left\{\left|\sum_k \left[\frac{\sum_{j=0}^k \lambda_j t_j (x_j^p - x_j)}{T_k}\right] y_k\right| : \delta(\Phi, y) \leq 1\right\} \leq \varepsilon.$$

It follows that $(x^p - x) \in \ell_M(R^t, \Lambda)$. Since (x^p) is in the space $\ell_M(R^t, \Lambda)$ and $\ell_M(R^t, \Lambda)$ is a linear space, we have $x = (x_k) \in \ell_M(R^t, \Lambda)$.

(iii) From the above proof, one can easily conclude that $\|x^p\|_M^{R^t} \to 0$ implies that $x_k^p \to 0$ for each $p \in \mathbb{N}$ which leads us to the desired result.

Therefore, the proof of the theorem is completed. □

Proposition 2.4. $\ell_M(R^t, \Lambda)$ *is a normed linear space under the norm* $\|.\|_{(M)}^{R^t}$ *defined by*

$$\|x\|_{(M)}^{R^t} = \inf\left\{\rho > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right) \leq 1\right\}. \tag{2.3}$$

Proof. Clearly $\|x\|_{(M)}^{R^t} = 0$ if $x = 0$. Now, suppose that $\|x\|_{(M)}^{R^t} = 0$. Then, we have

$$\|x\|_{(M)}^{R^t} = \inf\left\{\rho > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right) \leq 1\right\} = 0.$$

This yields the fact for a given $\varepsilon > 0$ that there exists some $\rho_\varepsilon \in (0, \varepsilon)$ such that

$$\sup_{k\in\mathbb{N}} M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho_\varepsilon T_k}\right) \leq 1$$

which implies that

$$M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_\varepsilon T_k}\right)\le 1$$

for all $k\in\mathbb{N}$. Thus,

$$M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\varepsilon T_k}\right)\le M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_\varepsilon T_k}\right)\le 1$$

for all $k\in\mathbb{N}$. Suppose $\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\varepsilon T_k}\neq 0$ for some $k\in\mathbb{N}$. Then, $\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\varepsilon T_k}$ $\to\infty$ as $\varepsilon\to 0$. It follows that $M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\varepsilon T_k}\right)\to\infty$ as $\varepsilon\to 0$ for some $k\in\mathbb{N}$, which is a contradiction. Therefore, $\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\varepsilon T_k}=0$ for all $k\in\mathbb{N}$. It follows that $\lambda_k t_k x_k=0$ for all $k\in\mathbb{N}$. Hence $x=0$, since (λ_k) is a sequence of non-zero scalars and $t=(t_k)$ be a sequence of non-negative real numbers with $t_0>0$.

Let $x=(x_k)$ and $y=(y_k)$ be any two elements of $\ell_M(R^t,\Lambda)$. Then, there exists $\rho_1,\rho_2>0$ such that

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_1 T_k}\right)\le 1 \text{ and } \sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_2 T_k}\right)\le 1.$$

Let $\rho=\rho_1+\rho_2$. Then, by the convexity of M, we have

$$\begin{aligned}M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j (x_j+y_j)\right|}{\rho T_k}\right) &\le \frac{\rho_1}{\rho_1+\rho_2}\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_1 T_k}\right)\\ &+ \frac{\rho_2}{\rho_1+\rho_2}\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j y_j\right|}{\rho_2 T_k}\right)\le 1.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x+y\|_{(M)}^{R^t} &= \inf\left\{\rho>0:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j (x_j+y_j)\right|}{\rho T_k}\right)\le 1\right\}\\ &\le \inf\left\{\rho_1>0:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho_1 T_k}\right)\le 1\right\}\end{aligned}$$

$$+\ \inf\left\{\rho_2>0:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j y_j\right|}{\rho_2 T_k}\right)\le 1\right\}$$

which gives that $\|x+y\|_{(M)}^{R^t}\le \|x\|_{(M)}^{R^t}+\|y\|_{(M)}^{R^t}$.

Finally, let α be any scalar and define r by $r=\rho/|\alpha|$. Then,

$$\begin{aligned}\|\alpha x\|_{(M)}^{R^t} &= \inf\left\{\rho>0:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j \alpha x_j\right|}{\rho T_k}\right)\le 1\right\}\\ &= \inf\left\{r|\alpha|>0:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{r T_k}\right)\le 1\right\}=|\alpha|\|x\|_{(M)}^{R^t}.\end{aligned}$$

This completes the proof. □

Proposition 2.4 inspires us to define the following sequence space.

Definition 2.5. For any Orlicz function M, we define

$$\ell_M'(R^t,\Lambda):=\left\{x=(x_k)\in\omega:\sum_k M\left(\frac{\left|\sum_{j=0}^{k}\lambda_j t_j x_j\right|}{\rho T_k}\right)<\infty \text{ for some } \rho>0\right\}.$$

Now, we can give the corresponding proposition on the space $\ell_M'(R^t,\Lambda)$ to the Proposition 2.3.

Proposition 2.6. *The following statements hold:*

(i) $\ell_M'(R^t,\Lambda)$ *is a normed linear space under the norm* $\|x\|_{(M)}^{R^t}$ *defined by* (2.3).

(ii) $\ell_M'(R^t,\Lambda)$ *is a Banach space under the norm defined by* (2.3).

(iii) $\ell_M'(R^t,\Lambda)$ *is a* BK*-space under the norm defined by* (2.3).

Proof. (i) Since the proof is similar to the proof of Proposition 2.4, we omit the detail.

(ii) Let (x^p) be any Cauchy sequence in the space $\ell_M'(R^t,\Lambda)$. Let $\delta>0$ be fixed and $r>0$ be given such that $0<\varepsilon<1$ and $r\delta\ge 1$. Then, there exists a positive integer n_0 such that $\|x^p-x^q\|_{(M)}^{R^t}<\varepsilon/r\delta$ for all $p,q\ge n_0$.

Using the definition of the norm given by (2.3), we get

$$\inf\left\{\rho>0:\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j^q)\right|}{\rho T_k}\right)\le 1\right\}<\frac{\varepsilon}{r\delta}$$

for all $p,q\ge n_0$. This implies that

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j^q)\right|}{\|x^p-x^q\|_{(M)}^{R^t} T_k}\right)\le 1$$

for all $p,q\ge n_0$. It follows that

$$M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j^q)\right|}{\|x^p-x^q\|_{(M)}^{R^t} T_k}\right)\le 1$$

for all $p,q\ge n_0$ and for all $k\in\mathbb{N}$. For $r>0$ with $M(r\delta/2)\ge 1$, we have

$$M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j^q)\right|}{\|x^p-x^q\|_{(M)}^{R^t} T_k}\right)\le M\left(\frac{r\delta}{2}\right)$$

for all $p,q\ge n_0$ and for all $k\in\mathbb{N}$. Since M is non-decreasing , we have

$$\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j^q)\right|}{T_k}\le\frac{r\delta}{2}\cdot\frac{\varepsilon}{r\delta}=\frac{\varepsilon}{2}$$

for all $p,q\ge n_0$ and for all $k\in\mathbb{N}$. Hence, $\{\lambda_k t_k x_k^p\}_k$ is a Cauchy sequence in $\mathbb{C}$ for all $k\in\mathbb{N}$ which implies that it is a convergent sequence in $\mathbb{C}$ for all $k\in\mathbb{N}$. Let $\lim_{p\to\infty}\lambda_k t_k x_k^p=x_k$ for each $k\in\mathbb{N}$. Using the continuity of an Orlicz function and modulus, we can have

$$\inf\left\{\rho>0:\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j(x_j^p-x_j)\right|}{\rho T_k}\right)\le 1\right\}<\varepsilon$$

for all $p\ge n_0$, as $q\to\infty$. It follows that $(x^p-x)\in\ell_M'(R^t,\Lambda)$. Since x^p is in the space $\ell_M'(R^t,\Lambda)$ and $\ell_M'(R^t,\Lambda)$ is a linear space, we have $x=(x_k)\in\ell_M'(R^t,\Lambda)$.

(iii) From the above proof, one can easily conclude that $\|x^p\|_{(M)}^{R^t}\to 0$ as $p\to\infty$, which implies that $x_k^p\to 0$ as $k\to\infty$ for each $p\in\mathbb{N}$. This leads us to the desired result. □

Definition 2.7. For any Orlicz function M, we define

$$h_M(R^t,\Lambda) := \left\{x=(x_k)\in\omega : \sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right)<\infty \text{ for each } \rho>0\right\}.$$

Clearly $h_M(R^t,\Lambda)$ is a subspace of $\ell'_M(R^t,\Lambda)$. Here and after we shall write $\|.\|$ instead of $\|.\|_{(M)}^{R^t}$ provided it does not lead to any confusion. The topology $h_M(R^t,\Lambda)$ is induced by $\|.\|$.

Proposition 2.8. *The inequality* $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\|x\|_{(M)}^{R^t} T_k}\right) \leq 1$ *holds for all* $x=(x_k)\in \ell'_M(R^t,\Lambda)$.

Proof. This is immediate from the definition of the norm $\|x\|_{(M)}^{R^t}$ defined by (2.3). □

Proposition 2.9. *Let M be an Orlicz function. Then, $(h_M(R^t,\Lambda),\|.\|)$ is an AK-BK space.*

Proof. First we show that $h_M(R^t,\Lambda)$ is an AK-space. Let $x=(x_k)\in h_M(R^t,\Lambda)$. Then, for each $\varepsilon\in(0,1)$, we can find n_0 such that

$$\sum_{k\geq n_0} M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\varepsilon T_k}\right) \leq 1.$$

Define the n^{th} section $x^{[n]}$ of a sequence $x=(x_k)$ by $x^{[n]}=\sum_{k=0}^n x_k e^k$. Hence for $n\geq n_0$, it holds

$$\begin{aligned}\|x-x^{[n]}\| &= \inf\left\{\rho>0 : \sum_{k\geq n_0} M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right)\leq 1\right\}\\ &\leq \inf\left\{\rho>0 : \sum_{k\geq n} M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right)\leq 1\right\} < \varepsilon.\end{aligned}$$

Thus, we can conclude that $h_M(R^t,\Lambda)$ is an AK-space.

Next to show that $h_M(R^t,\Lambda)$ is a BK-space, it is enough to show $h_M(R^t,\Lambda)$ is a closed subspace of $\ell'_M(R^t,\Lambda)$. For this, let (x^n) be a sequence in $h_M(R^t,\Lambda)$ such that $\|x^n-x\|\to 0$ as $n\to\infty$ where $x=(x_k)\in\ell'_M(R^t,\Lambda)$.

To complete the proof we need to show that $x = (x_k) \in h_M(R^t, \Lambda)$, i.e.,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right) < \infty \text{ for all } \rho > 0.$$

There is l corresponding to $\rho > 0$ such that $\|x^l - x\| \le \rho/2$. Then, using the convexity of M, we have by Proposition 2.8 that

$$\begin{aligned}
&\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right)\\
&= \sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j t_j x_j^l - 2\left(\left|\sum_{j=0}^k \lambda_j t_j x_j^l\right| - \left|\sum_{j=0}^k \lambda_j t_j x_j\right|\right)\right.}{2\rho T_k}\right)\\
&\le \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j t_j x_j^l\right|}{\rho T_k}\right) + \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j t_j (x_j^l - x_j)\right|}{\rho T_k}\right)\\
&\le \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j t_j x_j^l\right|}{\rho T_k}\right) + \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j t_j (x_j^l - x_j)\right|}{\|x^l - x\| T_k}\right)\\
&< \infty.
\end{aligned}$$

Hence, $x = (x_k) \in h_M(R^t, \Lambda)$ and consequently $h_M(R^t, \Lambda)$ is a BK-space. □

Proposition 2.10. *Let M be an Orlicz function. If M satisfies the Δ_2-condition at 0, then $\ell'_M(R^t, \Lambda)$ is an AK-space.*

Proof. We shall show that $\ell'_M(R^t, \Lambda) = h_M(R^t, \Lambda)$ if M satisfies the Δ_2-condition at 0. To do this it is enough to prove that $\ell'_M(R^t, \Lambda) \subset h_M(R^t, \Lambda)$. Let $x = (x_k) \in \ell'_M(R^t, \Lambda)$. Then for some $\rho > 0$,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right) < \infty.$$

This implies that

$$\lim_{k\to\infty} M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{\rho T_k}\right) = 0. \tag{2.4}$$

Choose an arbitrary $l > 0$. If $\rho \le l$, then $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j t_j x_j\right|}{l T_k}\right) < \infty$. Now, let $l < \rho$ and put $k = \rho/l$. Since M satisfies Δ_2-condition at 0, there exists

$R \equiv R_k > 0$ and $r \equiv r_k > 0$ with $M(kx) \leq RM(x)$ for all $x \in (0, r]$. By (2.4), there exists a positive integer n_1 such that

$$M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho T_k}\right) < p(\frac{r}{2})\frac{r}{2} \qquad \text{for all } k \geq n_1.$$

We claim that $\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho T_k} \leq r$ for all $k \geq n_1$. Otherwise, we can find $d > n_1$ with $\frac{\left|\sum_{j=0}^{d} \lambda_j t_j x_j\right|}{\rho T_d} > r$ and thus

$$M\left(\frac{\left|\sum_{j=0}^{d} \lambda_j t_j x_j\right|}{\rho T_d}\right) \geq \int_{r/2}^{\frac{\left|\sum_{j=0}^{d} \lambda_j t_j x_j\right|}{\rho T_d}} p(t)dt > p(\frac{r}{2})\frac{r}{2},$$

a contradiction. Hence, our claim is true. Then, we can find that

$$\sum_{k \geq n_1} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{l T_k}\right) \leq R \sum_{k \geq n_1} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho T_k}\right).$$

Hence,

$$\sum_{k} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{l T_k}\right) < \infty \text{ for all } l > 0.$$

This completes the proof. □

Proposition 2.11. *Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent, then $\ell'_{M_1}(R^t, \Lambda) = \ell'_{M_2}(R^t, \Lambda)$ and the identity map*

$$I : \left(\ell'_{M_1}(R^t, \Lambda), \|.\|_{M_1}^{R^t}\right) \to \left(\ell'_{M_2}(R^t, \Lambda), \|.\|_{M_2}^{R^t}\right)$$

is a topological isomorphism.

Proof. Let α, β and b be constants from (1.4). Since M_1 and M_2 are equivalent, it is obvious that (1.4) holds. Let us take any $x = (x_k) \in \ell'_{M_2}(R^t, \Lambda)$. Then,

$$\sum_{k} M_2\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho T_k}\right) < \infty \text{ for some } \rho > 0.$$

Hence, for some $l \geq 1$, $\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{l\rho T_k} \leq b$ for all $k \in \mathbb{N}$. Therefore,

$$\sum_k M_1\Big(\frac{\alpha|\sum_{j=0}^{k} \lambda_j t_j x_j|}{l\rho T_k}\Big) \leq \sum_k M_2\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{\rho T_k}\Big)$$

which shows that the inclusion

$$\ell'_{M_2}(R^t, \Lambda) \subset \ell'_{M_1}(R^t, \Lambda) \tag{2.5}$$

holds. One can easily see in the same way that the inclusion

$$\ell'_{M_1}(R^t, \Lambda) \subset \ell'_{M_2}(R^t, \Lambda) \tag{2.6}$$

also holds. By combining the inclusions (2.5) and (2.6), we conclude that $\ell'_{M_1}(R^t, \Lambda) = \ell'_{M_2}(R^t, \Lambda)$.

For simplicity in notation, let us write $\|.\|_1$ and $\|.\|_2$ instead of $\|.\|_{M_1}^{R^t}$ and $\|.\|_{M_2}^{R^t}$, respectively. For $x = (x_k) \in \ell'_{M_2}(R^t, \Lambda)$, we get

$$\sum_k M_2\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{\|x\|_2 T_k}\Big) \leq 1.$$

One can find $\mu > 1$ with

$$\frac{b}{2}\mu p_2(\frac{b}{2}) \geq 1$$

where p_2 is the kernel associated with M_2. Hence,

$$M_2\Big(\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{\|x\|_2 T_k}\Big) \leq \frac{b}{2}\mu p_2(\frac{b}{2})$$

for all $k \in \mathbb{N}$. This implies that

$$\frac{|\sum_{j=0}^{k} \lambda_j t_j x_j|}{\mu\|x\|_2 T_k} \leq b \qquad \text{for all } k \in \mathbb{N}.$$

Therefore,

$$\sum_k M_1\Big(\frac{\alpha|\sum_{j=0}^{k} \lambda_j t_j x_j|}{\mu\|x\|_2 T_k}\Big) < 1.$$

Hence, $\|x\|_1 \leq (\mu/\alpha)\|x\|_2$. Similarly, we can show that $\|x\|_2 \leq \beta\gamma\|x\|_1$ by choosing γ with $\gamma\beta > 1$ such that $\gamma\beta(b/2)p_1(b/2) \geq 1$. Thus,

$$\frac{\alpha}{\mu}\|x\|_1 \leq \|x\|_2 \leq \beta\gamma\|x\|_1$$

which establish that I is a topological isomorphism. □

Proposition 2.12. *Let M be an Orlicz function and p be the corresponding kernel. If $p(x) = 0$ for all x in $[0, b]$, where b is some positive number, then the spaces $\ell'_M(R^t, \Lambda)$ and $h_M(R^t, \Lambda)$ are topologically isomorphic to the spaces $\ell_\infty(R^t, \Lambda)$ and $c_0(R^t, \Lambda)$, respectively; where $\ell_\infty(R^t, \Lambda)$ and $c_0(R^t, \Lambda)$ are defined by*

$$\ell_\infty(R^t, \Lambda) = \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} \frac{1}{T_k}\sum_{j=0}^{k} |\lambda_j t_j x_j| < \infty\right\}$$

and

$$c_0(R^t, \Lambda) = \left\{x = (x_k) \in \omega : \lim_{k\to\infty} \frac{1}{T_k}\sum_{j=0}^{k} |\lambda_j t_j x_j| = 0\right\}.$$

It is easy to see that the spaces $\ell_\infty(R^t, \Lambda)$ and $c_0(R^t, \Lambda)$ are the Banach spaces under the norm

$$\|x\|_\infty^{R^t} = \sup_{k\in\mathbb{N}} \frac{1}{T_k}\sum_{j=0}^{k} |\lambda_j t_j x_j|.$$

Proof. Let $p(x) = 0$ for all x in $[0, b]$. If $y \in \ell_\infty(R^t, \Lambda)$, then we can find $\rho > 0$ such that $\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k} \leq b$ for all $k \in \mathbb{N}$. Hence, $\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k}\right) < \infty$. That is to say that $y \in \ell'_M(R^t, \Lambda)$. On the other hand, let $y \in \ell'_M(R^t, \Lambda)$. Then, for some $\rho > 0$, we have

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k}\right) < \infty.$$

Therefore, $\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k} \leq K < \infty$ for a constant $K > 0$ and for all $k \in \mathbb{N}$ which yields that $y \in \ell_\infty(R^t, \Lambda)$. Hence, $y \in \ell_\infty(R^t, \Lambda)$ if and only if $y \in \ell'_M(R^t, \Lambda)$. We can easily find x_1 such that $M(x_1) \geq 1$. Let $y \in \ell_\infty(R^t, \Lambda)$

and

$$\alpha = \|y\|_\infty = \sup_{k\in\mathbb{N}} \frac{1}{T_k}\sum_{j=0}^{k} |\lambda_j t_j y_j| > 0.$$

For every $\varepsilon \in (0,\alpha)$, we can determine d with $\sum_{j=0}^{d} \frac{|\lambda_j t_j y_j|}{T_d} > \alpha - \varepsilon$ and so

$$\sum_k M\left(\frac{|\sum_{j=0}^{k} \lambda_j t_j y_j| x_1}{\alpha T_k}\right) \geq M\left(\frac{\alpha - \varepsilon}{\alpha} x_1\right).$$

Since M is continuous, $\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right| x_1}{\alpha T_k}\right) \geq 1$, and so $\|y\|_\infty \leq x_1 \|y\|$, otherwise

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right| x_1}{\|y\| T_k}\right) > 1$$

which contradicts Proposition 2.8. Again,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right| x_1}{\alpha T_k}\right) = 0$$

which gives that $\|y\| \leq \|y\|_\infty / x_1$. That is to say that the identity map $I : (\ell'_M(R^t,\Lambda), \|.\|) \to (\ell_\infty(R^t,\Lambda), \|.\|)$ is a topological isomorphism.

For the last part, let $y \in h_M(R^t,\Lambda)$. Then, for any $\varepsilon > 0$, $\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{T_k} \leq \varepsilon x_1$ for all sufficiently large k, where x_1 is a positive number such that $p(x_1) > 0$. Hence, $y \in c_0(R^t,\Lambda)$. Conversely, let $y \in c_0(R^t,\Lambda)$. Then, for any $\rho > 0$, $\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k} < \frac{x_1}{2}$ for all sufficiently large k. Thus, $\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho T_k}\right) < \infty$ for all $\rho > 0$ and so $y \in h_M(R^t,\Lambda)$. Hence, $h_M(R^t,\Lambda) = c_0(R^t,\Lambda)$ and this step completes the proof. □

Proposition 2.13. *$c_0(R^t,\Lambda), c(R^t,\Lambda)$ and $\ell_\infty(R^t,\Lambda)$ are convex sets.*

Proof. We prove the Theorem for $c_0(R^t,\Lambda)$ and for other cases it will follow on applying similar arguments.

Let $x, y \in c_0(R^t,\Lambda)$. Then, there exists $\rho_1, \rho_2 > 0$ such that

$$\lim_{k\to\infty} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho_1 T_k}\right) = 0 \text{ and } \lim_{k\to\infty} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho_2 T_k}\right) = 0.$$

For $\mu = 0$ or $\mu = 1$, the result is obvious. Let $0 < \mu < 1$. Considering $\rho = \max\{|\mu|\rho_1, |1-\mu|\rho_2\}$, we have

$$\begin{aligned} & M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j[\mu x_j + (1-\mu)y_j]\right|}{2\rho T_k}\right) \\ \leq\ & \frac{1}{2}M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j(\mu x_j)\right|}{\rho T_k}\right) + \frac{1}{2}M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j[(1-\mu)y_j]\right|}{\rho T_k}\right) \\ \leq\ & \frac{1}{2}M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j x_j\right|}{\rho_1 T_k}\right) + \frac{1}{2}M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j t_j y_j\right|}{\rho_2 T_k}\right). \end{aligned}$$

This completes the proof. □

Prior to giving our final two consequences concerning the α-dual of the spaces $\ell'_M(R^t, \Lambda)$ and $h_M(R^t, \Lambda)$, we present the following easy lemma without proof.

Lemma 2.14. *For any Orlicz function M, $\Lambda x = (\lambda_k x_k) \in \ell_\infty$ whenever $x = (x_k) \in \ell'_M(R^t, \Lambda)$.*

Proposition 2.15. *Let M be an Orlicz function and p be the corresponding kernel of M. Define the sets D_1 and D_2 by*

$$D_1 := \left\{a = (a_k) \in \omega : \sum_k \left|\frac{a_k}{\lambda_k}\right| < \infty\right\}$$

and

$$D_2 := \left\{s = (s_k) \in \omega : \sup_{k\in\mathbb{N}} |\lambda_k s_k| < \infty\right\}.$$

If $p(x) = 0$ for all x in $[0, d]$, where d is some positive number, then the following statements hold:

(i) *Köthe-Toeplitz dual of $\ell'_M(R^t, \Lambda)$ is the set D_1.*
(ii) *Köthe-Toeplitz dual of D_1 is the set D_2.*

Proof. Since the proof of Part (ii) is similar to that of the proof of Part (i), to avoid the repetition of the similar statements we prove only Part (i).

Let $a = (a_k) \in D_1$ and $x = (x_k) \in \ell'_M(R^t, \Lambda)$. Then, since

$$\sum_k |a_k x_k| = \sum_k |a_k \lambda_k^{-1}||\lambda_k x_k| \leq \sup_{k\in\mathbb{N}} |\lambda_k x_k| \sum_k |a_k \lambda_k^{-1}| < \infty,$$

applying Lemma 2.14, we have $a = (a_k) \in \{\ell'_M(R^t, \Lambda)\}^\alpha$. Hence, the inclusion

$$D_1 \subset \{\ell'_M(R^t, \Lambda)\}^\alpha \tag{2.7}$$

holds.

Conversely, suppose that $a = (a_k) \in \{\ell'_M(R^t, \Lambda)\}^\alpha$. Then, $(a_k x_k) \in \ell_1$, the space of all absolutely convergent series, for every $x = (x_k) \in \ell'_M(R^t, \Lambda)$. So, we can take $x_k = \lambda_k^{-1}$ for all $k \in \mathbb{N}$ because $x = (x_k) \in \ell'_M(R^t, \Lambda)$ by Proposition 2.12 whenever $x = (x_k) \in \ell_\infty(R^t, \Lambda)$. Therefore, $\sum_k |a_k \lambda_k^{-1}| = \sum_k |a_k x_k| < \infty$ and we have $a = (a_k) \in D_1$. This leads us to the inclusion

$$\{\ell'_M(R^t, \Lambda)\}^\alpha \subset D_1. \tag{2.8}$$

By combining the inclusion relations (2.7) and (2.8), we have $\{\ell'_M(R^t, \Lambda)\}^\alpha = D_1$. □

Proposition 2.15 (ii) shows that $\{\ell'_M(R^t, \Lambda)\}^{\alpha\alpha} \neq \ell'_M(R^t, \Lambda)$ which leads us to the consequence that $\ell'_M(R^t, \Lambda)$ is not perfect under the given conditions.

Proposition 2.16. *Let M be an Orlicz function and p be the corresponding kernel of M and the set D_1 be defined as in the Proposition 2.15. If $p(x) = 0$ for all x in $[0, b]$, where b is a positive number, then the Köthe-Toeplitz dual of $h_M(R^t, \Lambda)$ is the set D_1.*

Proof. Let $a = (a_k) \in D_1$ and $x = (x_k) \in h_M(R^t, \Lambda)$. Then, since

$$\sum_k |a_k x_k| = \sum_k |a_k \lambda_k^{-1}||\lambda_k x_k| \leq \sup_{k \in \mathbb{N}} |\lambda_k x_k| \sum_k |a_k \lambda_k^{-1}| < \infty,$$

we have $a = (a_k) \in \{h_M(R^t, \Lambda)\}^\alpha$. Hence, the inclusion

$$D_1 \subset \{h_M(R^t, \Lambda)\}^\alpha \tag{2.9}$$

holds.

Conversely, suppose that $a = (a_k) \in \{h_M(R^t, \Lambda)\}^\alpha \backslash D_1$. Then, there exists a strictly increasing sequence (n_i) of positive integers n_i such that

$$\sum_{k=n_i+1}^{n_{i+1}} |a_k||\lambda_k|^{-1} > i.$$

Define $x = (x_k)$ by

$$x_k := \begin{cases} \lambda_k^{-1}\mathrm{sgn}\dfrac{a_k}{i}, & n_i < k \leq n_{i+1}, \\ 0, & 0 \leq k < n_0, \end{cases}$$

for all $k \in \mathbb{N}$. Then, since $x = (x_k) \in c_0(R^t, \Lambda)$ and so by Proposition 2.12 $x = (x_k) \in h_M(R^t, \Lambda)$. Therefore, we have

$$\begin{aligned} \sum_k |a_k x_k| &= \sum_{k=n_0+1}^{n_1} |a_k x_k| + \cdots + \sum_{k=n_i+1}^{n_{i+1}} |a_k x_k| + \cdots \\ &= \sum_{k=n_0+1}^{n_1} |a_k \lambda_k^{-1}| + \cdots + \frac{1}{i}\sum_{k=n_i+1}^{n_{i+1}} |a_k \lambda_k^{-1}| + \cdots \\ &> 1 + \cdots + 1 + \cdots = \infty, \end{aligned}$$

which contradicts the hypothesis. Hence, $a = (a_k) \in D_1$. This leads us to the inclusion

$$\{h_M(R^t, \Lambda)\}^\alpha \subset D_1. \tag{2.10}$$

By combining the inclusion relations (2.9) and (2.10), we obtain the desired result $\{h_M(R^t, \Lambda)\}^\alpha = D_1$. This completes the proof. □

3. Conclusion

The general aim of this study is to fill a gap in literature by extending certain Orlicz sequence spaces and to investigate some topological properties.

The Orlicz difference sequence spaces $\ell_M(\Delta, \Lambda)$ and $\widetilde{\ell}_M(\Delta, \Lambda)$ were recently been studied by H. Dutta [21]. Quite recently, generalized Orlicz difference sequence spaces $c_0(M, \Delta^m), c(M, \Delta^m)$ and $\ell_\infty(M, \Delta^m)$ have been examined by the same author in [22]. Of course, the sequence spaces introduced in this paper can be redefined as a domain of a suitable matrix in the Orlicz sequence space ℓ_M. Indeed, if we define the infinite matrix

$R^t(\lambda) = \{r^t_{nk}(\lambda)\}$ via the multiplier sequence $\Lambda = (\lambda_k)$ by

$$r^t_{nk}(\lambda) := \begin{cases} \dfrac{\lambda_k t_k}{T_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, then the sequence spaces $\ell'_M(R^t, \Lambda), c_0(R^t, \Lambda)$ and $\ell_\infty(R^t, \Lambda)$ represent the domain of the matrix $R^t(\lambda)$ in the sequence spaces ℓ_M, c_0 and ℓ_∞, respectively. Nevertheless, the present results does not compare with the results obtained by [23]. But our results are more general and more comprehensive than the corresponding results of Dutta and Başar [23], since the spaces $\ell_M(R^t, \Lambda), \widetilde{\ell}_M(R^t, \Lambda), \ell'_M(R^t, \Lambda)$ and $h_M(R^t, \Lambda)$ reduce in the cases $\lambda_k = 1$ and $t_k = 1$ to the $\ell_M(C, \Lambda), \widetilde{\ell}_M(C, \Lambda), \ell'_M(C, \Lambda)$ and $h_M(C, \Lambda)$, respectively, where $C = (c_{nk})$ is the matrix of Cesáro of order one.

Acknowledgment

We wish to thank the referees for their valuable suggestions and comments which improved the paper considerably.

References

1. F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, 2011.

2. B. Choudary, S. Nanda, *Functional Analysis with Applications*, John Wiley Sons Inc., New Delhi, 1989.

3. J. Musielak, *On Orlicz sequence spaces*, Lecture Notes in Mathematics 1083, Springer-Verlag, 1983.

4. J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

5. S. Chen, Geometry of Orlicz spaces, *Dissertationes Math.*, **356** (1996), 1-204.

6. Y. Gribanov, On the theory of ℓ_M spaces, *Uc. Zap. Kazansk un-ta*, **117** (1957), 62-65, (in Russian).

7. P. K. Kamthan, Bases in a certain class of Frechet spaces, *Tamkang J. Math.*, **7** (1976), 41-49.

8. P. K. Kamthan and M. Gupta, *Sequence spaces and series*, Marcel Deccer Inc., New York and Basel, 1981.

9. M. A. Krasnoselskii and Y. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen, Netherlands, 1961.

10. G. Köthe and O. Toeplitz, Linear Raume mit unendlichvielen koordinaten and Ringe unenlicher Matrizen, *J. Reine Angew. Math.*, **171** (1934), 193-226.

11. I. J. Maddox, *Elements of Functional Analysis*, 2nd edition, The University Press, Cambridge, 1988.

12. B. C. Tripathy, A class of difference sequences related to the p-normed space ℓ^p, *Demonstratio Math.*, **36** (2003), no. 4, 867-872.

13. M. Mursaleen, Kuldip Raj and Sunil K. Sharma, Some spaces of difference sequences and lacunary statistical convergence in n-normed space defined by sequence of Orlicz functions, *Miskolc Math. Notes*, **16** (2015), no.1, 283-304.

14. A. Alotaibi, M. Mursaleen, Sunil K. Sharma and S. A. Mohiuddine, Sequence spaces of fuzzy numbers defined by a Musielak-Orlicz function, *Filomat*, **29** (2015), no. 7, 1461-1468.

15. A. Alotaibi, M. Mursaleen and Kuldip Raj, Double sequence spaces by means of Orlicz functions, *Abstract Appl. Anal.*, Volume 2014, Article ID 260326, 8 Pages.

16. M. Mursaleen, Sunil K. Sharma, S. A. Mohiuddine and A. Kiliçman, New difference sequence spaces defined by Musielak-Orlicz function, *Abstract Appl. Anal.*, Volume 2014, Article ID 691632, 9 Pages.

17. A. Alotaibi, M. Mursaleen and Sunil K. Sharma, Double sequence spaces over n-normed spaces defined by a sequence of Orlicz functions, *J. Ineq. Appl.*, 2014, 2014:216.

18. M. Mursaleen, Sunil K. Sharma and A. Kiliçman, Sequence spaces defined by Musielak-Orlicz function over n-normed spaces, *Abstract Appl. Anal.*, Volume 2013, Article ID 364743, 10 Pages.

19. M. Et, M. Mursaleen and M. Işik, On a class fuzzy sets defined by Orlicz functions, *Filomat*, **27** (2013), no. 5, 789-796.

20. G. Goes, S. Goes, Sequences of bounded variation and sequences Fourier coefficients, *Math. Z.*, **118** (1970), 93-102.

21. H. Dutta, On Orlicz difference sequence spaces, *SDU J. Sci.*, **5** (2010), 119-136.

22. H. Dutta, Generalized difference sequence spaces defined by Orlicz functions and their Köthe-Toeplitz and null duals, *Thai J. Math.*, **8**(1) (2010), 11-19.

23. H. Dutta and F. Başar, A generalization of Orlicz sequence spaces by Cesáro mean of order one, *Acta Math. Univ. Comenianae*, **80**(2) (2011), 185-200.