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CONVERGENCE RATE OF THE GLIMM SCHEME

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Abstract

In this paper we prove that there exists a random sequence θ_i for the Glimm scheme such that the approximate solution $u^\epsilon(t)$ converges to the exact semigroup solution $S_t\bar{u}$ of the strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \qquad u(t=0) = \bar{u}$$

as follows: for all $T \geq 0$ it holds

$$\lim_{\varepsilon \to 0} \frac{\|u^\varepsilon(T) - S_T\bar{u}\|_1}{\sqrt{\varepsilon}|\log \varepsilon|} = 0.$$

This result is the extension of the analysis of [8] to the general case, when no assumptions on the flux f are made besides strict hyperbolicity. As a corollary, we obtain a deterministic version of the Glimm scheme for the general system case, extending the analysis of [14].

The analysis requires an extension of the quadratic interaction estimates obtained in [3] in order to analyze interaction occurring during an interval of time.

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1. Introduction

A strict hyperbolic system of conservation laws in one space dimension (see [5]) is a system of PDEs of the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}^n$ is the unknown and $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth (C^3) map, called *flux*, defined on a neighborhood Ω of a compact set $K \subseteq \mathbb{R}^n$ and satisfying the strict hyperbolicity condition, i.e. the Jacobian $Df(u)$ of f has n distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u) \tag{1.2}$$

in each point $u \in \Omega$ of its domain. Throughout this paper, we will assume w.l.o.g. that $0 \in K \subseteq \Omega$ and

$$\lambda_k(u) \in [0,1] \qquad \text{for all } k \text{ and for all } u. \tag{1.3}$$

This can always be achieved by a change of variable in the (t,x)-plane. As it is customary, denote by $r_1(u),\dots,r_n(u)$ the right eigenvalues (normalized to 1) associated to $\lambda_1(u),\dots,\lambda_n(u)$ respectively:

$$Df(u)r_k(u) = \lambda_k(u)r_k(u), \qquad \text{for all } k = 1,\dots,n \text{ and for all } u \in \Omega.$$

Equation (1.1) is usually coupled with an initial datum

$$u(t=0) = \bar{u}, \tag{1.4}$$

where $\bar{u} : \mathbb{R} \to \mathbb{R}^n$ is a given map, with sufficiently small total variation. W.l.o.g. we assume also that $\bar{u}$ has compact support.

It is well known that classical (smooth) solutions to the Cauchy problem (1.1)$-$(1.4) are in general not defined on the whole time interval $[0,\infty)$, even if the initial datum is smooth, because they develop discontinuities in finite time. On the other hand, the notion of distributional solution is too weak to guarantee the uniqueness. For this reasons the notion of solution which is typically used is the following one.

Definition 1.1. A map $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}^n$ belonging to L^1_{loc} is said to be *a weak solution* of the Cauchy problem (1.1)$-$(1.4) if:

(1) u satisfies equation (1.1) in the sense of distributions;
(2) u is continuous as a map $[0, \infty) \to L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$;
(3) at time $t = 0$, $u(0, x) = \bar{u}(x)$;
(4) u satisfies some additional admissibility criteria, which follow from physical or stability considerations and guarantee the uniqueness of the solution.

Many admissibility criteria have been proposed in the literature: just to name a few, the Lax-Liu condition on shocks (see [10, 12, 13]), the entropy condition (see [11]), the vanishing viscosity criterion (see [2]). We do not want to enter into details: the interested reader can refer to the cited literature.

1.1. The Riemann problem

The basic ingredient to solve the Cauchy problem (1.1)$-$(1.4) is the solution of the Riemann problem, i.e. the Cauchy problem when the initial datum has the simple form

$$u(0, x) = \bar{u}(x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x \geq 0. \end{cases} \tag{1.5}$$

The solution of the Riemann problem (1.1)$-$(1.5) was obtained first by P. Lax in 1957 [10], under the assumption that each characteristic field is either *genuinely non linear* (GNL), i.e

$$\nabla \lambda_k(u) \cdot r_k(u) \neq 0 \qquad \text{for any } u,$$

or *linearly degenerate* (LD), i.e.

$$\nabla \lambda_k(u) \cdot r_k(u) = 0 \qquad \text{for any } u.$$

In this case, if $|u^R - u^L| \ll 1$, using the Implicit Function Theorem, one can find intermediate states $u^L = \omega_0, \omega_1, \ldots, \omega_n = u^R$ such that each pair of adjacent states (ω_{k-1}, ω_k) can be connected by either a shock or a rarefaction wave or a contact discontinuity of the k-th family. The complete solution is now obtained by piecing together the solutions of the n Riemann problems (ω_{k-1}, ω_k) on different sectors of the (t, x)-plane.

In the general case (here and in the rest of the paper, by *general case* we mean that no assumption on f is made besides strict hyperbolicity), the solution to the Riemann problem (u^L, u^R) was obtained by S. Bianchini and A. Bressan in [2]. They first construct, for any left state u^L and for any family $k = 1, \dots, n$, a curve $s \to T^k_s u^L$ of *admissible right states*, defined for $s \in \mathbb{R}$ small enough, such that the Riemann problem $(u^L, T^k_s u^L)$ can be solved by (countable many) admissible shocks (in the sense of being limits of travelling profiles for the viscosity approximation), contact discontinuities and rarefactions waves. Then, as in the GNL/LD case, the global solution of (u^L, u^R) is obtained by piecing together the solutions of n Riemann problems, one for each family: namely by using the Implicit Function Theorem to write

$$u^R = T^n_{s_n} \circ \cdots \circ T^1_{s_1} u^L$$

and solve each Riemann problem (ω_{k-1}, ω_k) with admissible waves of the k-th family, where

$$\omega_k = T^k_{s_k} \circ \cdots \circ T^1_{s_1} u^L.$$

In Section 2.1 we briefly recall the construction of the admissible curves $s \mapsto T^k_s u^L$.

1.2. Glimm approximate solutions in the GNL/LD case

The first result about existence of solutions to the Cauchy problem (1.1)$-$(1.4) can be found in the celebrated paper by J. Glimm [9] in 1965, in which the existence of solutions is proved under the assumption that each characteristic field is either GNL or LD. In [9], for all $\varepsilon > 0$ an approximate solution $u^\varepsilon(t,x)$ is constructed by recursion as follows. First of all, take a sampling sequence $\{\vartheta_i\}_{i \in \mathbb{N}} \subseteq [0,1]$. The algorithm starts choosing, at time $t = 0$, an approximation $\bar{u}^\varepsilon$ of the initial datum $\bar{u}$, such that $\bar{u}^\varepsilon$ is compactly supported, right continuous, piecewise constant with jumps located at points $t = m\varepsilon$, $m \in \mathbb{Z}$. We can thus separately solve the Riemann problems located at $(t,x) = (0, m\varepsilon)$, $m \in \mathbb{Z}$. Thanks to (1.3), the solution $u^\varepsilon(t,x)$ can now be prolonged up to time $t = \varepsilon$. At $t = \varepsilon$ a restarting procedure is used. The value of u^ε at time ε is redefined as

$$u^\varepsilon(\varepsilon+, x) := u^\varepsilon(\varepsilon-, m\varepsilon + \vartheta_1\varepsilon), \qquad \text{if } x \in [m\varepsilon, (m+1)\varepsilon). \tag{1.6}$$

The solution $u(\varepsilon,\cdot)$ is now again piecewise constant with compact support, with discontinuities on points of the form $x = m\varepsilon$, $m \in \mathbb{Z}$. If the sizes of the jumps are sufficiently small, we can again solve the Riemann problem at each point $(t,x) = (\varepsilon, m\varepsilon)$, $m \in \mathbb{Z}$ and thus prolong the solution up to time 2ε, where again the restarting procedure (1.6) is used, with ϑ_2 instead of ϑ_1. The above procedure can be repeated on any time interval $[i\varepsilon,(i+1)\varepsilon]$, $i \in \mathbb{N}$, as far as the size of the jump at each grid point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}, m \in \mathbb{Z}$, remains small enough: this is the case whenever

$$\text{Tot.Var.}(u^\varepsilon(t);\mathbb{R}) \ll 1. \tag{1.7}$$

In order to prove (1.7), Glimm introduced a uniformly bounded decreasing functional

$$t \mapsto Q^{\text{Glimm}}(t) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u})^2,$$

such that at any time $i\varepsilon$, $i \in \mathbb{N}$,

$$\begin{aligned} &\text{Tot.Var.}(u^\varepsilon(i\varepsilon+);\mathbb{R}) - \text{Tot.Var.}(u(i\varepsilon-);\mathbb{R}) \\ &\leq \mathcal{O}(1)\big(Q^{\text{Glimm}}(i\varepsilon-) - Q^{\text{Glimm}}(i\varepsilon+)\big). \end{aligned} \tag{1.8}$$

Here and in the following $\mathcal{O}(1)$ denotes a constant which depends only on the flux f and on the sampling sequence $\{\vartheta_i\}_i$. As an immediate consequence, we get $\text{Tot.Var.}(u^\varepsilon(t);\mathbb{R}) \leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(0);\mathbb{R}) \ll 1$ and thus the solution $u^\varepsilon(t,x)$ can be defined on the whole (t,x)-plane $[0,\infty)\times\mathbb{R}$. The uniform bound on the $\text{Tot.Var.}(u^\varepsilon(t);\mathbb{R})$ yields a compactness on the family $\{u^\varepsilon\}_\varepsilon$: we can thus extract a converging subsequence, which turns out to be, for almost every sampling sequence $\{\vartheta_i\}_i$, a weak admissible solution of the Cauchy problem (1.1)–(1.4).

In 1977 T.-P. Liu [14] gave a deterministic version of Glimm's result, showing that if the sampling sequence is *equidistributed*, i.e. for all $\lambda \in [0,1]$,

$$\lim_{j\to\infty} \frac{\sharp\{i \in \mathbb{N} \mid 1 \leq i \leq j \text{ and } \vartheta_i \in [0,\lambda]\}}{j} = \lambda,$$

then the subsequence extracted from $\{u^\varepsilon\}_\varepsilon$ converges to a weak admissible solution of (1.1)–(1.4).

The analysis of the stability in L^1 of the solution of (1.1)−(1.4) w.r.t the initial datum $\bar{u}$ led to the introduction of the notion of *standard Riemann semigroup*.

Definition 1.2. A *standard Riemann semigroup* for the system of conservation laws (1.1) is a map $S : \mathcal{D} \times [0, \infty) \to \mathcal{D}$, defined on a domain $\mathcal{D} \subseteq L^1(\mathbb{R}; \mathbb{R}^n)$ containing all functions with sufficiently small total variation, with the following properties:

(1) for some Lipschitz constants L, L',

$$\|S_t\bar{u} - S_s\bar{v}\|_1 \leq L\|\bar{u} - \bar{v}\|_1 + L'|t - s|, \quad \text{for all } \bar{u}, \bar{v} \in \mathcal{D},\ t, s \geq 0; \tag{1.9}$$

(2) if $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small $S_t\bar{u}$ coincides with the solution of (1.1)−(1.4), which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

If it exists, the standard Riemann semigroup is unique [4].

In the GNL/LD case it is proved (see, among others, [6, 7, 16]) that the standard Riemann semigroup exists and that at any time $t \geq 0$ the solution $u(t)$ obtained as limit of Glimm approximations $u^\varepsilon(t)$, for the initial datum $\bar{u}$, coincides with the semigroup trajectory $S_t\bar{u}$. We will discuss in the next section the general case.

Relying on the existence of the standard Riemann semigroup for GNL/LD systems, in 1998 A. Bressan and A. Marson [8] further improved the Glimm sampling method, constructing an equidistributed sequence $\{\vartheta_i\}$, satisfying the additional assumption:

$$\sup_{\lambda \in [0,1]} \left| \lambda - \frac{\sharp\{i \in \mathbb{N} \mid j_1 \leq i < j_2 \text{ and } \vartheta_i \in [0, \lambda]\}}{j_2 - j_1} \right| \leq C \cdot \frac{1 + \log(j_2 - j_1)}{j_2 - j_1}. \tag{1.10}$$

Using this sequence, they were able to prove that the rate of convergence of the Glimm approximate solutions $u^\varepsilon(t)$ to the semigroup weak admissible solution $u(t) = S_t\bar{u}$ at every fixed time t is given by

$$\lim_{\varepsilon \to 0} \frac{\left\|u^\varepsilon(t, \cdot) - S_t\bar{u}\right\|_{L^1}}{\left|\log \varepsilon\right| \sqrt{\varepsilon}} = 0. \tag{1.11}$$

1.3. Glimm approximate solutions in the general case

All the results cited in the previous section were obtained under the assumption that each characteristic field is either GNL or LD. In this section we consider now the general case, when this assumption is removed and the only requirement is that the system is strictly hyperbolic (1.2).

The problem of finding a suitable decreasing potential to bound the increase of $t \mapsto \text{Tot.Var.}(u^\varepsilon(t);\mathbb{R})$ for a Glimm approximate solution u^ε (see (1.8)) was solved first by T.-P. Liu in [15] for fluxes f with a finite number of inflection points. Later, in [1], Bianchini solved the problem for general fluxes, introducing the cubic functional

$$t \mapsto Q^{\text{cubic}}(t) := \sum_{k=1}^{n} \iint |\sigma_k(t,s_k) - \sigma(t,s'_k)| ds_k ds'_k \le \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(t))^3,$$

where s_k, s'_k are two waves of the k-th family in the approximate solution at time t and $\sigma_k(t,s_k), \sigma_k(t,s'_k)$ denote their speed (see Section 2.4 for a precise definition). In [2] Bianchini and Bressan also proved that every strictly hyperbolic f admits a standard Riemann semigroup S_t of *vanishing viscosity solutions* with small total variation obtained as the (unique) limit of solutions to the viscous parabolic approximations

$$u_t + f(u)_x = \mu u_{xx},$$

when the viscosity $\mu \to 0$. The semigroup S is defined on

$$\mathcal{D} := \Big\{ u \in L^1(\mathbb{R};\mathbb{R}^n) \;\Big|\; \text{Tot.Var.}(u) \ll 1,\ \lim_{x\to-\infty} u(x) \in K \Big\}$$

and satisfies the Lipschitz condition

$$\|S_t\bar{u} - S_s\bar{v}\|_1 \le L\|\bar{u}-\bar{v}\|_1 + L'|t-s|, \quad \text{for any } \bar{u},\bar{v} \in \mathcal{D}, \quad t,s \ge 0. \tag{1.12}$$

Aim of this paper is to prove that the same rate of convergence (1.11) obtained by Bressan and Marson in the GNL/LD case holds also in the general case, when no assumption on f is made except its strictly hyperbolicity. In particular we prove the following theorem.

Theorem 1.3. *Consider the Cauchy problem* (1.1)−(1.4) *and assume that the system* (1.1) *is strictly hyperbolic. Let* u^ε *be a Glimm approximate solution with mesh size* $\varepsilon > 0$ *and sampling sequence satisfying* (1.10), *and denote by* $t \mapsto S_t\bar{u}$ *the semigroup of vanishing viscosity solutions. Then for every fixed time* $T \in [0, +\infty)$ *the following limit holds:*

$$\lim_{\varepsilon \to 0} \frac{\big\|u^\varepsilon(T, \cdot) - S_T\bar{u}\big\|_1}{\sqrt{\varepsilon}|\log \varepsilon|} = 0. \tag{1.13}$$

1.4. Bressan's and Marson's technique

We recall now the technique used by A. Bressan and A. Marson in [8] to prove Theorem 1.3 in the GNL/LD case. In particular we wish to highlight which is the point in Bressan's and Marson's proof which can not be easily extended to the general case, where no assumption of f is made except its strict hyperbolicity, and whose detailed proof is given in this paper, using the tools introduced by the authors in [3].

Bressan's and Marson's technique is as follows. Thanks to the Lipschitz property of the semigroup (1.9), in order to estimate the distance

$$\big\|u^\varepsilon(T, \cdot) - S_T\bar{u}\big\|_{L^1},$$

we can partition the time interval $[0, T]$ in subintervals $J_a := [t_a, t_{a+1}]$ and estimate the error

$$\big\|u^\varepsilon(t_{a+1}) - S_{t_{a+1}-t_a}u^\varepsilon(t_a)\big\|_{L^1} \tag{1.14}$$

on each interval J_a. The error (1.14) on J_r comes from two different sources:

(1) first of all there is an error due to the algorithm itself: indeed, in a Glimm approximate solution, roughly speaking, we give each wave either speed 0 or speed 1 (according to the sampling sequence $\{\vartheta_i\}_i$), while in the exact solution it would have a speed in $[0, 1]$, but not necessarily equal to 0 or 1;

(2) secondly, there is an error due to the fact that some waves can be created at times $t > t_a$, some waves can be canceled at times $t < t_{a+1}$ and, above all, some waves, which are present both at time t_a and at time t_{a+1}, can change their speeds, when they interact with other waves.

The first error source is estimated by choosing the intervals J_a sufficiently large in order to use estimate (1.10) with $j_2 - j_1 \gg 1$.

The second error source can be estimated (choosing the intervals J_a not too large) if we are able to (uniformly) bound the change in speed of the waves present in the approximate solution. In the GNL/LD case, this was achieved by Liu in [14], where he provided a wave tracing algorithm which splits each wavefront in the approximate solution into a finite number of discrete waves, whose trajectories can be traced and whose changes in speed at any interaction time are bounded by the corresponding decrease of the functional Q^{Glimm}. In particular, using Liu's wave tracing, Bressan and Marson prove that for any $i_1, i_2 \in \mathbb{N}$, on the time interval $[t_1, t_2]$, $t_1 = i_1\varepsilon$, $t_2 = i_2\varepsilon$, it holds

$$\begin{aligned}&\big\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\big\|_1 \\ &\leq \mathcal{O}(1)\bigg[\Big(Q^{\mathrm{Glimm}}(t_2) - Q^{\mathrm{Glimm}}(t_1)\Big) + \frac{1+\log(i_2-i_1)}{i_2-i_1} + \varepsilon\bigg](t_2-t_1). \quad (1.15)\end{aligned}$$

As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_a in such a way that the errors in (1) and (2) have approximately the same order of magnitude. In particular, the estimate (1.13) is obtained by choosing $|J_a| \approx \sqrt{\varepsilon}\log|\log\varepsilon|$.

Estimate (1.15) is precisely the point in Bressan's and Marson's proof which can not be easily extended to the general case, because the functional Q^{Glimm} is not of help in this case. Improving the results recently obtained by the authors in [3], in this paper a suitable functional

$$\Upsilon : [0,+\infty) \to [0,+\infty), \quad \Upsilon(0) \leq \mathcal{O}(1)\mathrm{Tot.Var.}(u_0),$$

is constructed, such that for any $i_1, i_2 \in \mathbb{N}$, $i_1 < i_2$,

$$\big\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\big\|_1 \leq \mathcal{O}(1)\bigg[\Big(\Upsilon(t_2) - \Upsilon(t_1)\Big) + \frac{1+\log(i_2-i_1)}{i_2-i_1}\bigg](t_2-t_1). \quad (1.16)$$

In order to prove (1.16), one could be tempted to use the well known semigroup inequality (see [5])

$$\|u^\varepsilon(t_2) - S_{t_2-t_1}u^\varepsilon(t_1)\|_1 \leq L\int_{t_1}^{t_2} \limsup_{h\to 0} \frac{\big\|u^\varepsilon(t+h) - S_h u^\varepsilon(t)\big\|_1}{h} dt.$$

However, for a Glimm solution u^ε this estimate can not be directly applied, because it does not take into account the error due to the restarting procedure. To go beyond this difficulty, in the same spirit as in [8], we will introduce in Section 3 a "wavefront" map

$$\psi : [t_1, t_2] \times \mathbb{R} \to \mathbb{R}^n$$

with the following properties:

$$\psi(t_2, x) = u^\varepsilon(t_2, x), \tag{1.17a}$$

$$\big\|S_{t_2-t_1}\psi(t_1) - \psi(t_2)\big\|_1 \leq \mathcal{O}(1)\bigg[\Big(\Upsilon(t_1)-\Upsilon(t_2)\Big)+\frac{1+\log(i_2-i_1)}{i_2-i_1}\bigg](t_2-t_1), \tag{1.17b}$$

$$\big\|\psi(t_1) - u^\varepsilon(t_1)\big\|_1 \leq \mathcal{O}(1)\Big(\Upsilon(t_1) - \Upsilon(t_2)\Big)(t_2 - t_1). \tag{1.17c}$$

Clearly (1.16) is an immediate consequence of (1.17) and the Lipschitz continuity of the semigroup S_t.

Remark 1.4. Notice that all the functionals $Q^{\text{Glimm}}, Q^{\text{cubic}}, \Upsilon$ are defined on the approximate solution u^ε, or, in other words, they depend on ε, even if we do not write this dependence explicitly. What is important, is that they are decreasing and uniformly (i.e. without any reference to ε) bounded at $t = 0$.

1.5. Proof of Theorem 1.3

We conclude this introduction proving Theorem 1.3 in the general case, assuming that estimate (1.16) holds and using Bressan's and Marson's techniques. Fix $T, \varepsilon > 0$, say $T = \bar{i}\varepsilon + \varepsilon'$ for some integer $\bar{i}$ and some $\varepsilon' \in [0, \varepsilon)$. In connection with a constant $\delta \geq \varepsilon$ (whose precise value will be specified later), we construct a partition of the interval $[0, \bar{i}\varepsilon]$ into finitely many subintervals $J_a = [t_a, t_{a+1}]$, inserting the points $t_a = i_a\varepsilon$ inductively as follows. Set $i_0 := 0$. If the integers $i_0 < i_1 < \cdots < i_a < \bar{i}$ have already been defined, then

(i) if $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^\varepsilon\big((i_a + 1)\varepsilon\big) \leq \delta$, let i_{a+1} be the largest integer $\leq \bar{i}$ such that $(i_{a+1} - i_a)\varepsilon \leq \delta$ and $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^\varepsilon(i_{a+1}\varepsilon) \leq \delta$;

(ii) if $\Upsilon^\varepsilon(i_a\varepsilon) - \Upsilon^a\big((i_a + 1)\varepsilon\big) > \delta$, define $i_{a+1} := i_a + 1$.

Clearly $i_A = \bar{i}$ for some integer $A \leq \bar{i}$. Call $\mathcal{A}', \mathcal{A}''$ respectively the set of indices a for which the alternative (i), (ii) holds. Observe that the cardinalities of these sets can be bounded by

$$\sharp\mathcal{A}' \leq \mathcal{O}(1)\frac{T}{\delta}, \quad \sharp\mathcal{A}'' \leq \mathcal{O}(1)\frac{\text{Tot.Var.}(u_0)^2}{\delta} \leq \mathcal{O}(1)\frac{T}{\delta} \tag{1.18}$$

for $\delta \ll 1$. On each subinterval J_a, $a \in \mathcal{A}'$ we can apply (1.16), thus obtaining

$$\begin{aligned} &\left\|u^\varepsilon(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u^\varepsilon(i_a\varepsilon)\right\|_1 \\ &\leq \mathcal{O}(1)\left[\Big(\Upsilon^\varepsilon(i_{a+1}\varepsilon) - \Upsilon^\varepsilon(i_a\varepsilon)\Big) + \frac{1+\log(i_{a+1}-i_a)}{i_{a+1}-i_a} + \varepsilon\right](i_{a+1}-i_a)\varepsilon. \end{aligned} \tag{1.19}$$

On the other hand, on each interval J_a with $a \in \mathcal{A}''$, the 1-Lipschitz continuity of $u^\varepsilon : [0,\infty) \to L^1(\mathbb{R};\mathbb{R}^n)$ implies that

$$\left\|u^\varepsilon(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u^\varepsilon(i_a\varepsilon)\right\|_1 \leq (i_{a+1}-i_a)\varepsilon = \varepsilon. \tag{1.20}$$

Using the Lipschitz property (1.12) of the semigroup we get

$$\begin{aligned} \left\|u^\varepsilon(\bar{i}\varepsilon) - S_{\bar{i}\varepsilon}u^\varepsilon(0)\right\| &\leq \sum_{a=0}^{A-1}\left\|S_{(\bar{i}-i_{a+1})\varepsilon}u(i_{a+1}\varepsilon) - S_{(\bar{i}-i_a)\varepsilon}u(i_a\varepsilon)\right\|_1 \\ &\leq L\sum_{a=0}^{A-1}\left\|u(i_{a+1}\varepsilon) - S_{(i_{a+1}-i_a)\varepsilon}u(i_a\varepsilon)\right\|_1 \\ \text{(by (1.19)–(1.20))} \quad &\leq \mathcal{O}(1)\left\{\sum_{a\in\mathcal{A}'}\left[\Big(\Upsilon^\varepsilon(i_{a+1}\varepsilon) - \Upsilon^\varepsilon(i_a\varepsilon)\Big)\right.\right. \\ &\qquad \left.\left. + \frac{1+\log(i_{a+1}-i_a)}{i_{a+1}-i_a} + \varepsilon\right](i_{a+1}-i_a)\varepsilon + \sum_{a\in\mathcal{A}''}\varepsilon\right\} \\ \text{(by Points (i), (ii) above)} \quad &\leq \mathcal{O}(1)\left\{\sum_{a\in\mathcal{A}'}\left(\delta^2 + \varepsilon + \varepsilon\log\frac{\delta}{\varepsilon} + \varepsilon\delta\right) + \sum_{a\in\mathcal{A}''}\varepsilon\right\} \\ \text{(by (1.18))} \quad &\leq \mathcal{O}(1)T\left(\delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon}{\delta}\log\frac{\delta}{\varepsilon} + \varepsilon\right). \end{aligned}$$

Hence

$$\left\|u^\varepsilon(T) - S_T u_0\right\| \leq \left\|u^\varepsilon(T) - u^\varepsilon(\bar{i}\varepsilon)\right\| + \left\|u^\varepsilon(\bar{i}\varepsilon) - S_{\bar{i}\varepsilon}u^\varepsilon(0)\right\|$$

$$+\big\|S_{\bar{i}\varepsilon}u^\varepsilon(0) - S_{\bar{i}\varepsilon}u_0\big\| + \big\|S_{\bar{i}\varepsilon}u_0 - S_T u_0\big\|$$
$$\leq \mathcal{O}(1)\max\{1,T\}\Big(\delta + \frac{\varepsilon}{\delta} + \frac{\varepsilon}{\delta}\log\frac{\delta}{\varepsilon} + \varepsilon\Big). \qquad (1.21)$$

Since (1.21) holds for any $\delta \geq \varepsilon$, choosing $\delta(\varepsilon) := \sqrt{\varepsilon}$, we finally obtain (1.13).

1.6. Notations

- For $s \in \mathbb{R}$, define

$$\mathbf{I}(s) := \begin{cases} (0,s] & \text{if } s \geq 0, \\ [s,0) & \text{if } s < 0. \end{cases}$$

- Let X be any set and let $f : \mathbf{I}(s') \to X$, $g : s' + \mathbf{I}(s'') \to X$;
  - if $s's'' \geq 0$ and $f(s') = g(s')$, define

$$f \cup g : \ \mathbf{I}(s'+s'') \to X,$$
$$\big(f \cup g\big)(x) \ := \ \begin{cases} f(x) & \text{if } x \in \mathbf{I}(s'), \\ g(x) & \text{if } x \in s' + \mathbf{I}(s''); \end{cases} \qquad (1.22)$$

  - if $s's'' < 0$, define

$$f \vartriangle g : \ \mathbf{I}(s'+s'') \to X,$$
$$(f \vartriangle g)(x) \ := \ \begin{cases} f(x) & \text{if } |s'| \geq |s''|, \ x \in \mathbf{I}(s'+s''), \\ g(x) & \text{if } |s'| < |s''|, \ x \in \mathbf{I}(s'+s''). \end{cases} \qquad (1.23)$$

- For a continuous real valued function f, we denote its convex envelope in the interval $[a,b]$ as $\text{conv}_{[a,b]} f$.
- Given a totally ordered set $(A, \preceq)$, we define a partial pre-ordering on 2^A setting, for $I, J \subseteq A$,

$$I \prec J \text{ if and only if for } a \in I, b \in J \text{ it holds } a \prec b.$$

  We will also write $I \preceq J$ if either $I \prec J$ or $I = J$, i.e. we add the diagonal to the relation, making it a partial ordering.
- The L^∞ norm of a map $g : [a,b] \to \mathbb{R}^n$ will be denoted either by $\|g\|_\infty$ or by $\|g\|_{L^\infty([a,b])}$, if we want to stress the domain of g; similar notation for the L^1-norm.

- Given a C^1 map $g : \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq \mathbb{R}$, possibly made by a single point, let us define the Rankine-Hugoniot speed

$$\sigma^{\mathrm{rh}}(g, I) := \begin{cases} \frac{g(\sup I) - g(\inf I)}{\sup I - \inf I} & \text{if } I \text{ is not a singleton,} \\ \frac{dg}{du}(I) & \text{if } I \text{ is a singleton.} \end{cases}$$

2. Summary of the Paper [3] with a Modified Version of the Quadratic Potential

In [3] an estimate on the change of the speeds of the infinitesimal waves present in a Glimm approximate solution u^ε is provided. This estimate is achieved in two steps. First of all it is proved that at each grid point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$, the change in speed of the waves interacting at $(i\varepsilon, m\varepsilon)$ is bounded by a quantity $\mathtt{A}(i\varepsilon, m\varepsilon)$, called *amount of interaction*. Then it is shown that there exists an uniformly bounded, decreasing functional $t \mapsto \Upsilon(t)$ such that at each time $i\varepsilon$

$$\sum_{m \in \mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \leq \mathcal{O}(1)\big(\Upsilon(i\varepsilon-) - \Upsilon(i\varepsilon+)\big).$$

The functional $\Upsilon(t)$ is defined as the sum of some already known decreasing functionals (see Section 2.4 below) and of a new quadratic functional $t \mapsto \mathfrak{Q}(t)$, whose definition requires a careful analysis of waves collisions. Aim of this section is to summarize the main results present in the cited paper [3], providing meanwhile a stronger definition of the functional $\mathfrak{Q}(t)$. This stronger definition is needed to prove estimate (1.16) in Section 5 and thus Theorem 1.3.

2.1. Entropic self similar solution to the Riemann problem

As we pointed out in Section 1.1, the crucial point to solve the Riemann problem (1.1)−(1.5) is to find, for any left state u^L, a curve $s \mapsto T^k_s u^L$ of admissible right state, defined for $|s| \ll 1$, such that the Riemann problem $(u^L, T^k_s u^L)$ can be solved by (countable many) admissible shocks (in the sense of limit of viscosity approximations), contact discontinuities and rarefaction waves. In the GNL/LD case the admissible curve $s \mapsto T^k_s u^L$ coincides with the rarefaction curve for $s \geq 0$ and with the shock curve for $s \leq 0$ (see

[5]). In the general case, however, the situation is much more difficult and the problem was completely solved by Bianchini and Bressan in [2]. Here we describe just the main points of their construction, in order to recall the notations we will need.

First of all, for any index $k \in \{1, \dots, n\}$, through a Center Manifold technique, one can find a neighborhood of the point $(0, 0, \lambda_k(0))$ of the form

$$\mathcal{D}_k := \Big\{(u, v_k, \sigma_k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \;\Big|\; |u| \le \rho, |v_k| \le \rho, |\sigma_k - \lambda_k(0)| \le \rho\Big\}$$

for some $\rho > 0$ (depending only on f) and a smooth vector field

$$\tilde{r}_k : \mathcal{D}_k \to \mathbb{R}^n, \qquad \tilde{r}_k = \tilde{r}_k(u, v_k, \sigma_k),$$

satisfying

$$\tilde{r}_k(u, 0, \sigma_k) = r_k(u), \qquad \left|\frac{\partial \tilde{r}_k}{\partial \sigma_k}(u, v_k, \sigma_k)\right| \le \mathcal{O}(1)\big|v_k\big|. \tag{2.1}$$

We will call $\tilde{r}_k$ the *k-generalized eigenvector*. The characterization of $\tilde{r}_k$ is that

$$\mathcal{D}_k \ni (u, v_k, \sigma_k) \mapsto \big(u, v_k\tilde{r}_k, \sigma_k\big) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

is a parameterization of a center manifold near the equilibrium $(0, 0, \lambda_k(0)) \in \mathcal{D}_k$ for the ODE of traveling waves

$$\big(A(u) - \sigma\mathbb{I}\big)u_x = u_{xx} \qquad \Longleftrightarrow \qquad \begin{cases} u_x = v \\ v_x = (A(u) - \sigma\mathbb{I})v \\ \sigma_x = 0 \end{cases}$$

where $A(u) = Df(u)$, the Jacobian matrix of the flux f, and $\mathbb{I}$ is the identity $n \times n$ matrix.

Associated to the generalized eigenvectors, we can define smooth functions $\tilde{\lambda}_k : \mathcal{D}_k \to \mathbb{R}$ by

$$\tilde{\lambda}_k(u, v_k, \sigma_k) := \big\langle l_k(u), A(u)\tilde{r}_k(u, v_k, \sigma_k)\big\rangle.$$

We will call $\tilde{\lambda}_k$ the *k-generalized eigenvalue*. By (2.1) and the definition of

$\tilde{\lambda}_k$, we can get

$$\tilde{\lambda}_k(u,0,\sigma_k) = \lambda_k(u), \qquad \left|\frac{\partial \tilde{\lambda}_k}{\partial \sigma_k}(u,v_k,\sigma_k)\right| \le \mathcal{O}(1)|v_k|. \tag{2.2}$$

For the construction of the generalized eigenvectors and eigenvalues and the proof of (2.1), (2.2), see Section 4 of [2].

Then, by a fixed point technique one can now prove that there exist $\eta > 0$ (depending only on f), such that for

$$k \in \{1,\dots,n\}, \qquad u^L \in B(0,\rho/2), \qquad 0 \le s < \eta,$$

there is a curve

$$\begin{aligned} \gamma \ : \ [0,s] \ &\to \qquad\qquad \mathcal{D}_k \\ \tau \quad &\mapsto \ \gamma(\tau) = (u(\tau), v_k(\tau), \sigma_k(\tau)) \end{aligned}$$

such that $u, v_k \in C^{1,1}([0,s])$, $\sigma_k \in C^{0,1}([0,s])$ and this curve is the unique solution to the system

$$\begin{cases} u(\tau) = u^L + \displaystyle\int_0^\tau \tilde{r}_k(\gamma(\varsigma))d\varsigma \\ v_k(\tau) = f_k(\gamma;\tau) - \operatorname{conv}_{[0,s]} f_k(\gamma;\tau) \\ \sigma_k(\tau) = \dfrac{d}{d\tau}\operatorname{conv}_{[0,s]} f_k(\gamma;\tau) \end{cases} \tag{2.3}$$

where

$$f_k(\gamma;\tau) := \int_0^\tau \tilde{\lambda}_k(\gamma(\varsigma))d\varsigma \tag{2.4}$$

and $\operatorname{conv}_{[0,s]} f_k$ is the convex envelope of f_k in the interval $[0,s]$:

$$\operatorname{conv}_{[a,b]} g(u) := \sup\left\{ h(u) \ \middle|\ h : [a,b] \to \mathbb{R} \text{ is convex and } h \le g \right\}.$$

In the case $s < 0$ a completely similar result holds, replacing the convex envelope with the concave one.

If we want to stress the dependence of the curve γ on u^L and s we will use the notation

$$\gamma(u^L,s)(\tau)=\Big(u(u^L,s)(\tau),v_k(u^L,s)(\tau),\sigma_k(u^L,s)(\tau)\Big).$$

Finally the curve of admissible right states $(-\eta,\eta)\ni s\mapsto T^k_s u^L$ is defined as $T^k_s u^L:=u(u^L,s)(s)$.

2.2. Elementary estimates on the merging of two Riemann problems

Consider two contiguous Riemann problem

$$u^M=T^n_{s'_n}\circ\cdots\circ T^1_{s'_1}u^L,\qquad u^R=T^n_{s''_n}\circ\cdots\circ T^1_{s''_1}u^M, \tag{2.5}$$

and the Riemann problem obtained joining them,

$$u^R=T^n_{s_n}\circ\cdots\circ T^1_{s_1}u^L.$$

In particular the curves of the incoming Riemann problems are

$$\gamma'_1=(u'_1,v'_1,\sigma'_1):=\gamma_1(u^L,s'_1),\ \ \gamma'_k=(u'_k,v'_k,\sigma'_k):=\gamma_k\big(u'_{k-1}(s'_{k-1}),s'_k\big)\quad \text{for } k=2,\dots,n,$$

$$\gamma''_1=(u''_1,v''_1,\sigma''_1):=\gamma_1(u^M,s''_1),\quad \gamma''_k=(u''_k,v''_k,\sigma''_k):=\gamma_k\big(u''_{k-1}(s''_{k-1}),s''_k\big)\quad \text{for } k=2,\dots,n,$$

while the outcoming ones are

$$\gamma_1=(u_1,v_1,\sigma_1):=\gamma_1(u^L,s_1),\gamma_k=(u_k,v_k,\sigma_k):=\gamma_k\big(u_{k-1}(s_{k-1}),s_k\big)\quad \text{for } k=2,\dots,n.$$

We will denote by f'_k,f''_k,f_k the reduced fluxes associated by (2.4) to $\gamma'_k,\gamma''_k,\gamma_k$ respectively; for simplicity, we will assume that γ''_k and f''_k are defined on $s'_k+\mathbf{I}(s''_k)$, instead of $\mathbf{I}(s''_k)$ and $f''_k(s'_k)=f'_k(s'_k)$: indeed, it is clear that adding a constant to $\tilde f_k$ does not vary system (2.3).

Fix an index $k \in \{1, \dots n\}$ and consider the points (Figure 1)

$$u_1^L := u^L, \quad u_k^L := T^{k-1}_{s''_{k-1}} \circ T^{k-1}_{s'_{k-1}} \circ \cdots \circ T^1_{s''_1} \circ T^1_{s'_1} u^L, \quad k \geq 2$$

$$u_k^M := T^k_{s'_k} u_k^L, \quad u_k^R := T^k_{s''_k} u_k^M, \quad k = 1, \dots, n.$$

By definition, the Riemann problem between u_k^L and u_k^M is solved by wavefronts of the k-th family with total strength s'_k and the Riemann problem between u_k^M and u_k^R is solved by wavefront of the k-th family with total strength s''_k. Denote by $\tilde{\gamma}'_k = (\tilde{u}'_k, \tilde{v}'_k, \tilde{\sigma}'_k)$ the curve which solves the Riemann problem $[u_k^L, u_k^M]$ and by $\tilde{f}'_k$ the associated reduced flux (see (2.4)).
Similarly, let $\tilde{\gamma}''_k = (\tilde{u}''_k, \tilde{v}''_k, \tilde{\sigma}''_k)$ be the curve solving the Riemann problem $[u_k^M, u_k^R]$ and let $\tilde{f}''_k$ be the associated reduced flux. Clearly, $\tilde{\gamma}'_k$, $\tilde{f}'_k$ are defined on $\mathbf{I}(s_k)$, while, since we are going to perform the patching (1.22), (1.23), we will assume as above that $\tilde{\gamma}''_k$ and $\tilde{f}''_k$ are defined on $s'_k + \mathbf{I}(s''_k)$ (instead of $\mathbf{I}(s''_k)$) and that $\tilde{f}''_k(s'_k) = \tilde{f}'_k(s'_k)$.

As in [3], define the following quantities, called *amounts of interaction.*

Definition 2.1. The quantity

$$\mathtt{A}^{\mathrm{trans}}(u^L, u^M, u^R) := \sum_{1 \leq h < k \leq n} |s'_k||s''_h|$$

is called the *transversal amount of interaction* associated to the two Riemann problems (2.5).

For $s'_k > 0$, we define *cubic amount of interaction of the k-th family* for the two Riemann problems (u^L, u^M), (u^M, u^R) as follows:

(1) if $s''_k \geq 0$,

$$\begin{aligned}\mathtt{A}_k^{\mathrm{cubic}}(u^L, u^M, u^R) :=& \int_0^{s'_k} \Big[\mathrm{conv}_{[0,s'_k]} f'_k(\tau) - \mathrm{conv}_{[0,s'_k+s''_k]} \big(f'_k \cup f''_k\big)(\tau)\Big] d\tau \\ &+ \int_{s'_k}^{s'_k+s''_k} \Big[\mathrm{conv}_{[s'_k,s''_k]} f''_k(\tau) - \mathrm{conv}_{[0,s'_k+s''_k]} \big(f'_k \cup f''_k\big)(\tau)\Big] d\tau;\end{aligned}$$

(2) if $-s'_k \leq s''_k < 0$

$$\mathtt{A}_k^{\mathrm{cubic}}(u^L, u^M, u^R) := \int_0^{s'_k+s''_k} \Big[\mathrm{conv}_{[0,s'_k+s''_k]} f'_k(\tau) - \mathrm{conv}_{[0,s'_k]} f'_k(\tau)\Big] d\tau$$

$$+\int_{s'_k+s''_k}^{s'_k}\Big[\mathrm{conc}_{[s'_k+s''_k,s'_k]}f'_k(\tau)-\mathrm{conv}_{[0,s'_k]}f'_k(\tau)\Big]d\tau;$$

(3) if $s''_k < -s'_k$,

$$\mathtt{A}_k^{\mathrm{cubic}}(u^L,u^M,u^R) := \int_{s'_k+s''_k}^{0}\Big[\mathrm{conc}_{[s'_k+s''_k,s'_k]}f''_k(\tau)-\mathrm{conc}_{[s'_k+s''_k,0]}f''_k(\tau)\Big]d\tau$$
$$+\int_{0}^{s'_k}\Big[\mathrm{conc}_{[s'_k+s''_k,s'_k]}f''_k(\tau)-\mathrm{conv}_{[0,s'_k]}f''_k(\tau)\Big]d\tau.$$

Here $\mathrm{conc}_{[a,b]}g$ denotes the *concave envelope* of a function f in the interval $[a,b]$:

$$\mathrm{conc}_{[a,b]}g(u) := \inf\Big\{h(u)\ \Big|\ h:[a,b]\to\mathbb{R} \text{ is concave and } h\geq g\Big\}.$$

Similar definitions can be given if $s'_k < 0$, interchanging convex envelopes with concave.

The *amount of cancellation of the k-th family* is defined by

$$\mathtt{A}_k^{\mathrm{canc}}(u^L,u^M,u^R) := \begin{cases} 0 & \text{if } s'_k s''_k \geq 0,\\ \min\{|s'_k|,|s''_k|\} & \text{if } s'_k s''_k < 0.\end{cases}$$

The *amount of creation of the k-th family* is defined by

$$\mathtt{A}_k^{\mathrm{cr}}(u^L,u^M,u^R) := \Big[|s_k| - |s'_k+s''_k|\Big]^+.$$

If $s'_k s''_k \geq 0$, we define the *quadratic amount of interaction of the k-family* associated to the two Riemann problems (2.5) by

$$\mathtt{A}_k^{\mathrm{quadr}}(u^L,u^M,u^R) := \begin{cases} \tilde f'_k(s'_k)-\mathrm{conv}_{[0,s'_k+s''_k]}\big(\tilde f'_k\cup\tilde f''_k\big)(s'_k) & \text{if } s'_k>0, s''_k>0,\\ \mathrm{conc}_{[s'_k+s''_k,0]}\big(\tilde f'_k\cup\tilde f''_k\big)(s'_k)-\tilde f'_k(s'_k) & \text{if } s'_k<0, s''_k<0,\\ 0 & \text{if } s'_k s''_k\leq 0.\end{cases}$$

Finally we define the *total amount of interaction* associated to the two Rie-

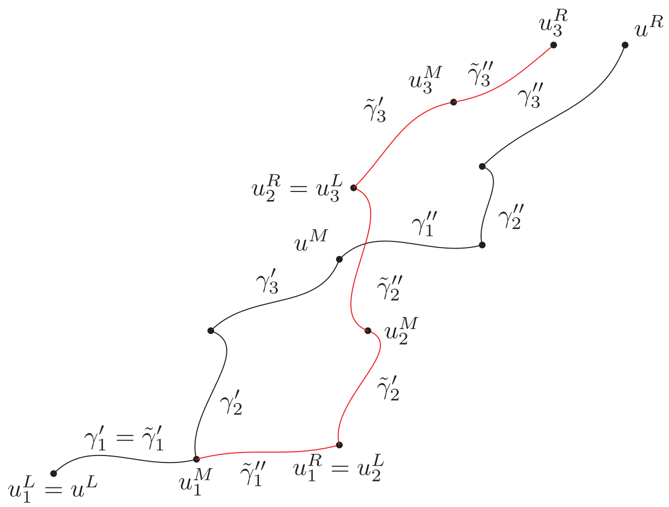

Figure 1: Elementary curves of two interacting Riemann problems before and after transversal interactions.

mann problems (2.5) as

$$\begin{aligned} \mathtt{A}(u^L, u^M, u^R) \; := \; & \mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{h=1}^{n} \Big(\mathtt{A}_h^{\text{quadr}}(u^L, u^M, u^R) \\ & + \mathtt{A}_h^{\text{canc}}(u^L, u^M, u^R) + \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R) \Big). \end{aligned}$$

It is well known (see [1]) that

$$\sum_{k=1}^{n} \big| s_k - (s_k' + s_k'') \big| \le \mathcal{O}(1) \Big[\mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{k=1}^{n} \mathtt{A}_k^{\text{cubic}}(u^L, u^M, u^R) \Big].$$

and thus

$$\mathtt{A}_k^{\text{cr}}(u^L, u^M, u^R) \le \mathtt{A}^{\text{trans}}(u^L, u^M, u^R) + \sum_{h=1}^{n} \mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R).$$

The distance between incoming and outgoing Riemann problems can be estimated as follows (see [3], Theorem 3.3).

Theorem 2.2. *For any $k = 1, \dots, n$,*

- *if* $s'_k s''_k \geq 0$*, then*

$$\left.\begin{array}{r} \big\|(u'_k \cup u''_k) - u_k\big\|_{L^\infty(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \big\|(v'_k \cup v''_k) - v_k\big\|_{L^\infty(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \big\|(\sigma'_k \cup \sigma''_k) - \sigma_k\big\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \left\|\left(\dfrac{d^2 f'_k}{d\tau^2} \cup \dfrac{d^2 f''_k}{d\tau^2}\right) - \dfrac{d^2 f_k}{d\tau^2}\right\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \end{array}\right\} \leq \mathcal{O}(1)\mathtt{A}(u^L, u^M, u^R);$$

- *if* $s'_k s''_k < 0$*, then*

$$\left.\begin{array}{r} \big\|(u'_k \vartriangle u''_k) - u_k\big\|_{L^\infty(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \big\|(v'_k \vartriangle v''_k) - v_k\big\|_{L^\infty(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \big\|(\sigma'_k \vartriangle \sigma''_k) - \sigma_k\big\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \\ \left\|\left(\dfrac{d^2 f'_k}{d\tau^2} \vartriangle \dfrac{d^2 f''_k}{d\tau^2}\right) - \dfrac{d^2 f_k}{d\tau^2}\right\|_{L^1(\boldsymbol{I}(s'_k+s''_k)\cap \boldsymbol{I}(s_k))} \end{array}\right\} \leq \mathcal{O}(1)\mathtt{A}(u^L, u^M, u^R);$$

Remark 2.3. In the statement of Theorem 3.3 in [3] only the inequalities about u, σ, $\frac{d^2 f_k}{d\tau^2}$ are explicitly proved, while the ones about v are not. However it is not difficult to see that the proof used for u, σ and $\frac{d^2 f_k}{d\tau^2}$ can be adapted also to v.

2.3. Lagrangian representation for the Glimm approximate solution u^ε

In this section we recall the notion, introduced in [3], of *Lagrangian representation* of an approximate solution u^ε (ϵ above) to the Cauchy problem (1.1)−(1.4) obtained by the Glimm scheme, and we state the theorem about the existence of a Lagrangian representation satisfying some useful additional properties. At the end of the section we introduce some notions related to the Lagrangian representation; in particular, the notion of *effective flux* $\mathtt{f}^{\mathrm{eff}}_k(t)$ *of the* k*-th family at time* t.

Let us first introduce some notation related to the Glimm approximate solution u^ε. For any grid point $(i\varepsilon, m\varepsilon)$, $i \geq 0$, $m \in \mathbb{Z}$, set

$$u^{i,m} := u^\varepsilon(i\varepsilon, m\varepsilon),$$

and assume that the Riemann problem $(u^{i,m-1}, u^{i,m})$ is solved by

$$u^{i,m} = T^n_{s_n^{i,m}} \circ \cdots \circ T^1_{s_1^{i,m}} u^{i,m-1},$$

moreover denote by

$$\sigma_k^{i,m} : \mathbf{I}(s_k^{i,m}) \to \mathbb{R}, \qquad k = 1, \dots, n,$$

the speed function of the k-th wavefront solving the Riemann problem $(u^{i,m-1}, u^{i,m})$.

Let us introduce also the following notation for the transversal, cubic and quadratic amounts of interaction and for the amounts of creation and cancellation related to the two Riemann problems $(u^{i,m-1}, u^{i-1,m-1})$, $(u^{i-1,m-1}, u^{i,m})$ which interact at grid point $(i\varepsilon, m\varepsilon)$:

$$\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) := \mathtt{A}^{\text{trans}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

and for $k = 1, \dots, n$,

$$\mathtt{A}_k^{\text{cubic}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{cubic}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

$$\mathtt{A}_k^{\text{canc}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{canc}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

$$\mathtt{A}_k^{\text{cr}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{cr}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}),$$

$$\mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon) := \mathtt{A}_k^{\text{quadr}}(u_{i,m-1}, u_{i-1,m-1}, u_{i,m}).$$

We now introduce the notion of Lagrangian representation. Given a piecewise constant approximate solution u^ε constructed by the Glimm scheme (see Section 1.2, for any time $t \geq 0$ define the quantities

$$L_k^+(t) := \sum_{m\in\mathbb{Z}} \left[s_k^{i,m}\right]^+, \qquad L_k^-(t) := -\sum_{m\in\mathbb{Z}} \left[s_k^{i,m}\right]^-, \quad \text{if } t \in [i\varepsilon, (i+1)\varepsilon).$$

It is easy to see that $|L_k^+(t)| + |L_k^-(t)| \leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(t))$.

Definition 2.4. A *Lagrangian representation* for u^ε is a set $\mathcal{W}$ called *the set of waves*, together with

- the maps

$$\begin{aligned} &\text{family} : \mathcal{W} \to \{1,\dots,n\} && \text{the family of the wave } w \in \mathcal{W}, \\ &\mathcal{S} : \mathcal{W} \to \{\pm 1\} && \text{the sign of the wave } w \in \mathcal{W}, \\ &\mathtt{t}^{\mathrm{cr}} : \mathcal{W} \to [0,+\infty) && \text{the creation time of the wave } w \in \mathcal{W}, \\ &\mathtt{t}^{\mathrm{canc}} : \mathcal{W} \to (0,+\infty] && \text{the cancellation time of the wave } w \in \mathcal{W}, \end{aligned}$$

- a relation, which we will denote by $\leq$,
- the map, called *position function*,

$$\mathtt{x} : \big\{(t,w) \in [0,\infty) \times \mathcal{W} \;\big|\; \mathtt{t}^{\mathrm{cr}}(w) \leq t < \mathtt{t}^{\mathrm{canc}}(w)\big\} \to \mathbb{R},$$

which satisfy the conditions (1)$-$(4) below.

For convenience, set

$$\begin{aligned} \mathcal{W}_k &:= \big\{w \in \mathcal{W} \;\big|\; \text{family}(w) = k\big\}, \\ \mathcal{W}_k(t) &:= \big\{w \in \mathcal{W}_k \;\big|\; \mathtt{t}^{\mathrm{cr}}(w) \leq t < \mathtt{t}^{\mathrm{canc}}(w)\big\}, \\ \mathcal{W}_k^{\pm}(t) &:= \big\{w \in \mathcal{W}_k(t) \;\big|\; \mathcal{S}(w) = \pm 1\big\}. \end{aligned}$$

The additional conditions to be satisfied by a Lagrangian representation are the following:

(1) for any family k, time t, sign ± 1, the relation $\leq$ is a total order both on $\mathcal{W}_k^+(t)$ and on $\mathcal{W}_k^-(t)$; if $\mathcal{I} \subseteq \mathcal{W}_k^{\pm}(t)$ is an interval in the order set $(\mathcal{W}_k^{\pm}(t), \leq)$, we will say that $\mathcal{I}$ is *an interval of waves (i.o.w.) at time t;*

(2) the map $\mathtt{x}$ satisfies:
   (a) for fixed time t, $\mathtt{x}(t,\cdot) : \mathcal{W}_k(t) \to \mathbb{R}$ is increasing;
   (b) for fixed $w \in \mathcal{W}$, the map $\mathtt{x}(\cdot,w) : [\mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{canc}}(w)) \to \mathbb{R}$ is Lipschitz;
   (c) for any point $(\bar{t},\bar{x}) \in [0,+\infty) \times \mathbb{R}$, all the waves in

$$\mathcal{W}_k(\bar{t},\bar{x}) := \mathtt{x}(\bar{t})^{-1}(\bar{x}) \cap \mathcal{W}_k$$

   have the same sign;

(3) there exist maps $\Phi_k(t) : \mathcal{W}_k(t) \to \mathbf{I}\big(L_k^-(t)\big) \cup \mathbf{I}\big(L_k^+(t)\big)$ such that $\Phi_k(t)|_{\mathcal{W}_k^+(t)} : \mathcal{W}_k^+(t) \to \mathbf{I}\big(L_k^+(t)\big)$ is an isomorphism of ordered sets, while $\Phi_k(t)|_{\mathcal{W}_k^-(t)} : \mathcal{W}_k^-(t) \to \mathbf{I}\big(L_k^-(t)\big)$ is an antisomorphism of ordered sets;

(4) there exist maps $\hat{\gamma}_k(t) : \mathcal{W}_k(t) \to \mathcal{D}_k \subseteq \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$, $\hat{\gamma}_k(t) = \big(\hat{u}_k(t), \hat{v}_k(t), \hat{\sigma}_k(t)\big)$, such that

(a) for any $\bar{x} \in \mathbb{R}$, setting

$$u^L := \lim_{x \to \bar{x}^-} u^\varepsilon(t,x), \qquad u^R := \lim_{x \to \bar{x}^+} u^\varepsilon(t,x),$$

the collection of curves

$$\Big\{ \Phi_k(t)\big(\mathcal{W}_k(t,\bar{x})\big) \ni \tau \mapsto \hat{\gamma}_k\big(t, \Phi_k(t)^{-1}(\tau)\big) \Big\}_{k=1,\dots,n},$$

solves the Riemann problem (u^L, u^R);

(b) for any $w \in \mathcal{W}_k^\pm(i\varepsilon)$, if $\mathtt{t}^{\mathrm{canc}}(w) \geq (i+1)\varepsilon$, then for any time $t \in [i\varepsilon, (i+1)\varepsilon)$ it holds

$$\mathtt{x}(t,w) = \begin{cases} \mathtt{x}(i\varepsilon, w) & \text{if } \vartheta_{i+1} \geq \hat{\sigma}_k(i\varepsilon, w), \\ \mathtt{x}(i\varepsilon, w) + (t - i\varepsilon) & \text{if } \vartheta_{i+1} < \hat{\sigma}_k(i\varepsilon, w). \end{cases}$$

The following theorem is taken from [3, Theorem 4.1].

Theorem 2.5. *There exists at least one Lagrangian representation for the approximate solution u^ε constructed by the Glimm scheme, which moreover satisfies the following conditions: for any grid point $(i\varepsilon, m\varepsilon) \in \mathbb{N}\varepsilon \times \mathbb{Z}\varepsilon$,*

(a) *the set $\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon)$ is an i.o.w. both at time $(i-1)\varepsilon$ and at time $i\varepsilon$, while the set $\mathcal{W}_k(i\varepsilon, m\varepsilon) \setminus \mathcal{W}_k((i-1)\varepsilon)$ is an i.o.w. at time $i\varepsilon$;*

(b) *the map*

$$\Phi_k((i-1)\varepsilon)(\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon)) \xrightarrow{\Phi_k(i\varepsilon)\circ\Phi_k((i-1)\varepsilon)^{-1}} \Phi_k(i\varepsilon)(\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon))$$

is an affine map with Lipschitz constant equal to 1.

Definition 2.6. Fix $\bar{t} \geq 0$. Let $\mathcal{I} \subseteq \mathcal{W}_k(\bar{t})$ be an interval of waves at time $\bar{t}$. Set $I := \Phi_k(\bar{t})(\mathcal{I})$. By Property (3) of the Definition of Lagrangian representation, I is an interval in $\mathbb{R}$ (possibly made by a single point). Let us define:

- the *Rankine-Hugoniot speed given to the interval of waves $\mathcal{I}$ by a function* $g : \mathbb{R} \to \mathbb{R}$ as

$$\sigma^{\mathrm{rh}}(g, \mathcal{I}) := \begin{cases} \frac{g(\sup I) - g(\inf I)}{\sup I - \inf I} & \text{if } I \text{ is not a singleton,} \\ g'(I) & \text{if } I \text{ is a singleton;} \end{cases}$$

- for any $w \in \mathcal{I}$, the *entropic speed given to the wave w by the Riemann problem $\mathcal{I}$ and the flux function g* as

$$\sigma^{\mathrm{ent}}(g, \mathcal{I}, w) := \begin{cases} \dfrac{d}{d\tau} \mathrm{conv}_{\,I} g\Big(\Phi_k(\bar{t})(w)\Big) & \text{if } \mathcal{S}_k(w) = +1, \\ \dfrac{d}{d\tau} \mathrm{conc}_{\,I} g\Big(\Phi_k(\bar{t})(w)\Big) & \text{if } \mathcal{S}_k(w) = -1. \end{cases}$$

If $\sigma^{\mathrm{rh}}(g, \mathcal{I}) = \sigma^{\mathrm{ent}}(g, \mathcal{I}, w)$ for any $w \in \mathcal{I}$, we will say that $\mathcal{I}$ is *entropic* w.r.t. the function g.

We will also say that *the Riemann problem $\mathcal{I}$ with flux function g divides* w, w' if $\sigma^{\mathrm{ent}}(g, \mathcal{I}, w) \neq \sigma^{\mathrm{ent}}(g, \mathcal{I}, w')$.

Definition 2.7. For each family $k = 1, \dots, n$ and for each time $t \geq 0$ define the *effective flux of the k-th family at time t* as any $C^{1,1}$ function

$$\mathtt{f}_k^{\mathrm{eff}}(t, \cdot) : [L_k^-, L_k^+] \to \mathbb{R}$$

whose second derivative satisfies the following relation:

$$\frac{\partial^2 \mathtt{f}_k^{\mathrm{eff}}(t, \cdot)}{\partial \tau^2}(\tau) := \frac{d\tilde{\lambda}_k(\hat{\gamma}_k(t, w))}{d\tau},$$

for $\mathcal{L}^1$-a.e. $\tau \in [L_k^-, L_k^+]$, where $w = \Phi_k(t)^{-1}(\tau)$ and $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure on $\mathbb{R}$.

2.4. Glimm-type functionals

We have already observed (see Sections 1.2, 1.3) that the main tool to get a priori estimates on the Glimm approximate solutions is to find suitable decreasing functional. Here we recall the definitions of some Glimm-type functional, which we will use throughout the paper.

Definition 2.8. Define the *total variation along curves* as

$$V(t) := \sum_{k=1}^{n} \sum_{m\in\mathbb{Z}} |s_k^{i,m}|, \qquad \text{for any } t \in [i\varepsilon, (i+1)\varepsilon).$$

Define the *transversal interaction functional* as

$$Q^{\text{trans}}(t) := \sum_{k=1}^{n} \sum_{h=1}^{k-1} \sum_{m>m'} |s_k^{i,m'}||s_h^{i,m}|, \qquad \text{for any } t \in [i\varepsilon, (i+1)\varepsilon).$$

Define the *cubic interaction functional* as

$$Q^{\text{cubic}}(t) := \sum_{k=1}^{n} \sum_{m,m'\in\mathbb{Z}} \int_{\mathbf{I}(s_k^{i,m})} \int_{\mathbf{I}(s_k^{i,m'})} \big|\sigma_k^{i,m}(\tau) - \sigma_k^{i,m'}(\tau')\big| d\tau' d\tau.$$

The following statements hold: for the proofs, see [5], [1].

Proposition 2.9. *There exists a constant $C > 0$, depending only of the flux f, such that for any time $t \geq 0$*

$$\frac{1}{C}\text{Tot.Var.}(u(t)) \leq V(t) \leq C\text{Tot.Var.}(u(t)).$$

Theorem 2.10. *The following hold:*

(1) *the functionals $t \mapsto V(t), Q^{\text{trans}}(t), Q^{\text{cubic}}(t)$ are constant on each interval $[i\varepsilon, (i+1)\varepsilon)$;*
(2) *they are bounded by powers of the* Tot.Var.$(u(t))$ *as follows:*

$$\begin{aligned} V(t) &\leq \mathcal{O}(1)\text{Tot.Var.}(u(t)), \\ Q^{\text{trans}}(t) &\leq \mathcal{O}(1)\text{Tot.Var.}(u(t))^2, \\ Q^{\text{cubic}}(t) &\leq \mathcal{O}(1)\text{Tot.Var.}(u(t))^3; \end{aligned}$$

(3) *there exist constants $c_1, c_2, c_3 > 0$, depending only on the flux f, such that for any $i \in \mathbb{N}$, defining*

$$Q^{\text{known}}(t) := c_1 V(t) + c_2 Q^{\text{trans}}(t) + c_3 Q^{\text{cubic}}(t),$$

it holds

$$\sum_{m\in\mathbb{Z}}\left[\mathtt{A}^{\mathrm{trans}}(i\varepsilon,m\varepsilon)+\sum_{k=1}^{n}\Big(\mathtt{A}_k^{\mathrm{canc}}(i\varepsilon,m\varepsilon)+\mathtt{A}_k^{\mathrm{cubic}}(i\varepsilon,m\varepsilon)\Big)\right] \\ \leq\ Q^{\mathrm{known}}((i-1)\varepsilon)-Q^{\mathrm{known}}(i\varepsilon). \tag{2.6}$$

2.5. Analysis of waves collisions

This section corresponds to [3, Section 5]. Here however we introduce a new definition of *characteristic interval* associated to a pair of waves (w,w') and a new definition of the partition of this interval. These new definitions provide the correct setting to define the new quadratic interaction potential which we are going to introduce in Section 2.6 and which will be used in Section 5 to prove estimate (1.16) and thus Theorem 1.3.

We first introduce the following equivalence relation $\bowtie$: for any fixed time $\bar{t}\in[i\varepsilon,(i+1)\varepsilon)$ and for any couple of waves $w,w'\in\mathcal{W}_k(t)$, we set $w\bowtie w'$ if and only if

$$\mathtt{t}^{\mathrm{cr}}(w)=\mathtt{t}^{\mathrm{cr}}(w')\text{ and }\mathtt{x}(t,w)=\mathtt{x}(t,w')\quad\text{ for any }t\in\big[\mathtt{t}^{\mathrm{cr}}(w),(i+1)\varepsilon\big]. \tag{2.7}$$

and we denote the equivalence classes as

$$\mathcal{E}(\bar{t},w)\ :=\ \Big\{z\in\mathcal{W}_k(\bar{t})\ \Big|\ \mathtt{t}^{\mathrm{cr}}(z)=\mathtt{t}^{\mathrm{cr}}(w)\text{ and }\mathtt{x}(t,w)=\mathtt{x}(t,z) \\ \text{for any }t\in\big[\mathtt{t}^{\mathrm{cr}}(w),(i+1)\varepsilon\big)\Big\}.$$

Definition 2.11. Let $\bar{t}$ be a fixed time and let $w,w'\in\mathcal{W}_k(\bar{t})$. We say that

- w,w' *interact at time* $\bar{t}$ if $\mathtt{x}(\bar{t},w)=\mathtt{x}(\bar{t},w')$;
- w,w' *have already interacted at time* $\bar{t}$ if there is $t\leq\bar{t}$ such that w,w' interact at time t;
- w,w' *have never interacted at time* $\bar{t}$ if for all $t\leq\bar{t}$, they do not interact at time t.
- w,w' *will interact after time* $\bar{t}$ if there is $t>\bar{t}$ such that w,w' interact at time t.

- w, w' *are joined in the real solution at time* $\bar{t}$ if there is a right neighborhood of $\bar{t}$, say $[\bar{t}, \bar{t}+\zeta)$, such that they interact at any time $t \in [\bar{t}, \bar{t}+\zeta)$;

- w, w' *are divided in the real solution at time* $\bar{t}$ if they are not joined at time $\bar{t}$.

Remark 2.12. It $\bar{t} \neq i\varepsilon$ for each $i \in \mathbb{N}$, then two waves are divided in the real solution if and only if they have different position. If $\bar{t} = i\varepsilon$, they are divided if there exists a time $t > \bar{t}$, arbitrarily close to $\bar{t}$, such that w, w' have different positions at time t.

Definition 2.13. Fix a time $\bar{t}$ and two k-waves $w, w' \in \mathcal{W}_k(\bar{t})$, $w < w'$. Assume that w, w' are divided in the real solution at time $\bar{t}$. Define the *time of last splitting* $\mathtt{t}^{\mathrm{split}}(\bar{t}, w, w')$ (if w, w' have already interacted at time $\bar{t}$) and the *time of next interaction* $\mathtt{t}^{\mathrm{int}}(\bar{t}, w, w')$ (if w, w' will interact after time $\bar{t}$) by the formulas

$$\begin{aligned} \mathtt{t}^{\mathrm{split}}(\bar{t}, w, w') &:= \max\big\{t < \bar{t} \mid \mathtt{x}(t, w) = \mathtt{x}(t, w')\big\}, \\ \mathtt{t}^{\mathrm{int}}(\bar{t}, w, w') &:= \min\big\{t > \bar{t} \mid \mathtt{x}(t, w) = \mathtt{x}(t, w')\big\}. \end{aligned}$$

Given two k-waves $w, w' \in \mathcal{W}_k$ and given a time $t \in [0, \infty)$, we define the property

$$\mathtt{p}(t, w, w') \ : \ \text{“either } w, w' \in \mathcal{W}_k(t) \text{ and they are divided at time } t \text{ in the real solution or at least one between } w, w' \text{ does not belong to } \mathcal{W}_k(t)\text{”}.$$

Definition 2.14. Let $t_1 \leq t_2$, be two times. Let $w, w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they have the same sign and that they satisfy $\mathtt{p}(t_1, w, w')$. We define the *characteristic interval* $\mathcal{I}(t_1, t_2, w, w')$ *of* w, w' *at time* t_2 *starting from time* t_1 as follows. Assume first that $t_2 = i\varepsilon$ for some $i \in \mathbb{N}$.

(1) If at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$,

but they have never interacted at time t_1, then

$$\mathcal{I}(t_1,t_2,w,w')=\begin{cases} \big\{z\in\mathcal{W}_k(t_2)\mid \mathcal{S}(z)=\mathcal{S}(w) \text{ and } z<\mathcal{E}(t_2,w')\big\}\cup\mathcal{E}(t_2,w') \\ \hfill \text{if } \mathtt{t}^{\text{cr}}(w)\le\mathtt{t}^{\text{cr}}(w'), \\ \\ \mathcal{E}(t_2,w)\cup\big\{z\in\mathcal{W}_k(t_2)\mid \mathcal{S}(z)=\mathcal{S}(w) \text{ and } z>\mathcal{E}(t_2,w)\big\} \\ \hfill \text{if } \mathtt{t}^{\text{cr}}(w)>\mathtt{t}^{\text{cr}}(w'); \end{cases}$$

(2) If $w,w'\in\mathcal{W}_k(t_1)$ and they have already interacted at time t_1, we have to distinguish two cases:

(a) if $t_1=\mathtt{t}^{\text{split}}(t_1,w,w')$, then argue by recursion:

- if $t_2=t_1=\mathtt{t}^{\text{split}}(t_1,w,w')$, set

$$\mathcal{I}(t_1,t_2,w,w'):=\mathcal{W}(t_1,\mathtt{x}(t_1,w))=\mathcal{W}(t_1,\mathtt{x}(t_1,w'));$$

- if $t_2=i\varepsilon>(i-1)\varepsilon\ge t_1=\mathtt{t}^{\text{split}}(t_1,w,w')$, define $\mathcal{I}(t_1,t_2,w,w')$ as the smallest interval in $(\mathcal{W}_k^{\pm}(t_2),\le)$ which contains $\mathcal{I}(t_1,(i-1)\varepsilon,w,w')\cap\mathcal{W}_k(t_2)$, i.e.

$$\mathcal{I}(t_1,t_2,w,w'):=\Big\{z\in\mathcal{W}_k(t_2)\;\Big|\;\mathcal{S}(z)=\mathcal{S}(w)=\mathcal{S}(w') \\ \text{and } \exists\, y,y'\in\mathcal{I}(t_1,(i-1)\varepsilon,w,w')\cap\mathcal{W}_k(t_2) \text{ such that } y\le z\le y'\Big\}.$$

(b) if $t_1>\mathtt{t}^{\text{split}}(t_1,w,w')$, set

$$\mathcal{I}(t_1,t_2,w,w')=\mathcal{I}(\mathtt{t}^{\text{split}}(t_1,w,w'),t_2,w,w').$$

Finally set

$$\mathcal{I}(t_1,t_2,w,w'):=\mathcal{I}(t_1,i\varepsilon,w,w') \qquad \text{for } t_2\in[i\varepsilon,(i+1)\varepsilon).$$

As in [3], we define now a partition $\mathcal{P}(t_1,t_2,w,w')$ of the characteristic interval $\mathcal{I}(t_1,t_2,w,w')$, with the properties that each element of $\mathcal{P}(t_1,t_2,w,w')$ is an interval of waves at time t_2, entropic w.r.t. the flux $\mathtt{f}_k^{\text{eff}}(t_2)$ of Definition 2.7.

Definition 2.15. As before, let $t_1\le t_2$, be two times. Let $w,w'\in\mathcal{W}_k(t_2)$ be two k-waves. Assume that they have the same sign and that they satisfy $\mathtt{p}(t_1,w,w')$. Assume first that $t_2=i\varepsilon$, $i\in\mathbb{N}$.

(1) If at least one between w, w' does not belong to $\mathcal{W}_k(t_1)$ or $w, w' \in \mathcal{W}_k(t_1)$, but they have never interacted at time t_1, then the equivalence classes of the partition $\mathcal{P}(t_1, t_2, w, w')$ are singletons.

(2) Assume now that w, w' have already interacted at time t_1; we distinguish two cases:

(a) if $t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, argue by recursion:

- if $t_2 = t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{P}(t_1, t_2, , w, w')$ is given by the equivalence relation

$$z \sim z' \iff \begin{cases} z, z' \text{ are not divided by the Riemann problem} \\ \mathcal{W}_k(t_1, \mathtt{x}(t_1, w)) \text{ with flux function } \mathtt{f}_k^{\mathrm{eff}}(t_1, \cdot); \end{cases}$$

- if $t_2 = i\varepsilon > (i-1)\varepsilon \geq t_1 = \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{P}(t_1, t_2, w, w')$ is given by the equivalence relation

$$z \sim z' \iff \begin{cases} \left[\begin{array}{l} z, z' \text{ belong to the same} \\ \text{equivalence class } \mathcal{J} \in \mathcal{P}(t_1, (i-1)\varepsilon, w, w') \\ \text{and the Riemann problem } \mathcal{J} \cap \mathcal{W}_k(t_2) \\ \text{with flux } \mathtt{f}_k^{\mathrm{eff}}(t_2, \cdot) \text{ does not divide them} \end{array} \right] \\ \text{or} \\ \Big[\mathtt{t}^{\mathrm{cr}}(z) = \mathtt{t}^{\mathrm{cr}}(z') = t_2 \text{ and } z = z' \Big]. \end{cases}$$

It is not difficult to see that the previous definition is well posed, since $\mathcal{J} \cap \mathcal{W}(i\varepsilon)$ is an interval of waves at time $i\varepsilon$.

(b) if $t_1 > \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, set

$$\mathcal{P}(t_1, t_2, w, w') = \mathcal{P}(\mathtt{t}^{\mathrm{split}}(t_1, w, w'), t_2, w, w').$$

Finally extend the definition of $\mathcal{P}(t_1, t_2, w, w')$ for any time $t_2 \in [i\varepsilon, (i+1)\varepsilon)$, setting

$$\mathcal{P}(t_1, t_2, w, w') = \mathcal{P}(t_1, i\varepsilon, w, w') \qquad \text{for any } \bar{t} \in [i\varepsilon, (i+1)\varepsilon).$$

We collect now the main results about the characteristic interval and its partition. In this paper the definitions of the characteristic interval $\mathcal{I}(t_1, t_2, w, w')$ and of the associated partition $\mathcal{P}(t_1, t_2, w, w')$ are different from the analog definitions given in [3]. However the results we present now can be proved with the same techniques as in [3, Section 5]. For this reason we just state the results, omitting the proofs.

The following proposition corresponds to [3, Proposition 5.12] and can be proved in a similar way.

Proposition 2.16. *Let $t_1 \leq t_2$, be two times. Let $w, w' \in \mathcal{W}_k(t_2)$ be two k-waves. Assume that they have the same sign and that they satisfy $\mathtt{p}(t_1, w, w')$. Let $\mathcal{J} \in \mathcal{P}(t_1, t_2, w, w')$. Then $\mathtt{x}(t_2, \cdot)$ is constant on $\mathcal{J}$ and $\mathcal{J}$ is an entropic interval of waves at time t_2 w.r.t. the flux function $\mathtt{f}_k^{\mathrm{eff}}(t_2, \cdot)$.*

Definition 2.17. Let A, B two sets, $A \subseteq B$. Let $\mathcal{P}$ be a partition of B. We say that $\mathcal{P}$ *can be restricted to* A if for any $C \in \mathcal{P}$, either $C \subseteq A$ or $C \subseteq B \setminus A$. We also write

$$\mathcal{P}|_A := \big\{ C \in \mathcal{P} \;\big|\; C \subseteq A \big\}.$$

Clearly $\mathcal{P}$ can be restricted to A if and only if it can be restricted to $B \setminus A$.

The following proposition is the equivalent to [3, Proposition 5.14] and can be proved in an analogous way.

Proposition 2.18. *Let $t_1 \leq t_2$, be two times. Let $w, w', z, z' \in \mathcal{W}_k(t_2)$ be two k-waves, $z \leq w < w' \leq z'$. Assume that they have the same sign and that they satisfy both $\mathtt{p}(t_1, w, w')$ and $\mathtt{p}(t_1, z, z')$. Then $\mathcal{P}(t_1, t_2, z, z')$ can be restricted both to $\mathcal{I}(t_1, t_2, z, z') \cap \mathcal{I}(t_1, t_2, w, w')$ and to $\mathcal{I}(t_1, t_2, z, z') \setminus \mathcal{I}(t_1, t_2, w, w')$.*

The following proposition is the equivalent to [3, Proposition 5.15] and can be proved in an analogous way.

Proposition 2.19. *Let $t_1 \leq t_2$, be two times. Let $w, w', z, z' \in \mathcal{W}_k(t_2)$ be two k-waves, $z \leq w < w' \leq z'$. Assume that they have the same sign and that they satisfy both $\mathtt{p}(t_1, w, w')$ and $\mathtt{p}(t_1, z, z')$.*

(1) *If $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1, if $z, z' \in \mathcal{I}(t_1, t_2, , w, w')$ and if $\mathtt{t}^{\mathrm{cr}}(z), \mathtt{t}^{\mathrm{cr}}(z') \leq \mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then $\mathcal{I}(t_1, t_2, , z, z') = \mathcal{I}(t_1, t_2, , w, w')$ and $\mathcal{P}(t_1, t_2, z, z') = \mathcal{P}(t_1, t_2, w, w')$.*
(2) *If $w, w' \in \mathcal{W}_k(t_1)$ and they have already interacted at time t_1, but at least one wave between z, z' is created after $\mathtt{t}^{\mathrm{split}}(t_1, w, w')$, then z, z' have never interacted at time t_1.*

(3) *If either* $w, w' \in \mathcal{W}_k(t_1)$ *and they have never interacted at time* t_1*, or if at least one between* w, w' *does not belong to* $\mathcal{W}_k(t_1)$*, then the partition* $\mathcal{P}(t_1, t_2, z, z')$ *is made of singletons.*

2.6. New quadratic potential

Let $t \in [0, +\infty)$ be a fixed time and let $w, w' \in \mathcal{W}_k(t)$ be two k-waves having the same sign. In this section we introduce the weight $\mathfrak{q}_k(t, w, w')$ of the pair of waves w, w' at time t; as we have already pointed out, the definition we present here is different (and stronger) from the one we gave in [3]. We proceed as follows.

First of all, fix three times $t_1 \leq t_2 \leq t_3$. Assume that $w, w' \in \mathcal{W}_k(t_2) \cap \mathcal{W}_k(t_3)$. Assume also that $\mathtt{p}(t_1, w, w')$ holds and that $t_3 \in \mathbb{N}\varepsilon$. We set

$$\mathfrak{q}_k(t_1, t_2, t_3, w, w') := \frac{\pi_k(t_1, t_2, t_3, w, w')}{d_k(t_1, t_2, t_3, w, w')},$$

where $\pi_k(t_1, t_2, t_3, w, w')$, $d_k(t_1, t_2, t_3, w, w')$ are defined as follows. Let

$$\begin{aligned} &\mathcal{J}, \mathcal{J}' \in \mathcal{P}(t_1, t_2, w, w'), \text{ such that } w \in \mathcal{J}, w' \in \mathcal{J}', \\ &\mathcal{K}, \mathcal{K}' \in \mathcal{P}(t_1, t_3, w, w'), \text{ such that } w \in \mathcal{K}, w' \in \mathcal{K}', \end{aligned} \tag{2.9}$$

be the elements of the partition of $\mathcal{I}(t_1, t_2, w, w')$ and $\mathcal{I}(t_1, t_3, w, w')$ containing w, w' respectively. Set

$$\mathcal{G} := \mathcal{K} \cup \big\{z \in \mathcal{J} \mid z > \mathcal{K}\big\}, \qquad \mathcal{G}' := \mathcal{K}' \cup \big\{z \in \mathcal{J}' \mid z < \mathcal{K}'\big\}, \tag{2.10}$$

and

$$\mathcal{B} := \mathcal{K} \cup \Big\{z \in \mathcal{W}_k(t_2) \mid \mathcal{S}(z) = \mathcal{S}(w) = \mathcal{S}(w') \text{ and } \mathcal{K} < z < \mathcal{K}'\Big\} \cup \mathcal{K}'.$$

Using a version of [3, Lemma 5.11] adapted to our new definition of the characteristic intervals and partitions, one can easily prove that $\mathcal{G}, \mathcal{G}'$ are i.o.w.s at time t_2. We can thus define

$$\pi_k(t_1, t_2, t_3, w, w') := \Big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), \mathcal{G}')\Big]^+$$

and

$$d_k(t_1, t_2, t_3, w, w') := \mathcal{L}^1\big(\Phi_k(t_2)(\mathcal{B})\big).$$

Remark 2.20. It is easy to see that $\mathfrak{q}_k(t_1, t_2, t_3, w, w')$ is uniformly bounded: in fact,

$$0 \leq \mathfrak{q}_k(t_1, t_2, t_3, w, w') = \frac{\pi_k(t_1, t_2, t_3, w, w')}{d_k(t_1, t_2, t_3, w, w')} \leq \|D^2 \mathtt{f}_k^{\mathrm{eff}}(t_2)\|_\infty \leq \mathcal{O}(1).$$

Fix now two times $t_1 \leq t_2$ such that $w, w' \in \mathcal{W}_k(t_2)$ and $\mathtt{p}(t_1, w, w')$ holds. Define

$$\mathfrak{q}_k(t_1, t_2, w, w') := \sup_{\substack{t_3 \geq t_2 \\ t_3 \in \mathbb{N}\varepsilon \\ w, w' \in \mathcal{W}_k(t_3)}} \mathfrak{q}_k(t_1, t_2, t_3, w, w'). \tag{2.11}$$

Finally, for any fixed time t and for any $w, w' \in \mathcal{W}_k(t)$, define

$$\mathfrak{q}_k(t, w, w') := \begin{cases} \mathfrak{q}_k(t, t, w, w'), & \text{if } w, w' \text{ are divided in} \\ & \text{the real solution at time } t_2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.12}$$

Remark 2.21. Notice that the definition of the weight $\mathfrak{q}(t, w, w')$ is different and stronger from the old definition of the weight we gave in [3] and which we will denote by $\mathfrak{q}^{\mathrm{old}}(t, w, w')$. Indeed,

$$\mathfrak{q}_k^{\mathrm{old}}(t, w, w') = \begin{cases} \mathfrak{q}_k(t, t, \mathtt{t}^{\mathrm{int}}(t, w, w') - \varepsilon, w, w') & \text{if } w, w' \text{ are divided at time } t \\ & \text{and will interact after time } t, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathfrak{q}_k^{\mathrm{old}}(t, w, w') \leq \mathfrak{q}_k(t, w, w') \tag{2.13}$$

As in [3], we can finally define the functional $\mathfrak{Q}_k(t)$ as

$$\mathfrak{Q}_k(t) := \mathfrak{Q}_k^+(t) + \mathfrak{Q}_k^-(t),$$

where

$$\mathfrak{Q}_k^+(t) := \int_0^{L_k^+(t)} d\tau \int_\tau^{L_k^+(t)} d\tau' \mathfrak{q}_k\big(t, \Phi_k(t)^{-1}(\tau), \Phi_k(t)^{-1}(\tau')\big)$$

and

$$\mathfrak{Q}_k^-(t) := \int_{L_k^-(t)}^0 d\tau \int_\tau^0 d\tau' \mathtt{q}_k\big(t, \Phi_k(t)^{-1}(\tau'), \Phi_k(t)^{-1}(\tau)\big).$$

Remark 2.22. Clearly $\mathfrak{Q}_k(t)$ is constant on the time intervals $[i\varepsilon, (i+1)\varepsilon)$ and it changes its value only at times $i\varepsilon$, $i \in \mathbb{N}$.

This functional $\mathfrak{Q}_k$, whose definition is different from the one in [3], still satisfies [3, Theorem 6.3]. We state now this theorem and we give a brief sketch of how its proof in [3] can be adapted to the new setting.

Theorem 2.23. *For any $i \in \mathbb{N}$, $i \geq 1$, it holds*

$$\begin{aligned}
&\mathfrak{Q}_k(i\varepsilon) - \mathfrak{Q}_k((i-1)\varepsilon) \\
&\quad \leq -\sum_{m\in\mathbb{Z}} \mathtt{A}_k^{\mathrm{quadr}}(i\varepsilon, m\varepsilon) + \mathcal{O}(1)\text{Tot.Var.}(u(0);\mathbb{R}) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon).
\end{aligned} \tag{2.14}$$

Sketch of the Proof. The proof is analogous to the proof of [3, Theorem 6.3]. We will not enter into details. Some notations, which will be used again later, are introduced here.

First of all observe that it is sufficient to prove inequality (2.14) separately for $\mathfrak{Q}_k^+$ and $\mathfrak{Q}_k^-$. Let us thus concentrate our attention of $\mathfrak{Q}_k^+$, since the analysis on $\mathfrak{Q}_k^-$ is completely similar. For any $m \in \mathbb{Z}$, set

$$\begin{aligned}
J_m^L &:= \Phi_k((i-1)\varepsilon)\Big(\Big\{w \in \mathcal{W}_k^+((i-1)\varepsilon) \;\Big|\; \mathtt{x}((i-1)\varepsilon, w) = (m-1)\varepsilon, \\
&\qquad \mathtt{x}(i\varepsilon, w) = m\varepsilon\Big\}\Big), \\
J_m^R &:= \Phi_k((i-1)\varepsilon)\Big(\Big\{w \in \mathcal{W}_k^+((i-1)\varepsilon) \;\Big|\; \mathtt{x}((i-1)\varepsilon, w) = m\varepsilon, \\
&\qquad \mathtt{x}(i\varepsilon, w) = m\varepsilon\Big\}\Big), \\
J_m &:= J_m^L \cup J_m^R, \\
K_m &:= \Phi_k(i\varepsilon)\Big(\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k^+(i\varepsilon)\Big), \\
S_m &:= \Phi_k((i-1)\varepsilon)\Big(\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon)\Big), \\
T_m &:= \Phi_k(i\varepsilon)\Big(\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon)\Big).
\end{aligned} \tag{2.15}$$

Observe that if $\tau, \tau' \in J_m^L$ (or $\tau, \tau' \in J_m^R$), then $w := \Phi_k^{-1}((i-1)\varepsilon)(\tau)$ and $w' := \Phi_k^{-1}((i-1)\varepsilon)(\tau')$ are not divided in the real solution at time $(i-1)\varepsilon$ and thus $\mathtt{q}_k((i-1)\varepsilon, w, w') = 0$.

Similarly, if $\tau, \tau' \in K_m$, $\tau < \tau'$, setting again $w := \Phi_k^{-1}(i\varepsilon)(\tau)$, $w' := \Phi_k^{-1}(i\varepsilon)(\tau')$ then either w, w' are not divided at time $i\varepsilon$, and thus $\mathfrak{q}_k(i\varepsilon, w, w') = 0$, or they are divided at time $i\varepsilon$, i.e. they have different positions at times $t \in (i\varepsilon, (i+1)\varepsilon)$. In this second case, by definition $\mathtt{t}^{\mathrm{split}}(i\varepsilon, w, w') = i\varepsilon$; for any fixed $j \in \mathbb{N}$, $j \geq i$, with $w, w' \in \mathcal{W}_k(j\varepsilon)$, with notations similar to (2.9)$-$(2.10), denote by

$$\mathcal{J}, \mathcal{J}' \in \mathcal{P}(i\varepsilon, i\varepsilon, w, w'), \text{ such that } w \in \mathcal{J}, w' \in \mathcal{J}',$$
$$\mathcal{K}, \mathcal{K}' \in \mathcal{P}(i\varepsilon, j\varepsilon, w, w'), \text{ such that } w \in \mathcal{K}, w' \in \mathcal{K}'.$$

the element of the partition containing w, w' at time $i\varepsilon$ and at time $j\varepsilon$ respectively, and set

$$\mathcal{G} := \mathcal{K} \cup \big\{z \in \mathcal{J} \bigm| z > \mathcal{K}\big\}, \qquad \mathcal{G}' := \mathcal{K}' \cup \big\{z \in \mathcal{J}' \bigm| z < \mathcal{K}'\big\}.$$

Using the monotonicity properties of the derivative of the convex envelope and the fact that the element of the partition $\mathcal{P}(i\varepsilon, i\varepsilon, w, w')$ are entropic w.r.t. the function $\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon)$, we obtain

$$0 \geq \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon), \mathcal{J}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon), \mathcal{J}') \geq \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon), \mathcal{G}) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(i\varepsilon), \mathcal{G}').$$

Thus $\pi_k(i\varepsilon, i\varepsilon, j\varepsilon, w, w') = 0 = \mathfrak{q}_k(i\varepsilon, i\varepsilon, j\varepsilon, w, w')$, for any $j \geq i$ such that $w, w' \in \mathcal{W}_k(j\varepsilon)$. Hence, by (2.11) and (2.12),

$$\mathfrak{q}_k(i\varepsilon, w, w') = \mathfrak{q}_k(i\varepsilon, i\varepsilon, w, w') = \sup_{\substack{j \geq i \\ w, w' \in \mathcal{W}_k(j\varepsilon)}} \mathfrak{q}_k(i\varepsilon, i\varepsilon, j\varepsilon, w, w') = 0.$$

We can thus perform the following computation:

$$\begin{aligned}
&\mathfrak{Q}_k^+(i\varepsilon) - \mathfrak{Q}_k^+((i-1)\varepsilon) \\
&\leq \sum_{m < m'} \bigg[\iint_{T_m \times T_{m'}} \mathfrak{q}_k\Big(i\varepsilon, \Phi_k(i\varepsilon)^{-1}(\tau), \Phi_k(i\varepsilon)^{-1}(\tau')\Big) d\tau d\tau' \\
&\quad + \iint_{(K_m \times K_{m'}) \setminus (T_m \times T_{m'})} \mathfrak{q}_k\Big(i\varepsilon, \Phi_k(i\varepsilon)^{-1}(\tau), \Phi_k(i\varepsilon)^{-1}(\tau')\Big) d\tau d\tau' \\
&\quad - \iint_{S_m \times S_{m'}} \mathfrak{q}_k\Big((i-1)\varepsilon, \Phi_k((i-1)\varepsilon)^{-1}(\tau), \Phi_k((i-1)\varepsilon)^{-1}(\tau')\Big) d\tau d\tau' \bigg]
\end{aligned}$$

$$- \sum_{m\in\mathbb{Z}} \iint_{J_m^L \times J_m^R} \mathfrak{q}_k\Big((i-1)\varepsilon, \Phi_k((i-1)\varepsilon)^{-1}(\tau), \Phi_k((i-1)\varepsilon)^{-1}(\tau')\Big) d\tau d\tau'.$$

Now the tree terms in the r.h.s. of the last inequality are estimated separately as follows.

1. The integral over *pairs of waves such that at least one of them is created at time $i\varepsilon$* is estimated exactly in the same way as is [3, Section 6.3]:

$$\begin{aligned} \sum_{m<m'} \iint_{(K_m \times K_{m'}) \setminus (T_m \times T_{m'})} & \mathfrak{q}_k(i\varepsilon) d\tau d\tau' \\ \leq \ & \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon). \end{aligned} \tag{2.16}$$

2. The variation of the integral over *pairs of waves which exist both at time $(i-1)\varepsilon$ and at time $i\varepsilon$ and which do not interact at time $i\varepsilon$* is estimated by

$$\begin{aligned} \sum_{m<m'} \bigg[\iint_{T_m \times T_{m'}} & \mathfrak{q}_k(i\varepsilon) d\tau d\tau' - \iint_{S_m \times S_{m'}} \mathfrak{q}_k((i-1)\varepsilon) d\tau d\tau' \bigg] \\ \leq \ & \mathcal{O}(1)\text{Tot.Var.}(u(0)) \sum_{r\in\mathbb{Z}} \mathtt{A}(i\varepsilon, r\varepsilon). \end{aligned} \tag{2.17}$$

   in the following way:

   (a) first one adapts the proof of [3, Lemma 6.6] to show that for any $t_1 \leq (i-1)\varepsilon < i\varepsilon \leq t_3$, for any pair of waves $w, w' \in \mathcal{W}_k(i\varepsilon) \cap \mathcal{W}_k(t_3)$, if $\mathtt{p}(t_1, w, w')$ holds, setting $m\varepsilon := \mathtt{x}(i\varepsilon, w) \leq \mathtt{x}(i\varepsilon, w') =: m'\varepsilon$, we have

$$\Big| d_k\big(t_1, i\varepsilon, t_3, w, w'\big) - d_k\big(t_1, (i-1)\varepsilon, t_3, w, w'\big)\Big| \leq \mathcal{O}(1) \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon),$$

$$\pi_k\big(t_1, i\varepsilon, t_3, w, w'\big) - \pi_k\big(t_1, (i-1)\varepsilon, t_3, w, w'\big) \leq \mathcal{O}(1) \sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon),$$

   and thus

$$\mathfrak{q}_k(t_1, i\varepsilon, t_3, w, w') - \mathfrak{q}_k(t_1, (i-1)\varepsilon, t_3, w, w')$$

$$\leq \mathcal{O}(1)\frac{1}{\big|\Phi_k(i\varepsilon)(w') - \Phi_k(i\varepsilon)(w)\big|}\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon). \qquad (2.18)$$

(b) then one observes that $\mathtt{t}^{\text{split}}(i\varepsilon, w, w') = \mathtt{t}^{\text{split}}((i-1)\varepsilon, w, w')$, since $\mathtt{x}(i\varepsilon, w) \neq \mathtt{x}(i\varepsilon, w')$;

(c) finally one uses the new definition of $\mathfrak{q}_k$, (2.11)–(2.12) to prove that

$$\begin{aligned}&\mathfrak{q}_k(i\varepsilon, w, w') - \mathfrak{q}_k((i-1)\varepsilon, w, w')\\ &\quad\leq \mathcal{O}(1)\frac{1}{\big|\Phi_k(i\varepsilon)(w') - \Phi_k(i\varepsilon)(w)\big|}\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon),\end{aligned}$$

and then one concludes by the elementary estimate

$$\begin{aligned}&\sum_{m<m'} \frac{1}{\big|\Phi_k(i\varepsilon)(w') - \Phi_k(i\varepsilon)(w)\big|}\sum_{r=m}^{m'} \mathtt{A}(i\varepsilon, r\varepsilon)\\ &\quad\leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(t))\sum_r \mathtt{A}(i\varepsilon, r\varepsilon).\end{aligned}$$

3. Finally the estimate on the *pairs of waves which are divided at time* $(i-1)\varepsilon$ *and are interacting at time* $i\varepsilon$:

$$\begin{aligned}&-\sum_{m\in\mathbb{Z}}\iint_{J_m^L\times J_m^R} \mathfrak{q}_k((i-1)\varepsilon)d\tau d\tau'\\ &\quad\leq -\sum_{\substack{m\in\mathbb{Z}\\ \mathcal{S}(\mathcal{W}_k(i\varepsilon,m\varepsilon))=1}} \mathtt{A}_k^{\text{quadr}}(i\varepsilon, m\varepsilon)\\ &\qquad+\mathcal{O}(1)\text{Tot.Var.}(u(0))\sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon), \qquad (2.19)\end{aligned}$$

is an immediate consequence of the analogous estimate [3, Inequality (6.9)] and of the fact that the new definition of $\mathfrak{q}_k$ is "stronger" than the old one, inequality (2.13).

It is easy to see that inequality (2.14) in the statement of Theorem 2.23 follows from (2.16), (2.17), (2.19). □

As an immediate consequence of the previous theorem and of estimate (2.6), we get the following corollary.

Corollary 2.24. *There exists a constant $C = C(f) > 0$, depending only on f such that the functional*

$$t \mapsto \Upsilon(t) := \mathfrak{Q}(t) + CQ^{\text{known}}(t)$$

is uniformly bounded at $t = 0$ by

$$\Upsilon(0) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}),$$

it is decreasing and at each time step $i\varepsilon$, $i \in \mathbb{N}$, it decreases at least of

$$\frac{1}{2}\sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon) \leq \Upsilon((i-1)\varepsilon) - \Upsilon(i\varepsilon). \tag{2.20}$$

3. The Wavefront Map ψ

We have seen in Section 1.4 that a key point to prove Theorem 1.3 on the rate of convergence of the Glimm scheme is to construct, for any $i_1, i_2 \in \mathbb{N}$, a map

$$\psi : [i_1\varepsilon, i_2\varepsilon] \times \mathbb{R} \to \mathbb{R}^n$$

which satisfies the Properties in (1.17). In this section we first explicitly define the map ψ, which trivially satisfies Property (1.17a), and we construct a Lagrangian representation for the map ψ; then we state Theorem 3.3, on the variation in time of the speed of the waves in ψ, whose proof will be the subject of Sections 4 and 5; finally, using Theorem 3.3, we prove that ψ satisfies also Properties (1.17b) and (1.17c).

3.1. Definition of ψ

We start with the explicit definition of ψ. This map ψ is constructed more or less as in [8], with some slight modification. Set for simplicity $t_1 := i_1\varepsilon$ and $t_2 := i_2\varepsilon$. The definition of ψ is given backward in time, starting from time t_2 and going backward to time t_1. First of all we set $\psi(t_2, x) := u^\varepsilon(t_2, x)$ for any $x \in \mathbb{R}$, so that Property (1.17a) is trivially satisfied. Then we define two Riemann solvers, a *starting* RS and a *transversal* RS: both act backward in time and produce a self-similar wavefront solution, with a finite number of wavefronts. The *starting* RS is used at time $t_2 = i_2\varepsilon$ to define ψ on a left neighborhood $[\tilde{t}, t_2]$ of t_2. Then, anytime two wavefronts collide at some time $\bar{t} \in (t_1, t_2)$, assuming that ψ is defined on the time interval $[\bar{t}, t_2]$, we use the *transversal* RS to prolong ψ on a left neighborhood of $\bar{t}$.

The *starting* Riemann Solver. This is the Riemann Solver used at time $t = t_2$. It is defined as follows. For $m, r \in \mathbb{Z}$, $m \in [r - (i_2 - i_1), r]$, set

$$\begin{aligned} \check{s}_k^{m \rightsquigarrow r} &:= \mathcal{S}\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big) \\ &\quad \mathcal{L}^1\bigg(\Phi_k(i_1\varepsilon)\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big)\bigg) \\ &= \mathcal{S}\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big) \\ &\quad \mathcal{L}^1\bigg(\Phi_k(i_2\varepsilon)\Big(\mathcal{W}_k(i_1\varepsilon, m\varepsilon) \cap \mathcal{W}_k(i_2\varepsilon, r\varepsilon)\Big)\bigg). \end{aligned} \tag{3.1}$$

Notice that, by the monotonicity of the map $w \to \mathtt{x}(t, w)$, if $\check{s}_k^{m \rightsquigarrow r}, \check{s}_{k'}^{m \rightsquigarrow r'} \neq 0$ and $r < r'$, then $k \leq k'$. Fix now $r \in \mathbb{Z}$ and for $m \in [r - (i_2 - i_1), r]$ set

$$\begin{aligned} \psi^{r-(i_2-i_1) \rightsquigarrow r} &:= T^n_{s_n^{r-(i_2-i_1)\rightsquigarrow r}} \circ \cdots \circ T^1_{s_1^{r-(i_2-i_1)\rightsquigarrow r}}\big(u^{i_2, r-1}\big), \\ \psi^{m \rightsquigarrow r} &:= T^n_{s_n^{m\rightsquigarrow r}} \circ \cdots \circ T^1_{s_1^{m\rightsquigarrow r}}\big(\psi^{m-1\rightsquigarrow r}\big). \end{aligned}$$

The (backward) solution to the Riemann problem $(u^{i_2, r-1}, u^{i_2, r})$ is now defined as follows: for any $m = r - (i_2 - i_1), \dots, r$ there is a *physical* wavefront traveling with speed

$$\check{\lambda}^{m \rightsquigarrow r} := \frac{r\varepsilon - m\varepsilon}{i_2\varepsilon - i_1\varepsilon} \tag{3.2}$$

which connects the left state $\psi^{m-1\rightsquigarrow r}$ with the right state $\psi^{m \rightsquigarrow r}$; moreover, there is one more *non-physical* wavefront, traveling with speed equal to $\check{\lambda} := -1$ connecting $\psi^{r \rightsquigarrow r}$ to $u^{i_2, r}$.

The *transversal* Riemann solver. This RS is used every time two (or more) wavefronts collide at a time in (t_1, t_2). We assume w.l.o.g. that every collision involves exactly two wavefronts: the rules can be easily extended to the case of several simultaneous collisions, because the outcome does not depend on the order of the collisions. Assume thus that at point $(\bar{t}, \bar{x})$, $\bar{t} \in (t_1, t_2)$ two wavefronts collide. We have to distinguish two cases.

Case 1*: both the colliding wavefronts are physical.* Assume that before the collision the first wavefront is traveling with speed λ' and it is connecting

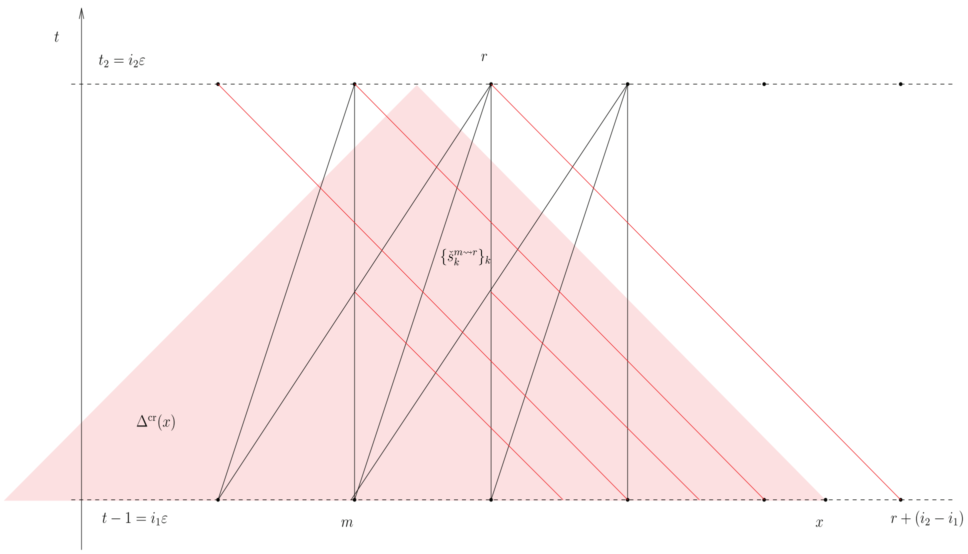

Figure 2: The wavefronts of the function ψ: the pink region $\Delta^{\rm cr}(x)$ is used in the proof of Proposition 3.6.

the states

$$\psi^M = T^n_{s'_n} \circ \cdots \circ T^1_{s'_1} \psi^L,$$

while the second wavefront is traveling with speed $\lambda' < \lambda''$ and it is connecting the states

$$\psi^R = T^n_{s''_n} \circ \cdots \circ T^1_{s''_1} \psi^M.$$

Notice that, by the monotonicity of the map $w \mapsto \mathtt{x}(t, w)$, there exists $\bar{k} \in \{1, \dots, n\}$ such that $s''_1, \dots, s''_{\bar{k}} = 0$ and $s'_{\bar{k}+1}, \dots, s'_n = 0$. Hence the interaction at $(\bar{t}, \bar{x})$ is purely transversal. The (backward) Riemann problem (ψ^L, ψ^R) at point $(\bar{t}, \bar{x})$ is now solved as follows. Define the intermediate states

$$\tilde{\psi}^M := T^n_{s''_n} \circ \cdots \circ T^1_{s''_{\bar{k}+1}} \psi^L, \qquad \tilde{\psi}^R := T^n_{s'_{\bar{k}}} \circ \cdots \circ T^1_{s'_1} \psi^M,$$

The solution for times $t \leq \bar{t}$ around the point $(\bar{t}, \bar{x})$ is made by a *physical* wavefront traveling with speed λ'' connecting ψ^L and $\tilde{\psi}^M$; a *physical* wavefront traveling with speed λ' connecting $\tilde{\psi}^M$ and $\tilde{\psi}^R$; a *non-physical* wavefront traveling with speed $\check{\lambda} = -1$ connecting $\tilde{\psi}^R$ and ψ^R.

Case 2: one of the two colliding wavefronts is non-physical. Assume that the non-physical wavefront is connecting ψ^L with ψ^M, while the physical wavefront is traveling with speed λ and it is connecting

$$\psi^R = T^n_{s_n} \circ \cdots \circ T^1_{s_1} \psi^M.$$

Define the intermediate state

$$\tilde{\psi}^M := T^n_{s_n} \circ \cdots \circ T^1_{s_1} \psi^L.$$

The solution around $(\bar{t}, \bar{x})$ for times $t \leq \bar{t}$ is now made by a physical wavefront traveling with speed λ connecting ψ^L with $\tilde{\psi}^M$ and by a non-physical wavefront traveling with speed $\check{\lambda} = -1$ and connecting $\tilde{\psi}^M$ with ψ^R.

It is not difficult to see that the definition of ψ is well posed.

3.2. Lagrangian representation for ψ

In the same spirit as in Section 2.3 we introduce now a sort of Lagrangian representation for the wavefront solution ψ. We are not interested here in defining a general notion of Lagrangian representation, since the map ψ is a map *ad hoc* constructed to get estimate (1.15).

First of all, let us analyze the physical waves. For any $k = 1, \dots, n$ the set of the physical waves of the k-th family in ψ is the set $\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$.

Set, for any $k = 1, \dots, n$,

$$\begin{aligned} \check{L}^{\pm}_k &:= \mathcal{L}^1\Big(\Phi_k(i_2\varepsilon)\Big(\mathcal{W}^{\pm}_k(i_1\varepsilon) \cap \mathcal{W}^{\pm}_k(i_2\varepsilon)\Big)\Big) \\ &= \mathcal{L}^1\Big(\Phi_k(i_1\varepsilon)\Big(\mathcal{W}^{\pm}_k(i_1\varepsilon) \cap \mathcal{W}^{\pm}_k(i_2\varepsilon)\Big)\Big). \end{aligned}$$

Define also the position map for the physical waves in ψ and follows:

$$\begin{aligned} \mathtt{y} &: [t_1, t_2] \times \bigcup_{k=1}^n \big(\mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)\big) \to \mathbb{R}, \\ \mathtt{y}(t, w) &:= \mathtt{x}(t_2, w) - \frac{\mathtt{x}(t_2, w) - \mathtt{x}(t_1, w)}{t_2 - t_1}(t_2 - t). \end{aligned}$$

Notice that $\mathtt{y}$ takes values in the discontinuity points of ψ, it is increasing in w and affine in t.

The analog of the collection of the maps $\{\Phi_k(t)\}_{t\in[0,\infty)}$ (see Definition 2.4) for ψ is the map

$$\Psi_k : \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2) \to \big[-\check{L}_k^-, 0\big) \cap \big(0, +\check{L}_k^+\big]$$

defined by

$$\begin{aligned}\Psi_k(w) \;:=\; & \mathcal{S}(w)\mathcal{L}^1\bigg(\Phi_k(t_1)\Big(\big\{w' \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2) \;\big|\; \mathcal{S}(w') = \mathcal{S}(w) \text{ and} \\ & w' \le w\big\}\Big)\bigg) \\ =\; & \mathcal{S}(w)\mathcal{L}^1\bigg(\Phi_k(t_2)\Big(\big\{w' \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2) \;\big|\; \mathcal{S}(w') = \mathcal{S}(w) \text{ and} \\ & w' \le w\big\}\Big)\bigg).\end{aligned}$$

The restriction $\Psi : \mathcal{W}_k^+(t_1) \cap \mathcal{W}_k^+(t_2) \to \mathbf{I}(\check{L}_k^+)$ is an isomorphism of ordered sets, while the restriction $\Psi : \mathcal{W}_k^-(t_1) \cap \mathcal{W}_k^-(t_2) \to \mathbf{I}(\check{L}_k^-)$ is an anti-isomorphism of ordered sets.

Notice that while the maps $\Phi_k(t)$ for u^ε depends on the time, the map Ψ_k for ψ does not, since the total amount of physical waves in ψ is constant in time.

We define also the maps $\check{\gamma}_k(t,\cdot) := (\check{u}_k(t,\cdot), \check{v}_k(t,\cdot), \check{\sigma}_k(t,\cdot))$ and the effective flux $\check{\mathtt{f}}_k^{\text{eff}}(t,\cdot)$ at any time $t \in [t_1, t_2)$ as follows. Fix a time t; assume first that no wavefront collision takes place at time t. Fix any point $x \in \mathbb{R}$. Assume that

$$u(t,x) = T^n_{s_n} \circ \cdots \circ T^1_{s_1} u(t, x-);$$

denote by $\{\gamma_k\}_k$, $\gamma_k = (u_k, v_k, \sigma_k) : \mathbf{I}(s_k) \to \mathbb{R}^{n+2}$ the collection of curves which solve the Riemann problem $(u(t,x-), u(t,x+))$ and by $f_k : \mathbf{I}(s_k) \to \mathbb{R}$ the associated reduced flux. Since

$$\Psi_k|_{\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k} : \mathtt{y}(t)^{-1}(x) \cap \mathcal{W}_k \to a + \mathbf{I}(s_k)$$

is an (anti)isomorphism of ordered sets for some $a \in \mathbb{R}$, we can define

$$\check{\gamma}_k(t,\cdot) : \mathtt{y}(t)^{-1}(\bar{x}) \cap \mathcal{W}_k \to \mathcal{D}_k \subseteq \mathbb{R}^{n+2}, \qquad \check{\gamma}_k(t,w) := \gamma_k(\Psi_k(w) - a).$$

Using the fact that, for fixed time t, the position map $\mathtt{y}$ takes values in the discontinuity points of ψ, $\check{\gamma}_k(t,w)$ is defined for any k-wave w.

We also define

$$\check{\mathtt{f}}^{\mathrm{eff}}_k : [-\check{L}^-_k, \check{L}^+_k] \to \mathbb{R}$$

as any $C^{1,1}$ map such that

$$\frac{d^2\check{\mathtt{f}}^{\mathrm{eff}}_k(t)}{d\tau^2}(\tau) = \frac{d\tilde{\lambda}\big(\check{\gamma}(t,w)\big)}{d\tau}, \quad \text{with } \tau = \Psi_k(w).$$

Now, if $t = t_2$ or if t is a time when a collision between two wavefronts takes place, we extend the definitions of $\check{\gamma}_k(t)$ and $\check{\mathtt{f}}^{\mathrm{eff}}_k(t)$ in order to have left-continuous in time maps.

Remark 3.1. We usually want our maps to be right-continuous in time. In this case, however, we are using backward-in-time Riemann solvers, and thus it is quite natural to require that $t \mapsto \gamma_k(t)$ is left-continuous in time.

Finally, we define the *wavefront speed* of a wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ as

$$\check{\lambda}(w) := \frac{\mathtt{x}(i_2\varepsilon, w) - \mathtt{x}(i_1\varepsilon, w)}{i_2\varepsilon - i_1\varepsilon} = \frac{\mathtt{y}(i_2\varepsilon, w) - \mathtt{y}(i_1\varepsilon, w)}{i_2\varepsilon - i_1\varepsilon},$$

which coincides with (3.2).

As for the Glimm approximate solution u^ε, we say that a set $\mathcal{I} \subseteq \mathcal{W}^{\pm}_k(t_1) \cap \mathcal{W}^{\pm}_k(t_2)$ is an *interval of waves for* ψ if $\mathcal{I}$ is an interval in the ordered set $\big(\mathcal{W}^{\pm}_k(t_1) \cap \mathcal{W}^{\pm}_k(t_2), \leq\big)$. The following definition is the analog of Definition 2.6.

Definition 3.2. Fix $\bar{t} \in [t_1, t_2]$. Let $\mathcal{I} \subseteq \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$ be an interval of waves for ψ. Set $I := \Psi_k(\mathcal{I})$. Since the restriction of Ψ_k to positive (resp. negative) waves is an isomorphism (resp. anti-isomorphism) of ordered sets, I is an interval in $\mathbb{R}$ (possibly made by a single point). Let us define:

- the *Rankine-Hugoniot speed given to the interval of waves $\mathcal{I}$ by a function $g : \mathbb{R} \to \mathbb{R}$* as

$$\sigma^{\mathrm{rh}}(g, \mathcal{I}) := \begin{cases} \frac{g(\sup I) - g(\inf I)}{\sup I - \inf I} & \text{if } I \text{ is not a singleton,} \\ g'(I) & \text{if } I \text{ is a singleton;} \end{cases}$$

- for any $w \in \mathcal{I}$, the *entropic speed given to the wave w by the Riemann problem $\mathcal{I}$ and the flux function g* as

$$\sigma^{\text{ent}}(g, \mathcal{I}, w) := \begin{cases} \dfrac{d}{d\tau}\text{conv}_{\,I} g\Big(\Psi_k(w)\Big) & \text{if } \mathcal{S}_k(w) = +1, \\ \dfrac{d}{d\tau}\text{conc}_{\,I} g\Big(\Psi_k(w)\Big) & \text{if } \mathcal{S}_k(w) = -1. \end{cases}$$

If $\sigma^{\text{rh}}(g, \mathcal{I}) = \sigma^{\text{ent}}(g, \mathcal{I}, w)$ for any $w \in \mathcal{I}$, we will say that $\mathcal{I}$ is *entropic* w.r.t. the function g. We will also say that *the Riemann problem $\mathcal{I}$ with flux function g divides w, w'* if $\sigma^{\text{ent}}(g, \mathcal{I}, w) \neq \sigma^{\text{ent}}(g, \mathcal{I}, w')$.

Let us now analyze the non-physical waves. The set of non-physical wavefront is defined as

$$\mathcal{W}_0 := \Big\{(t, x) \;\Big|\; \text{in } (t, x) \text{ a non-physical wavefront is generated}\Big\}.$$

We are labeling each non-physical wavefront with the point in the (t, x) plane in which it is generated.

Since the speed of the non-physical wavefronts is strictly less than the speed of any physical wave, we will refer to the set of non-physical wavefronts also as the set of waves of the 0-th family.

Clearly $\mathcal{W}_0$ is a finite set. For any non-physical wavefronts $\alpha = (\bar{t}, \bar{x}) \in \mathcal{W}_0$, we define its creation time $\mathtt{t}^{\text{cr}}(\alpha) := \bar{t}$ and its position $\mathtt{y}(t, \alpha) = \bar{x} - (t - \bar{t})$. Moreover, if t is any time when no collision between wavefronts takes place, we define the *strength* of the non-physical wavefront α as

$$s(t, \alpha) := \Big|\psi\big(t, \mathtt{y}(t, \alpha)+\big) - \psi\big(t, \mathtt{y}(t, \alpha)-\big)\Big|;$$

then, as usual, we extend the definition to all times in $(t_1, t_2]$ in order to have a left-continuous in time map. Finally define

$$\mathcal{W}_0(t) := \Big\{\alpha \in \mathcal{W}_0 \;\Big|\; \mathtt{t}^{\text{cr}}(\alpha) \geq t\Big\}.$$

We will call $\mathcal{W}_0(t_2)$ the set of *primary* non-physical wavefronts and $\mathcal{W}_0 \setminus \mathcal{W}_0(t_2)$ the set of *secondary* non-physical wavefronts.

3.3. The main theorem on ψ

In this section we state the main theorem about physical and non-physical waves in ψ, which will be proved in Sections 4 and 5, and, using this theorem, we prove estimates (1.17b) and (1.17c).

Theorem 3.3. *With the same notations as before,*

(1) *the following bounds on physical waves hold:*

$$\left.\begin{array}{l} \displaystyle\int_{-\check L_k^-}^{\check L_k^+} \Big\{\mathrm{Tot.Var.}\Big(\check u_k\big(\cdot,\Psi^{-1}(\tau)\big);(t_1,t_2)\Big) \\ \quad + \Big|\big(\check u_k(t_2,\cdot)-\hat u_k(t_2,\cdot)\big)\circ\Psi_k^{-1}(\tau)\Big|\Big\}d\tau \\ \\ \displaystyle\int_{-\check L_k^-}^{\check L_k^+} \Big\{\mathrm{Tot.Var.}\Big(\check v_k\big(\cdot,\Psi^{-1}(\tau)\big);(t_1,t_2)\Big) \\ \quad + \Big|\big(\check v_k(t_2,\cdot)-\hat v_k(t_2,\cdot)\big)\circ\Psi_k^{-1}(\tau)\Big|\Big\}d\tau \\ \\ \displaystyle\int_{-\check L_k^-}^{\check L_k^+} \Big\{\mathrm{Tot.Var.}\Big(\check \sigma_k\big(\cdot,\Psi^{-1}(\tau)\big);(t_1,t_2)\Big) \\ \quad + \Big|\big(\check \sigma_k(t_2,\cdot)-\hat \sigma_k(t_2,\cdot)\big)\circ\Psi_k^{-1}(\tau)\Big|\Big\}d\tau \end{array}\right\} \leq \mathcal{O}(1)\Big[\Upsilon(t_1)-\Upsilon(t_2)\Big],$$

where $(\hat u_k,\hat v_k,\hat\sigma_k)$ is the curve solving the exact Riemann problems at time t_2 (i.e. with all waves in $\mathcal{W}(t_2)\cap(i_2\varepsilon,m\varepsilon)$, $m\in\mathbb{Z}$).

(2) *the following bound on non-physical waves holds:*

$$\sum_{\alpha\in\mathcal{W}_0}\Big[\mathrm{Tot.Var.}\Big(s(\cdot,\alpha);\big(t_1,\mathtt{t}^{\mathrm{cr}}(\alpha)\big)\Big)+s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\Big]\leq\mathcal{O}(1)\Big[\Upsilon(t_1)-\Upsilon(t_2)\Big].$$

As an immediate consequence, we get the following corollary. For any $k=1,\dots,n$, for any physical wave $w\in\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)$ and for any $t\in(t_1,t_2]$, set

$$\begin{aligned}\check r_k(t,w) &:= \tilde r_k\Big(\check u_k(t,w),\check v_k(t,w),\check\sigma_k(t,w)\Big),\\ \hat r_k(t,w) &:= \tilde r_k\Big(\hat u_k(t,w),\hat v_k(t,w),\hat\sigma_k(t,w)\Big).\end{aligned}$$

Corollary 3.4. *It holds*

$$\int_{-\check{L}_k^-}^{\check{L}_k^+}\Big\{\text{Tot.Var.}\Big(\check{r}_k\big(\cdot,\Psi^{-1}(\tau)\big);(t_1,t_2)\Big)+\Big|\big(\check{r}_k(t_2,\cdot)-\hat{r}_k(t_2,\cdot)\big)\circ\Psi_k^{-1}(\tau)\Big|\Big\}d\tau$$
$$\leq \mathcal{O}(1)\Big[\Upsilon(t_1)-\Upsilon(t_2)\Big].$$

As we have already said, the proof of Theorem 3.3 is the subject of Sections 4 and 5. We now use Theorem 3.3 and Corollary 3.4 to prove estimates (1.17b)−(1.17c) and thus complete the proof of Theorem 1.3.

Proposition 3.5 (Estimate (1.17b))**.** *It holds*

$$\big\|S_{t_2-t_1}\psi(t_1)-\psi(t_2)\big\|_1 \leq \mathcal{O}(1)\bigg[\Big(\Upsilon(t_1)-\Upsilon(t_2)\Big)+\frac{1+\log(i_2-i_1)}{i_2-i_1}\bigg](t_2-t_1).$$

Proof. We make use the semigroup estimate

$$\big\|\psi(t_2)-S_{t_2-t_1}\psi(t_1)\ \big\|_1 \leq L\int_{t_1}^{t_2}\limsup_{h\to 0}\frac{\big\|\psi(t+h)-S_h\psi(t)\big\|_1}{h}dt. \tag{3.3}$$

Since the map ψ is piecewise constant at any fixed time t, it is not hard to see that the integrand on the r.h.s. can be estimated as

$$\limsup_{h\to 0}\frac{\big\|\psi(t+h)-S_h\psi(t)\big\|_1}{h}$$
$$\leq \sum_{k=1}^{n}\int_{\Psi_k\big(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)\big)}\Big|\check{\lambda}\big(\Psi^{-1}(\tau)\big)-\check{\sigma}\Big(t,\Psi^{-1}(\tau)\Big)\Big|d\tau+2\sum_{\alpha\in\mathcal{W}_0(t)}s(t,\alpha).$$

For the term concerning the non-physical waves, we easily obtain

$$\begin{aligned}\sum_{\alpha\in\mathcal{W}_0(t)}s(t,\alpha)\ &\leq \sum_{\alpha\in\mathcal{W}_0(t)}\big|s(t,\alpha)-s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\big|+s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\\ &\leq \sum_{\alpha\in\mathcal{W}_0}\Big[\text{Tot.Var.}\Big(s(\cdot,\alpha);\big(t_1,\mathtt{t}^{\mathrm{cr}}(\alpha)\big)\Big)+s\big(\mathtt{t}^{\mathrm{cr}}(\alpha),\alpha\big)\Big]\\ \text{(by Theorem 3.3)}\ &\leq \mathcal{O}(1)\Big[\Upsilon(t_1)-\Upsilon(t_2)\Big].\end{aligned}$$

For the term concerning the physical waves, we argue as follows. Fix any

$\tau \in \Psi_k\big(\mathcal{W}(t_1) \cap \mathcal{W}(t_2)\big)$ and set $w := \Psi^{-1}(\tau)$.

$$\begin{aligned}
&\big|\check{\lambda}(w) - \check{\sigma}(t,w)\big| \\
&\quad \le \left|\check{\lambda}(w) - \frac{1}{i_2 - i_1}\sum_{i=i_1}^{i_2-1} \hat{\sigma}(i\varepsilon, w)\right| + \left|\frac{1}{i_2 - i_1}\sum_{i=i_1}^{i_2-1} \hat{\sigma}(i\varepsilon, w) - \hat{\sigma}(i_2\varepsilon, w)\right| \\
&\qquad + \left|\hat{\sigma}(i_2\varepsilon, w) - \check{\sigma}(t,w)\right| \\
&\quad \le \left|\check{\lambda}(w) - \frac{1}{i_2 - i_1}\sum_{i=i_1}^{i_2-1} \hat{\sigma}(i\varepsilon, w)\right| + \text{Tot.Var.}\left(\hat{\sigma}(\cdot, w); \left(t_1, t_2 + \frac{\varepsilon}{2}\right)\right) \\
&\qquad + \big|\hat{\sigma}(t_2, w) - \check{\sigma}(t_2, w)\big| + \text{Tot.Var.}\Big(\check{\sigma}(\cdot, w); \big(t_1, t_2\big)\Big). \qquad (3.4)
\end{aligned}$$

To estimate the first term of the last summation we use the same technique as in [14]. Define first the map

$$\omega \ : \ [0,1] \times [0,1] \to \mathbb{R}, \qquad \omega(\sigma, \vartheta) := \begin{cases} -\sigma & \text{if } \sigma \le \vartheta \\ 1 - \sigma & \text{if } \sigma > \vartheta. \end{cases}$$

Set

$$\sigma^{\min} := \min_{i = i_1, \dots, i_2 - 1} \hat{\sigma}(i\varepsilon, w), \qquad \sigma^{\max} := \max_{i = i_1, \dots, i_2 - 1} \hat{\sigma}(i\varepsilon, w),$$

and

$$\begin{aligned}
\mathcal{J} &:= \big\{ i \in [i_1, i_2 - 1] \ \big| \ \sigma^{\max} \le \vartheta_i \le \sigma^{\min} \big\}, \\
\mathcal{K} &:= \big\{ i \in [i_1, i_2 - 1] \ \big| \ \vartheta_i < \hat{\sigma}(i_1\varepsilon, w) \big\}.
\end{aligned}$$

We thus have

$$\begin{aligned}
&\left|\check{\lambda}(w) - \frac{1}{i_2 - i_1}\sum_{i=i_1}^{i_2-1} \hat{\sigma}(i\varepsilon, w)\right| \\
&\quad = \left|\frac{1}{i_2 - i_1}\sum_{i=i_1}^{i_2-1} \omega\big(\hat{\sigma}(i\varepsilon, w), \vartheta_i\big)\right| \\
&\quad = \frac{1}{i_2 - i_1}\left|\sum_{i=i_1}^{i_2-1} \Big[\omega\big(\hat{\sigma}(i\varepsilon, w), \vartheta_i\big) - \omega\big(\hat{\sigma}(i_1\varepsilon, w), \vartheta_i\big)\Big] + \omega\big(\hat{\sigma}(i_1\varepsilon, w), \vartheta_i\big)\right| \\
&\quad = \frac{1}{i_2 - i_1}\left|\sum_{i \notin \mathcal{J}} \Big(\hat{\sigma}(i_1\varepsilon, w) - \hat{\sigma}(i\varepsilon, w)\Big) + \sum_{i \in \mathcal{J}} \Big(\hat{\sigma}(i_1\varepsilon, w) - \hat{\sigma}(i\varepsilon, w) + a_i\Big)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i \notin \mathcal{K}} \Big(- \hat{\sigma}(i_1\varepsilon, w)\Big) + \sum_{i \in \mathcal{K}} \Big(1 - \hat{\sigma}(i_1\varepsilon, w)\Big)\Bigg| \\
&\quad \text{(Here } a_i \text{ is a number in } \{-1,0,1\}) \\
&= \frac{1}{i_2 - i_1}\Bigg| \sum_{i \notin \mathcal{J}} \Big(\hat{\sigma}(i_1\varepsilon, w) - \hat{\sigma}(i\varepsilon, w)\Big) + \sum_{i \in \mathcal{J}} \Big(\hat{\sigma}(i_1\varepsilon, w) - \hat{\sigma}(i\varepsilon, w) + a_i\Big) \\
&\quad - \hat{\sigma}(i_1\varepsilon, w)(i_2 - i_1) + \sharp\mathcal{K}\Bigg| \\
&\le \frac{1}{i_2 - i_1}\left(\sum_{i=i_1}^{i_2-1} \Big|\hat{\sigma}(i_1\varepsilon, w) - \hat{\sigma}(i\varepsilon, w)\Big| + \sharp\mathcal{J} + \Big|\sharp\mathcal{K} - \hat{\sigma}(i_1\varepsilon, w)(i_2 - i_1)\Big|\right) \\
&\le \left(2\Big|\hat{\sigma}^{\max} - \hat{\sigma}^{\min}\Big| + \left|\frac{\sharp\mathcal{J}}{i_2 - i_1} - \big(\hat{\sigma}^{\max} - \hat{\sigma}^{\min}\big)\right| + \left|\frac{\sharp\mathcal{K}}{i_2 - i_1} - \hat{\sigma}(i_1\varepsilon, w)\right|\right) \\
&\quad \text{(using (1.10))} \\
&\le \mathcal{O}(1)\left[\text{Tot.Var.}\left(\hat{\sigma}(\cdot, w); \left(t_1, t_2 + \frac{\varepsilon}{2}\right)\right) + \frac{1 + \log(i_2 - i_1)}{i_2 - i_1}\right]. \qquad (3.5)
\end{aligned}
$$

Using (3.4), (3.5), Corollary 2.24 and Theorem 3.3 we thus get

$$
\begin{aligned}
&\int_{\Psi_k\left(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)\right)} \Big|\check{\lambda}\big(\Psi^{-1}(\tau)\big) - \check{\sigma}\big(t, \Psi^{-1}(\tau)\big)\Big| d\tau \\
&\le \mathcal{O}(1) \int_{\Psi_k\left(\mathcal{W}_k(t_1)\cap\mathcal{W}_k(t_2)\right)} \Bigg\{ \frac{1 + \log(i_2 - i_1)}{i_2 - i_1} \\
&\quad + \text{Tot.Var.}\left(\hat{\sigma}\big(\cdot, \Psi_k^{-1}(\tau)\big); \left(t_1, t_2 + \frac{\varepsilon}{2}\right)\right) \\
&\quad + \Big|\hat{\sigma}\big(t_2, \Psi_k^{-1}(\tau)\big) - \check{\sigma}\big(t_2, \Psi_k^{-1}(\tau)\big)\Big| + \text{Tot.Var.}\Big(\check{\sigma}\big(\cdot, \Psi_k^{-1}(\tau)\big); \big(t_1, t_2\big)\Big)\Bigg\} d\tau \\
&\le \mathcal{O}(1)\left\{ \frac{1 + \log(i_2 - i_1)}{i_2 - i_1} + \Upsilon(t_1) - \Upsilon(t_2)\right\}.
\end{aligned}
$$

Therefore, using (3.3), integrating over all times $t \in [i_1\varepsilon, i_2\varepsilon]$ we get the conclusion. □

Proposition 3.6 (Estimate (1.17c))**.** *It holds*

$$
\big\|\psi(t_1) - u^{\varepsilon}(t_1)\big\|_1 \le \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big)(t_2 - t_1).
$$

Proof. Fix any $x \in \mathbb{R}$. Consider the segment on the (t,x)-plane joining (t_1, x) and $(t_2, x-(t_2-t_1))$. Assume that $x \notin \mathbb{Z}\varepsilon$ and that no non-physical wavefront travels on this segment (this holds for all but finitely many $x \in \mathbb{R}$). Define the set of k-waves which cross this segment in u^ε and in ψ respectively:

$$\begin{aligned}\mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) \;&:=\; \Big\{ w \in \mathcal{W}_k \;\big|\; \text{there exists } t =: \mathtt{t}^{\text{cross}}(u^\varepsilon, x, w) \in (t_1, t_2) \\ &\qquad \text{such that } \mathtt{x}(t,w) = x - (t - t_1) \Big\}, \\ \mathcal{W}_k^{\text{cross}}(\psi, x) \;&:=\; \Big\{ w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2) \;\big|\; \text{there exists} \\ &\qquad t =: \mathtt{t}^{\text{cross}}(\psi, x, w) \in (t_1, t_2) \\ &\qquad \text{such that } \mathtt{y}(t,w) = x - (t - t_1) \Big\}.\end{aligned}$$

Since, for any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$, $\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w)$ and $\mathtt{x}(t_2, w) = \mathtt{y}(t_2, w)$,

$$\mathcal{W}_k^{\text{cross}}(\psi, x) = \mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) \cap \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2).$$

Moreover, if a k-wave $w \in \mathcal{W}_k^{\text{cross}}(\psi, x)$, then its position at time t_1 must be

$$\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w) \in \big[x - 2(t_2 - t_1), x\big],$$

while if $w \in \mathcal{W}_k^{\text{cross}}(u^\varepsilon, x) \setminus \mathcal{W}_k^{\text{cross}}(\psi, x)$, then either it is created at some grid point in the triangle

$$\Delta^{\text{cr}}(x) := \Big[\big(t_1, x - 2(t_2 - t_1)\big), \big(t_2, x - (t_2 - t_1)\big), \big(t_1, x\big) \Big]$$

or it is canceled at some grid point in the triangle

$$\Delta^{\text{canc}}(x) := \Big[\big(t_2, x - (t_2 - t_1)\big), \big(t_1, x\big), \big(t_2, x + (t_2 - t_1)\big) \Big].$$

Since $\psi(t_2) = u^\varepsilon(t_2)$, we can now write

$$\begin{aligned}&\big|\psi(t_1, x) - u^\varepsilon(t_1, x)\big| \\ &= \Big| \Big[\psi\big(t_1, x\big) - \psi\big(t_2, x - (t_2 - t_1)\big) \Big] - \Big[u^\varepsilon\big(t_1, x\big) - u^\varepsilon\big(t_2, x - (t_2 - t_1)\big) \Big] \Big| \\ &= \Bigg| \sum_{k=1}^{n} \int_{\Psi_k\left(\mathcal{W}_k^{\text{cross}}(\psi, x)\right)} \Big\{ \check{r}_k\Big(\mathtt{t}^{\text{cross}}\big(\psi, x, \Psi_k^{-1}(\tau)\big), \Psi_k^{-1}(\tau) \Big) \\ &\quad - \hat{r}_k\Big(\mathtt{t}^{\text{cross}}\big(u^\varepsilon, x, \Psi^{-1}(\tau)\big), \Psi^{-1}(\tau) \Big) \Big\} d\tau \Bigg|\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}(1)\Big\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{cr}}(x)}}\mathtt{A}^{\mathrm{cr}}(i\varepsilon,m\varepsilon)+\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{canc}}(x)}}\mathtt{A}^{\mathrm{canc}}(i\varepsilon,m\varepsilon)\Big\}\\
\leq\ &\sum_{k=1}^{n}\int_{\Psi_k\left(\mathcal{W}_k^{\mathrm{cross}}(\psi,x)\right)}\Big\{\Big|\check r_k\Big(\mathtt{t}^{\mathrm{cross}}\big(\psi,x,\Psi_k^{-1}(\tau)\big),\Psi_k^{-1}(\tau)\Big)-\check r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)\Big|\\
&+\Big|\check r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)-\hat r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)\Big|\\
&+\Big|\hat r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)-\hat r_k\Big(\mathtt{t}^{\mathrm{cross}}\big(u^{\varepsilon},x,\Psi^{-1}(\tau)\big),\Psi^{-1}(\tau)\Big)\Big|\Big\}d\tau\\
&+\mathcal{O}(1)\Big\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{cr}}(x)}}\mathtt{A}^{\mathrm{cr}}(i\varepsilon,m\varepsilon)+\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{canc}}(x)}}\mathtt{A}^{\mathrm{canc}}(i\varepsilon,m\varepsilon)\Big\}\\
\leq\ &\sum_{k=1}^{n}\int_{\Psi_k\left(\mathtt{x}^{-1}([x-2(t_2-t_1),x])\right)}\Big\{\Big|\mathrm{Tot.Var.}\Big(\check r_k\big(\cdot,\Psi_k^{-1}(\tau)\big);(t_1,t_2)\Big)\\
&+\Big|\check r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)-\hat r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)\Big|\\
&+\Big|\mathrm{Tot.Var.}\Big(\hat r_k\big(\cdot,\Psi_k^{-1}(\tau)\big);(t_1,t_2)\Big)\Big|\Big\}d\tau\\
&+\mathcal{O}(1)\Big\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{cr}}(x)}}\mathtt{A}^{\mathrm{cr}}(i\varepsilon,m\varepsilon)+\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{canc}}(x)}}\mathtt{A}^{\mathrm{canc}}(i\varepsilon,m\varepsilon)\Big\}.
\end{aligned}
$$

Hence, integrating over all $x\in\mathbb{R}$, we get

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\big|\psi(t_1,x)-u^{\varepsilon}(t_1,x)\big|dx\\
&\leq\int_{-\infty}^{+\infty}\Big\{\sum_{k=1}^{n}\int_{\Psi_k\left(\mathtt{x}^{-1}([x-2(t_2-t_1),x])\right)}\Big[\Big|\mathrm{Tot.Var.}\Big(\check r_k\big(\cdot,\Psi_k^{-1}(\tau)\big);(t_1,t_2)\Big)\\
&\quad+\Big|\check r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)-\hat r_k\Big(t_2,\Psi_k^{-1}(\tau)\Big)\Big|\\
&\quad+\Big|\mathrm{Tot.Var.}\Big(\hat r_k\big(\cdot,\Psi_k^{-1}(\tau)\big);(t_1,t_2)\Big)\Big|\Big]d\tau\\
&\quad+\mathcal{O}(1)\Big\{\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{cr}}(x)}}\mathtt{A}^{\mathrm{cr}}(i\varepsilon,m\varepsilon)+\sum_{\substack{(i,m)\in\mathbb{N}\times\mathbb{Z}\\(i\varepsilon,m\varepsilon)\in\Delta^{\mathrm{canc}}(x)}}\mathtt{A}^{\mathrm{canc}}(i\varepsilon,m\varepsilon)\Big\}\Big\}dx
\end{aligned}
$$

(using Fubini's Theorem and Corollaries 2.24 and 3.4)

$$\leq \mathcal{O}(1)\Big[\Upsilon(t_1) - \Upsilon(t_2)\Big](t_2 - t_1),$$

which is what we wanted to get. □

4. Analysis of the Interactions in ψ

In this and next section we prove Theorem 3.3. We will follow the same technique we used in [3]. In particular this section is devoted to study the *local* part of the theorem: we introduce a suitable notion of amount of interaction and we prove that at any interaction the variation of $\hat{u}_k, \hat{v}_k, \hat{\sigma}_k$ is bounded by such amount of interaction.

In the next section, we will prove the *global* part of the theorem, i.e. that the sum of all the amounts of interactions is bounded by the decrease of Υ in the time interval $[t_1, t_2]$.

The crucial point is that the new definition of the functional $\mathfrak{Q}$ we gave in Section 2.6 is the one we need to prove Theorem 3.3, as we will see in the next section.

4.1. Amounts of interaction at the final time t_2

Instead of defining immediately the amounts of interactions at any point $(i_2\varepsilon, r\varepsilon)$, $r \in \mathbb{Z}$, it is more convenient (to avoid too heavy notations) to consider first a more abstract situation, and then apply it to our analysis.

Fix a left state u^L, a right state u^R and a collection of A vectors

$$\mathbf{s}^a = (s_1^a, \ldots, s_n^a) \in \mathbb{R}^n, \quad a = 0, 1, \ldots, A.$$

The Riemann problem (u^L, u^R) is solved by the collection of curves $\{\gamma_k\}_{k=1,\ldots,n}$, where

$$\gamma_k : \mathbf{I}(s_k) \to \mathcal{D} \subseteq \mathbb{R}^{n+2}, \quad \gamma_k = (u_k, v_k, \sigma_k),$$

and denote by $f_k : \mathbf{I}(s_k) \to \mathbb{R}$ the associated reduced fluxes.

Assume that for any fixed $k = 1, \ldots, n$,

- all the s_k^a, $a \in \{1, \ldots, A\}$, and s_k have the same sign;

- $\Big|\sum_{a=1}^A s_k^a\Big| \leq |s_k|$.

Observe that our assumptions describe precisely the collisions taking place at any point $(i_2\varepsilon, m\varepsilon)$, $m \in \mathbb{Z}$.

Set $I_k^a := \sum_{b<a} s_k^b + \mathbf{I}(s_k^a)$. Let $\Theta_k : \mathbf{I}\big(\sum_{a=1}^A s_k^a\big) \to \mathbf{I}(s_k)$ be any increasing map such that for each $a = 0, 1, \dots, A$, $\Theta_k|_{I_k^a}$ is an affine map with slope equal to 1. Denote by Θ_k^{-1} its pseudo-inverse, which turns out to be a continuous map. Set $J_k^a := \big\{\tau \in \mathbf{I}(s_k) \bigm| \Theta_k^{-1}(\tau) \in I_k^a\big\}$.

Set $u^0 := u^L$ and for any $a = 1, \dots, A$,

$$u^a := T^n_{s_n^a} \circ \cdots \circ T^1_{s_1^a} u^{a-1}.$$

Assume that the Riemann problem (u^{a-1}, u^a) is solved by the collection of curves $\{\gamma_k^a\}_{k=1,\dots,n}$, with $\gamma_k^a = (u_k^a, v_k^a, \sigma_k^a)$. Assume moreover that, for any k and a, γ_k^a is defined on I_k^a.

We can now define:

- the *transversal amount of interaction* as

$$\mathtt{B}^{\text{trans}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R) := \sum_{a=0}^{A} \sum_{b=a+1}^{A} \sum_{k=1}^{n} \sum_{h=1}^{k-1} |s_k^a||s_h^b|;$$

- the *quadratic amount of interaction of the k-th family* as

$$\begin{aligned} &\mathtt{B}_k^{\text{quadr}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R) \\ &:= \begin{cases} \Big\| \frac{d}{d\tau}\text{conv}_{\,\mathbf{I}(s_k)} f_k - \bigcup_{a=0}^A \frac{d}{d\tau}\text{conv}_{\,J_k^a} f_k \Big\|_1 & \text{if } s_k \geq 0, \\ \Big\| \frac{d}{d\tau}\text{conc}_{\,\mathbf{I}(s_k)} f_k - \bigcup_{a=0}^A \frac{d}{d\tau}\text{conc}_{\,I_k^a} f_k \Big\|_1 & \text{if } s_k < 0; \end{cases} \end{aligned}$$

- the *amount of creation of the k-th family* as

$$\mathtt{B}_k^{\text{cr}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R) := \Big| s_k - \sum_{a=1}^{A} s_k^a \Big|;$$

- the *global amount of interaction* as

$$\mathtt{B}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R)$$

$$:= \mathtt{B}^{\text{trans}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R) + \sum_{k=1}^{n} \Big[\mathtt{B}_k^{\text{quadr}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R) + \mathtt{B}_k^{\text{cr}}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R)\Big].$$

We have used the letter $\mathtt{B}$ instead of $\mathtt{A}$ to distinguish these amounts of interaction from the amounts of interactions concerning two merging Riemann problems, already introduced in Section 2.2.

Proposition 4.1. *For any $k = 1, \dots, n$, the following inequalities hold*

$$\left.\begin{array}{l} \left\| \bigcup_{a=1}^{A} u_k^a - u_k \circ \Theta_k - \big(u_k^1(0) - u_k(0)\big) \right\|_\infty \\ \left\| \bigcup_{a=1}^{A} v_k^a - v_k \circ \Theta_k \right\|_\infty \\ \left\| \bigcup_{a=1}^{A} \sigma_k^a - \sigma_k \circ \Theta_k \right\|_1 \end{array}\right\} \leq \mathcal{O}(1)\mathtt{B}(u^L, \mathbf{s}_1, \dots, \mathbf{s}_A, u^R).$$

The proof can be achieved using the same techniques as in [3, Section 3] and for this reason it is omitted here.

Recall now the definition of $\check{s}_k^{m\rightsquigarrow r}$ in (3.1) and define the vector

$$\check{\mathbf{s}}^{m\rightsquigarrow r} := \big(\check{s}_1^{m\rightsquigarrow r}, \dots, \check{s}_n^{m\rightsquigarrow r}\big).$$

Applying the previous definitions to the collisions taking place at time $t_2 = i_2\varepsilon$, we can define, for any $r \in \mathbb{Z}$,

$$\begin{aligned} \mathtt{B}^{\text{trans}}(i_2\varepsilon, r\varepsilon) &:= \mathtt{B}^{\text{trans}}(u^{i_2,r-1}, \check{\mathbf{s}}^{r-(i_2-i_1)\rightsquigarrow r}, \dots, \check{\mathbf{s}}^{r\rightsquigarrow r}, u^{i_2,r}), \\ \mathtt{B}_k^{\text{quadr}}(i_2\varepsilon, r\varepsilon) &:= \mathtt{B}_k^{\text{quadr}}(u^{i_2,r-1}, \check{\mathbf{s}}^{r-(i_2-i_1)\rightsquigarrow r}, \dots, \check{\mathbf{s}}^{r\rightsquigarrow r}, u^{i_2,r}), \quad k = 1, \dots, n \\ \mathtt{B}^{\text{cr}}(i_2\varepsilon, r\varepsilon) &:= \mathtt{B}_k^{\text{cr}}(u^{i_2,r-1}, \check{\mathbf{s}}^{r-(i_2-i_1)\rightsquigarrow r}, \dots, \check{\mathbf{s}}^{r\rightsquigarrow r}, u^{i_2,r}), \quad k = 1, \dots, n, \\ \mathtt{B}(i_2\varepsilon, r\varepsilon) &:= \mathtt{B}(u^{i_2,r-1}, \check{\mathbf{s}}^{r-(i_2-i_1)\rightsquigarrow r}, \dots, \check{\mathbf{s}}^{r\rightsquigarrow r}, u^{i_2,r}). \end{aligned}$$

Applying Proposition 4.1, we obtain the following corollary.

Corollary 4.2. *It holds*

$$\left.\begin{array}{l}\left\|\big(\check{u}_k(t_2-,\cdot)-\hat{u}_k(t_2,\cdot)\big)\circ\Psi_k^{-1}-\big(\check{u}_k(t_2-,0)-\hat{u}_k(t_2,0)\big)\right\|_{L^\infty\left([-\check{L}_k^-,\check{L}_k^+]\right)}\\ \left\|\big(\check{v}_k(t_2-,\cdot)-\hat{v}_k(t_2,\cdot)\big)\circ\Psi_k^{-1}\right\|_{L^\infty\left([-\check{L}_k^-,\check{L}_k^+]\right)}\\ \left\|\big(\check{\sigma}_k(t_2-,\cdot)-\hat{\sigma}_k(t_2,\cdot)\big)\circ\Psi_k^{-1}\right\|_{L^1\left([-\check{L}_k^-,\check{L}_k^+]\right)}\end{array}\right\}$$
$$\leq \mathcal{O}(1)\sum_{r\in\mathbb{Z}}\mathtt{B}(i_2\varepsilon,r\varepsilon).$$

4.2. Amounts of interaction at times $t\in(t_1,t_2)$

Let $t\in(t_1,t_2)$ and let (t,x) be a point where two wavefronts collide. As in Section 3.1, we have to distinguish two cases.

Case 1*: both the colliding wavefronts are physical.* Assume that before the collision the first wavefront is traveling with speed λ' and it is connecting the states

$$\psi^M=T^n_{s'_{\bar{k}}}\circ\cdots\circ T^1_{s'_1}\psi^L,$$

while the second wavefront is traveling with speed $\lambda'<\lambda''$ and it is connecting the states

$$\psi^R=T^n_{s''_n}\circ\cdots\circ T^1_{s''_{\bar{k}}}\psi^M.$$

We have already observed that the interaction at $(\bar{t},\bar{x})$ is purely transversal. Define thus the *(transversal) amount of interaction* at (t,x) as

$$\mathtt{B}^{\mathrm{trans}}(t,x):=\sum_{k=1}^{\bar{k}}\sum_{h=\bar{k}+1}^{n}|s'_k||s''_h|.$$

Case 2*: one of the two colliding wavefronts is non-physical.* Assume that the non-physical wavefront α is connecting ψ^L with ψ^M, while the physical wavefront is traveling with speed λ and it is connecting

$$\psi^R=T^n_{s_n}\circ\cdots\circ T^1_{s_1}\psi^M.$$

Also in this case the interaction is purely transversal. Define thus the *amount* *of interaction* at (t,x) as

$$\mathtt{B}(t,x) := \mathtt{B}^{\text{trans}}(t,x) := s(t+,\alpha)\sum_{k=1}^{n}|s_k| = |\psi^M - \psi^L|\sum_{k=1}^{n}|s_k|.$$

The following proposition covers both the case of a collision between physical wavefronts and the case of a collision between a physical and a non-physical wavefront.

Proposition 4.3. *The following hold.*

(1) *For any $k = 1,\dots,n$, for the k-physical waves $\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k$ located at (t,x) in the wavefront map ψ, we have*

$$\left.\begin{array}{l}\Big\|\big(\check{u}_k(t+,\cdot)-\check{u}_k(t-,\cdot)\big)\circ\Psi_k^{-1}-\big(\check{u}_k(t+,0)-\hat{u}_k(t-,0)\big)\Big\|_{L^\infty\left(\Psi_k(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)\right)}\\ \Big\|\big(\check{v}_k(t+,\cdot)-\check{v}_k(t-,\cdot)\big)\circ\Psi_k^{-1}\Big\|_{L^\infty\left(\Psi_k(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)\right)}\\ \Big\|\big(\check{\sigma}_k(t+,\cdot)-\check{\sigma}_k(t-,\cdot)\big)\circ\Psi_k^{-1}\Big\|_{L^1\left(\Psi_k(\mathtt{y}(t)^{-1}(x)\cap\mathcal{W}_k)\right)}\end{array}\right\}$$
$$\leq \mathcal{O}(1)\mathtt{B}^{\text{trans}}(t,x).$$

(2) *If both wavefronts interacting at (t,x) are physical, denoting by α the non-physical wavefront generated at (t,x), its initial strength can be estimated by*

$$\big|s\big(\mathtt{t}^{\text{cr}}(\alpha),\alpha\big)\big| \leq \mathcal{O}(1)\mathtt{B}^{\text{trans}}(t,x).$$

(3) *If one of the two wavefronts interacting at (t,x) is a non-physical wavefront α, the variation of the strength of α can be estimated by*

$$|s(t+,\alpha)-s(t-,\alpha)| \leq \mathcal{O}(1)\mathtt{B}^{\text{trans}}(t,x).$$

The proof of this proposition can again be obtained with the same techniques as in [3, Section 3], and thus it is omitted here.

5. Estimates on the Amounts of Interaction in ψ

In this section we prove the following theorem, which is the *global* part of the proof of Theorem 3.3. The proof of this theorem is the last step in order to complete the proof of the convergence rate of the Glimm scheme, Theorem 1.3.

Theorem 5.1. *The sum of all amounts of interaction in the time interval* $(t_1, t_2]$ *is bounded by the decrease of the functional* Υ *in the same time interval, i.e.*

$$\sum_{r\in\mathbb{Z}} \mathtt{B}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x)\ \text{int. pt.}\\ t\in(t_1,t_2)}} \mathtt{B}^{\text{trans}}(t,x) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

The proof is a direct consequence of the following three propositions.

Proposition 5.2 (Transversal amounts of interactions)**.** *It holds*

$$\sum_{r\in\mathbb{Z}} \mathtt{B}^{\text{trans}}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x)\ \text{int. pt.}\\ t\in(t_1,t_2)}} \mathtt{B}^{\text{trans}}(t,x) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

Proof. Since for any wave $w \in \mathcal{W}_k(t_1) \cap \mathcal{W}_k(t_2)$,

$$\mathtt{x}(t_1, w) = \mathtt{y}(t_1, w), \qquad \mathtt{x}(t_2, w) = \mathtt{y}(t_2, w),$$

and thus the waves which have to cross in ψ also cross in u^ε, it is not difficult to see that

$$\begin{aligned} \sum_{r\in\mathbb{Z}} \mathtt{B}^{\text{trans}}(i_2\varepsilon, r\varepsilon) + \sum_{\substack{(t,x)\ \text{int. pt.}\\ t\in(t_1,t_2)}} \mathtt{B}^{\text{trans}}(t,x) &\leq \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon) \\ \text{(by (2.20))} &\leq \mathcal{O}(1)\big(\Upsilon(i_2\varepsilon) - \Upsilon(t_1)\big), \end{aligned}$$

which is what we wanted to prove. □

Proposition 5.3 (Amounts of creation)**.** *It holds*

$$\sum_{r\in\mathbb{Z}} \mathtt{B}^{\text{cr}}_k(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big).$$

Proof. It is fairly easy to see that

$$\sum_{r\in\mathbb{Z}} \mathtt{B}^{\text{cr}}_k(i_2\varepsilon, r\varepsilon) \leq \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}^{\text{cr}}(i_2\varepsilon, m\varepsilon),$$

and thus, again using (2.20), we get the conclusion. □

Proposition 5.4 (Quadratic amounts of interaction)**.** *It holds*

$$\sum_{r \in \mathbb{Z}} \mathtt{B}_k^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1)\big(\Upsilon(t_1) - \Upsilon(t_2)\big). \tag{5.1}$$

The proof of this proposition is much more difficult than the previous two. However, the technique we will use is the same we used in [3] to prove estimate (2.19) on the decreasing part of the functional $\mathfrak{Q}(t)$. Here, however, the new definition of the functional $\mathfrak{Q}(t)$ we presented in Section 2.6 plays a crucial role, since, with the old definition (the one in [3]), the decrease of $\mathfrak{Q}$ in the time interval $[t_1, t_2]$ is not big enough to prove (5.1).

Proof. Introduce first the following sets:

$$\begin{aligned}
\mathcal{E}_r :=& \Big\{(w, w') \in \mathcal{W}_k(i_2\varepsilon, r\varepsilon) \times \mathcal{W}_k(i_2\varepsilon, r\varepsilon) \ \Big|\ w < w', \\
& \qquad\qquad \mathtt{x}(t_1, w) < \mathtt{x}(t_1, w')\Big\}, r \in \mathbb{Z}, \\
\mathcal{F}_r :=& \Big\{(w, w') \in \mathcal{W}_k(i_2\varepsilon, r\varepsilon) \times \mathcal{W}_k(i_2\varepsilon, r\varepsilon) \ \Big|\ w < w', \\
& \qquad\qquad \max\big\{\mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{cr}}(w')\big\} > t_1\Big\}, r \in \mathbb{Z},
\end{aligned} \tag{5.2}$$

$$\mathcal{E} := \bigcup_{r \in \mathbb{Z}} \mathcal{E}_r, \qquad \mathcal{F} := \bigcup_{r \in \mathbb{Z}} \mathcal{F}_r, \tag{5.3}$$

$$\mathcal{E}^i := \Big\{(w, w') \in \mathcal{E} \ \Big|\ \mathtt{t}^{\mathrm{int}}(t_1, w, w') = i\varepsilon\Big\}, \quad i = i_1 + 1, \dots, i_2.$$

We need now the following four lemmas, which conclude the proof of the proposition.

Lemma 5.5. *For any $r \in \mathbb{Z}$,*

$$\mathtt{B}_k^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon) \leq \mathcal{O}(1) \iint_{(\Psi_k \times \Psi_k)(\mathcal{E}_r \cup \mathcal{F}_r)} \mathfrak{q}_k\Big(t_1, t_2, t_2, \Psi_k^{-1}(\tau), \Psi_k^{-1}(\tau')\Big) d\tau d\tau'.$$

Proof. We assume for the sake of simplicity that the k-waves interacting at $(i_2\varepsilon, r\varepsilon)$ are positive, the negative case being completely similar. We divide the proof in several steps.

Step 1. Set $u^L := u^{i_2, r-1}$, $u^R := u^{i_2, r}$ and

$$s_k^a := s_k^{r-(i_2-i_1)+a}$$

for any $a = 0, 1, \dots, i_2 - i_1 =: A$. As in Section 4.1, let

$$\mathbf{s}^a := (s_1^a, \dots, s_n^a),$$

let $\{\gamma_k\}_{k=1,\dots,n}$, $\gamma_k : \mathbf{I}(s_k) \to \mathcal{D} \subseteq \mathbb{R}^{n+2}$ be the collection of curves which solve the Riemann problem (u^L, u^R) and let f_k be the associated reduced flux. Define also

$$\Theta_k := \Phi_k(t_2) \circ \Psi_k^{-1}|_{\Psi_k\big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon) \cap \mathcal{W}_k(t_1)\big)}.$$

It is not difficult to see that there exists two real numbers $\zeta, \zeta' \in \mathbb{R}$ such that

$$\begin{aligned}\Psi_k\Big(\mathcal{W}_k\big(i_2\varepsilon, r\varepsilon\big) \cap \mathcal{W}_k\big(i_1\varepsilon, (r - (i_2 - i_1) + a)\varepsilon\big)\Big) &= \zeta + \sum_{b<a} s_k^b + \mathbf{I}(s_k^a) =: I_k^a, \\ \Phi_k(t_2)\big(\mathcal{W}_k(i_2\varepsilon, r\varepsilon)\big) &= \zeta' + \mathbf{I}(s_k),\end{aligned}$$

and

$$\Theta_k : \zeta + \mathbf{I}\Big(\sum_{a=1}^{A} s_k^a\Big) \to \zeta' + \mathbf{I}(s_k)$$

is an increasing map and for each $a = 0, 1, \dots, A$ the restriction $\Theta_k|_{I_k^a}$ is an affine map with slope equal to 1. We are thus exactly in the situation described in Section 4.1 and therefore we can define the intervals $J_k^a := \big\{\tau \in \zeta' + \mathbf{I}(s_k) \;\big|\; \Theta_k^{-1}(\tau) \in I_k^a\big\}$. Notice, moreover, that the effective flux $\mathtt{f}_k^{\mathrm{eff}}(t_2)$ at time t_2 and the flux f_k associated to the Riemann problem (u^L, u^R) coincide up to affine functions, i.e.

$$\frac{d^2}{d\tau^2}\mathrm{conv}_{\,\zeta'+\mathbf{I}(s_k)}\mathtt{f}_k^{\mathrm{eff}}(t_2)(\zeta' + \tau) = \frac{d^2}{d\tau^2}\mathrm{conv}_{\,\mathbf{I}(s_k)} f_k(\tau), \quad \tau \in \mathbf{I}(s_k).$$

Hence, by the properties of the convex envelope, we can compute the quadratic amount of interaction $\mathtt{B}^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon)$ using the effective flux $\mathtt{f}_k^{\mathrm{eff}}(t_2)$ instead of f_k:

$$\mathtt{B}_k^{\mathrm{quadr}}(i_2\varepsilon, r\varepsilon) := \left\| \frac{d}{d\tau}\mathrm{conv}_{\bigcup_{a=0}^{A} J_k^a}\mathtt{f}_k^{\mathrm{eff}}(t_2) - \bigcup_{a=0}^{A} \frac{d}{d\tau}\mathrm{conv}_{\,J_k^a}\mathtt{f}_k^{\mathrm{eff}}(t_2) \right\|_1.$$

By triangular inequality, it is enough to prove that for any $b = 1, \dots, A$,

$$\left\| \frac{d}{d\tau}\mathrm{conv}_{\bigcup_{a=0}^{b} J_k^a}\mathtt{f}_k^{\mathrm{eff}}(t_2) - \Big(\frac{d}{d\tau}\mathrm{conv}_{\bigcup_{a=0}^{b-1}}\mathtt{f}_k^{\mathrm{eff}}(t_2) \cup \frac{d}{d\tau}\mathrm{conv}_{\,J_k^b}\mathtt{f}_k^{\mathrm{eff}}(t_2) \Big) \right\|_1$$

$$\leq \iint_{(\bigcup_{a=0}^{b-1} J_k^a)\times J_k^b} \mathtt{q}_k\Big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)d\tau d\tau'. \tag{5.4}$$

The technique we use to prove (5.4) is the same as in [3, Proposition 6.9].

Step 2. Set

$$\tau_M := \sup \bigcup_{a=0}^{b-1} J_k^a = \inf J_k^b,$$

and

$$\begin{aligned}\tau_L &:= \max\Big\{\tau \in \bigcup_{a=0}^{b-1} J_k^a \;\Big|\; \mathrm{conv}_{\bigcup_{a=0}^{b-1} J_k^a} \mathtt{f}_k^{\mathrm{eff}}(t_2)(\tau) = \mathrm{conv}_{\bigcup_{a=0}^{b} J_k^a} \mathtt{f}_k^{\mathrm{eff}}(t_2)(\tau)\Big\},\\ \tau_R &:= \min\Big\{\tau \in J_k^b \;\Big|\; \mathrm{conv}_{J_k^b} \mathtt{f}_k^{\mathrm{eff}}(t_2)(\tau) = \mathrm{conv}_{\bigcup_{a=0}^{b} J_k^a} \mathtt{f}_k^{\mathrm{eff}}(t_2)(\tau)\Big\}.\end{aligned}$$

W.l.o.g. we assume that $\tau_L < \tau_M < \tau_R$, otherwise there is nothing to prove.

It is quite easy to see that

$$\begin{aligned}\mathtt{B}_k^{\mathrm{quadr}}(i\varepsilon, r\varepsilon) &= \frac{1}{\tau_R-\tau_L}\Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_L,\tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_M,\tau_R]\big)\Big]\\ &\quad \times\mathcal{L}^2\big((\tau_L,\tau_M]\times(\tau_M,\tau_R]\big),\end{aligned}$$

and thus it is sufficient to prove that

$$\begin{aligned}&\frac{1}{\tau_R-\tau_L}\Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_L,\tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_M,\tau_R]\big)\Big]\mathcal{L}^2\big((\tau_L,\tau_M]\times(\tau_M,\tau_R]\big)\\ &\leq \int_{\tau_L}^{\tau_M}\int_{\tau_M}^{\tau_R} \mathtt{q}_k\Big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)d\tau d\tau'.\end{aligned} \tag{5.5}$$

Observe that, by Proposition 2.16,

$$d\big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\big) \leq \tau_R - \tau_L;$$

hence (5.5) will follow if we prove that

$$\begin{aligned}&\Big[\sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_L,\tau_M]\big) - \sigma^{\mathrm{rh}}\big(\mathtt{f}_k^{\mathrm{eff}}(t_2),(\tau_M,\tau_R]\big)\Big]\mathcal{L}^2\big((\tau_L,\tau_M]\times(\tau_M,\tau_R]\big)\\ &\quad\leq \int_{\tau_L}^{\tau_M}\int_{\tau_M}^{\tau_R} \pi_k\big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\big)d\tau d\tau'.\end{aligned} \tag{5.6}$$

Step 3. Let

$$\mathcal{L} := \Psi_k^{-1}\big((\tau_L, \tau_M]\big), \qquad \mathcal{R} := \Psi_k^{-1}\big((\tau_M, \tau_R]\big).$$

We will identify waves through the equivalence relation $\bowtie$, already introduced in (2.7): for any couple of waves $w, w' \in \mathcal{L} \cup \mathcal{R}$, set $w \bowtie w'$ if and only if

$$\mathtt{t}^{\mathrm{cr}}(w) = \mathtt{t}^{\mathrm{cr}}(w') \text{ and } \mathtt{x}(t,w) = \mathtt{x}(t,w') \quad \text{for any } t \in \Big[\mathtt{t}^{\mathrm{cr}}(w), i\varepsilon\Big).$$

The sets

$$\widehat{\mathcal{L}} := \mathcal{L} \;/\bowtie, \qquad \widehat{\mathcal{R}} := \mathcal{R} \;/\bowtie$$

are finite and totally ordered by the order $\leq$ of $\mathcal{W}_k(t_2)$. Moreover for any $\xi \in \widehat{\mathcal{L}}$, $\xi' \in \widehat{\mathcal{R}}$, let $w \in \xi$, $w' \in \xi'$ and set

$$\mathcal{I}(t_1, t_2, \xi, \xi') \;:=\; \mathcal{I}(t_1, t_2, w, w'), \quad \mathcal{P}(t_1, t_2, \xi, \xi') := \mathcal{P}(t_1, t_2, w, w'),$$

and

$$\widehat{\mathcal{I}}(t_1, t_2, \xi, \xi') \;:=\; \mathcal{I}(t_1, t_2, \xi, \xi') \;/\bowtie.$$

It is not hard to see that the above definitions are well posed and that $\widehat{\mathcal{I}} \subseteq \widehat{\mathcal{L}} \cup \widehat{\mathcal{R}}$.

Now we partition the rectangle $\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$ in sub-rectangles, as follows. For any non empty rectangle $\widehat{\mathcal{C}} := \widehat{\mathcal{L}}_{\mathcal{C}} \times \widehat{\mathcal{R}}_{\mathcal{C}} \subseteq \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$, define (see Figure 3)

$$\Pi_0(\widehat{\mathcal{C}}) := \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big],$$

$$\Pi_1(\widehat{\mathcal{C}}) := \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big],$$

$$\Pi_2(\widehat{\mathcal{C}}) := \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big],$$

$$\Pi_3(\widehat{\mathcal{C}}) := \Big[\widehat{\mathcal{L}}_{\mathcal{C}} \setminus \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big] \times \Big[\widehat{\mathcal{R}}_{\mathcal{C}} \cap \widehat{\mathcal{I}}(t_1, t_2, \max \widehat{\mathcal{L}}_{\mathcal{C}}, \min \widehat{\mathcal{R}}_{\mathcal{C}})\Big].$$

Clearly $\Big\{\Pi_0(\widehat{\mathcal{C}}), \Pi_1(\widehat{\mathcal{C}}), \Pi_2(\widehat{\mathcal{C}}), \Pi_3(\widehat{\mathcal{C}})\Big\}$ is a disjoint partition of $\widehat{\mathcal{C}}$.

For any set A, denote by $A^{<\mathbb{N}}$ the set of all finite sequences taking values in A. We assume that $\emptyset \in A^{<\mathbb{N}}$, called the *empty sequence*. There is a natural

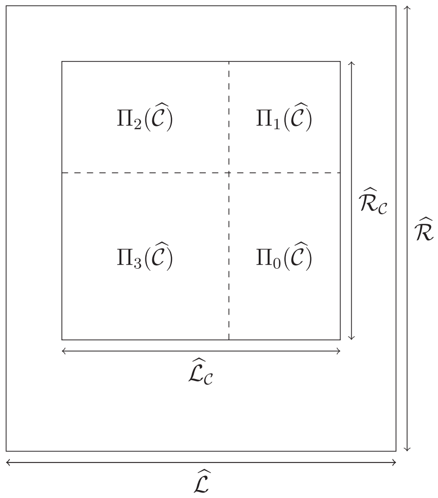

Figure 3: Partition of $\widehat{C} := \widehat{\mathcal{L}}_{\mathcal{C}} \times \widehat{\mathcal{R}}_{\mathcal{C}}$.

ordering $\trianglelefteq$ on $A^{<\mathbb{N}}$: given $\alpha, \beta \in A^{<\mathbb{N}}$,

$$\alpha \trianglelefteq \beta \quad \Longleftrightarrow \quad \beta \text{ is obtained from } \alpha \text{ by adding a finite sequence.}$$

A subset $D \subseteq A^{<\mathbb{N}}$ is called a *tree* if for any $\alpha, \beta \in A^{<\mathbb{N}}$, $\alpha \trianglelefteq \beta$, if $\beta \in D$, then $\alpha \in D$.

Define a map $\widehat{\Psi} : \{0,1,2,3\}^{<\mathbb{N}} \longrightarrow 2^{\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}}$, by setting

$$\widehat{\Psi}_\alpha = \begin{cases} \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}, & \text{if } \alpha = \emptyset, \\ \Pi_{z_n} \circ \cdots \circ \Pi_{z_1}(\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}), & \text{if } \alpha = (z_1, \dots, z_L) \in \{0,1,2,3\}^{<\mathbb{N}} \setminus \{\emptyset\}. \end{cases}$$

For $\alpha \in \{0,1,2,3\}^{<\mathbb{N}}$, let $\widehat{\mathcal{L}}_\alpha, \widehat{\mathcal{R}}_\alpha$ be defined by the relation $\widehat{\Psi}_\alpha = \widehat{\mathcal{L}}_\alpha \times \widehat{\mathcal{R}}_\alpha$. Define a tree D in $\{0,1,2,3\}^{<\mathbb{N}}$ setting

$$D := \{\emptyset\} \ \cup \ \bigg\{ \alpha = (z_1, \dots, z_L) \in \{0,1,2,3\}^{<\mathbb{N}} \ \bigg| \ L \in \mathbb{N}, \\ \widehat{\Pi}_\alpha \neq \emptyset, \ z_l \neq 0 \text{ for } l = 1, \dots, L-1 \bigg\}.$$

See Figure 4.

Since $\Pi_0(\Pi_0(\widehat{\mathcal{C}})) = \Pi_0(\widehat{\mathcal{C}})$ for any $\widehat{\mathcal{C}} \subseteq \widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$, this implies, together with the fact that $\widehat{\mathcal{L}} \times \widehat{\mathcal{R}}$ is a finite set, that D is a finite tree.

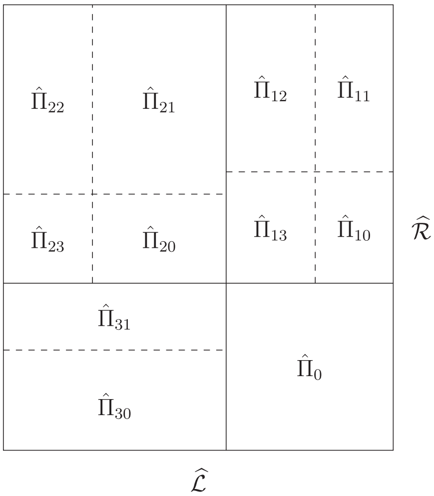

Figure 4: Partition of $\mathcal{L} \times \mathcal{R}$ using the tree D.

For any $\alpha \in D$, set

$$\mathcal{L}_\alpha := \bigcup_{\xi \in \widehat{\mathcal{L}}_\alpha} \xi, \qquad \mathcal{R}_\alpha := \bigcup_{\xi' \in \widehat{\mathcal{R}}_\alpha} \xi',$$
$$L_\alpha := \Psi_k(\mathcal{L}_\alpha), \qquad R_\alpha := \Psi_k(\mathcal{R}_\alpha).$$

The idea of the proof is to show that, for each $\alpha \in D$, on the rectangle $L_\alpha \times R_\alpha$ it holds

$$\begin{aligned} &\big[\sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), L_\alpha) - \sigma^{\mathrm{rh}}(\mathtt{f}_k^{\mathrm{eff}}(t_2), R_\alpha)\big]\mathcal{L}^2(L_\alpha \times R_\alpha) \\ &\leq \int_{L_\alpha \times R_\alpha} \pi_k(t_1, t_2, \tau, \tau')d\tau d\tau'. \end{aligned} \tag{5.7}$$

The conclusion will follow just considering that $\emptyset \in D$ and $L_\emptyset = (\tau_L, \tau_M]$, $R_\emptyset = (\tau_M, \tau_R]$.

Step 4. Using Propositions 2.16, 2.18, 2.19, it is possible to prove that 5.7 holds for each $\alpha = (z_1, \dots z_L) \in D$ such that $z_L = 0$.

This is a major part of the proof, in which the partitions $\mathcal{P}(t_1, t_2, w, w')$ are widely used, but we don't prove this step explicitly, since its proof can be obtained adapting the proofs of [3, Lemmas 6.10-6.11].

Step 5. We prove now that (5.7) holds for any $\alpha \in D$ by (inverse) induction on the tree. If α is a leaf of the tree, then, by definition, the last component of α is equal to zero, and thus (5.7) has already been proved in Step 4. If α is not a leaf, then

$$\widehat{\Psi}_\alpha = \widehat{\Psi}_{\alpha 0} \cup \widehat{\Psi}_{\alpha 1} \cup \widehat{\Psi}_{\alpha 2} \cup \widehat{\Psi}_{\alpha 3}$$

and thus

$$L_\alpha \times R_\alpha = \Big(L_{\alpha 0} \times R_{\alpha 0}\Big) \cup \Big(L_{\alpha 1} \times R_{\alpha 1}\Big) \cup \Big(L_{\alpha 2} \times R_{\alpha 2}\Big) \cup \Big(L_{\alpha 3} \times R_{\alpha 3}\Big).$$

The estimate (5.7) holds on $L_{\alpha 0} \times R_{\alpha 0}$ by Step 4, while it holds on $L_{\alpha a} \times R_{\alpha a}$, $a = 1, 2, 3$, by inductive assumption. Hence we can write

$$\begin{aligned}
&\big[\sigma^{\rm rh}(\mathtt{f}^{\rm eff}_k(t_2), L_\alpha) - \sigma^{\rm rh}(\mathtt{f}^{\rm eff}_k(t_2), R_\alpha)\big] \mathcal{L}^2(L_\alpha \times R_\alpha) \\
&\quad = \iint_{L_\alpha \times R_\alpha} \left[\frac{d\mathtt{f}^{\rm eff}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\rm eff}_k(t_2)}{d\tau}(\tau')\right] d\tau d\tau' \\
&\quad = \sum_{a=0}^{3} \iint_{L_{\alpha a} \times R_{\alpha a}} \left[\frac{d\mathtt{f}^{\rm eff}_k(t_2)}{d\tau}(\tau) - \frac{d\mathtt{f}^{\rm eff}_k(t_2)}{d\tau}(\tau')\right] d\tau d\tau' \\
&\quad = \sum_{a=0}^{3} \big[\sigma^{\rm rh}(\mathtt{f}^{\rm eff}_k(t_2), L_{\alpha a}) - \sigma^{\rm rh}(\mathtt{f}^{\rm eff}_k(t_2), R_{\alpha a})\big] \mathcal{L}^2(L_{\alpha a} \times R_{\alpha a}) \\
&\quad \leq \sum_{a=0}^{3} \iint_{L_{\alpha a} \times R_{\alpha a}} \pi_k\big(t_1, t_2, \Psi_k^{-1}(\tau), \Psi_k^{-1}(\tau')\big) d\tau d\tau' \\
&\quad = \iint_{L_{\alpha a} \times R_{\alpha a}} \pi_k\big(t_1, t_2, \Psi_k^{-1}(\tau), \Psi_k^{-1}(\tau')\big) d\tau d\tau'.
\end{aligned}$$

As already observed, for $\alpha = \emptyset$, we get inequality (5.6), thus concluding the proof of the lemma. □

Lemma 5.6. *It holds*

$$\begin{aligned}
&\iint_{(\Psi_k \times \Psi_k)(\mathcal{F})} \mathfrak{q}_k \Big(t_1, t_2, t_2, \Psi_k^{-1}(\tau), \Psi_k^{-1}(\tau')\Big) d\tau d\tau' \\
&\qquad \leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \sum_{m \in \mathbb{Z}} \mathtt{A}^{\rm cr}_k(i\varepsilon, m\varepsilon).
\end{aligned}$$

Proof. The proof is an easy consequence of the definition (5.2)–(5.3) of the sets $\mathcal{F}_r, \mathcal{F}$ and the fact that the weights $\mathfrak{q}_k$ are uniformly bounded, Remark 2.20. □

Lemma 5.7. *It holds*

$$\iint_{(\Psi_k\times\Psi_k)(\mathcal{E})} \mathtt{q}_k\Big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big) \\ -\mathtt{q}_k\Big(\mathtt{t}^{\mathrm{int}}\big(t_1,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\big)-\varepsilon,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)d\tau d\tau' \\ \le\ \mathcal{O}(1)\sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon).$$

Proof. Fix $(w,w')\in\mathcal{E}$. Observe that for any $i=i_1,\dots,i_2$,

$$\Big|\Phi_k(i\varepsilon)(w')-\Phi_k(i\varepsilon)(w)\Big|\ge\Big|\Psi_k(i\varepsilon)(w')-\Psi_k(i\varepsilon)(w)\Big|, \tag{5.8}$$

since Ψ takes into account only the waves which are in $\mathcal{W}_k(i_1\varepsilon)\cap\mathcal{W}_k(i_2\varepsilon)$. Then notice that

$$\begin{aligned}\mathtt{q}\Big(\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,w,w'\Big) &=\mathtt{q}\Big(\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,w,w'\Big)\\ &=\mathtt{q}\Big(t_1,\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,w,w'\Big)\\ &\ge\mathtt{q}\Big(t_1,\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,t_2,w,w'\Big).\end{aligned}$$

Hence

$$\begin{aligned}\Delta\mathtt{q}_k(w,w') &=\mathtt{q}\Big(t_1,t_2,t_2,w,w'\Big)-\mathtt{q}\Big(\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,w,w'\Big)\\ &\le\mathtt{q}\Big(t_1,t_2,t_2,w,w'\Big)-\mathtt{q}\Big(t_1,\mathtt{t}^{\mathrm{int}}(t_1,w,w')-\varepsilon,t_2,w,w'\Big)\\ &\le\sum_{i=\mathtt{t}^{\mathrm{int}}(t_1,w,w')/\varepsilon}^{i_2}\Big[\mathtt{q}\Big(t_1,i\varepsilon,t_2,w,w'\Big)-\mathtt{q}\Big(t_1,(i-1)\varepsilon,t_2,w,w'\Big)\Big]\\ &\text{(by (2.18))}\\ &\le\mathcal{O}(1)\sum_{i=\mathtt{t}^{\mathrm{int}}(t_1,w,w')/\varepsilon}^{i_2}\frac{1}{|\Phi_k(i\varepsilon)(w')-\Phi_k(i\varepsilon)(w)|}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon)\\ &\text{(by (5.8))}\\ &\le\mathcal{O}(1)\frac{1}{|\Psi_k(w')-\Psi_k(w)|}\sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon).\end{aligned}$$

Therefore

$$\begin{aligned}&\iint_{(\Psi_k\times\Psi_k)(\mathcal{E})} \mathtt{q}_k\Big(t_1,t_2,t_2,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)\\&\qquad - \mathtt{q}_k\Big(\mathtt{t}^{\rm int}\big(t_1,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\big)-\varepsilon,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)\\&\quad\le \mathcal{O}(1)\sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon)\iint_{(\Psi_k\times\Psi_k)(\mathcal{E})}\frac{d\tau d\tau'}{|\tau'-\tau|}\\&\quad\le \mathcal{O}(1)\mathcal{L}^2\Big(\big(\Psi_k\times\Psi_k\big)(\mathcal{E})\Big)\sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon)\\&\quad\le \mathcal{O}(1)\sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\mathtt{A}(i\varepsilon,m\varepsilon).\end{aligned}$$

□

Lemma 5.8. *It holds*

$$\begin{aligned}&\iint_{(\Psi_k\times\Psi_k)(\mathcal{E})} \mathtt{q}_k\bigg(\mathtt{t}^{\rm int}\Big(t_1,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)-\varepsilon,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\bigg)d\tau d\tau'\\&\le\ \mathcal{O}(1)\big(\Upsilon(t_1)-\Upsilon(t_2)\big).\end{aligned}$$

Proof. It holds

$$\begin{aligned}&\iint_{(\Psi_k\times\Psi_k)(\mathcal{E})} \mathtt{q}_k\bigg(\mathtt{t}^{\rm int}\Big(t_1,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\Big)-\varepsilon,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\bigg)d\tau d\tau'\\&=\ \sum_{i=i_1+1}^{i_2}\iint_{(\Psi_k\times\Psi_k)(\mathcal{E}^i)}\mathtt{q}_k\bigg((i-1)\varepsilon,\Psi_k^{-1}(\tau),\Psi_k^{-1}(\tau')\bigg)d\tau d\tau'\\&=\ \sum_{i=i_1+1}^{i_2}\iint_{\big(\Phi_k((i-1)\varepsilon)\times\Phi_k((i-1)\varepsilon)\big)(\mathcal{E}^i)}\mathtt{q}_k\bigg((i-1)\varepsilon,\Phi_k((i-1)\varepsilon^{-1}(\tau),\\&\qquad\Phi_k((i-1)\varepsilon)^{-1}(\tau')\bigg)d\tau d\tau'\end{aligned}$$

(see (2.15))

$$\le\ \sum_{i=i_1+1}^{i_2}\sum_{m\in\mathbb{Z}}\iint_{J_m^L\times J_m^R}\mathtt{q}((i-1)\varepsilon)d\tau d\tau'$$

(using (2.16)–(2.17) and the fact that for waves w,w' interacting at time $i\varepsilon$,

$\mathfrak{q}(i\varepsilon, w, w') = 0)$.

$$\leq \sum_{i=i_1+1}^{i_2} \Big(\mathfrak{Q}((i-1)\varepsilon) - \mathfrak{Q}(i\varepsilon)\Big) + \mathcal{O}(1)\text{Tot.Var.}(\bar{u}) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)$$

(since Q^{known} is decreasing in time)

$$\leq \sum_{i=i_1+1}^{i_2} \Big(\mathfrak{Q}((i-1)\varepsilon) - \mathfrak{Q}(i\varepsilon)\Big) + C\Big(Q^{\text{known}}((i-1)\varepsilon) - Q^{\text{known}}(i\varepsilon)\Big)$$
$$+\mathcal{O}(1)\text{Tot.Var.}(\bar{u}) \sum_{i=i_1+1}^{i_2} \sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)$$

(by the definition of Υ and Corollary 2.24)

$$\leq \mathcal{O}(1) \sum_{i=i_1+1}^{i_2} \Big(\Upsilon((i-1)\varepsilon) - \Upsilon(i\varepsilon)\Big)$$
$$= \mathcal{O}(1)\Big(\Upsilon(t_1) - \Upsilon(t_2)\Big).$$

The conclusion of the proof of Proposition 5.4 is an immediate consequence of the previous four lemmas, Corollary 2.24 and Proposition 5.3. □

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