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QUANTITATIVE ESTIMATE OF THE STATIONARY NAVIER-STOKES EQUATIONS AT INFINITY AND UNIQUENESS OF THE SOLUTION

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This work is dedicated to Professor Tai-Ping Liu for his 70th birthday.

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Abstract

In this paper we are interested in the asymptotic behavior of incompressible fluid around a bounded obstacle. Under certain a priori decaying assumptions, we derive a quantitative estimate of the decaying rate of the difference of any two velocity functions at infinity. This quantitative estimate gives us a sufficient condition, expressed in terms of integrability, to guarantee that the solution of the Navier-Stokes equations is unique.

1. Introduction

Let D be a bounded domain in $\mathbb{R}^n$ and $\Omega=\mathbb{R}^n\setminus\bar{D}$ with $n\geq 2$. Without loss of generality, we let 0 belong to interior of D. Assume that Ω is filled with an incompressible fluid described by the stationary Navier-Stokes equations

$$\begin{cases}-\Delta u+u\cdot\nabla u+\nabla p=f \quad \text{in} \quad \Omega,\\ \nabla\cdot u=0 \quad \text{in} \quad \Omega.\end{cases} \tag{1.1}$$

We are interested in the following question: let u_1 and u_2 be two solutions of (1.1) satisfying some pre-described assumptions such as boundedness or

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decaying conditions, then find a sufficient condition which guarantees that $u_1 \equiv u_2$ in Ω. In this paper, we answer this question by deriving a minimal decay rate of $u_1 - u_2$ at infinity if $u_1 \not\equiv u_2$.

This question is motivated by the following problem. It was shown by Finn [1] that when $n = 3$ and $f = 0$, if $u|_{\partial D} = 0$ and $u = o(|x|^{-1})$, then u is trivial. Inspired by Finn's result, we would like to ask the following question: when $n = 3$, if we know a priori that $u = O(|x|^{-1})$, what is the minimal decaying rate of any nontrivial u satisfying (1.1)? It should be remarked that the boundary value of u on ∂B is irrelevant in this problem. Moreover, the asymptotic behavior $u = O(|x|^{-1})$ characterizes the so-called physically reasonable solutions introduced by Finn [2].

To answer the main question of the paper, we simply subtract two equations for u_1 and u_2 and obtain

$$\begin{cases} -\Delta v + v \cdot \nabla v + v \cdot \nabla u_2 + u_2 \cdot \nabla v + \nabla p_v = 0 \quad \text{in} \quad \Omega, \\ \nabla \cdot v = 0 \quad \text{in} \quad \Omega, \end{cases}$$

where $v = u_1 - u_2$ and $p_v = p_1 - p_2$. Therefore, to solve the problem, it suffices to consider the generalized Navier-Stokes equations

$$\begin{cases} -\Delta v + v \cdot \nabla v + v \cdot \nabla \alpha + \alpha \cdot \nabla v + \nabla p = 0 \quad \text{in} \quad \Omega, \\ \nabla \cdot v = 0 \quad \text{in} \quad \Omega \end{cases} \tag{1.2}$$

with $\nabla \cdot \alpha = 0$. To describe the main theorem, we denote

$$I(x) = \int_{|y-x|<1} |v(y)|^2 dy$$

and

$$M(t) = \inf_{|x|=t} I(x).$$

Then we prove that

Theorem 1.1. *Let $v \in (H^1_{loc}(\Omega))^n$ be a nontrivial solution of (1.2) with an appropriate $p \in H^1_{loc}(\Omega)$. Assume that for $0 \le \kappa_1 < \frac{1}{4}$, $0 \le \kappa_2 < \frac{1}{2}$, $0 < \delta \le \frac{1}{8}$ and $\lambda \ge 1$*

$$\begin{cases} |v(x)| + |\alpha(x)| + |\nabla v(x)| \le \lambda(1 + |x|^2)^{-\kappa_1 - \delta} \\ |\nabla \alpha(x)| \le \lambda(1 + |x|^2)^{-\kappa_2 - \delta}. \end{cases} \tag{1.3}$$

Then there exist $\tilde{t}$ depending on λ, n, $\kappa_1, \kappa_2, \delta$ and positive constants C_1 such that

$$M(t) \geq \exp\left(-C_1 t^{\kappa} \log t\right) \quad for \quad t \geq \tilde{t}, \tag{1.4}$$

where $\kappa = \max\{2-4\kappa_1, 2-2\kappa_2\}$ and the constant C_1 depends on λ, n and

$$\left|\log\left(\min\{\inf_{\tilde{t}<|x|<\tilde{t}^{(1-\delta)^{-1}}} \int_{|y-x|<1} |v(y)|^2 dy, 1\}\right)\right|.$$

It is interesting to compare Theorem 1.1 with the result obtained in [5] where we showed that for the standard stationary Navier-Stokes equations (i.e., $\alpha = 0$ in (1.2)) if v is bounded (for $n = 2$) or C^1 bounded (for $n \geq 3$) in Ω, then

$$M(t) \geq \exp(-Ct^{2+}).$$

We can immediately deduce several consequences from Theorem 1.1. Assume that $n = 3$ and $f = O(|x|^{-3})$ at infinity. Let u_1, u_2 be two solutions of (1.1) satisfying $u_1 = O(|x|^{-1})$ and $u_2 = O(|x|^{-1})$. It was proved by Sverak and Tsai [7] that both ∇u_1 and ∇u_2 are $O(|x|^{-2})$. So we can choose $\kappa_1 = 3/16$, $\kappa_2 = 3/8$ (then $\kappa = 5/4$), and fix $\delta = 1/8$ in Theorem 1.1. Due to Sverak and Tsai's result, we can also relax condition (1.3). Setting $v = u_1 - u_2$ and $\alpha = v_1$, we obtain from Theorem 1.1 that

Corollary 1.2. *Let $u_1, u_2 \in (H^1_{loc}(\Omega))^3$ be solutions of (1.1) with appropriate pressures $p_1, p_2 \in H^1_{loc}(\Omega)$. Assume that $f(x) = O(|x|^{-3})$, $u_1(x) = O(|x|^{-1})$, and $u_2 = O(|x|^{-1})$, at infinity. Then there exist $\tilde{t}$ and positive constant s_1 such that*

$$\inf_{|x|=t} \int_{|y-x|<1} |(u_1-u_2)(y)|^2 dy \geq \exp\left(-s_1 t^{5/4} \log t\right) \quad for \quad t \geq \tilde{t},$$

where s_1 depends linearly on

$$\left|\log\left(\min\{\inf_{\tilde{t}<|x|<\tilde{t}^{8/7}} \int_{|y-x|<1} |(u_1-u_2)(y)|^2 dy, 1\}\right)\right|.$$

Corollary 1.2 immediately implies the following qualitative uniqueness results.

Corollary 1.3. *Let $u_1, u_2 \in (H^1_{loc}(\Omega))^3$ be solutions of* (1.1) *with appropriate pressures $p_1, p_2 \in H^1_{loc}(\Omega)$. Assume that $f(x) = O(|x|^{-3})$, $u_1(x) = O(|x|^{-1})$, and $u_2 = O(|x|^{-1})$, at infinity. Then there exist R and positive constant s_1 such that if*

$$\int_{\Omega\cap\{|x|\geq R\}} \exp(s|x|^{5/4}\log|x|)|(u_1-u_2)(x)|^2 dx < \infty$$

for all $s > s_1$, then $u_1 \equiv u_2$ in Ω, where s_1's dependence is described in Corollary 1.2.

In particular, let $u_2 = 0$ and $f = 0$, we have that

Corollary 1.4. *Let $n = 3$, $f = 0$, and $u \in (H^1_{loc}(\Omega))^3$ be a solution of* (1.1) *with an appropriate $p \in H^1_{loc}(\Omega)$. Assume that $u(x) = O(|x|^{-1})$. Then there exist R and positive constants s_1 such that if*

$$\int_{\Omega\cap\{|x|\geq R\}} \exp(s|x|^{5/4}\log|x|)|u(x)|^2 dx < \infty$$

for all $s > s_1$, then $u \equiv 0$ in Ω, where s_1 depends linearly on the quantity

$$\left|\log\left(\min\{\inf_{R<|x|<R^{\frac{8}{7}}}\int_{|y-x|<1}|u(y)|^2dy, 1\}\right)\right|.$$

As in [5], we prove our result along the line of Carleman's method. Some useful techniques used in [5] are collected in the next Section. The proof of the main theorem is given in Section 3.

2. Reduced system and Carleman estimates

Fixing x_0 with $|x_0| = t >> 1$, we define

$$w(x) = (at)v(atx+x_0),\ \tilde{\alpha}(x) = (at)\alpha(at+x_0),\ \text{and}\ \tilde{p}(x) = (at)^2p(atx+x_0),$$

where r_1 is the constant given in Lemma 2.1 and $a \geq 8/r_1$ which will be determined in the proof of Theorem 1.1. Likewise, we denote

$$\Omega_t := B_{\frac{1}{a}-\frac{1}{20at^{\delta}}}(0) = \{x : |x| < \frac{1}{a} - \frac{1}{20at^{\delta}}\}.$$

From (1.2), it is easy to get that

$$\begin{cases} -\Delta w + w\cdot\nabla w + w\cdot\nabla\tilde{\alpha} + \tilde{\alpha}\cdot\nabla w + \nabla\tilde{p} = 0 \quad \text{in} \quad \Omega_t, \\ \nabla\cdot w = 0 \quad \text{in} \quad \Omega_t. \end{cases} \tag{2.1}$$

In view of (1.3), we have that

$$\begin{cases} \|\tilde{\alpha}\|_{L^\infty(\Omega_t)} + \|w\|_{L^\infty(\Omega_t)} \le C_0 a\lambda t^{1-2\kappa_1-\delta}, \\ \|\nabla w\|_{L^\infty(\Omega_t)} \le C_0 a^2\lambda t^{2-2\kappa_1-\delta}, \\ \|\nabla\tilde{\alpha}\|_{L^\infty(\Omega_t)} \le C_0 a^2\lambda t^{2-2\kappa_2-\frac{3}{4}\delta}, \end{cases} \tag{2.2}$$

where we can choose $C_0 = (20)^{5/4}$.

To prove Theorem1.1, we use the reduced system containing the vorticity equation derived in [5]. Let us define the vorticity q of the velocity w by

$$q = \operatorname{curl} w := \frac{1}{\sqrt{2}}(\partial_i w_j - \partial_j w_i)_{1\le i,j\le n}.$$

The formal transpose of curl is given by

$$(\operatorname{curl}^\top v)_{1\le i\le n} := \frac{1}{\sqrt{2}}\sum_{1\le j\le n}\partial_j(v_{ij} - v_{ji}),$$

where $v = (v_{ij})_{1\le i,j\le n}$. It is easy to see that

$$\Delta w = \nabla(\nabla\cdot w) - \operatorname{curl}^\top\operatorname{curl} w$$

(see, for example, [6] for a proof), which implies

$$\Delta w + \operatorname{curl}^\top q = 0 \quad \text{in} \quad \Omega_t. \tag{2.3}$$

Next we observe that

$$\begin{aligned} w\cdot\nabla\tilde{\alpha} + \tilde{\alpha}\cdot\nabla w &= \nabla(w\cdot\tilde{\alpha}) - \sqrt{2}(\operatorname{curl} w)\tilde{\alpha} - \sqrt{2}(\operatorname{curl}\tilde{\alpha})w \\ &= \nabla(w\cdot\tilde{\alpha}) - \sqrt{2}q\tilde{\alpha} - \sqrt{2}(\operatorname{curl}\tilde{\alpha})w \end{aligned}$$

and in particular

$$w\cdot\nabla w = \nabla(\frac{1}{2}|w|^2) - \sqrt{2}(\operatorname{curl} w)w = \nabla(\frac{1}{2}|w|^2) - \sqrt{2}qw.$$

Thus, applying curl on the first equation of (2.1), we have that

$$-\Delta q+Q(q)(w+\tilde{\alpha})+q(\nabla w+\nabla\tilde{\alpha})^{\top}-(\nabla w+\nabla\tilde{\alpha})q^{\top}-\mathrm{div}F=0 \text{ in } \Omega_t, \quad (2.4)$$

where

$$(Q(q)w)_{ij} = \sum_{1\le k\le n}(\partial_j q_{ik}-\partial_i q_{jk})w_k$$

and

$$(\mathrm{div}F)_{ij}Z = \sum_{k=1}^{n}\partial_k F_{ijk}$$

with

$$F_{ijk} = \sum_{1\le m\le n}\Big((\mathrm{curl}\ \tilde{\alpha})_{jm}w_m\delta_k^i-(\mathrm{curl}\ \tilde{\alpha})_{im}w_m\delta_k^j\Big).$$

Putting together (2.3), (2.4), and using (1.3), to prove the main theorem, it suffices to consider

$$\begin{cases}\Delta q+A(x)\cdot\nabla q+B(x)q+\mathrm{div}F=0 \quad \text{in} \quad \Omega_t,\\ \Delta w+\mathrm{curl}^{\top}q=0 \quad \text{in} \quad \Omega_t,\end{cases} \quad (2.5)$$

where A is a $(3,2)$ tensor and B is a $(2,2)$ tensor with

$$\|A\|_{L^\infty(\Omega_t)} \le C_0\lambda a t^{1-2\kappa_1-\delta},\ \|B\|_{L^\infty(\Omega_t)}\le C_0\lambda a^2t^{2-2\kappa_1-\delta}+C_0\lambda a^2 t^{2-2\kappa_2-\frac{3}{4}\delta},$$

and

$$|F(x)|\le C_0\lambda a^2t^{2-2\kappa_2-\frac{3}{4}\delta}|w(x)|, \quad \forall\ x\in\Omega_t.$$

Our proof relies on appropriate Carleman estimates. Here we need two Carleman estimates with weights $\varphi_\beta=\varphi_\beta(x)=\exp(-\beta\tilde{\psi}(x))$, where $\beta>0$ and $\tilde{\psi}(x)=\log|x|+\log((\log|x|)^2)$.

Lemma 2.1. *There exist a sufficiently small number $r_1>0$ depending on n and a sufficiently large number $\beta_1>3$, a positive constant C, depending on n such that for all $v\in U_{r_1}$ and $f=(f_1,\cdots,f_n)\in(U_{r_1})^n$, $\beta\ge\beta_1$, we have that*

$$\int\varphi_\beta^2(\log|x|)^2(\beta|x|^{4-n}|\nabla v|^2+\beta^3|x|^{2-n}|v|^2)dx$$
$$\le C\int\varphi_\beta^2(\log|x|)^4|x|^{2-n}[(|x|^2\Delta v+|x|\mathrm{div}f)^2+\beta^2\|f\|^2]dx, \quad (2.6)$$

where $U_{r_1}=\{v\in C_0^\infty(\mathbb{R}^n\setminus\{0\}):\mathrm{supp}(v)\subset B_{r_1}\}$.

Lemma 2.1 is a modified form of [4, Lemma 2.4]. For the sake of brevity, we omit the proof here. Replacing β of Lemma 2.1 with $\beta+1$ and choosing $f=0$ implies

Lemma 2.2. *There exist a sufficiently small number $r_1>0$, a sufficiently large number $\beta_1>1$, a positive constant C, such that for all $v\in U_{r_1}$ and $\beta\geq\beta_1$, we have*

$$\int \varphi_\beta^2(\log|x|)^{-2}|x|^{-n}(\beta|x|^2|\nabla v|^2+\beta^3|v|^2)dx\leq C\int \varphi_\beta^2|x|^{-n}(|x|^4|\Delta v|^2)dx. \tag{2.7}$$

In addition to Carleman estimates, we also need the following interior estimate.

Lemma 2.3. *For any $0<a_1<a_2$ such that $B_{a_2}\subset\Omega_t$ for $t>1$, let $X=B_{a_2}\backslash\bar{B}_{a_1}$ and $d(x)$ be the distant from $x\in X$ to $\mathbb{R}^n\backslash X$. Then we have*

$$\begin{aligned}&\int_X d(x)^2|\nabla w|^2dx+\int_X d(x)^4|\nabla q|^2dx+\int_X d(x)^2|q|^2dx\\&\quad\leq C\left(1+a^2t^{-\frac{3\delta}{2}}\right)^2\int_X|w|^2dx,\end{aligned} \tag{2.8}$$

where the constant C depends on n,λ.

The proof of this lemma is similar to that given in [5].

3. Proof of Theorem 1.1

This section is devoted to the proof of the main theorem, Theorem 1.1. Since $(w,p)\in(H^1(\Omega_t))^{n+1}$, the regularity theorem implies $w\in H^2_{loc}(\Omega_t)$. Therefore, to use estimate (2.7), we simply cut-off w. So let $\chi(x)\in C_0^\infty(\mathbb{R}^n)$ satisfy $0\leq\chi(x)\leq 1$ and

$$\chi(x)=\begin{cases}0, & |x|\leq\frac{1}{8at},\\ 1, & \frac{1}{4at}<|x|<\frac{1}{a}-\frac{3}{20at^\delta},\\ 0, & |x|\geq\frac{1}{a}-\frac{2}{20at^\delta}.\end{cases}$$

It is easy to see that for any multiindex α

$$\begin{cases}|D^\alpha\chi| = O((at)^{|\alpha|}) & \text{if} \quad \frac{1}{8at} \le |x| \le \frac{1}{4at},\\ |D^\alpha\chi| = O((at^\delta)^{|\alpha|}) & \text{if} \quad \frac{1}{a} - \frac{3}{20at^\delta} \le |x| \le \frac{1}{a} - \frac{2}{20at^\delta}.\end{cases} \tag{3.1}$$

To apply Carleman estimates above, it suffices to take $1/a \le r_1$. Now applying (2.7) to χw gives

$$\int (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta|x|^2|\nabla(\chi w)|^2 + \beta^3|\chi w|^2)dx \le C\int \varphi_\beta^2|x|^{-n}|x|^4|\Delta(\chi w)|^2dx. \tag{3.2}$$

Here and after, C and $\tilde{C}$ denote general constants whose value may vary from line to line. The dependence of C and $\tilde{C}$ will be specified whenever necessary. Next applying (2.6) to $v = \chi q$ and $f = |x|\chi F$ yields that

$$\int \varphi_\beta^2(\log|x|)^2(|x|^{4-n}\beta|\nabla(\chi q)|^2 + |x|^{2-n}\beta^3|\chi q|^2)dx \le C\int \varphi_\beta^2(\log|x|)^4|x|^{2-n}[(|x|^2\Delta(\chi q)+|x|\mathrm{div}(|x|\chi F))^2+\beta^2\||x|\chi F\|^2]dx. \tag{3.3}$$

Combining $\beta\times$(3.2) and (3.3), we obtain that

$$\begin{aligned}&\int_W (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta^2|x|^2|\nabla w|^2 + \beta^4|w|^2)dx\\ &\quad + \int_W (\log|x|)^2\varphi_\beta^2|x|^{-n}(\beta|x|^4|\nabla q|^2 + |x|^2\beta^3|q|^2)dx\\ &\le C\beta\int \varphi_\beta^2|x|^{-n}|x|^4|\Delta(\chi w)|^2dx\\ &\quad + C\int \varphi_\beta^2(\log|x|)^4|x|^{2-n}[\big(|x|^2\Delta(\chi q) + |x|\mathrm{div}(|x|\chi F)\big)^2\\ &\quad + \beta^2\||x|\chi F\|^2]dx,\end{aligned} \tag{3.4}$$

where W denotes the domain $\{x : \frac{1}{4at} < |x| < \frac{1}{a} - \frac{3}{20at^\delta}\}$. To simplify the notations, we denote $Y = \{x : \frac{1}{8at} \le |x| \le \frac{1}{4at}\}$ and $Z = \{x : \frac{1}{a} - \frac{3}{20at^\delta} \le |x| \le \frac{1}{a} - \frac{2}{20at^\delta}\}$. By (2.4) and estimates (3.1), we deduce from (3.4) that

$$\int_W (\log|x|)^{-2}\varphi_\beta^2|x|^{-n}(\beta^2|x|^2|\nabla w|^2 + \beta^4|w|^2)dx$$

$$
\begin{aligned}
&+\int_W (\log|x|)^2\varphi_\beta^2|x|^{-n}(\beta|x|^4|\nabla q|^2+|x|^2\beta^3|q|^2)dx\\
\le\ & C\beta\int_W \varphi_\beta^2|x|^{-n}|x|^4|\nabla q|^2dx\\
&+Ca^2t^{2-4\kappa_1-2\delta}\int_W(\log|x|)^4\varphi_\beta^2|x|^{-n}|x|^6|\nabla q|^2dx\\
&+Ca^4t^{4-4\kappa_1-2\delta}\int_W(\log|x|)^4\varphi_\beta^2|x|^{-n}|x|^6|q|^2dx\\
&+C\beta^2a^4t^{4-4\kappa_2-\frac{3}{4}\delta}\int_W(\log|x|)^4\varphi_\beta^2|x|^{-n}|x|^4|w|^2dx\\
&+C(at)^4\beta\int_{Y\cup Z}\varphi_\beta^2|x|^{-n}|\tilde{U}|^2dx\\
&+C(at)^4\beta^2\int_{Y\cup Z}(\log|x|)^4\varphi_\beta^2|x|^{2-n}|\tilde{U}|^2dx, \qquad (3.5)
\end{aligned}
$$

where $|\tilde{U}(x)|^2=|x|^4|\nabla q|^2+|x|^2|q|^2+|x|^2|\nabla w|^2+|w|^2$ and C depends on n, λ.

Now we can choose $a>a_0\ge 8/r_1$ such that $(\log|x|)^2\ge 2C$ for all $x\in W$. Then the first term on the right hand side of (3.5) can be absorbed by the left hand side of (3.5). Now, let $\beta\ge\beta_2=t^\kappa$ and choose $t\ge t_0$ with t_0 depending on a,λ,δ such that the second term to the fourth term on the right hand side of (3.5) can be removed. With the choices described above, we obtain from (3.5) that

$$
\begin{aligned}
&\beta^4(b_1)^{-n}(\log b_1)^{-2}\varphi_\beta^2(b_1)\int_{\frac{1}{at}<|x|<b_1}|w|^2dx\\
&\le\ \beta^4\int_W(\log|x|)^{-2}\varphi_\beta^2|x|^{-n}|w|^2dx\\
&\le\ C\beta(at)^4\int_{Y\cup Z}(\log|x|)^4\varphi_\beta^2|x|^{-n}|\tilde{U}|^2dx\\
&\le\ C\beta^2(at)^4(\log b_2)^4b_2^{-n}\varphi_\beta^2(b_2)\int_Y|\tilde{U}|^2dx\\
&\quad+C\beta^2(at)^4(\log b_3)^4b_3^{-n}\varphi_\beta^2(b_3)\int_Z|\tilde{U}|^2dx, \qquad (3.6)
\end{aligned}
$$

where $b_1=\frac{1}{a}-\frac{8}{20at^\delta}$, $b_2=\frac{1}{8at}$ and $b_3=\frac{1}{a}-\frac{3}{20at^\delta}$.

Using (2.8), we can control $|\tilde{U}|^2$ terms on the right hand side of (3.6).

Indeed, let $X = Y_1 := \{x : \frac{1}{16at} \le |x| \le \frac{1}{2at}\}$, then we can see that

$$d(x) \ge C|x| \quad \text{for all} \quad x \in Y,$$

where C an absolute constant. Therefore, (2.8) implies

$$\begin{aligned}
&\int_Y \left(|x|^2|\nabla w|^2 + |x|^4|\nabla q|^2 + |x|^2|q|^2\right) dx \\
&\le C\int_{Y_1} \left(d(x)^2|\nabla w|^2 + d(x)^4|\nabla q|^2 + d(x)^2|q|^2\right) dx \\
&\le C\left(1 + a^2 t^{-\frac{3\delta}{2}}\right)^2 \int_{Y_1} |w|^2 dx \\
&\le Ca^4 \int_{Y_1} |w|^2 dx. \qquad (3.7)
\end{aligned}$$

Here C depends on n, λ. On the other hand, let $X = Z_1 := \{x : \frac{1}{2a} \le |x| \le \frac{1}{a} - \frac{1}{20at^\delta}\}$, then

$$d(x) \ge Ct^{-\delta}|x| \quad \text{for all} \quad x \in Z,$$

where C another absolute constant. Thus, it follows from (2.8) that

$$\begin{aligned}
&\int_Z \left(|x|^2|\nabla w|^2 + |x|^4|\nabla q|^2 + |x|^2|q|^2\right) dx \\
&\le Ct^{4\delta}\int_{Z_1} \left(d(x)^2|\nabla w|^2 + d(x)^4|\nabla q|^2 dx + d(x)^2|q|^2\right) dx \\
&\le Ct^{4\delta}\left(1 + a^2 t^{-\frac{3\delta}{2}}\right)^2 \int_{Z_1} |w|^2 dx \\
&\le C(at)^4 \int_{Z_1} |w|^2 dx. \qquad (3.8)
\end{aligned}$$

Combining (3.6), (3.7), and (3.8) leads to

$$\begin{aligned}
&b_1^{-2\beta-n}(\log b_1)^{-4\beta-2}\int_{\frac{1}{2at}<|x|<b_1} |w|^2 dx \\
&\le Ca^8 t^4 (\log b_2)^4 b_2^{-n}\varphi_\beta^2(b_2)\int_{Y_1} |w|^2 dx \\
&\quad + C(at)^8(\log b_3)^4 b_3^{-n}\varphi_\beta^2(b_3)\int_{Z_1} |w|^2 dx. \qquad (3.9)
\end{aligned}$$

Notice that (3.9) holds for all $\beta \geq \beta_2$.

Changing $2\beta + n$ to β, (3.9) becomes

$$\begin{aligned} & b_1^{-\beta}(\log b_1)^{-2\beta+2n-2}\int_{\frac{1}{2at}<|x|<b_1}|w|^2dx \\ & \leq\; Ca^8t^4b_2^{-\beta}(\log b_2)^{-2\beta+2n+4}\int_{Y_1}|w|^2dx \\ & \quad +C(at)^8b_3^{-\beta}(\log b_3)^{-2\beta+2n+4}\int_{Z_1}|w|^2dx. \end{aligned} \tag{3.10}$$

Dividing $b_1^{-\beta}(\log b_1)^{-2\beta+2n-2}$ on the both sides of (3.10) and noting $\beta \geq n+2 > n-1$, i.e., $2\beta - 2n + 2 > 0$, we have for $t \geq t_1 \geq t_0$ that

$$\begin{aligned} & \int_{|x+\frac{b_4x_0}{t}|<\frac{1}{at}}|w(x)|^2dx \\ \leq\; & \int_{\frac{1}{2at}<|x|<b_1}|w(x)|^2dx \\ \leq\; & Ca^8t^4(\log(8at))^6(b_1/b_2)^\beta\int_{Y_1}|w|^2dx \\ & +C(at)^8(b_1/b_3)^\beta(\log b_3)^6[\log b_1/\log b_3]^{2\beta-2n+2}\int_{Z_1}|w|^2dx \\ \leq\; & Ca^8t^4(\log(8at))^6(8t)^\beta\int_{|x|<\frac{1}{at}}|w(x)|^2dx \\ & +C(at)^8(\log b_3)^6(b_1/b_5)^\beta\int_{Z_1}|w(x)|^2dx, \end{aligned} \tag{3.11}$$

where $b_4 = \frac{1}{a} - \frac{1}{at^\delta}$ and $b_5 = \frac{1}{a} - \frac{6}{20at^\delta}$. In deriving the third inequality above, we use the fact that

$$0 \leq (\frac{b_5}{b_3})(\frac{\log b_1}{\log b_3})^2 = 1 - \frac{1}{2t^\delta\log a} - \frac{3}{20t^\delta} + O(t^{-2\delta}) \leq 1$$

for all $t \geq t_2 \geq t_1$ and $a > a_1 = \max\{1, a_0\}$, where t_2 depends on t_1, δ, and a. From now on we fix a, which depends only on n and r_1. Recall that r_1 is a function of n. Therefore, t_2 depends on n, λ, and δ. Having fixed constant a, $|\log b_3|$ can be bounded by a positive constant. Thus, (3.11) is reduced to

$$\int_{|x+\frac{b_4x_0}{t}|<\frac{1}{at}}|w(x)|^2dx \;\leq\; Ct^4(\log t)^6(8t)^\beta\int_{|x|<\frac{1}{at}}|w(x)|^2dx$$

$$+Ct^8(b_1/b_5)^\beta \int_{Z_1} |w(x)|^2 dx, \tag{3.12}$$

where C depends on n and λ.

From (3.12), (2.2), the definition of $w(x)$, the change of variables $y = atx + x_0$, and $x_0 = ty_0$, we have that

$$\begin{aligned} I(t^{1-\delta}y_0) &\leq Ct^4(\log t)^6(8t)^\beta \int_{|y-x_0|<1} |u(y)|^2 dy + Ct^{8-\frac{3\delta}{2}} \left(\frac{t^\delta}{t^\delta + \frac{1}{10}}\right)^\beta \\ &\leq C(8t)^{\beta+10} I(ty_0) + Ct^8 \left(\frac{t^\delta}{t^\delta + \frac{1}{10}}\right)^\beta \\ &\leq C(8t)^{2\beta} I(ty_0) + Ct^8 \left(\frac{t^\delta}{t^\delta + \frac{1}{10}}\right)^\beta \end{aligned} \tag{3.13}$$

provided $\beta \geq \beta_2$. For simplicity, by denoting

$$A(t) = 2\log 8t, \quad B(t) = \log(\frac{t^\delta + \frac{1}{10}}{t^\delta}),$$

(3.13) becomes

$$I(t^{1-\delta}y_0) \leq C\Big\{\exp(\beta A(t)) I(ty_0) + t^8 \exp(-\beta B(t))\Big\}. \tag{3.14}$$

Now, we consider two cases. If

$$\exp(\beta_2 A(t)) I(ty_0) \geq t^8 \exp(-\beta_2 B(t)),$$

then we have

$$I(x_0) = I(ty_0) \geq t^8 \exp(-\beta_2(A(t) + B(t))) = t^8 (8t)^{-2\beta_2} \left(\frac{t^\delta + \frac{1}{10}}{t^\delta}\right)^{-\beta_2},$$

that is

$$I(ty_0) \geq t^{-2\beta_2+8} = t^{-2t^\kappa+8} \geq \exp(-2t^\kappa \log t) \tag{3.15}$$

for any fixed $t \geq t_2$. Note that we have used the relation $\beta_2 = t^\kappa$ in (3.15).

On the other hand, if

$$\exp(\beta_2 A(t))I(ty_0) < t^8 \exp(-\beta_2 B(t)),$$

then we can pick a $\tilde{\beta} > \beta_2$ such that

$$\exp(\tilde{\beta} A(t))I(ty_0) = t^8 \exp(-\tilde{\beta} B(t)). \tag{3.16}$$

Solving $\tilde{\beta}$ from (3.16) and using (3.14), we have that

$$\begin{aligned} I(t^{1-\delta}y_0) &\leq C\exp(\tilde{\beta}A(t))I(ty_0) \\ &= C\left(I(ty_0)\right)^{\tau}\left(t^8\right)^{1-\tau} \\ &\leq Ct^8\left(I(ty_0)\right)^{\tau}, \end{aligned} \tag{3.17}$$

where $\tau = \frac{B(t)}{A(t)+B(t)}$.

It is time to prove Theorem 1.1. Let $|x_0| = t$ for $t \geq t_2^{\frac{1}{1-\delta}}$ and $y_0 = \frac{x_0}{t}$, then we can write

$$t = \mu^{\left((1-\delta)^{-s}\right)} \tag{3.18}$$

for some positive integer s and $t_2 \leq \mu < t_2^{\frac{1}{1-\delta}} \leq t_2^2$. For simplicity, we define $d_j = \mu^{\left((1-\delta)^{-j}\right)}$ and $\tau_j = \frac{B(d_j)}{A(d_j)+B(d_j)}$ for $j = 1, 2 \cdots s$. Define

$$J = \{1 \leq j \leq s : \exp(d_j^{\kappa} A(d_j))I(d^j y_0) \geq d_j^8 \exp(-d_j^{\kappa} B(d_j))\}.$$

Now, we divide it into two cases. If $J = \emptyset$, we only need to consider (3.17). Using (3.17) iteratively starting from $t = d_1$, we have that

$$\begin{aligned} I(\mu y_0) &\leq C(d_1^8)\left(I(d_1 y_0)\right)^{\tau_1} \\ &\leq C^s (d_1 d_2 \cdots d_s)^8 \left(I(x_0)\right)^{\tau_1\tau_2\cdots\tau_s}. \end{aligned} \tag{3.19}$$

By (3.18) and (3.19), we obtain that

$$\begin{aligned} I(\mu y_0) &\leq C^{(\log\log t/|\log(1-\delta)|)} t^{8/\delta}\left(I(x_0)\right)^{\tau_1\tau_2\cdots\tau_s} \\ &\leq t^{\tilde{C}_0/\delta}\left(I(x_0)\right)^{\tau_1\tau_2\cdots\tau_s}, \end{aligned} \tag{3.20}$$

where $\tilde{C}_0$ depends on λ, n. It is easily to see that

$$\frac{1}{\tau_j}=\frac{2\log(8d_j)+\log(1+0.1d_j^{-\delta})}{\log(1+0.1d_j^{-\delta})}\le\frac{4\log(8d_j)}{\log(1+0.1d_j^{-\delta})}\le 160d_j^{\delta}\log(d_j),$$

and thus

$$\begin{aligned}\frac{1}{\tau_1\tau_2\cdots\tau_s} &\le (160\log\mu\log t)^s(d_1\cdots d_s)^{\delta}\\ &\le t\omega(t),\end{aligned}\tag{3.21}$$

where $\omega(t)=(\log t)^{4\log(\log t)}$. Raising both sides of (3.20) to the power $\frac{1}{\tau_1\tau_2\cdots\tau_s}$ and using (3.21), we obtain that

$$\begin{aligned}(\min\{I(\mu y_0),1\})^{t\omega(t)} &\le I(\mu y_0)^{\frac{1}{\tau_1\tau_2\cdots\tau_s}}\\ &\le e^{(\tilde{C}_0/\delta)t\omega(t)}\left(I(x_0)\right).\end{aligned}\tag{3.22}$$

Next, if $J\neq\emptyset$, let l be the largest integer in J. Then from (3.15) we have

$$I(d_l y_0)\ge d_l^{-2d_l^{\kappa}+8}.\tag{3.23}$$

Iterating (3.17) starting from $t=d_{l+1}$ yields

$$\begin{aligned}I(d_l y_0) &\le C^{s-l}(d_{l+1}\cdots d_s)^8\left(I(x_0)\right)^{\tau_{l+1}\cdots\tau_s}\\ &\le C^{(\log\log t/|\log(1-\delta)|)}(t/d_l)^{8/\delta}\left(I(x_0)\right)^{\tau_{l+1}\cdots\tau_s}\\ &\le t^{\tilde{C}_0/\delta}\left(I(x_0)\right)^{\tau_{l+1}\cdots\tau_s}.\end{aligned}\tag{3.24}$$

It is enough to assume $I(d_l y_0)<1$. Repeating the computations in (3.21), we can see that

$$\frac{1}{\tau_{l+1}\cdots\tau_s}\le(t/d_l)\omega(t).\tag{3.25}$$

Hence, combining (3.23), (3.24) and using (3.25), we get that

$$t^{-\tilde{C}_3t^{\kappa}\log(t)}\le e^{(\tilde{C}_0/\delta)t\omega(t)}\left(I(x_0)\right),\tag{3.26}$$

where $\tilde{C}_3$ is an absolute constant. The proof is complete in view of (3.15), (3.22) and (3.26).

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