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A DYNAMICAL SYSTEMS APPROACH TO MATHEMATICAL MODELING OF TORNADOES

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Dedicated to Tai-Ping Liu on the occasion of his seventieth birthday.

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Abstract

We present a derivation, based on asymptotic analysis of Arsen'yev's second order ordinary differential equation, for the propagation of a boundary layer where thermal inversion occurs. The equation we obtained has one more free parameter U_0 than Arsen'yev's original ODE. Our analysis suggests that this ODE is only a first order closure in what could be a hierarchal system of ODEs. The derivation and analysis of the hierarchal system would provide an appealing generalization of Arsen'yev's ODE and could be the basis of further comparison with experimental data.

1. Introduction

In a recent paper Arsen'yev [1] has given a description of tornado formation and propagation via a model analogous to shallow water wave theory. Moreover Arsen'yev's predictions when compared with actual tornado data are extremely accurate (within 10%) and hence provide a strong motivation for further mathematical analysis of the derivation of his model. Indeed that is the goal of this paper. Here, just as Arsen'yev, we begin with the incompressible Navier-Stokes equations with the additional contribution of

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a turbulent stress that is phenomenologically modeled. However, unlike Arsen'yev, we find it convenient to work in a dimensionless formulation thus allowing us to be precise when neglecting small terms. This approach allows us to see that our derived theory is fully non-linear and no linear approximations are made. Furthermore this allows us delineate exactly where our model differs from that of Arsen'yev.

In fact the model of Arsen'yev belongs to a class of systems which are open dissipative systems that are far from equilibrium. In such systems energy and/or mass can flow through open boundaries. Such models are actively studied in physics and in particular geophysics and have lead the development of both theory and computational tools for the analysis of tornadoes, squall storms, and ocean storm surges [2, 3, 4, 5].

Our physical set up of the model is identical to Arsen'yev. We consider fluid motion (with velocity in Cartesian coordinates (u, v, w)) between the earth and a thin thermal inversion layer which is modeled as a propagating surface $z = \zeta(x, t)$ and hence the analogy with shallow water wave theory. Integration of the Navier-Stokes equations in the $z-$direction from the surface of the earth to the propagating surface allows us to find averaged equations of motion which upon scaling lead to equations for $\zeta(x, t)$ and the space averaged momentum in the x-direction $S(x, t) = \int_{\zeta(x,t)}^{1} u\, dz$. However the system for ζ, S is not closed and an extra dependent variable $u(x, \zeta(x, t), t)$ appears in our equations. Arsen'yev has suggested in his Section 5 that this relation can be taken as $S(x, t) \approx u(x, \zeta(x, t), t)$ (in dimensionless variables) which he refers to as the "slab model". Of course this is nothing more than a rectangular approximation to the integral defining $S(x, t)$ when $\zeta(x, t)$ is small and there is no-slip boundary condition at $z = 1$. Here we see that a higher order slab model with $\int_{\zeta(x,t)}^{1} u^2 dz \approx u^2(x, \zeta(x, t), t)$ suffices to close our system and provide a new system of ordinary differential equations when u, S, ζ are represented as traveling waves. The system is almost identical to that of Arsen'yev with an additional free parameter entering governing ordinary differential equations.

As the end result of any new theory is comparison with existing theory, i.e. in our case it is Arsen'yev theory, our final section will provide both such a comparison and in addition a comparison with the observed tornado data that Arsen'yev has provided in his paper.

2. The Basic Model

The basic model relies on the concept of an inversion layer in the atmosphere and is taken from the paper of Arsen'yev [1]. In simple terms the issue follows from how we measure the air temperature above the surface of the earth. In common experience the temperature decreases as we increase the distance from the earth's surface. However in unusual circumstances in an inversion layer the reverse is true and the temperature increases as the distance increases. Pictorially this is represented in Figure 1 of [1] . There the inversion layer is represented as a surface $z = \varsigma(x, t)$, $z = 1$ denotes the surface of the earth, and $z = 0$ denotes the unperturbed layer.

The underlying dynamics are

1. between the earth's surface $z = 1$and $z = \varsigma(x, t)$ cold air will rise meeting the warmer inversion layer hence causing (via the ideal gas law) a pressure difference pushing the cold air out and down;
2. above the inversion layer warm air arrives at the top of the inversion layer where the air is colder again causing an opposite pressure difference that of 1). Now the air is pushed out and up.
3. This difference is pressures induces a turbulent shear stress E_{13} on the layer $z = \varsigma(x, t)$. It is this turbulent shear stress that is constitutive modeled at both $z = \varsigma(x, t)$ and $z = 1$ and provides an additional input to the classical incompressible Navier- Stokes equations.

3. The Basic Equations

As noted in the Introduction our goal is to provide an asymptotic derivation of Arsen'yev's theory as well as obtain a more general mathematical result than he had obtained. In this regard we recall the incompressible Navier- Stokes equations in dimensionless form.

Let (u, v, w) denote the components of velocity with respect to the Cartesian space coordinates (x, y, z), p the pressure , t denote time, Re denote the dimensionless Reynolds number $\mathrm{Re} = VL/\nu$ where $V$is a typical velocity magnitude, L a typical length, and ν is the kinematic viscosity. We note that an original set of dimensional coordinates (x^*, y^*, z^*) has been made dimensionless by division by $H, (x^*, y^*, z^*)/H = (x, y, z), t^*/T = t$ where H is the dimensional distance between the earth and the unperturbed inversion layer and T is a typical time interval of approximately 30 seconds. This explains why the surface of the earth has been set at $z = 1$.

In addition we represent the dimensionless force of gravity by the inverse Froude number $1/\ Fr$ where $Fr = V^2/gL$. Then the Navier-Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{\partial E_{13}}{\partial z} \quad (1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \quad (1b)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) - \frac{\partial E_{13}}{\partial x} + \frac{1}{Fr} \quad (1c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (1d)$$

On the boundary $z = 1$ we will impose no-slip boundary conditions

$$u = v = w = 0, \quad (2)$$

while on the boundary $z = \zeta(x, t)$ we impose the kinematic boundary condition,

$$w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}. \quad (3)$$

In the subsequent analysis we will assume no dependence on y.

4. Analysis of the Continuity Equation

If there is no y dependence, the continuity equation (1d) becomes $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$. Integrate from $z = \zeta(x, t)$ to $z = 1$ to obtain

$$\int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial x} dz - w(x, \zeta(x, t), t) = 0$$

and via the kinematic boundary condition we then have

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} = \int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial x} dz.$$

Define $S(x,t) = \int_{\zeta(x,t)}^{1} u\, dz$. Then from Leibnitz's rule we have

$$\frac{\partial S}{\partial x} = \int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial x} dz - u(x, \zeta(x,t), t)\frac{\partial \zeta}{\partial x}. \tag{4}$$

Now substitute the above formula for $\frac{\partial \zeta}{\partial t}$ to obtain

$$\frac{\partial \zeta}{\partial t} = \frac{\partial S}{\partial x}. \tag{5}$$

5. Analysis of the Momentum Equation in the $x-$direction

Integrate the balance of linear momentum equation (1a) to obtain

$$\begin{aligned}\int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} dz &= \int_{\zeta(x,t)}^{1} -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}) dz \\ &\quad -E_{13}(x,t,1) + E_{13}(x,t,\zeta(x,t)). \end{aligned} \tag{6}$$

Now integrate by parts, use the continuity equation and (3) to find

$$\begin{aligned}\int_{\zeta(x,t)}^{1} w\frac{\partial u}{\partial z} dz &= -w(x,t,\zeta(x,t))u(x,t,\zeta(x,t)) + \int_{\zeta(x,t)}^{1} u\frac{\partial u}{\partial x} dz \\ &= -(\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x})u(x,t,\zeta(x,t)) + \int_{\zeta(x,t)}^{1} u\frac{\partial u}{\partial x} dz \end{aligned} \tag{7}$$

and hence

$$\begin{aligned}&\int_{\zeta(x,t)}^{1} \{\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x}\} dz - (\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x})u(x,t,\zeta(x,t)) \\ &= \int_{\zeta(x,t)}^{1} \{-\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2})\} dz - E_{13}(x,t,1) + E_{13}(x,t,\zeta(x,t)). \end{aligned} \tag{8}$$

Leibnitz's rule tells us

$$\frac{\partial S}{\partial t} = \int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial t} dz - u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial t}$$

and we insert this expression into (8). This yields

$$\int_{\zeta(x,t)}^{1} 2u\frac{\partial u}{\partial x} dz + \frac{\partial S}{\partial t} + u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial t} - (\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x})u(x,t,\zeta(x,t))$$

$$= \int_{\zeta(x,t)}^{1} -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2})\}dz - E_{13}(x,t,1) + E_{13}(x,t,\zeta(x,t)) \quad (9)$$

which simplifies to

$$\int_{\zeta(x,t)}^{1} \{2u\frac{\partial u}{\partial x}\}dz + \frac{\partial S}{\partial t} - \frac{\partial \zeta}{\partial x}u^2(x,t,\zeta(x,t))$$
$$= \int_{\zeta(x,t)}^{1} \{-\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2})\}dz - E_{13}(x,t,1) + E_{13}(x,t,\zeta(x,t)). \quad (10)$$

Recall from (4) that

$$\frac{\partial S}{\partial x} = \int_{\zeta(x,t)}^{1} \frac{\partial u}{\partial x}dz - u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x}.$$

One additional differentiation gives

$$\frac{\partial^2 S}{\partial x^2} = \int_{\zeta(x,t)}^{1} \frac{\partial^2 u}{\partial x^2}dz - \left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x} - \frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x})$$

or

$$\int_{\zeta(x,t)}^{1} \frac{\partial^2 u}{\partial x^2}dz = \frac{\partial^2 S}{\partial x^2} + \left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x})). \quad (11)$$

We can check (11) if we make the following computations. Note

$$\int_{\zeta(x,t)}^{1} \frac{\partial^2 u}{\partial x^2}dz = -\int_{\zeta(x,t)}^{1} \frac{\partial^2 w}{\partial x \partial z}dz = -\left[\frac{\partial w}{\partial x}\right]_{z=\zeta(x,t)}^{z=1} = \left[\frac{\partial w}{\partial x}(x,z,t)\right]_{z=\zeta(x,t)}$$

where we have used the continuity equation and the no-slip boundary condition at $z = 1$. Next use the continuity equation to see

$$\begin{aligned} \frac{\partial}{\partial x}w(x,\zeta(x,t),t) &= \left[\frac{\partial}{\partial x}w(x,z,t)\right]_{z=\zeta(x,t)} + \left[\frac{\partial}{\partial z}w(x,z,t)\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x} \\ &= \left[\frac{\partial}{\partial x}w(x,z,t)\right]_{z=\zeta(x,t)} - \left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x}. \end{aligned}$$

Insert the kinematic boundary condition (3) into left most term to find

$$\frac{\partial}{\partial x}\Big(\frac{\partial \zeta}{\partial t} + u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x}\Big) = \left[\frac{\partial}{\partial x}w(x,z,t)\right]_{z=\zeta(x,t)} - \left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x}.$$

Perform the indicated differentiations to see

$$\frac{\partial^2\zeta}{\partial t\partial x}+\frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x})$$
$$=\left[\frac{\partial}{\partial x}w(x,z,t)\right]_{z=\zeta(x,t)}-\left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x}$$
$$\frac{\partial^2\zeta}{\partial t\partial x}+\frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x})+\left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x}$$
$$=\left[\frac{\partial}{\partial x}w(x,z,t)\right]_{z=\zeta(x,t)}$$

and we have

$$\int_{\zeta(x,t)}^{1}\frac{\partial^2 u}{\partial x^2}dz=\frac{\partial^2\zeta}{\partial t\partial x}+\frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x})+\left[\frac{\partial}{\partial x}u(x,z,t)\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x}. \tag{12}$$

Thus we see (11) and (12) are equivalent if $\frac{\partial^2\zeta}{\partial t\partial x}=\frac{\partial^2 S}{\partial x^2}$ and we know this equality holds.

Now return to (10) and we have

$$\int_{\zeta(x,t)}^{1}2u\frac{\partial u}{\partial x}dz+\frac{\partial S}{\partial t}-\frac{\partial\zeta}{\partial x}u^2(x,t,\zeta(x,t))$$
$$=\int_{\zeta(x,t)}^{1}\{-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}(\frac{\partial^2 u}{\partial z^2})\}dz+\frac{1}{\mathrm{Re}}\{\frac{\partial^2 S}{\partial x^2}+\left[\frac{\partial u(x,z,t)}{\partial x}\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x}$$
$$+\frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x})\}-E_{13}(x,t,1)+E_{13}(x,t,\zeta(x,t)). \tag{13}$$

Finally write

$$\int_{\zeta(x,t)}^{1}2u\frac{\partial u}{\partial x}dz=\frac{\partial}{\partial x}\int_{\zeta(x,t)}^{1}u^2dz+u^2(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x},$$

so that (13) becomes

$$\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}\int_{\zeta(x,t)}^{1}u^2dz$$
$$=\int_{\zeta(x,t)}^{1}-\frac{\partial p}{\partial x}dz+\frac{1}{\mathrm{Re}}\Big\{\frac{\partial^2 S}{\partial x^2}+\left[\frac{\partial u(x,z,t)}{\partial x}\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x}+\frac{\partial}{\partial x}\Big(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x}\Big)$$

$$+\frac{\partial u}{\partial z}(x,1,t)-\frac{\partial u}{\partial z}(x,\zeta(x,t),t)\Big\}-E_{13}(x,t,1)+E_{13}(x,t,\zeta(x,t)). \tag{14}$$

7. Scaling and the Balance of Linear Momentum in the $z-$direction

Since we are interested in both a comparatively thin boundary over a long section of the earth's surface and motion with large Reynolds number we scale the independent values as

$$x'=x\sqrt{\text{Re}},\quad t'=t\sqrt{\text{Re}},\quad z'=\frac{\lambda}{H}z \tag{15}$$

where λ is a typical dimensional length along the surface of the earth , i.e. in the $x-$direction. In this case we expect $\frac{H}{\lambda}$ to be small. Substitute (15) into (1c) and we obtain

$$\sqrt{\text{Re}}\{\frac{\partial w}{\partial t'}+u\frac{\partial w}{\partial x'}-w\frac{\partial u}{\partial x'}\}=-\frac{\lambda}{H}\frac{\partial p}{\partial z'}+\frac{\partial^2 w}{\partial x'^2}+\frac{1}{\text{Re}}(\frac{\lambda}{H})^2(\frac{\partial^2 w}{\partial z'^2})-\sqrt{\text{Re}}\frac{\partial E_{13}}{\partial x}+\frac{1}{Fr} \tag{16}$$

Multiply both sides of (16) by $\frac{H}{\lambda}$ and we see that if

$$\frac{H}{\lambda}\sqrt{\text{Re}}\ll 1,\quad \frac{H}{\lambda}\ll 1,\quad \frac{1}{\text{Re}}\frac{\lambda}{H}\ll 1, \tag{17}$$

to leading order we will have the static atmosphere balance relation

$$\frac{\partial p}{\partial z'}=\frac{1}{Fr}\frac{H}{\lambda} \tag{18}$$

as long as $\frac{1}{Fr}\frac{H}{\lambda}$ is of order one. Examination of (17) tells us that we must satisfy $\frac{1}{\text{Re}}\ll\frac{H}{\lambda}\ll 1/\sqrt{\text{Re}}$ or $\sqrt{\text{Re}}\ll\frac{\lambda}{H}\ll\text{Re}$ which is consistent with our large Reynolds number assumption. This list of assumptions motivates us to introduce data given by Arsen'yev in Section 6 of [1] and compare our assumptions with his data. Arsen'yev takes

$\nu=67.32m^2/\sec$

$H=980m$

$L=H$

$\lambda=7101.82m$

$V=31m/\sec$

$\mathrm{Re} = \frac{VL}{\nu} = 31 \cdot 980/67.32 = 451.27..$

$\sqrt{\mathrm{Re}} = 21.24, 1/\sqrt{\mathrm{Re}} = .047..$

$\frac{1}{Fr} = \frac{9.8m/\sec^2 \cdot 980m}{(31m/\sec)^2} = \frac{9.8 \cdot 980}{961} = 9.99..$

These values yield

$\frac{H}{\lambda} = \frac{980}{7101.82} = .137.. =$

$\frac{1}{\mathrm{Re}}\frac{\lambda}{H} = \frac{1}{902.55} \cdot \frac{1}{.137} = \frac{1}{123.64..} = .008...$

$\frac{1}{Fr}\frac{H}{\lambda} = 9.99 \cdot .137 = 1.37 \ldots$

where the desired relation $\sqrt{\mathrm{Re}} \ll \frac{\lambda}{H} \ll \mathrm{Re}$ requires

$$20,815m \ll \lambda \ll 441,980m. \tag{19}$$

Hence for our asymptotic analysis the choice of $\lambda = 7101.82$ suggested by Arsen'yev is too low and more reasonable figure appears to be a value satisfying (19). For example multiplying Arsen'yev value of λ by 5 would allow a better approximation to (17) with $\frac{1}{Fr}\frac{H}{\lambda} = 9.99 \cdot .137/5 = 1.37/5 \cdots = .27 \ldots$ We thus continue with our analysis based on the assumption that (17) is satisfied where a larger value of λ has been used. In terms of the original unscaled variables (18) gives

$$p = \frac{1}{Fr}(z - \zeta(x,t)) + p_0(x,t) \tag{20}$$

where $p_0(x,t)$ is the pressure on $z = \zeta(x,t)$ but for convenience we take p_0 to be a constant (as did Arsen'yev in his computations) and thus have $p = \frac{1}{Fr}(z - \zeta(x,t)) + p_0$. This of course yields $\frac{\partial p}{\partial x} = -\frac{1}{Fr}\frac{\partial \zeta}{\partial x}$.

8. Rescaled Momentum Equation in the $x-$direction

Substitution of (20) into (14) gives us

$$\begin{aligned}\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\int_{\zeta(x,t)}^{1} u^2 dz = \frac{1}{Fr}\frac{\partial \zeta}{\partial x}(1 - \zeta(x,t)) + \frac{1}{\mathrm{Re}}\Big\{\frac{\partial^2 S}{\partial x^2} + \left[\frac{\partial u(x,z,t)}{\partial x}\right]_{z=\zeta(x,t)}\frac{\partial \zeta}{\partial x} \\ + \frac{\partial}{\partial x}(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x}) + \frac{\partial u}{\partial z}(x,1,t) - \frac{\partial u}{\partial z}(x,\zeta(x,t),t)\Big\} \\ -E_{13}(x,t,1) + E_{13}(x,t,\zeta(x,t)). \end{aligned} \tag{21}$$

In terms of the scaled variables we then have

$$\begin{aligned}
&\sqrt{\mathrm{Re}}\Big\{\frac{\partial S}{\partial t'}+\frac{\partial}{\partial x'}\int_{\zeta(x,t)}^{1}u^2dz\Big\}\\
&=\sqrt{\mathrm{Re}}\frac{1}{Fr}\frac{\partial\zeta}{\partial x'}(1-\zeta(x,t))+\frac{\partial^2 S}{\partial x'^2}+\left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x'}\\
&\quad+\frac{\partial}{\partial x'}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x'})+\frac{1}{\mathrm{Re}}\frac{\lambda}{H}\frac{\partial u}{\partial z'}(x,1,t)-\frac{1}{\mathrm{Re}}\frac{\lambda}{H}\frac{\partial u}{\partial z'}(x,\zeta(x,t),t)\\
&\quad-E_{13}(x,t,1)+E_{13}(x,t,\zeta(x,t)).
\end{aligned}\tag{22}$$

Since we have assumed $\frac{1}{\mathrm{Re}}\frac{\lambda}{H}<<1$ delete the terms multiplying this quantity in (22). In addition we assume $|\zeta|<<1$ so that to leading order we have

$$\begin{aligned}
&\sqrt{\mathrm{Re}}\{\frac{\partial S}{\partial t'}+\frac{\partial}{\partial x'}\int_{\zeta(x,t)}^{1}u^2dz\}\\
&=\sqrt{\mathrm{Re}}\frac{1}{Fr}\frac{\partial\zeta}{\partial x'}+\frac{\partial^2 S}{\partial x'^2}+\left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}\frac{\partial\zeta}{\partial x'}+\frac{\partial}{\partial x'}(u(x,\zeta(x,t),t)\frac{\partial\zeta}{\partial x'})\\
&\quad-E_{13}(x,t,1)+E_{13}(x,t,\zeta(x,t)).
\end{aligned}\tag{23}$$

At this point to continue further we must place constitutive assumptions on $E_{13}(x,t,1),E_{13}(x,t,\zeta(x,t))$. The constitutive equation for $E_{13}(x,t,1)$ is given by

$$E_{13}(x,t,1)=f_*S,\quad\text{where}\quad f_*=\frac{3A}{H^2}\cdot\frac{1}{(1-n^2)}\tag{24}$$

where A is the coefficient of shear turbulent viscosity, H is still the height of the unperturbed layer, and $n=z_0/H$ with z_0a measure of the roughness of the surface of the earth say caused by the height of grass or crops. Arsen'yev in his paper takes $A=300m^2/\sec, n=0.05m/980m$. Hence n^2 is negligible compared to 1 and we take

$$f_*=\frac{3A}{H^2}=\frac{900m^2/\sec}{(980m)^2}\approx .001/\sec .$$

Unfortunately as we see Arsen'yev value for f_* is in dimensional units. To make the relation dimensionless we take out typical velocity $V=31m/\sec$and typical length $L=H=980m$ and hence find the typical time scale $T=L/V=(980/31)\sec=31.6\sec$ or roughly one half minute. Thus the dimensionless value of f_* is given $f_*\approx .001/\sec .\cdot 31.6\sec=.0316$.

The relation for $E_{13}(x,t,\zeta(x,t))$ is given by

$$E_{13}(x,t,\zeta(x,t)) = \frac{C_g}{H^2k^2} \cdot S^2.$$

Set $\alpha = \frac{C_g}{H^2k^2}$and note that Arsen'yev takes $C_g = .02, k = 1, H = 980m$. So we have the dimensional value of $E_{13}(x,t,\zeta(x,t)) = .02S^2/H^2$ and since the typical length $L = H$ we get the dimensionless relation $E_{13}(x,t,\zeta(x,t)) = (02S^2/H^2) \cdot H^2$, and hence $E_{13}(x,t,\zeta(x,t)) = 02S^2$ with the dimensionless value of $\alpha = .02$.

We insert these relations into (23) and find

$$\begin{aligned}&\sqrt{\mathrm{Re}}\left\{\frac{\partial S}{\partial t'} + \frac{\partial}{\partial x'}\int_{\zeta(x,t)}^{1} u^2 dz - \frac{1}{Fr}\frac{\partial \zeta}{\partial x'}\right\} \\ &= \frac{\partial^2 S}{\partial x'^2} + \left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}\frac{\partial \zeta}{\partial x'} + \frac{\partial}{\partial x'}\left(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x'}\right) - f_*S + \alpha S^2. \quad (25)\end{aligned}$$

As is readily seen, (25) contains the non-local term $\int_{\zeta(x,t)}^{1} u^2 dz$ and hence we have little choice but to make a crude local approximation. The simplest such approximation would be to replace the integral with its rectangular approximation, that is,

$$\begin{aligned}\int_{\zeta(x,t)}^{1} u^2 dz &\approx (1-\zeta(x,t))(u^2(x,\zeta(x,t),t) - u^2(x,1,t)) \\ &= (1-\zeta(x,t))u^2(x,\zeta(x,t),t)\end{aligned}$$

where we have used the no-slip boundary condition at $z = 1$. We again recall $|\zeta| << 1$ so that our local approximation is now$\int_{\zeta(x,t)}^{1} u^2 dz \approx u^2(x,\zeta(x,t),t)$ and substitute this into (25) to obtain

$$\begin{aligned}&\sqrt{\mathrm{Re}}\left\{\frac{\partial S}{\partial t'} + \frac{\partial}{\partial x'}u^2(x,\zeta(x,t),t) - \frac{1}{Fr}\frac{\partial \zeta}{\partial x'}\right\} \\ &= \frac{\partial^2 S}{\partial x'^2} + \left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}\frac{\partial \zeta}{\partial x'} + \frac{\partial}{\partial x'}\left(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x'}\right) - f_*S + \alpha S^2. \quad (26)\end{aligned}$$

The final issue we must address in this section is the appearance of the unpleasant $\left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}$ term in (25).

We impose here yet one more boundary condition on the free surface $z = \zeta(x,t)$:

$$\left[\frac{\partial u(x,z,t)}{\partial z}\right]_{z=\zeta(x,t)} = 0, \tag{27}$$

i.e that the tangential component of the velocity $u(x,z,t)$ has no appreciable change as we go across the free boundary. We then compute via the chain rule

$$\frac{\partial u(x,\zeta(x,t),t)}{\partial x'} = \left[\frac{\partial u(x,z,t)}{\partial z}\right]_{z=\zeta(x,t)} \frac{\partial \zeta}{\partial x'} + \left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)}$$

and thus (27) gives us

$$\frac{\partial u(x,\zeta(x,t),t)}{\partial x'} = \left[\frac{\partial u(x,z,t)}{\partial x'}\right]_{z=\zeta(x,t)} \tag{28}$$

which in turn yields

$$\sqrt{\mathrm{Re}}\left\{\frac{\partial S}{\partial t'} + \frac{\partial}{\partial x'}u^2(x,\zeta(x,t),t) - \frac{1}{Fr}\frac{\partial \zeta}{\partial x'}\right\}$$
$$= \frac{\partial^2 S}{\partial x'^2} + \frac{\partial u(x,\zeta(x,t),t)}{\partial x'}\frac{\partial \zeta}{\partial x'} + \frac{\partial}{\partial x'}\left(u(x,\zeta(x,t),t)\frac{\partial \zeta}{\partial x'}\right) - f_* S + \alpha S^2. \tag{29}$$

One additional remark may be of interest. We have made the approximation $\int_{\zeta(x,t)}^1 u^2 dz \approx u^2(x,\zeta(x,t),t)$ to eliminate the non-local term. But if one multiplies the balance of linear momentum equation (1a) by u and performs the integration to compute $\frac{\partial}{\partial t}\int_{\zeta(x,t)}^1 u^2 dz$ then we can repeat the process and get the moment equation for $\int_{\zeta(x,t)}^1 u^2 dz$ in terms of $\frac{\partial}{\partial x}\int_{\zeta(x,t)}^1 u^3 dz$. Continuing in this way we may compute a hierarchy of moment equations. But inevitably a closure rule must be enforced. Arsen'yev used $\int_{\zeta(x,t)}^1 u\,dz \approx u(x,\zeta(x,t),t)$, we used $\int_{\zeta(x,t)}^1 u^2 dz \approx u^2(x,\zeta(x,t),t)$,and so at some point a rule of the form $\int_{\zeta(x,t)}^1 u^p dz \approx u^p(x,\zeta(x,t),t)$ for some p will come into play.

9. The Dynamical System

Inspection of equations (5), (29) shows it would be a closed system in

ζ, S if an additional relation for $u(x,t,\zeta(x,t))$ is imposed. We rewrite (5) in the scaled variables and display rest of the closed system

$$\frac{\partial \zeta}{\partial t'} = \frac{\partial S}{\partial x'}. \tag{30}$$

We look for a traveling wave solution of (29), (30)

$$S(x,t) = F(\xi), \quad u(x,\zeta(x,t),t) = U(\xi), \quad \xi = x' - ct' \tag{31}$$

Substitution into (30) gives $\frac{\partial \zeta}{\partial t'} = F'(\xi)$ and hence

$$\zeta(x,t) = -F(\xi)/c + \zeta_0 = -S(x,t)/c + \zeta_0, \quad \zeta_0 \quad \text{a constant.}$$

Next require

$$F(\xi) \to 0, \quad \zeta(\xi) \to 0 \quad \text{as} \quad \xi \to \infty \quad \text{and/or} \quad \xi \to -\infty \tag{32}$$

so that $\zeta_0 = 0$. Insert this relation into (29) and we find

$$\sqrt{\mathrm{Re}}\{-cF + U^2 + \frac{1}{cFr}F\}' = F'' \ - U'F'/c - \ (UF'/c)' - f_*F \ + \alpha F^2 \tag{33}$$

Since Re is taken to be large the only possibility is that U, F must balance by the relation $\{-cF + U^2 + \frac{1}{cFr}F\} = const.$ By (32) we see $U^2 \to const.$ as $\xi \to \infty$ and/or $\xi \to -\infty$. Thus a simple choice of $U^2(\xi)$ is $const.$ and $U = U_0$ and we have $\{-c + \frac{1}{cFr}\}F = 0$ which yields the "locking" condition

$$-c + \frac{1}{cFr} = 0, \quad c^2 = \frac{1}{Fr}. \tag{34}$$

Hence we may rewrite (33) as

$$(1 - U_0/c)F'' \ - f_*F \ + \alpha F^2 = 0. \tag{35}$$

This equation is classical and has the soliton solution

$$\begin{aligned} &S = F = \beta \sec h^2(\xi/\Delta), \quad \beta = \frac{3f_*}{2\alpha}\Delta = \left(\frac{4(1-\frac{U_o}{c})}{f_*}\right)^{\frac{1}{2}}, \\ &\zeta(x,t) = -F(\xi)/c = -S(x,t)/c, \quad c^2 = \frac{1}{Fr}, \end{aligned} \tag{36}$$

where we must take $(1 - U_0/c) > 0$. We do this by taking $U_0 \geq 0$ and $c = -\frac{1}{(Fr)^{1/2}}$.

Arsen'yev obtained this result with U_0 in (36) equal to zero. Here (36) has the additional parameter U_0 which will either steepen or flatten the graph shown Figure 3 of Arsen'yev's paper. Finally note that the data given in our earlier Section 5 gives $Fr = (9.99.)^{-1}$ and so to two decimal places we have $\zeta(\xi) = .32\ F(\xi)$.

10. Swirling Flow and the Cyclostrophic Approximation

First recall that the Navier-Stokes equations are Galilean invariant which means that the velocity $(\mathring{u}, \mathring{v}, \mathring{w})$ moving in the frame (ξ, y', z', t') of the traveling wave with velocity $(c, 0, 0)$ is related to the velocity (u, v, w) via the formula

$$(\mathring{u}, \mathring{v}, \mathring{w})(\xi, y', z', t') + (c, 0, 0) = (u, v, w)(x', y', z', t')$$

and both $(\mathring{u}, \mathring{v}, \mathring{w})$, (u, v, w) satisfy the Navier-Stokes equations in their respective frames. Next write the velocity in the moving frame in cylindrical coordinates $r = (\xi^2 + (y')^2)^{1/2}, \xi = r\cos\theta, y' = r\sin\theta, y = \frac{y'}{\sqrt{\mathrm{Re}}}, z = \frac{H}{\lambda} z'$

$$\begin{aligned}
\frac{\partial v_r}{\partial t'} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} &= -\frac{\partial p}{\partial r},\\
\frac{\partial v_\theta}{\partial t'} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} &= -\frac{1}{r}\frac{\partial p}{\partial \theta},
\end{aligned}$$

where the small terms multiplying $\frac{1}{\sqrt{\mathrm{Re}}}, \frac{H}{\lambda}$ are omitted. We are interested in finding a v_θ, v_r which are independent of t', that is flow which is steady with respect to the moving frame. To this end we set $\frac{\partial v_r}{\partial t'} = \frac{\partial v_\theta}{\partial t'} = 0$ to obtain

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{37a}$$

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}. \tag{37b}$$

Next recall that (20) tells us $p = \frac{1}{Fr}(z - \zeta(x, t)) + p_0$ which in the moving frame gives us $p = \frac{1}{Fr}(z - (Fr)^{1/2}F(\xi)) + p_0$. Use the chain rule to compute

$$\begin{aligned} \frac{\partial p}{\partial r} &= -\frac{F'(\xi)\cos\theta}{(Fr)^{1/2}}, \quad \frac{\partial p}{\partial \theta} = \frac{F'(\xi) r\sin\theta}{(Fr)^{1/2}}, \\ \frac{\partial v_\theta}{\partial \theta} &= -\frac{\partial v_\theta}{\partial \xi} r\sin\theta + \frac{\partial v_\theta}{\partial y'} r\cos\theta, \quad \frac{\partial v_r}{\partial \theta} = -\frac{\partial v_r}{\partial \xi} r\sin\theta + \frac{\partial v_r}{\partial y'} r\cos\theta. \end{aligned} \tag{38}$$

Recall we have assumed no y' dependence and hence $\frac{\partial v_\theta}{\partial \theta} = -\frac{\partial v_\theta}{\partial \xi} r\sin\theta$, $\frac{\partial v_r}{\partial \theta} = -\frac{\partial v_r}{\partial \xi} r\sin\theta$. Substitution of these relations into (37) gives

$$v_r \frac{\partial v_r}{\partial r} - v_\theta \frac{\partial v_r}{\partial \xi}\sin\theta - \frac{v_\theta^2}{r} = \frac{F'(\xi)\cos\theta}{(Fr)^{1/2}} \tag{39a}$$

$$v_r \frac{\partial v_\theta}{\partial r} - v_\theta \frac{\partial v_\theta}{\partial \xi}\sin\theta + \frac{v_r v_\theta}{r} = \frac{-F'(\xi) r\sin\theta}{(Fr)^{1/2}} \cdot \frac{1}{r}. \tag{39b}$$

Notice (39b) admits the solution $v_r = 0$ when $\theta = 0$. Substitution into (39a) then gives the "cyclostrophic approximation" for our problem

$$v_\theta^2 = -\frac{\xi F'(\xi)}{(Fr)^{1/2}} \tag{40}$$

and hence we able to compute the swirling angular velocity v_θ on the line $y = 0$ from our traveling wave solution $F(\xi)$. Of course (40) only makes sense when $\xi F'(\xi) \le 0$ and hence we assume that our homoclinic orbit (35) is adjusted so that it has $\xi > 0$ ($\xi < 0$) when the homoclinic orbit is in the lower-half (upper-half) of the F, F' phase plane. A graph of v_θ is would have the shape given in Figure 5 of Arsen'yev's paper. Notice however that Arsen'yev's graph would match the observed meteorological data if his graph was flattened. As we have noted above the graph could be flattened by an appropriate choice of U_0.

The specific formula predicted by (40) is

$$v_\theta^2 = 2\xi\beta \mathrm{sech}^2(\xi/\Delta)\tanh(\xi/\Delta)/\Delta(Fr)^{1/2}.$$

If we substitute the values used by Arsen'yev we note that here he has used the viscosity $\nu = 3.96$ which changes the Reynolds number to $\mathrm{Re} = 7671$, $(\mathrm{Re})^{1/2} = 87.5$. The values $\beta = 2.37$, $1/(Fr)^{1/2} = (9.99)^{1/2}$, $\Delta = \frac{20(1-U_0)^{1/2}}{(3.16)^{1/2}}$

remain unchanged and so

$$v_\theta^2 = 14.978(\xi/\Delta)\mathrm{sech}^2(\xi/\Delta)\tanh(\xi/\Delta).$$

For purposes of comparison with Arsen'yev this would give a value

$$\begin{aligned} v_\theta &= [14.978(\xi/\Delta)\tanh(\xi/\Delta)]^{1/2}\mathrm{sech}(\xi/\Delta)(31m/\sec) \\ &= 119.97[(\xi/\Delta)\tanh(\xi/\Delta)]^{1/2}\mathrm{sech}(\xi/\Delta)m/\sec \qquad (41) \end{aligned}$$

where we have made v_θ dimensional by putting in our typical velocity. We can now observe first that since $v_\theta^2 = 2\xi\beta\mathrm{sech}^2(\xi/\Delta)\tanh(\xi/\Delta)/\Delta(Fr)^{1/2}$ the dimensional maximum value of v_θ^2 is given by

$$\max v_\theta^2 = \frac{\beta V^2}{(Fr)^{1/2}} \max\{2\sigma\mathrm{sech}^2(\sigma)\tanh(\sigma)\}$$

and the value of Δ is irrelevant for computing $\left[v_\theta^2\right]_{\max}$, that is, the only quantity that plays a role is $\frac{\beta V^2}{(Fr)^{1/2}}$. In addition a plot of the graph of $\sigma\mathrm{sech}^2(\sigma)\tanh(\sigma)$ shows the maximum occurs at approximately $\sigma = 1$ and hence at approximately $\xi = \Delta$. Return to the definition of $\xi = x' - ct' = (x - ct)(\mathrm{Re})^{1/2} = (\frac{x^*}{H} - c\frac{t^*}{T})(\mathrm{Re})^{1/2}$ we see that the statement $\xi = \Delta$ implies that the approximate distance from the zero point of the wave propagating with velocity c to the maximum value is given $(\frac{x^*}{H})(\mathrm{Re})^{1/2} = \Delta, x^* = \Delta H/(\mathrm{Re})^{1/2}$. Hence for Arsen'yev's data $x^* \approx \frac{20(1-U_0)^{1/2}}{87.5\cdot(3.16)^{1/2}} \cdot 980 = 126.55$ if the value $U_0 = 0$ is chosen. This value of x^* is quite close the one seen in Arsen'yev's Figure 5.

11. Conclusion

In the above discussion we have given a derivation based on asymtotic analysis of Arsen'yev's second order ordinary differential equation for the propagation of a boundary layer where thermal inversion occurs. The equation we obtain has one more free parameter U_0 than Arsen'yev's original ODE. Also we note that our analysis suggests that this ODE is only a first order closure in what could be a hierarchal system of ODEs. The derivation and analysis of the hierarchal system would provide an appealing generalization of Arsen'yev's ODE and could be the basis of further comparison with experimental data.

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