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SCALAR VISCOUS CONSERVATION LAWS IN $\mathbb{R}^2$

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Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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Abstract

In this paper, we follow the framework of construction of Green's function for Boltzmann equation in [12] to establish the pointwise structure for the stability of shock profile for scalar viscous conservation law in $\mathbb{R}^2$. We treat the 2-D viscous conservation law like a kinetic equation, and introduce the shock front-remainder decomposition to separate the effect coming from shock front. Based on this decomposition, we construct the Green's function and get the pointwise structure of shock front.

1. Introduction

The viscous conservation law in 2-d is

$$u_t + f(u)_x + g(u)_y = \triangle u.$$

Assume the flux function $f(u)$ is strictly convex:

$$f''(u) \geq a > 0. \tag{1.1}$$

For any given states $u_+ < u_-$, when $|u_+ - u_-| \leq \epsilon \ll 1$, there is a planar

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traveling wave solution $\phi(x-\sigma t)$ connecting the given states, [1]:

$$\begin{cases} -\sigma\phi' + f(\phi)_x = \phi'', \\ \lim\limits_{x\to\pm\infty} \phi(x) = u_\pm, \end{cases} \tag{1.2}$$

where the speed σ of the traveling wave is determined by the Rankine-Hugoniot condition:

$$\sigma = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

In the study of whole space problem, without loss of generality, one can assume the speed σ to be 0. Here, to be consistent with the convex condition (1.1) of f, the planar viscous shock wave $\phi(x)$ satisfies:

$$\phi'(x) < 0, \quad \phi'(x) = O(|u_+ - u_-|^2)e^{-(u_- - u_+)||x|/C}. \tag{1.3}$$

One interesting problem is whether the planar wave $\phi(x)$ is stable under a multi-dimensional perturbation:

$$\begin{cases} u_t + f(u)_x + g(u)_y = \triangle u, \\ u(x,y,0) = \phi(x) + u_0(x,y), \end{cases} \tag{1.4}$$

where the perturbation $u_0(x,y)$ satisfies smallness condition in certain norm.

Goodman gave a positive answer to this problem in [3, 4]. The stability is proved in [3] and the asymptotic behavior in L^1 is given in [4] under the assumptions $f(u) = \frac{u^2}{2}$ and $g(\phi(x)) \in L^2(\mathbb{R})$. Both of the results are proved by the anti-derivative method which is classical in the study of 1-D viscous shock for scalar conservation law given by II'in and Oleinik [5] and the one for systems of conservation laws initiated by Kawashima and Matsumura [7] and Goodman [2] and completed by Liu [10].

During the proof of stability, the shock front tracing is always important since it is corresponding to a slower decaying term and used to take the non-zero mass part preparing for construction of anti-derivative. In 1-D case, due to the conservation law, the shift of shock front is a constant; while in multi-D case, it is a function of y and t which is not possible to be determined only by initial data. In fact, the equation for shock front and the equation for anti-derivative are coupled together which makes the energy estimate far more complicate than 1-D case. In [3], the structure of the shock front is proposed after a formal expansion. The dominated part is supposed to satisfy a heat equation and thus converges to 0 as time goes to infinity. In [4], they consider the 2-D case with $f(u) = \frac{u^2}{2}$ and $g(\phi(x)) \in L^2(\mathbb{R})$ and obtain an expression for the asymptotic form of small perturbations. They give a rigorous proof to verify that the leading order is governed by a heat equation with the diffusion coefficient depending on forces transverse to the shock front.

In this paper, we go on with the interest on the structure of shock front which gives the leading order in the asymptotic form of small perturbations. Besides, we introduce a new understanding and separation of shock front and remainder. It may help us to create a new method which is more suitable for the multi-D case than the anti-derivative method.

To get a clear picture of the perturbation, we construct the Green's function for the pointwise structure. The difficulty lies in that the linear equation is with variable coefficients and it is difficult to apply Fourier analysis. We do not follow the classical thinking about approximated Green's function, [11, 16], but take the special structure of the planar wave into consideration and make an analogy between the problem and kinetic equation. We introduce the shock front-remainder decomposition similar to macro-micro decomposition for kinetic equation and adopt the methodology in construction of the Green's function for Boltzmann equation in [12] and [13] to overcome the difficulties due to the non-constant coefficients. Finally we give the pointwise structure of small perturbation with only convexity assumption on $f(u)$ and it is very interesting that the structure shows that the hyperbolic speed λ (the pointwise structure is given in (1.10)) is determined by certain average of $g(\phi(x))$:

$$\lambda = \frac{g(u_+) - g(u_-)}{u_+ - u_-}.$$

Rewrite the scalar conservation law (1.4) into a Boltzmann-like equation:

$$u_t + g(u)_y - u_{yy} = \mathbf{Q}(u) \equiv u_{xx} - f(u)_x.$$

Consider the equation:

$$\mathbf{Q}(u) = 0. \tag{1.5}$$

The solution is just the travelling wave solution $\phi(x)$ given by (1.2) with $\sigma = 0$.

Denote:

$$\mathbf{M}(x) = (u_+ - u_-)^{-1}\phi'(x), \tag{1.6}$$

and reconsider linearization of (1.4) by setting:

$$u(x,y,t) = \phi(x) + \sqrt{\mathbf{M}(x)}w(x,y,t).$$

The equation for the perturbation $w(x,y,t)$ is

$$w_t + g'(\phi(x))w_y - w_{yy} = \mathbf{L}w + N(w). \tag{1.7}$$

Here, the linear and nonlinear operators $\mathbf{L}$ and N are given by

$$\mathbf{L}w \equiv \mathbf{M}^{-1/2}\left(\left(\mathbf{M}^{1/2}w\right)_{xx} - \left(f'(\phi)\mathbf{M}^{1/2}w\right)_x\right), \tag{1.8}$$

and

$$\begin{aligned} N(w) &\equiv -\mathbf{M}^{-1/2}\left(N_1(w)\right)_x - \mathbf{M}^{-1/2}\left(N_2(w)\right)_y \\ &\equiv -\mathbf{M}^{-1/2}\left(f(\phi+\mathbf{M}^{1/2}w) - f(\phi) - f'(\phi)\mathbf{M}^{1/2}w\right)_x \\ &\quad -\mathbf{M}^{-1/2}\left(g(\phi+\mathbf{M}^{1/2}w) - g(\phi) - g'(\phi)\mathbf{M}^{1/2}w\right)_y. \end{aligned}$$

The Green's function $\mathbb{G}(x,y,t;x_*)$ for the equation (1.7) is given by

$$\begin{cases} \partial_t\mathbb{G} + g'(\phi(x))\partial_y\mathbb{G} - \partial_{yy}\mathbb{G} = \mathbf{L}\mathbb{G}, \\ \mathbb{G}(x,y,0;x_*) = \delta(x-x_*)\delta(y). \end{cases} \tag{1.9}$$

Here, $\delta(z)$ is the dirac delta function.

The main result of the paper is:

Theorem 1.1. *The Green's function $\mathbb{G}(x,y,t;x_*)$ of (1.9) satisfies:*

$$\mathbb{G}(x,y,t;x_*) = \sum_{j=0}^{5} v^j(x,y,t;x_*) + \mathbb{G}^{R5}(x,y,t;x_*),$$

where the singular waves $v^j(x,y,t;x_*)$ *are given as follows:*

$$v^j(x,y,t;x_*) \;=\; O(1)\begin{cases} \dfrac{e^{-t/C-\frac{(x-x_*)^2+(y-\lambda t)^2}{4t}}}{t^{(2-j)/2}}, & for\ j=0,1,\\[2ex] e^{-(|x|+|y|+t)/C}, & for\ j=2,3,4,5,\end{cases}$$

and

$$\left\|\partial_y^k \mathbb{G}^{R5}(\cdot,y,t;x_*)\right\|_{L^2_x} \;=\; O(1)\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_1(1+t)}}}{1+t^{1/2+k/2}}+e^{-(|y|+t)/C_1}\right), \tag{1.10}$$

with the constant λ *given by*

$$\lambda \equiv \frac{g(u_+)-g(u_-)}{u_+-u_-}.$$

Furthermore, the remainder decays faster:

$$\begin{aligned}&\left\|\mathbf{P}_1\mathbb{G}^{R5}(\cdot,y,t;x_*)\right\|_{L^2_x},\\ &\left\|\mathbb{G}^{R5}(\cdot,y,t;x_*)\mathbf{P}_1\right\|_{L^2_x} = O(1)\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_1(1+t)}}}{1+t}+e^{-(|y|+t)/C_1}\right),\end{aligned}$$

where the remainder operator $\mathbf{P}_1$ *is defined in* (2.1).

The rest of the paper is arranged as follows. In Section 2, we define the shock front-remainder decomposition which is similar to the macro-micro decomposition for the Boltzamann equation and show the nonpositivity of shock front operator. In Section 3, we study a model linear problem with a regular initial function. In Section 4, we design a Picard iteration to separate the singular and regular terms in the Green's function. The singular part can be put down directly while the regular part can be deduced into the model problem studied in Section 3. We also use the bootstrap to improve the L^2_x estimate into the weighted $L^\infty_{x,1/2}$ estimate and give the pointwise structure of the Green's function.

2. Preliminaries

In this section, we define the shock front-remainder decomposition which replaces the anti-derivative method in [3] and prove the non-positivity of the linear operator $\mathbf{L}$ as a preparation for the construction of Green's function.

2.1. Shock front-remainder decomposition

We denote L^2_x to be the normal Hilbert space for functions on x with given inner product:

$$\begin{cases} (h,v) \equiv \int_{\mathbb{R}} h(x)v(x)dx \text{ for } h,v \in L^2_x, \\ \|h\|_{L^2_x} \equiv \sqrt{(h,h)}. \end{cases}$$

The null space of $\mathbf{L}$ in the Hilbert space L^2_x is a one-dimensional vector space with basis χ:

$$\chi \equiv \mathbf{M}^{1/2}.$$

For any function $h \in L^2_x$, the shock front-remainder decomposition $(\mathbf{P}_0, \mathbf{P}_1)$ is defined as follows:

$$\begin{cases} h \equiv \mathbf{P}_0 h + \mathbf{P}_1 h, \\ \mathbf{P}_0 h \equiv (\chi, h)\chi, \\ \mathbf{P}_1 h \equiv h - \mathbf{P}_0 g. \end{cases} \tag{2.1}$$

The decomposition defined here reveals the seperation of wave front and remaining parts, the nature showed in [3]. We will show it later.

2.2. Non-positivity for linear operator L

Non-positivity is one of the most important properties of linear collision operator in kinetic theory. We will prove that it is also true for the linear operator $\mathbf{L}$ defined in (1.8).

Lemma 2.2. *The linearized collision operator* $\mathbf{L}$ *is non-positive definite.*

Proof. It can be obtained by straightforward computations:

$$\begin{aligned} (v, \mathbf{L}v) &\equiv \left(v, \mathbf{M}^{-1/2}\left(\left(\mathbf{M}^{1/2}v\right)_{xx} - \left(f'(\phi)\mathbf{M}^{1/2}v\right)_x\right)\right) \\ &= -\left(v_x, \mathbf{M}^{-1/2}\left(\left(\mathbf{M}^{1/2}v\right)_x - f'(\phi)\mathbf{M}^{1/2}v\right)\right) \end{aligned}$$

$$
\begin{aligned}
&\quad - \left(v, \left(\mathbf{M}^{-1/2}\right)_x \left(\left(\mathbf{M}^{1/2} v\right)_x - f'(\phi)\mathbf{M}^{1/2} v\right)\right) \\
&= -\left(v_x, v_x + \frac{1}{2}\mathbf{M}^{-1}\mathbf{M}_x v - f'(\phi) v\right) \\
&\quad - \left(v, -\frac{1}{2}\mathbf{M}^{-1}\mathbf{M}_x \left(v_x + \frac{1}{2}\mathbf{M}^{-1}\mathbf{M}_x v - f'(\phi) v\right)\right) \\
&= -(v_x, v_x) + (v_x, f'(\phi) v) - \frac{1}{2}\left(v, \mathbf{M}^{-1}\mathbf{M}_x \left(f'(\phi) - \frac{1}{2}\mathbf{M}^{-1}\mathbf{M}_x\right) v\right).
\end{aligned}
$$

The definition (1.6) for $\mathbf{M}(x)$ and the equation (1.5) for traveling wave $\phi(x)$ yield that

$$
\mathbf{M}_x = (u_+ - u_-)^{-1}\phi'' = (u_+ - u_-)^{-1} f'(\phi)\phi' = f'(\phi)\mathbf{M}.
$$

Thus, one has

$$
\begin{aligned}
(v, \mathbf{L}v) &= -(v_x, v_x) + (v_x, f'(\phi) v) - \frac{1}{4}\left(v, \left(f'(\phi)\right)^2 v\right) \\
&= -\left(v_x - \frac{f'(\phi) v}{2}\right)^2 \leq 0.
\end{aligned}
$$

□

3. A Linear Problem

Consider the linear equation in (1.7):

$$
\begin{cases}
v_t + g'(\phi(x)) v_y - v_{yy} = \mathbf{L}v, \\
v(x, y, 0) = v_0(x, y).
\end{cases}
\tag{3.1}
$$

3.1. Fourier analysis: Pointwise estimate for long-wave component

Take the Fourier transform in y-variable. It turns out that the coefficient $g'(\phi(x))$ becomes constant with respect to y-variable:

$$
\begin{cases}
\hat{v}(x, \eta, t) \equiv \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} e^{-i\eta y} v(x, y, t) dy, \\
\partial_t \hat{v} + i\eta g'(\phi(x))\hat{v} + \eta^2 \hat{v} - \mathbf{L}\hat{v} = 0, \\
\hat{v}(x, \eta, 0) = \hat{v}_0(x, \eta).
\end{cases}
$$

We can formally put down the solution $v(x,y,t)$ as follows:

$$\begin{aligned} \hat{v}(x,\eta,t) &= e^{(-i\eta g'(\phi(x))-\eta^2+\mathbf{L})t}\hat{v}_0(x,\eta), \\ v(x,y,t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\eta y+(-i\eta g'(\phi(x))-\eta^2+\mathbf{L})t}\hat{v}_0(x,\eta)d\eta. \end{aligned} \tag{3.2}$$

To get estimates from (3.2), one needs to study the spectrum of the operator $-i\eta g'(\phi(x))-\eta^2+\mathbf{L}$. Denote

$$\sigma(\eta) \equiv \Big\{\lambda\in\mathbb{C}: \text{ there exist non-trivial } e\in L^2_x \text{ such that } \big(-i\eta g'(\phi(x))-\eta^2+\mathbf{L}\big)\, e=\lambda e\Big\}.$$

One has the following description for $\sigma(\eta)$:

Lemma 3.1. *There exist $\kappa_0>0$ and $\kappa_1>0$ such that for $|\eta|>\kappa_0$,*

$$\sigma(\eta)\subset\{z\in\mathbb{C}: Re(z)\le -\kappa_1\}, \tag{3.3}$$

and for any $|\eta|\le\kappa_0$,

$$\begin{cases} \sigma(\eta)=-i\lambda\eta-A|\eta|^2+O(1)|\eta|^3, \\ A=1-(\mathbf{P}_1 g'(\phi)\chi,\mathbf{L}^{-1}\mathbf{P}_1 g'(\phi)\chi)>0, \end{cases} \tag{3.4}$$

and λ is the eigenvalue of $\mathbf{P}_0 g'(\phi)\mathbf{P}_0$:

$$\mathbf{P}_0 g'(\phi)\chi=\lambda\chi, \qquad \lambda=\frac{g(u_+)-g(u_-)}{u_+-u_-}. \tag{3.5}$$

Furthermore, for $|\eta|\ll 1$, there exist normalized holomorphic eigenvectors $\mathbf{e}(\eta)\in L^2_x$ of the operator $-i\eta g'(\phi(x))-\eta^2+\mathbf{L}$:

$$\begin{cases} \big(-i\eta g'(\phi(x))-\eta^2+\mathbf{L}\big)\,\mathbf{e}(\eta)=\sigma(\eta)\mathbf{e}(\eta), \\ \mathbf{e}(\eta)=\chi+\eta\mathbf{e}'(0)+O(1)|\eta|^2, \end{cases} \tag{3.6}$$

and

$$\mathbf{e}'(0)=i\mathbf{L}^{-1}\mathbf{P}_1 g'(\phi(x))\chi. \tag{3.7}$$

Proof. Statement (3.3) for $|\eta|>\kappa_0$ can be resulted from spectral gap at the origin due to the dissipation $-\eta^2$ in the operator and also the non-positivity of operator $\mathbf{L}$.

When η is small, the spectrum information comes from the perturbation theory, [6]. From (2.1), one has

$$\begin{cases} \mathbf{P}_0 g'(\phi(x))\chi = \lambda\chi, \\ \mathbf{P}_0\left(-i\eta g'(\phi(x)) - \eta^2 + \mathbf{L}\right)\chi = (-i\eta\lambda - \eta^2)\chi, \end{cases}$$

with λ given in (3.5). This suggests that $\sigma(\eta)$ and $\mathbf{e}(\eta)$ are small perturbations of $-i\eta\lambda - \eta^2$ and χ.

We consider the normalized eigenvector $\mathbf{e}(\eta)$ and $\sigma(\eta)$ in the following form:

$$\begin{cases} \mathbf{e}(\eta) = \chi + \mathbf{b}_1(\eta), \;\; \mathbf{P}_0\mathbf{b}_1 \equiv 0, \\ \sigma(\eta) = -i\eta(\lambda - i\eta + \rho(\eta)), \\ \mathbf{b}_1(0) = \rho(0) = 0. \end{cases} \tag{3.8}$$

To estimate the dependent variable ρ, one applies the shock front-remainder decomposition to $(-i\eta g'(\phi(x)) - \eta^2 + \mathbf{L})\mathbf{e} = \sigma\mathbf{e}$ to obtain

$$\begin{cases} \mathbf{P}_0\left(g'(\phi(x)) - i\eta\right)\mathbf{b}_1 = \rho\chi, \\ -i\eta\mathbf{P}_1 g'(\phi(x))(\chi + \mathbf{b}_1) + \mathbf{L}\mathbf{b}_1 = -i\eta(\lambda + \rho)\mathbf{b}_1. \end{cases} \tag{3.9}$$

The second equality gives rise to

$$\mathbf{b}_1 = i\eta\left[\mathbf{L} - i\eta\mathbf{P}_1 g'(\phi(x)) + i\eta(\lambda + \rho)\right]^{-1}\mathbf{P}_1 g'(\phi(x))\chi. \tag{3.10}$$

Substitue (3.10) into the first equation of (3.9) to result in

$$\rho\chi = i\eta\mathbf{P}_0\left(g'(\phi(x)) - i\eta\right)\left[\mathbf{L} - i\eta\mathbf{P}_1 g'(\phi(x)) + i\eta(\lambda + \rho)\right]^{-1}\mathbf{P}_1 g'(\phi(x))\chi.$$

This gives an equation for two variables (ρ, η):

$$\mathscr{G}(\rho,\eta) \equiv \rho - i\eta\Big(\chi, \mathbf{P}_0\left(g'(\phi(x)) - i\eta\right)\left[\mathbf{L} - i\eta\mathbf{P}_1 g'(\phi(x)) + i\eta(\lambda + \rho)\right]^{-1} \mathbf{P}_1 g'(\phi(x))\chi\Big) = 0. \tag{3.11}$$

Since $\partial_\rho\mathscr{G}(0,0) = 1 \neq 0$, by the implicit function theorem, the dependent variable ρ is holomorphic function of η when $|\eta| \ll 1$.

From the equation (3.11), one can also easily observe that

$$\rho = i\eta \left(\mathbf{P}_1 g'(\phi(x))\chi, \mathbf{L}^{-1}\mathbf{P}_1 g'(\phi(x))\chi\right) + O(1)|\eta|^2.$$

Substitue it into σ in (3.8) and one concludes that:

$$\sigma(\eta) = -i\lambda\eta + \eta^2 \left(-1 + \left(\mathbf{P}_1 g'(\phi(x))\chi, \mathbf{L}^{-1}\mathbf{P}_1 g'(\phi(x))\chi\right)\right) + O(1)|\eta|^3,$$

which yields (3.4). The non-positivity of $\left(\mathbf{P}_1 g'(\phi(x))\chi, \mathbf{L}^{-1}\mathbf{P}_1 g'(\phi(x))\chi\right)$ comes from Lemma 2.2.

The statements (3.6) and (3.7) are direct results of (3.8) and (3.10). □

Lemma 3.1 tells full information when η is small and only partial information for η large. This leads to different treatments due to the frequency η. We introduce the long wave-short wave decomposition as follows:

$$v(x,y,t) = v_L(x,y,t) + v_S(x,y,t),$$

$$\begin{cases} v_L(x,y,t) \equiv \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{|\eta|<\kappa_0} e^{i\eta y}\hat{v}(x,\eta,t)d\eta \\ \qquad\qquad\quad = \frac{1}{\sqrt{2\pi}} \displaystyle\int_{|\eta|<\kappa_0} e^{i\eta y + (-i\eta g'(\phi(x)) - \eta^2 + \mathbf{L})t}\hat{v}_0(x,\eta)d\eta, \\ v_S(x,y,t) \equiv \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{|\eta|>\kappa_0} e^{i\eta y}\hat{v}(x,\eta,t)d\eta \\ \qquad\qquad\quad = \frac{1}{\sqrt{2\pi}} \displaystyle\int_{|\eta|>\kappa_0} e^{i\eta y + (-i\eta g'(\phi(x)) - \eta^2 + \mathbf{L})t}\hat{v}_0(x,\eta)d\eta. \end{cases}$$

The following two lemmas come directly from Planchel equality and Lemma 3.1:

Lemma 3.2. *For any $s \geq 0$,*

$$\|v_L(\cdot,\cdot,t)\|_{H^s_y(L^2_x)} = O(1)\|v_0\|_{L^2_y(L^2_x)}.$$

Lemma 3.3. *There exists a constant $C > 0$ such that the short wave component $v_S(x,y,t)$ satisfies*

$$\|v_S(\cdot,\cdot,t)\|_{H^s_y(L^2_x)} \ \leq \ Ce^{-t/C}\|v_0\|_{H^s_y(L^2_x)},$$

$$\|v_S(\cdot,\cdot,t)\|_{L^\infty_y(L^2_x)} \;\le\; Ce^{-t/C}\left(\|v_0\|_{L^2_y(L^2_x)}\|\partial_y v_0\|_{L^2_y(L^2_x)}\right)^{1/2}.$$

Lemma 3.2 gives a rough estimate for the long wave component. To get the pointwise time and spatial structure, one restricts in the finite Mach number region and apply contour integral with the information in Lemma 3.1. Denote

$$\mathscr{R}_{\mathscr{M}} \equiv \left\{(y,t)\in\mathbb{R}\times\mathbb{R}_+ : \frac{|y|}{|\lambda| t}\le \mathscr{M}\right\}.$$

Lemma 3.4. *There exist positive constants C_0, C_1 such that for any $(y,t)\in\mathscr{R}_2$,*

$$\begin{cases} \|\mathbb{G}_L\|_{L^2_x} = O(1)\left(\dfrac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + e^{-t/C_1}\right), \\ \|\mathbf{P}_1\mathbb{G}_L\|_{L^2_x}, \|\mathbb{G}_L\mathbf{P}_1\|_{L^2_x} = O(1)\left(\dfrac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-t/C_1}\right), \end{cases}$$

where

$$\mathbb{G}_L(x,y,t) \equiv \int_{|\eta|<\kappa_0} e^{i\eta y + (-i\eta g'(\phi(x)) - \eta^2 + L)t}\, d\eta,$$

and λ is given in (3.5).

Remark 3.5. The proof is the same to that for Lemma 3.8 in [12] and we omit the details here. As a conclusion of Lemma 3.4, one has that

$$\begin{aligned} \|v_L(\cdot,\cdot,t)\|_{L^\infty_y(L^2_x)} &= \frac{1}{\sqrt{2\pi}}\left\|\int_{|\eta|<\kappa_0} e^{i\eta y+(-i\eta g'(\phi(x))-\eta^2+L)t}\hat{v}_0(x,\eta)\,d\eta\right\|_{L^\infty_y(L^2_x)} \\ &= O(1)\|\mathbb{G}_L\|_{L^\infty_y(L^2_x)}\|\hat{v}_0\|_{L^\infty_{x,\eta}} \\ &= O(1)\|\mathbb{G}_L\|_{L^\infty_y(L^2_x)}\|v_0\|_{L^1_y(L^\infty_x)}. \end{aligned} \tag{3.12}$$

3.2. Energy method: Spatial structure outside finite mach number region

Lemma 3.4 gives a pointwise structure inside the finite Mach number region and it remains to describe the solution outside the finite Mach number region. Since solution decays fast enough outside the cone, one applies weighted energy method to yield exponetially sharp estimates.

Lemma 3.6. *There exists a constant $C>0$ such that for any $|y|>2|\lambda|t$, the solution $v(x,y,t)$ for (3.1) satisfies*

$$\|v(\cdot,y,t)\|_{L^2_x}\le Ce^{-(|y|+t)/C}\left\|e^{|y|/4}v_0\right\|_{H^1_y(L^2_x)}.$$

Proof. The weighted function is chosen to be

$$\mathscr{E}(y,t)=e^{(|y|-a|\lambda|t)/M},$$

and it satisfies

$$\mathscr{E}_t=-\frac{a|\lambda|}{M}\mathscr{E},\quad \mathscr{E}_y=\frac{|y|}{My}\mathscr{E}. \tag{3.13}$$

Here, $1<a<2$ and M is sufficiently large.

Multiply (3.1) with $\mathscr{E}(y,t)v(x,y,t)$ and integrate the product with respect to (x,y) in $\mathbb{R}^2$:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^2}\mathscr{E}v\left(v_t+g'(\phi(x))v_y-v_{yy}-Lv\right)dxdy\\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\mathscr{E}v^2dxdy-\frac{1}{2}\int_{\mathbb{R}^2}\mathscr{E}_tv^2dxdy-\frac{1}{2}\int_{\mathbb{R}^2}\mathscr{E}_yg'(\phi(x))v^2dxdy\\
&\quad+\int_{\mathbb{R}^2}\mathscr{E}v_y^2dxdy+\int_{\mathbb{R}^2}\mathscr{E}_yvv_ydxdy-\int_{\mathbb{R}}\mathscr{E}(v,Lv)dy\\
&\ge \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\mathscr{E}v^2dxdy+\frac{1}{2}\frac{a|\lambda|}{M}\int_{\mathbb{R}^2}\mathscr{E}v^2dxdy+\int_{\mathbb{R}^2}\mathscr{E}v_y^2dxdy\\
&\quad-\frac{1}{2}\frac{1}{M}\int_{\mathbb{R}^2}\frac{|y|}{y}\mathscr{E}g'(\phi(x))v^2dxdy+\frac{1}{M}\int_{\mathbb{R}^2}\frac{|y|}{y}\mathscr{E}vv_ydxdy\\
&\ge \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\mathscr{E}v^2dxdy+\frac{1}{4}\frac{a|\lambda|}{M}\int_{\mathbb{R}^2}\mathscr{E}v^2dxdy+\frac{1}{2}\int_{\mathbb{R}^2}\mathscr{E}v_y^2dxdy.
\end{aligned}\tag{3.14}$$

In the last two inequalities, one uses (3.13), Lemma 2.2 and Holder inequality. One also uses that $g'(z)$ is continuous on $[u_+,u_-]$ and thus $|g'(z)|-|\lambda|\ll 1$. Thus, (3.14) yields

$$\int_{\mathbb{R}}\mathscr{E}\|v(\cdot,y,t)\|^2_{L^2_x}dy\le\int_{\mathbb{R}}e^{|y|/M}\|v_0(\cdot,y)\|^2_{L^2_x}dy. \tag{3.15}$$

Similarly, one has the estimates for the derivative:

$$\begin{aligned}
0 &= \int_{\mathbb{R}^2} \mathscr{E} v_y \left((v_y)_t + g'(\phi(x)) v_{yy} - v_{yyy} - L v_y \right) dxdy \\
&\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \mathscr{E} v_y^2 dxdy + \frac{1}{2}\frac{a}{M}\int_{\mathbb{R}^2} \mathscr{E} v_y^2 dxdy + \int_{\mathbb{R}^2} \mathscr{E} v_{yy}^2 dxdy \\
&\quad - \frac{1}{2}\frac{1}{M}\int_{\mathbb{R}^2} \frac{|y|}{y} \mathscr{E} g'(\phi(x)) v_y^2 dxdy + \frac{1}{M}\int_{\mathbb{R}^2} \frac{|y|}{y} \mathscr{E} v_y v_{yy} dxdy \\
&\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \mathscr{E} v_y^2 dxdy + \frac{1}{4}\frac{a}{M}\int_{\mathbb{R}^2} \mathscr{E} v_y^2 dxdy + \frac{1}{2}\int_{\mathbb{R}^2} \mathscr{E} v_{yy}^2 dxdy,
\end{aligned}$$

which yields

$$\int_{\mathbb{R}} \mathscr{E} \|v_y(\cdot, y, t)\|_{L_x^2}^2 dy \leq \int_{\mathbb{R}} e^{|y|/M} \|(v_0)_y(\cdot, y)\|_{L_x^2}^2 dy. \tag{3.16}$$

Combining (3.15), (3.16) and Sobolev's inequality, one has

$$\begin{aligned}
e^{(|y| - a|\lambda| t)/2} \|v(\cdot, y, t)\|_{L_x^2}^2 &\leq C \int_{\mathbb{R}^2} e^{|y|/M} \left(v_0^2 + (v_0)_y^2 \right) dxdy \\
&\leq C \left\| e^{|y|/2M} v_0 \right\|_{H_y^1(L_x^2)}^2 .
\end{aligned}$$

If one restricts the region to be $|y| > 2|\lambda| t$, then $|y| - a|\lambda| t > (|y| + t)/C$ for some positive constant C. Thus we finish the proof. □

3.3. Conclusion

Lemma 3.3, (3.12) and Lemma 3.6 yield the pointwise structure of the linear problem (3.1):

Lemma 3.7. *There exist positive constants C_0 and C_1 such that if the initial function $v_0(x, y)$ satisfies $e^{|y|/C_0} v_0 \in H_y^1(L_x^2)$ and $v_0 \in L_y^1(L_x^\infty)$, then the solution $v(x, y, t)$ of (3.1) satisfies*

$$\|v(\cdot, y, t)\|_{L_x^2} = O(1) \left(\frac{e^{-\frac{(y - \lambda t)^2}{C_1(1+t)}}}{\sqrt{1+t}} + e^{-(|y| + t)/C_1} \right), \tag{3.17}$$

where

$$\lambda = \frac{g(u_+) - g(u_-)}{u_+ - u_-}.$$

Furthermore, $\mathbf{P}_1 v(x,y,t)$ *decays faster*

$$\|\mathbf{P}_1 v(\cdot,y,t)\|_{L^2_x} = O(1)\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_1(1+t)}}}{1+t} + e^{-(|y|+t)/C_1}\right).$$

Remark 3.8. After a little modification of Lemma 3.4 and similar to (3.12), if the initial function $v_0(x,y)$ satisfies $e^{|y|/C_0}v_0 \in H^2_y(L^2_x)$ and $v_0 \in L^1_y(L^\infty_x)$, one can get the following estimate for the derivative v_y:

$$\|v_y(\cdot,y,t)\|_{L^2_x} = O(1)\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_1(1+t)}}}{1+t} + e^{-(|y|+t)/C_1}\right). \tag{3.18}$$

4. Pointwise Structure for Green's Function

4.1. Picard Iteration and Regularity Improvement

A main difference between the Green's function $\mathbb{G}(x,y,t;x_*)$ and the solution $v(x,y,t)$ of the linear problem (3.1) is that Green's function contains a singular structure inherited from initial dirac function. To separate the singular structure, we design the following iteration scheme based on a decomposition of operator $\mathbf{L}$ and the approximated profile $\bar{\phi}(x)$. Denote

$$\bar{\phi}(x) = \begin{cases} u_+, & \text{for } x > 0,\\ u_-, & \text{for } x < 0,\end{cases}$$

and decompose

$$\mathbf{L}v \;=\; v_{xx} - \frac{1}{2}(f'(\phi(x)))_x v - \frac{1}{4}(f'(\phi(x)))^2 v \equiv \mathbf{L}_K v + \mathbf{K}v,$$

where

$$\begin{aligned}\mathbf{L}_K v \;&\equiv\; v_{xx} - \frac{1}{4}(f'(\bar{\phi}(x)))^2 v,\\ \mathbf{K}v \;&\equiv\; -\frac{1}{2}(f'(\phi(x)))_x v - \frac{1}{4}\left((f'(\phi(x)))^2 - (f'(\bar{\phi}(x)))^2\right) v.\end{aligned}$$

Denote $\mathbb{G}_\pm(x,y,t)$ solving

$$\begin{cases}\partial_t\mathbb{G}_\pm+\lambda\partial_y\mathbb{G}_\pm-\partial_{yy}\mathbb{G}_\pm=\partial_{xx}\mathbb{G}_\pm-\dfrac{1}{4}(f'(u_\pm))^2\mathbb{G}_\pm,\\ \mathbb{G}_\pm(x,y,0)=\delta(x)\delta(y).\end{cases}$$

Then one has that

$$\mathbb{G}_\pm(x,y,t)=\frac{e^{-\frac{1}{4}(f'(u_\pm))^2t-\frac{x^2+(y-\lambda t)^2}{4t}}}{4\pi t}.$$

Denote $\chi(x)$ to be the smooth cut-off function

$$\chi(x)=\begin{cases}1, \text{ for } x>1,\\ 0, \text{ for } x<-1,\end{cases}$$

with $0\le\chi(x)\le 1$.

The Picard iteration starts with:

$$v^0(x,y,t;x_*)\equiv\chi(x)\mathbb{G}_+(x-x_*,y,t)+(1-\chi(x))\mathbb{G}_-(x-x_*,y,t), \tag{4.1}$$

and for $j\ge 0$,

$$\begin{cases}ER_{-1}\equiv 0,\\ er_j\equiv(\partial_t+\lambda\partial_y-\partial_{yy}-\mathbf{L}_K)\,v^j-ER_{j-1},\\ ER_j\equiv\mathbf{K}v^j-(g'(\phi(x))-\lambda\partial_y)\,v^j-er_j,\\ v^{j+1}(x,y,t;x_*)\equiv\chi(x)\left(\mathbb{G}_+(x,y,t)*_{(x,y,t)}ER_j\right)\\ \qquad\qquad+(1-\chi(x))\left(\mathbb{G}_-(x,y,t)*_{(x,y,t)}ER_j\right).\end{cases} \tag{4.2}$$

The iteration scheme yields the equations for $v^j(x,y,t;x_*)$:

$$\begin{cases}\partial_tv^0+\lambda\partial_yv^0-\partial_{yy}v^0=\mathbf{L}_Kv^0+er_0,\\ v^0(x,y,0;x_*)=\delta(x-x_*)\delta(y),\end{cases}$$

and for $j\ge 1$,

$$\begin{cases}\partial_tv^j+\lambda\partial_yv^j-\partial_{yy}v^j=\mathbf{L}_Kv^j+ER_{j-1}+er_j,\\ v^j(x,y,0;x_*)=0.\end{cases}$$

Set

$$\mathbb{G}^{RJ}(x,y,t;x_*)=\mathbb{G}(x,y,t;x_*)-\sum_{j=0}^{J}v^j(x,y,t;x_*)$$

and it solves

$$\begin{cases}\mathbb{G}_t^{RJ}+g'(\phi(x))\mathbb{G}_y^{RJ}-\mathbb{G}_{yy}^{RJ}=\mathbf{L}\mathbb{G}^{RJ}+ER_J,\\ \mathbb{G}^{RJ}(x,y,0;x_*)=0.\end{cases}$$

There are singularities in $\mathbb{G}(x,y,0;x_*)$ when $t=0$ and the scheme improves the regularities. The following estimates can be obtained by direct computations. Denote

$$H(x)\equiv\begin{cases}1,\ \text{for }\ -1<x<0,\\ 0,\ \text{otherwise},\end{cases}$$

and one has

$$\begin{aligned}
v^0(x,y,t;x_*) &= \chi(x)\frac{e^{-\frac{1}{4}(f'(u_+))^2t-\frac{(x-x_*)^2+(y-\lambda_+t)^2}{4t}}}{4\pi t}\\
&\quad+(1-\chi(x))\frac{e^{-\frac{1}{4}(f'(u_-))^2t-\frac{(x-x_*)^2+(y-\lambda_-t)^2}{4t}}}{4\pi t},\\
er_0(x,y,t;x_*) &= H(x)\chi(x)\Big(\left(g'(u_-)-g'(u_+)\right)\partial_y\\
&\quad+\frac{1}{4}\left((f'(u_-))^2-(f'(u_+))^2\right)\Big)\mathbb{G}_+(x-x_*,y,t)\\
&\quad+H(-x)(1-\chi(x))\Big(\left(g'(u_+)-g'(u_-)\right)\partial_y\\
&\quad+\frac{1}{4}\left((f'(u_+))^2-(f'(u_-))^2\right)\Big)\mathbb{G}_-(x-x_*,y,t)\\
&\quad-\chi''(x)\mathbb{G}_+-2\chi'(x)\partial_x\mathbb{G}_++\chi''(x)\mathbb{G}_-+2\chi'(x)\partial_x\mathbb{G}_-\\
&= O(1)\frac{e^{-t/C-\frac{(x-x_*)^2+(y-\lambda_Kt)^2}{4t}}}{t^{3/2}},\\
ER_0(x,y,t;x_*) &= O(1)\frac{e^{-t/C-\frac{(x-x_*)^2+(y-\lambda_Kt)^2}{4t}}}{t^{3/2}}.
\end{aligned}\tag{4.3}$$

Substitute (4.3) into the formula of v^{j+1} in (4.2) to yield

$$v^1(x,y,t;x_*) = O(1)\frac{e^{-t/C-\frac{(x-x_*)^2+(y-\lambda_K t)^2}{4t}}}{t^{1/2}},$$

and

$$er_1(x,y,t;x_*),\ ER_1(x,y,t;x_*) = O(1)\frac{e^{-t/C-\frac{(x-x_*)^2+(y-\lambda_K t)^2}{4t}}}{t}.$$

Go on with the iteration and finally, when $J=5$, the error term ER_J which serves as the source term for remaining function $\mathbb{G}^{RJ}$ is regular enough: for $k=0,1,2$,

$$\partial_y^k ER_5(x,y,t;x_*) = O(1)e^{-t/C-\frac{(x-x_*)^2+(y-\lambda_K t)^2}{4t}} = O(1)e^{-(|x|+|y|+t)/C},$$

and one can apply the conclusions (3.17) and (3.18) for the linear problem to yield Theorem 1.1.

4.2. Pointwise estimates in x variable

We apply a bootstrap argument to improve the previous estimates in Theorem 1.1 to get a pointwise structure in x-variable.

To focus on the problems considered here, we suppose

$$(f'(u_+))^2 = (f'(u_-))^2 \equiv d_K, \qquad g'(u_+) = g'(u_-) \equiv \lambda_K. \tag{4.4}$$

Then the two-sided problem

$$\begin{cases} \partial_t \mathbb{G}_K + g'(\bar\phi(x))\partial_y \mathbb{G}_K - \partial_{yy}\mathbb{G}_K = \partial_{xx}\mathbb{G}_K - \dfrac{1}{4}(f'(\bar\phi(x)))^2\mathbb{G}_K, \\ \mathbb{G}_K(x,y,0;x_*) = \delta(x-x_*)\delta(y), \end{cases} \tag{4.5}$$

is reduced to a problem with constant coefficients:

$$\begin{cases} \partial_t \mathbb{G}_K + \lambda_K\partial_y \mathbb{G}_K - \partial_{yy}\mathbb{G}_K = \partial_{xx}\mathbb{G}_K - \dfrac{1}{4}d_K\mathbb{G}_K, \\ \mathbb{G}_K(x,y,0) = \delta(x)\delta(y), \end{cases}$$

and one has that

$$\mathbb{G}_K(x,y,t) = \frac{e^{-\frac{1}{4}d_K t - \frac{x^2+(y-\lambda_K t)^2}{4t}}}{4\pi t}. \tag{4.6}$$

Remark 4.1. One can remove the assumption (4.4) and put down the solution formula for (4.5) by using the master relationship, [14].

Theorem 4.2. *For the bounded, compact-supported initial data* v_0 *satisfying*

$$\begin{cases} v_0(x,y) \equiv 0, \ \textit{for } |y| \geq 1, \\ |||v_0||| \equiv \|v_0\|_{L_y^\infty(L^\infty_{x,1/2})} \equiv \sup_{x,y\in\mathbb{R}} \left|\mathbf{M}^{-1/2} v_0(x,y)\right| < \infty, \end{cases} \tag{4.7}$$

there exists a positive constant C_1 *such that for all* $x \in \mathbb{R}$

$$\left\|\mathbf{P}_0 \mathbb{G}^t v_0\right\|_{L^\infty_{x,1/2}} = O(1)|||v_0||| \left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C} \right), \tag{4.8}$$

$$\left\|\mathbb{G}^t \mathbf{P}_1 v_0\right\|_{L^\infty_{x,1/2}} = O(1)|||v_0||| \left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-(|y|+t)/C} \right), \tag{4.9}$$

$$\left\|\mathbf{P}_1 \mathbb{G}^t v_0\right\|_{L^\infty_{x,1/2}} = O(1)|||v_0||| \left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-(|y|+t)/C} \right). \tag{4.10}$$

Here, we use the notation

$$\mathbb{G}^t v_0(x,y,t) \equiv \int_{\mathbb{R}^2} \mathbb{G}(x, y-y_*, t; x_*) v_0(x_*, y_*) dx_* dy_*.$$

Proof. A direct computation based on Theorem 1.1 and initial condition (4.7) yield

$$\left\|\mathbb{G}^t v_0\right\|_{L^2_x} = O(1)|||v_0||| \left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + e^{-(|y|+t)/C} \right).$$

From definition (2.1),

$$\left\|\mathbf{P}_0 \mathbb{G}^t v_0\right\|_{L^\infty_{x,1/2}} \equiv \left\|\mathbf{M}^{-1/2}(x) \left(\int_{\mathbb{R}} \mathbf{M}^{1/2}(x) \mathbb{G}^t v_0(x,y,t) dx \right) \mathbf{M}^{1/2}(x) \right\|_{L^\infty_x}$$

$$\begin{aligned} &= O(\epsilon^{-1/2})\left|\int_{\mathbb{R}} \mathbf{M}^{1/2}(x)\mathbb{G}^t v_0(x,y,t)dx\right| \\ &\le C(\epsilon^{-1/2})\left\|\mathbf{M}^{1/2}\right\|_{L^2_x}\left\|\mathbb{G}^t v_0\right\|_{L^2_x}, \end{aligned}$$

and thus one has (4.8).

To obtain (4.9), one applies bootstrap as follows:

$$\begin{cases} \partial_t h + \lambda_K \partial_y h - \partial_{yy} h = \mathbf{L}_K h + \mathbf{K}h - (g'(\phi(x)) - \lambda_K)\partial_y h, \\ h(x,y,0) = \mathbf{P}_1 v_0. \end{cases} \tag{4.11}$$

The solution $h(x,y,t) = \mathbb{G}^t \mathbf{P}_1 v_0$. Also, Theorem 1.1 and initial condition (4.7) result in the L^2_x estimate for the solution:

$$\|h(\cdot,y,t)\|_{L^2_x} = O(1)|||v_0|||\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-(|y|+t)/C}\right).$$

Since $\mathbf{K}h - (g'(\phi(x)) - \lambda_K)\partial_y h = O(1)\mathbf{M}(h + \partial_y h)$, from Schwartz inequality, one has

$$\begin{aligned} &\left\|\mathbf{K}h - (g'(\phi(x)) - \lambda_K)\partial_y h\right\|_{L^\infty_{x,1/2}} \\ &\le C\left\|\mathbf{M}^{1/2}\right\|_{L^2_x}\|h + \partial_y h\|_{L^2_x} \\ &= O(1)|||v_0|||\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-(|y|+t)/C}\right). \end{aligned} \tag{4.12}$$

Use (4.12) as a source term of (4.11) and apply the Green's function $\mathbb{G}_K$ given in (4.6):

$$\begin{aligned} \|h\|_{L^\infty_{x,1/2}} &= \Bigg\|\int_0^t \int_{\mathbb{R}^2} \mathbb{G}_K(x - x_*, y - y_*, t - s)\left(\mathbf{K}h - (g'(\phi(x)) - \lambda_K)\partial_y h\right) \\ &\qquad (x_*, y_*, s)dx_* dy_* ds\Bigg\|_{L^\infty_{x,1/2}} \\ &= O(1)|||v_0|||\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t} + e^{-(|y|+t)/C}\right). \end{aligned}$$

The proof of (4.10) comes from the Picard iteration of the following problem:

$$\begin{cases}\partial_t\bar{h}+\lambda_K\partial_y\bar{h}-\partial_{yy}\bar{h}=\mathbf{L}_K\bar{h}+\mathbf{K}\bar{h}-(g'(\phi(x))-\lambda_K)\partial_y\bar{h},\\ \bar{h}(x,y,0)=v_0.\end{cases}$$

Similar to (4.1) and (4.2), one decomposes:

$$\bar{h}(x,y,t) \;=\; \sum_{j=0}^{5} h^j(x,y,t)+h^{R5}(x,y,t),$$

with

$$\begin{aligned} h^0(x,y,t) &= \mathbb{G}_K *_{(x,y)} v_0,\\ h^j(x,y,t) &= \mathbb{G}_K *_{(x,y,t)} \left(\mathbf{K}-\left(g'(\phi(x))-\lambda_K\right)\partial_y\right)h^{j-1},\ \text{for } 1\le j\le 5.\end{aligned}$$

For $h^j(x,y,t)$, one has the solution formula. By using Green's function $\mathbb{G}_K$ and the initial condition (4.7), one has:

$$\left\|h^j\right\|_{L^\infty_{x,1/2}}=O(1)e^{-(|y|+t)/C}|||v_0|||,\ \text{for } j=0,1,\cdots,5.$$

The remainder part $h^{R5}(x,y,t)$ satisfies

$$\begin{cases}\partial_t h^{R5}+g'(\phi(x))\partial_y h^{R5}-\partial_{yy}h^{R5}=\mathbf{L}_K h^{R5}+\mathbf{K}h^{R5}+ER_{h,5},\\ h^{R5}(x,y,0)=0,\end{cases}$$

where $ER_{h,5}\equiv\mathbf{K}h^5-(g'(\phi(x))-\lambda_K)\partial_y h^5$ satisfying

$$\left\|\partial_y^k ER_{h,5}\right\|_{L^\infty_{x,1/2}}=O(1)e^{-(|y|+t)/C}|||v_0|||\ \text{for } k=0,1.$$

Thus, similar to the proof of (4.9), we can prove that

$$\left\|\mathbf{P}_1h^{R5}\right\|_{L^\infty_{x,1/2}}=O(1)|||v_0|||\left(\frac{e^{-\frac{(y-\lambda t)^2}{C_0(1+t)}}}{1+t}+e^{-(|y|+t)/C}\right),$$

and we finish the proof. $\square$

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