Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **10** (2015), No. 4, pp. 479-539

ASYMPTOTIC PROFILE OF SOLUTIONS TO A HYPERBOLIC CAHN-HILLIARD EQUATION

HIROSHI TAKEDA[1,a], YASUNORI MAEKAWA[2,b] AND SHUICHI KAWASHIMA[3,c]

[1]Faculty of Engineering, Fukuoka Institute of Technology, Fukuoka 811-0295, Japan.

[a]E-mail: h-takeda@fit.ac.jp

[2]Mathematical Institute, Tohoku University, Sendai 980-8578, Japan.

[b]E-mail: maekawa@m.tohoku.ac.jp

[3]Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan.

[c]E-mail: kawashim@math.kyushu-u.ac.jp

Abstract

We study the initial value problem for a hyperbolic Cahn-Hilliard equation in n-dimensional space. The dissipative structure of our linearized equation is of the regularity-loss type. We overcome the difficulty caused by the regularity-loss property by introducing a set of suitable time-weighted spaces and prove the global existence and optimal decay of solutions under smallness and enough regularity assumptions on the initial data. Moreover, we investigate the asymptotic behavior of our nonlinear solutions as $t \to \infty$. When $n \geq 3$, they are asymptotic to the linear diffusion wave expressed by the fundamental solution of the equation $v_t + \Delta^2 v = 0$. On the other hand, when $n = 1$ or $n = 2$, they are asymptotic to the nonlinear diffusion wave which can be expressed in terms of the self-similar solution of the equation $v_t + \Delta^2 v = \Delta v^{1+\frac{2}{n}}$.

1. Introduction

We consider the initial value problem of the following "hyperbolic Cahn-Hilliard equation" in the n-dimensional whole space $\mathbb{R}^n$:

$$(1 - \Delta)u_{tt} + \Delta^2 u + u_t = \Delta f(u). \tag{1.1}$$

Received March 23, 2015 and in revised form June 8, 2015.

AMS Subject Classification: 35L30, 35L76, 35B40, 35C06.

Key words and phrases: Hyperbolic Cahn-Hilliard equation, regularity-loss property, global existence, decay estimate, asymptotic behavior.

The initial data are given as

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$, and the nonlinear term $f(u)$ is a smooth function of u under consideration and satisfies $f(u) = O(u^\nu)$ for $u \to 0$, where $\nu = \max\{1 + \frac{2}{n}, 2\}$; notice that $\nu = 3$ for $n = 1$ and $\nu = 2$ for $n \geq 2$. We write

$$f(u) = g(u) + O(u^{\nu+1}) \tag{1.3}$$

for $u \to 0$, where $g(u) = au^\nu$ with a constant $a \neq 0$.

The linearized equation corresponding to (1.1) is given by

$$(1 - \Delta)u_{tt} + \Delta^2 u + u_t = 0. \tag{1.4}$$

This is the dissipative plate equation with the rotational inertia effects described by the term $-\Delta u_{tt}$. It was observed in [42] (cf. [32]) that the dissipative structure of the equation (1.4) is of the regularity-loss type which is characterized by the property

$$\operatorname{Re} \lambda(\xi) \leq -\frac{c|\xi|^4}{(1 + |\xi|^2)^3},$$

where $\lambda(\xi)$ denotes the eigenvalue of the equation obtained by taking Fourier transform of (1.4). The corresponding decay property of the linearized equation (1.4) will be reviwed in Section 2.

On the other hand, if we neglect the term $(1 - \Delta)u_{tt}$, then our equation (1.1) is reduced to

$$u_t + \Delta^2 u = \Delta f(u), \tag{1.5}$$

which is a "Cahn-Hilliard type equation". We note that the standard Cahn-Hilliard equation is the equation (1.5) with $f(u) = u(u^2 - 1)$ (see [1]); notice that this $f(u)$ does contain a linear part, which is excluded in our situation. However, our assumption for the nonlinearity (1.3) is close to the one of the equation (1.5). Namely, Evans-Galaktionov-Williams [5] considered (1.5) with the single power type nonlinear term $f(u) = |u|^{p-1}u$ for $p > 1$ as the limit case of $\gamma \to 0+$ of (1.5) with a standard double potential function

$f_\gamma(u) = |u|^{p-1}u - \gamma|u|^p u$. They studied the asymptotic behavior of the global and blow-up solutions.

We also note that the modification of the equation (1.5) are introduced in many directions to describe physical phenomena accurately (see e.g. [40] and references therein). Especially, a hyperbolic Cahn-Hilliard equation

$$\varepsilon u_{tt} + \Delta^2 u + u_t = \Delta f(u), \quad \varepsilon > 0, \tag{1.6}$$

is proposed from the observation of the experimental data (cf. [6]-[9]). The modified equation (1.6) has its own interesting aspects from mathematical point of view, since it is easy to guess that the properties of the solutions of (1.5) and (1.6) are totally different (see e.g. [11]-[16], [43]). Our equation (1.1) can be regarded as one of the modification of (1.6), with the regularity-loss structure. For reasons discussed in the above, we call our equation (1.1) as a hyperbolic Cahn-Hilliard equation, including a rough indication of the form of the equation itself.

The main purpose of this paper is to investigate the large-time asymptotic behavior of solutions to the problem (1.1), (1.2). First, we show the global existence and optimal decay of solutions under smallness and enough regularity assumptions on the initial data. When the regularity index s is large enough and the the norm $E_1 = \|u_0\|_{H^{s+1}\cap L^1} + \|u_1\|_{H^s\cap L^1}$ of the initial data is small, our global solution exists and satisfies the decay estimates

$$\|\partial_x^k u(t)\|_{H^{s+1-\sigma_0(k)-j}} \le CE_1(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{1.7}$$

$$\|\partial_x^k u(t)\|_{H^{s-\sigma_1(k,n)}} \le CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}} \tag{1.8}$$

for $0 \le j \le [\frac{n}{4}]$ and $k \ge 0$ with $\sigma_0(k)+j \le s+1$ in (1.7), and $0 \le k \le s_0$ and $\sigma_1(k,n) \le s$ in (1.8). Here $\sigma_0(k) = k + [\frac{k+1}{2}]$ and $\sigma_1(k,n) = k + [\frac{n+2k-1}{4}]$ are the indices introduced in [42] to describe the regularity-loss property, and $s_0 = [\frac{n}{2}] + 1$ is the standard regularity index for the nonlinear problem.

Moreover, we show that when $n \ge 3$, our solution u is asymptotic to the linear diffusion wave v_* defined by

$$v_*(x,t) := MG_0(x,t+1), \tag{1.9}$$

where $G_0(x,t) = \mathcal{F}^{-1}[e^{-|\xi|^4 t}](x)$ is the fundamental solution of the linear parabolic equation

$$v_t + \Delta^2 v = 0, \tag{1.10}$$

and the mass M is determined by $M := \int_{\mathbb{R}^n}(u_0+u_1)(x)dx$. We note that $G_0(x,t) = t^{-\frac{n}{4}} G_*(xt^{-\frac{1}{4}})$ with $G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)$. On the other hand, when $n=1$ or $n=2$, we show that the solution u is asymptotic to the nonlinear diffusion wave v^* which is defined as

$$v^*(x,t) := (t+1)^{-\frac{n}{4}} \Phi_M(x(t+1)^{-\frac{1}{4}}). \tag{1.11}$$

Here the function $t^{-\frac{n}{4}}\Phi_M(xt^{-\frac{1}{4}})$ is the self-similar solution to the semilinear parabolic equation

$$v_t + \Delta^2 v = \Delta g_*(v) \tag{1.12}$$

satisfying the mass condition $\int_{\mathbb{R}^n} \Phi_M(x)\,dx = M := \int_{\mathbb{R}^n}(u_0+u_1)(x)dx$, where $g_*(v) = av^{1+\frac{2}{n}}$. Note that $g_*(v) = g(v)$ when $n = 1, 2$.

The similar approximation theory based on the diffusion waves was first developed in [24] for hyperbolic-parabolic systems of conservation equations. Then the theory was extended to the hyperbolic relaxation systems of the discrete Boltzmann equations ([25]) and the hyperbolic-elliptic systems of radiating gases ([26]). See also [31, 22, 47, 23] for related sharp approximation results.

The study on the global existence and asymptotic behavior of solutions to dissipative hyperbolic-type equations has a long history. We refer to [34, 38, 37, 46] for damped wave equations and [49, 50, 44, 45] for higher order wave equations which are similar to our fourth order equation. Also we refer to [42, 29, 30] for various aspects of dissipation and regularity-loss property of the plate equation.

The decay property of the regularity-loss type which is similar to our case is known also for other interesting model systems. We refer to [39, 18, 19, 41, 35, 36] for the Timoshenko system, [17, 27] for a hyperbolic-elliptic system of radiating gas, [2, 48] for the compressible Euler-Maxwell system, and [3] for the Vlasov-Maxwell-Boltzmann system.

In our study it is also an important step to analyze the semilinear parabolic equations such as (1.5) and (1.12), for we finally need to show that the solution to the hyperbolic problem (1.1), (1.2) is approximated in large time by the solution to the parabolic equation (1.5) with the initial data $u_0 + u_1$. For a wide class of semilinear parabolic equations there is an extensive literature on the large time behavior of solutions. One heuristic but important principle is to look at the balance between the linearity and the nonlinearity in view of scaling. When the linear part, such as $\partial_t + \Delta^2$ in (1.5), is dominant in view of scaling, one can establish the large-time asymptotic expansion of solutions rather explicitly and systematically; e.g., see [21] for a recent general result. On the other hand, when the linearity balances with the nonlinearity in view of scaling, the large time behavior of solutions is often described by the self-similar solutions. Again there is a lot of work related to this issue, and here we only refer to [28] which established the abstract framework for the analysis of evolution equations in the presence of scaling invariance.

The contents of the paper are as follows. In Section 2 we study the linearized equation (1.4) and review the result of [42] on the temporal decay estimates and the regularity-loss property of the associated evolution operators. The first main result of this paper is stated in Section 3 (Theorem 3.1), where we show the global existence of solutions to (1.1), (1.2) and their decay estimates (1.7), (1.8). Moreover, in Corollary 3.4 we see that the solution is approximated by the solution to the linear hyperbolic equation (1.4) when $n \geq 3$. Section 4 is devoted to the study of the large time behavior of solutions to (1.4), from which we obtain the linear diffusion wave (1.9) as the asymptotic profile for solutions to (1.1) when $n \geq 3$; see Proposition 4.6. In Section 5 we consider the parabolic equation of Cahn-Hilliard type (1.5) and obtain the global solvability and the decay property of solutions to (1.5). Then, in Section 6 we return to the original hyperbolic problem (1.1), (1.2), and show that the solution is approximated by the solution to the parabolic equation (1.5). This is our second main result on the asymptotic behavior of solutions to (1.1), (1.2), stated as Theorem 6.1. It should be emphasized that there is no restriction on the dimension in Theorem 6.1. This is contrastive to Corollary 3.4 and Proposition 4.6, where we need the condition $n \geq 3$. In Section 7 we show the unique existence and the stability of self-similar solutions to the parabolic equation (1.12). Combining the result in Section 7 with Theorem 6.1, we finally conclude that, in the case $n = 1, 2$, the solution to (1.1), (1.2) asymptotically converges to the nonlinear diffusion wave (1.11); see Corollary 7.3.

Notations. We give some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\mathcal{F}[u](\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx,$$

and we denote its inverse transform by $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[v](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} v(\xi) e^{ix\cdot\xi} d\xi.$$

For a nonnegative integer k, ∂_x^k denotes the totality of all the k-th order derivatives with respect to $x \in \mathbb{R}^n$.

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. Let $m \in \mathbb{R}$. Then $L^p_m = L^p_m(\mathbb{R}^n)$ denotes the weighted L^p space with the norm

$$\|u\|_{L^p_m} = \|\langle x\rangle^m u\|_{L^p}, \tag{1.13}$$

where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. When $p = 2$, L^2_m is a Hilbert space whose inner product $\langle\cdot,\cdot\rangle_{L^2_m}$ is defined in the natural manner. Let s be a nonnegative integer. Then $W^{s,p} = W^{s,p}(\mathbb{R}^n)$ denotes the Sobolev space of L^p functions, equipped with the norm $\|\cdot\|_{W^{s,p}}$. When $p = 2$, we simply write as $H^s = W^{s,2}$. Also, $C^k(I; W^{s,p})$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $W^{s,p}$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is the Gauss symbol.

2. Linearized Equation

In this section we review the results obtained in [42] for the linearized equation

$$(1-\Delta)u_{tt} + \Delta^2 u + u_t = 0. \tag{2.1}$$

We first give the solution formula for the initial value probelm (2.1), (1.2), and then describe the decay propery of the equation (2.1). Finally, we give the decay estimates of the solution to the problem (2.1), (1.2).

2.1. Solution formula

We consider the linearized equation (2.1) with the initial data (1.2). Let us denote this linear solution by u_L. We have the solution formula

$$u_L(t) = G(t) * (u_0 + u_1) + H(t) * u_0, \tag{2.2}$$

where $*$ denotes the convolution with respect to $x \in \mathbb{R}^n$, and $G(x,t)$ and $H(x,t)$ are the "modified" fundamental solutions of (2.1); $G+H$ and H are the fundamental solutions corresponding to u_0 and u_1, respectively. These fundamental solutions are given explicitly in the Fourier space as follows:

$$\begin{aligned} \hat{G}(\xi,t) &= \frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}(e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}), \\ \hat{H}(\xi,t) &= \frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}\{(1+\lambda_+(\xi))e^{\lambda_-(\xi)t} - (1+\lambda_-(\xi))e^{\lambda_+(\xi)t}\}. \end{aligned} \tag{2.3}$$

Here $\lambda = \lambda_\pm(\xi)$ are the eigenvalues of the ordinary differential equation obtained by taking the Fourier transform of (2.1). We see that these eigenvalues are the solution of the characteristic equation $(1+|\xi|^2)\lambda^2+\lambda+|\xi|^4 = 0$, and are given explicitly in the form

$$\lambda_\pm(\xi) = \frac{1}{2(1+|\xi|^2)}\big\{-1 \pm \sqrt{1-4|\xi|^4(1+|\xi|^2)}\,\big\}. \tag{2.4}$$

It was observed in [42] that

$$\operatorname{Re}\lambda_\pm(\xi) \le -\frac{c|\xi|^4}{(1+|\xi|^2)^3}, \tag{2.5}$$

where c is a positive constant. More precisely, we see from (2.4) that

$$\lambda_+(\xi) = -|\xi|^4\big(1+O(|\xi|^2)\big), \qquad \lambda_-(\xi) = -1+O(|\xi|^2) \tag{2.6}$$

for $|\xi| \to 0$, and

$$\lambda_\pm(\xi) = -\frac{1}{2}|\xi|^{-2}\big(1+O(|\xi|^{-2})\big) \pm i|\xi|\big(1+O(|\xi|^{-2})\big) \tag{2.7}$$

for $|\xi| \to \infty$. In view of (2.6) and (2.7), we see that the estimate (2.5) is optimal. This inequality (2.5) shows that the dissipative structure of the equation (2.1) is of the regularity-loss type, which will be explained in the next subsection.

2.2. Decay property

In this subsection we review the decay property of the linearized equation (2.1). It was shown in [42] that the solution operators $G(t)*$ and $H(t)*$ appearing in the solution formula (2.2) verify the following decay estimates.

Lemma 2.1 ([42]). *Let $1 \le q \le 2$ and $k \ge 0$. Then we have the following decay estimates:*

$$\|\partial_x^k G(t) * \phi\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{k}{4}}\|\phi\|_{L^q} + C(1+t)^{-\frac{l+1}{2}}\|\partial_x^{(k+l)_+}\phi\|_{L^2}, \quad (2.8)$$

$$\|\partial_x^k H(t) * \psi\|_{L^2} \le C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{k}{4}-\frac{1}{2}}\|\psi\|_{L^q} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}\psi\|_{L^2}, \quad (2.9)$$

where $l+1 \ge 0$ and $(k+l)_+ = \max\{k+l, 0\}$ in (2.8), and $l \ge 0$ in (2.9).

Remark. Note that the above decay estimates are of the regularity-loss type. In fact, for example in (2.9), we have the decay rate $(1+t)^{-\frac{l}{2}}$ only by assuming the additional l-th order regularity on the initial data ψ. Such a decay property of the regularity-loss type was also investigated for other interesting systems. See [18, 19] for the dissipative Timoshenko system, [35, 36] for the Timoshenko system with heat conduction, [17, 27] for a hyperbolic-elliptic system of a radiating gas model, and [2, 48] for the Euler-Maxwell system.

For the proof of Lemma 2.1, we need to show the pointwise estimates for $\hat{G}$ and $\hat{H}$. By applying the standard energy method in the Fourier space for (2.1) together with the formula (2.2), we find that

$$|\hat{G}(\xi,t)| \le C(1+|\xi|^2)^{-\frac{1}{2}}e^{-c\eta(\xi)t}, \qquad |\hat{H}(\xi,t)| \le Ce^{-c\eta(\xi)t} \quad (2.10)$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\eta(\xi) = \frac{|\xi|^4}{(1+|\xi|^2)^3}$ is the same function appearing in (2.5). More precisely, we can show that there is a small positive constant r_0 such that

$$|\hat{G}(\xi,t)| \le Ce^{-c|\xi|^4 t}, \qquad |\hat{H}(\xi,t)| \le C|\xi|^2 e^{-c|\xi|^4 t} + Ce^{-ct} \quad (2.11)$$

for $|\xi| \le r_0$ and $t \ge 0$, and

$$|\hat{G}(\xi,t)| \le C|\xi|^{-1}e^{-c|\xi|^{-2}t}, \qquad |\hat{H}(\xi,t)| \le Ce^{-c|\xi|^{-2}t} \tag{2.12}$$

for $|\xi| \ge r_0$ and $t \ge 0$. The estimate for $\hat{H}$ in (2.11) is obtained from the expression (2.3) together with the asymptotic expansion (2.6), while the other estimates in (2.11) and (2.12) are the consequences of (2.10). The desired decay estimates in Lemma 2.1 follows from these pointwise estimates and the Plancherel theorem. For the details, see [42].

To state the next decay estimates for the solution operators $G(t)*$ and $H(t)*$, we introduce the indices

$$\sigma_0(k) = k + [\tfrac{k+1}{2}], \qquad \sigma_1(k,n) = k + [\tfrac{n+2k-1}{4}]. \tag{2.13}$$

These indices are first introduced in [42] to describe the regularity-loss property of the solution to our equation (2.1). With these indices, we have the following decay result.

Lemma 2.2 ([42]). *Let $s \ge 0$. We have the decay estimates*

$$\|\partial_x^k G(t) * \phi\|_{H^{s+1-\sigma_0(k)-j}} \le C(1+t)^{-\frac{j}{2}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}, \tag{2.14}$$
$$\|\partial_x^k H(t) * \psi\|_{H^{s+1-\sigma_0(k)-j}} \le C(1+t)^{-\frac{j}{2}-\frac{k}{4}}\|\psi\|_{H^{s+1}\cap L^1}, \tag{2.15}$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$ and $\sigma_0(k)+j \le s+1$ in (2.14), and $k \ge 0$, $0 \le j \le [\frac{n}{4}]+1$ and $\sigma_0(k)+j \le s+1$ in (2.15). Also, we have

$$\|\partial_x^k G(t) * \phi\|_{H^{s-\sigma_1(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}, \tag{2.16}$$
$$\|\partial_x^k H(t) * \psi\|_{H^{s-\sigma_1(k,n)-j}} \le C(1+t)^{-\frac{n}{8}-\frac{j}{2}-\frac{k}{4}}\|\psi\|_{H^{s+1}\cap L^1}, \tag{2.17}$$

where $k \ge 0$ and $\sigma_1(k,n) \le s$ in (2.16), and $k \ge 0$, $0 \le j \le 1$ and $\sigma_1(k,n)+j \le s$ in (2.17).

Proof. Although these decay estimates in Lemma 2.2 were shown in [42], for completeness, we here give the proof of (2.14) and (2.16) to show how the indices $\sigma_0(k)$ and $\sigma_1(k,n)$ appear in the decay estimates. First we prove (2.14). Let $k \ge 0$ and $h \ge 0$. We have from (2.8) with k replaced by $k+h$

and with $q = 1$ that

$$\|\partial_x^{k+h} G(t) * \phi\|_{L^2} \leq C(1+t)^{-\frac{n}{8}-\frac{k+h}{4}} \|\phi\|_{L^1} + C(1+t)^{-\frac{l+1}{2}} \|\partial_x^m \phi\|_{L^2}, \quad (2.18)$$

where $l+1 \geq 0$ and $m := (k+h+l)_+ \leq s$. Now, letting $0 \leq j \leq [\frac{n}{4}]$, we choose l in (2.18) as the smallest integer satisfying $\frac{l+1}{2} \geq \frac{j}{2} + \frac{k}{4}$, *i.e.*, $l \geq \frac{k}{2} + j - 1$. This implies $l = [\frac{k+1}{2}] + j - 1 = \sigma_0(k) - k + j - 1$. For this choice of l, we obtain from (2.18) that

$$\|\partial_x^{k+h} G(t) * \phi\|_{L^2} \leq C(1+t)^{-\frac{j}{2}-\frac{k}{4}} \|\phi\|_{H^s \cap L^1}$$

for $0 \leq h \leq s + 1 - \sigma_0(k) - j$. This proves (2.14).

To show (2.16), we choose l in (2.18) as the smallest integer satisfying $\frac{l+1}{2} \geq \frac{n}{8} + \frac{k}{4}$, *i.e.*, $l \geq \frac{n+2k}{4} - 1$. This leads to $l = [\frac{n+2k-1}{4}] = \sigma_1(k,n) - k$. For this choice of l, we obtain

$$\|\partial_x^{k+h} G(t) * \phi\|_{L^2} \leq C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|\phi\|_{H^s \cap L^1}$$

for $0 \leq h \leq s - \sigma_1(k,n)$, which proves (2.16). This completes the proof of Lemma 2.2. □

2.3. Linear decay

As a direct consequence of Lemma 2.2, we have the following decay estimates for the linear solution u_L given by the formula (2.2).

Proposition 2.3 ([42]). *Let $n \geq 1$ and let $s \geq 0$ be specified below. Suppose that $u_0 \in H^{s+1} \cap L^1$ and $u_1 \in H^s \cap L^1$, and put*

$$E_1 = \|u_0\|_{H^{s+1} \cap L^1} + \|u_1\|_{H^s \cap L^1}. \quad (2.19)$$

Then the linear solution u_L of the problem (2.1), (1.2)*, which is given by the formula* (2.2)*, satisfies the decay estimates*

$$\|\partial_x^k u_L(t)\|_{H^{s+1-\sigma_0(k)-j}} \leq CE_1(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \quad (2.20)$$

$$\|\partial_x^k u_L(t)\|_{H^{s-\sigma_1(k,n)}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \quad (2.21)$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s + 1$ in (2.20)*, and $k \geq 0$ and $\sigma_1(k,n) \leq s$ in* (2.21)*.*

We put $\phi = u_0 + u_1$ and $\psi = u_0$ in Lemma 2.2. Then our decay estimate (2.20) follows from (2.14) and (2.15), while (2.21) follows from (2.16) and (2.17) with $j = 0$.

3. Global Solution and Decay Estimates

In this section we consider the nonlinear problem (1.1), (1.2). By the Duhamel principle, we see that the solution u verifies the integral equation

$$u(t) = G(t)*(u_0+u_1)+H(t)*u_0+\int_0^t G(t-\tau)*(1-\Delta)^{-1}\Delta f(u)(\tau)d\tau, \quad (3.1)$$

where G and H are the fundamental solutions of (2.1). The equation (3.1) is simply written in the form

$$u(t) = u_L(t) + F[u](t), \quad (3.2)$$

where u_L is the linear solution given in (2.2) and the nonlinear term $F[u]$ is defind by

$$F[u](t) := \int_0^t G(t-\tau) * (1-\Delta)^{-1}\Delta f(u)(\tau)d\tau. \quad (3.3)$$

3.1. Global existence and decay

We want to solve the problem (3.2) by applying the fixed point theorem. To this end, we define the Banach space X as follows:

$$X = X_1 \cap X_2, \qquad \|u\|_X = \|u\|_{X_1} + \|u\|_{X_2}, \quad (3.4)$$

where X_1 and X_2 are also the Banach spaces

$$\begin{aligned} X_1 &= \{u \in C^0([0,\infty); H^{s+1});\ \|u\|_{X_1} < \infty\}, \\ X_2 &= \{u \in C^0([0,\infty); H^{s});\ \|u\|_{X_2} < \infty\}, \end{aligned}$$

with the norms

$$\|u\|_{X_1} := \sum_{j=0}^{[\frac{n}{4}]} \sum_{\sigma_0(k)+j\le s+1} \sup_{t\ge 0}(1+t)^{\frac{j}{2}+\frac{k}{4}}\|\partial_x^k u(t)\|_{H^{s+1-\sigma_0(k)-j}}, \quad (3.5)$$

$$\|u\|_{X_2} := \sum_{k=0}^{s_0} \sup_{t\geq 0} (1+t)^{\frac{n}{8}+\frac{k}{4}} \|\partial_x^k u(t)\|_{H^{s-\sigma_1(k,n)}}, \tag{3.6}$$

respectively, where $s_0 = [\frac{n}{2}] + 1$. Here we have assumed that $s \geq \sigma_1(s_0, n)$ for the regularity exponent s. Notice that $\sigma_1(s_0, n) = 2[\frac{n}{2}] + 1$, so that we have $\sigma_1(s_0, n) = n$ if n is odd, and $\sigma_1(s_0, n) = n + 1$ if n is even. Also, we note that $\sigma_1(k, n) \leq s$ for $k \geq 0$ implies that $\sigma_0(k) + j \leq s + 1$ for $k \geq 0$ and $0 \leq j \leq [\frac{n}{4}]$.

Now we can state our result on the global existence and decay of solution to the problem (1.1), (1.2) as follows.

Theorem 3.1 (Global existence and decay)**.** *Let $n \geq 1$ and $s \geq \sigma_1(s_0, n)$, where $s_0 = [\frac{n}{2}] + 1$. Suppose that $u_0 \in H^{s+1} \cap L^1$ and $u_1 \in H^s \cap L^1$, and define the norm E_1 of the initial data by* (2.19)*. Then there is a positive constant δ_1 such that if $E_1 \leq \delta_1$, then the problem* (1.1), (1.2) *admits a unique global solution $u \in X$ satisfying $\|u\|_X \leq CE_1$. More precisely, the solution u verifies the decay estimates*

$$\|\partial_x^k u(t)\|_{H^{s+1-\sigma_0(k)-j}} \leq CE_1(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{3.7}$$

$$\|\partial_x^k u(t)\|_{H^{s-\sigma_1(k,n)}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{3.8}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s + 1$ in (3.7)*, and $0 \leq k \leq s_0$ and $\sigma_1(k, n) \leq s$ in* (3.8).

3.2. Preliminary

We need some preparations for the proof of Theorem 3.1. First, we show the decay estimate for the term $G(t) * (1 - \Delta)^{-1}\Delta f$ in (3.1).

Lemma 3.2. *Let $k \geq 0$ and $1 \leq q \leq 2$. Then we have*

$$\begin{aligned}&\|\partial_x^k G(t) * (1-\Delta)^{-1}\Delta f\|_{L^2} \\ &\leq C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{k+2-j}{4}} \|\partial_x^j f\|_{L^q} + C(1+t)^{-\frac{l+1}{2}} \|\partial_x^{(k+l)_+} f\|_{L^2},\end{aligned} \tag{3.9}$$

where $l + 1 \geq 0$, $0 \leq j \leq k + 2$ and $(k + l)_+ = \max\{k + l, 0\}$.

This is a simple corollary of (2.8) and we omit the proof.

By virtue of the above lemma, we can show the decay estimate for the nonlinear term $F[u]$ in (3.3). To state its result, we observe that

$$\|u(t)\|_{L^\infty} \le C\|u\|_{X_2}(1+t)^{-\frac{n}{4}}. \tag{3.10}$$

This can be proved by applying the Gagliardo-Nirenberg inequality together with the definition of the norm $\|u\|_{X_2}$. In fact, we have

$$\|u\|_{L^\infty} \le C\|\partial_x^{s_0}u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta} \tag{3.11}$$

with $\theta = \frac{n}{2s_0}$, where $s_0 = [\frac{n}{2}]+1$. On the other hand, it follows from (3.6) that $\|\partial_x^k u(t)\|_{L^2} \le \|u\|_{X_2}(1+t)^{-\frac{n}{8}-\frac{k}{4}}$ for $0 \le k \le s_0$. Substituting this inequality with $k = s_0$ and $k = 0$ into (3.11) and using the fact that $\frac{s_0\theta}{4} = \frac{n}{8}$, we obtain the desired estimate (3.10).

The decay estimate for the nonlinear term $F[u]$ is now given as follows.

Lemma 3.3. *Let $s \ge \sigma_1(s_0,n)$ and suppose that u, $v \in X$. Then we have the following decay estimates:*

$$\begin{aligned}&\|\partial_x^k(F[u]-F[v])(t)\|_{H^{s+1-\sigma_0(k)-j}}\\ &\quad \le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}},\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\|\partial_x^k(F[u]-F[v])(t)\|_{H^{s-\sigma_1(k,n)}}\\ &\quad \le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}},\end{aligned} \tag{3.13}$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$ and $\sigma_0(k)+j \le s+1$ in (3.12), $k \ge 0$ and $\sigma_1(k,n) \le s$ in (3.13), and $\rho(t)$ is the function given below.

The function $\rho(t)$ in Lemma 3.3 is given by

$$\rho(t) = \begin{cases} 1 & n = 1,2 \\ (1+t)^{-\frac{1}{4}} & n = 3 \\ (1+t)^{-\frac{1}{2}}\log(2+t) & n = 4 \\ (1+t)^{-\frac{1}{2}} & n \ge 5 \end{cases} \tag{3.14}$$

Proof of Lemma 3.3. Let k, $h \ge 0$. We apply ∂_x^{k+h} to the difference

$F[u]-F[v]$ and take the L^2 norm to obatin

$$\begin{aligned}
&\|\partial_x^{k+h}(F[u]-F[v])(t)\|_{L^2}\\
&\le \int_0^t \|\partial_x^{k+h}G(t-\tau)*(1-\Delta)^{-1}\Delta(f(u)-f(v))(\tau)\|_{L^2}d\tau\\
&=\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t =: J_1+J_2.
\end{aligned}$$

For the term J_1, we apply (3.9) with k replaced by $k+h$, and with $q=1$ and $j=0$. This yields

$$\begin{aligned}
J_1 \;\le\; & C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+h+2}{4}}\|(f(u)-f(v))(\tau)\|_{L^1}d\tau\\
&+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l+1}{2}}\|\partial_x^m(f(u)-f(v))(\tau)\|_{L^2}d\tau =: J_{11}+J_{12}, \qquad (3.15)
\end{aligned}$$

where $l+1\ge 0$ and $m:=(k+h+l)_+\le s+1$. On the other hand, for the term J_2, we apply (3.9) with k replaced by $k+h$, and with $q=2$, $j=k+h$ and $l=0$. This gives

$$J_2\le C\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^{k+h}(f(u)-f(v))(\tau)\|_{L^2}d\tau. \qquad (3.16)$$

First we estimate the term J_{11}. Since $f(u)=O(u^\nu)$, we see that $\|f(u)-f(v)\|_{L^1}\le C\|(u,v)\|_{L^\infty}^{\nu-2}\|(u,v)\|_{L^2}\|u-v\|_{L^2}$. Therefore, using (3.6) and (3.8), we find that

$$\|(f(u)-f(v))(t)\|_{L^1}\le C\|(u,v)\|_{X_2}^{\nu-1}\|u-v\|_{X_2}(1+t)^{-\frac{n}{4}(\nu-1)}.$$

Consequently, we can estimate the term J_{11} as

$$\begin{aligned}
J_{11} \;\le\; & C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+h+2}{4}}(1+\tau)^{-\frac{n}{4}(\nu-1)}d\tau\\
\le\; & C\|(u,v)\|_X^{\nu-1}\|u-v\|_X(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{n}{4}(\nu-1)}d\tau\\
\le\; & C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \qquad (3.17)
\end{aligned}$$

for $h\ge 0$, where $\rho(t)$ is given in (3.14). Here we used the fact that $\frac{n}{4}(\nu-1)=$

$\frac{1}{2}$ for $n = 1, 2$ and $\frac{n}{4}(\nu - 1) = \frac{n}{4}$ for $n \geq 3$.

Next we estimate the term J_{12}. When we prove (3.12), we let $0 \leq j \leq [\frac{n}{4}]$ and choose l in (3.15) as the smallest integer satisfying $\frac{l+1}{2} \geq \frac{j}{2} + \frac{k}{4} + \frac{1}{2}$, *i.e.*, $l \geq j + \frac{k}{2}$. This lead to $l = [\frac{k+1}{2}] + j = \sigma_0(k) - k + j$. For this choice of l, the requirement $m := (k+h+l)_+ \leq s+1$ implies that $0 \leq h \leq s+1-\sigma_0(k)-j$. Also, we have

$$\begin{aligned}
&\|\partial_x^m(f(u) - f(v))\|_{L^2} \\
&\leq C\|(u,v)\|_{L^\infty}^{\nu-2}\{\|(u,v)\|_{L^\infty}\|\partial_x^m(u-v)\|_{L^2} + \|\partial_x^m(u,v)\|_{L^2}\|(u-v)\|_{L^\infty}\} \\
&\leq C\|(u,v)\|_{L^\infty}^{\nu-2}\{\|(u,v)\|_{L^\infty}\|u-v\|_{H^{s+1}} + \|(u,v)\|_{H^{s+1}}\|(u-v)\|_{L^\infty}\}.
\end{aligned}$$

Therefore, using (3.5), (3.6) and (3.11), we find that

$$\begin{aligned}
&\|\partial_x^m(f(u) - f(v))(t)\|_{L^2} \\
&\leq\ C\|(u,v)\|_{X_2}^{\nu-2}\{\|(u,v)\|_{X_2}\|u-v\|_{X_1} + \|(u,v)\|_{X_1}\|u-v\|_{X_2}\}(1+t)^{-\frac{n}{4}(\nu-1)} \\
&\leq\ C\|(u,v)\|_X^{\nu-1}\|u-v\|_X(1+t)^{-\frac{n}{4}(\nu-1)}. \qquad (3.18)
\end{aligned}$$

Consequently, similarly as in (3.17), we can estimate J_{12} as

$$\begin{aligned}
J_{12}\ &\leq\ C\|(u,v)\|_X^{\nu-1}\|u-v\|_X \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{j}{2}-\frac{k}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{4}(\nu-1)}d\tau \\
&\leq\ C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}} \qquad (3.19)
\end{aligned}$$

for $0 \leq h \leq s+1-\sigma_0(k)-j$, where $0 \leq j \leq [\frac{n}{4}]$, and $\rho(t)$ is in (3.14).

On the other hand, when we prove (3.13), we choose l in (3.15) as the smallest integer satisfying $\frac{l+1}{2} \geq \frac{n}{8} + \frac{k}{4} + \frac{1}{2}$, *i.e.*, $l \geq \frac{n+2k}{4}$. This leads to $l = [\frac{n+2k-1}{4}] + 1 = \sigma_1(k,n) - k + 1$. For this choice of l, the requirement $m := (k+h+l)_+ \leq s+1$ implies $0 \leq h \leq s - \sigma_1(k,n)$. In this case, we also have (3.18). Therefore we obtain

$$\begin{aligned}
J_{12}\ &\leq\ C\|(u,v)\|_X^{\nu-1}\|u-v\|_X \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}(1+\tau)^{-\frac{n}{4}(\nu-1)}d\tau \\
&\leq\ C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \qquad (3.20)
\end{aligned}$$

for $0 \leq h \leq s - \sigma_1(k,n)$, where $\rho(t)$ is in (3.14).

Finally we estimate the term J_2. We see that

$$\begin{aligned}&\|\partial_x^{k+h}(f(u)-f(v))\|_{L^2}\\&\le C\|(u,v)\|_{L^\infty}^{\nu-2}\{\|(u,v)\|_{L^\infty}\|\partial_x^k(u-v)\|_{H^h}+\|\partial_x^k(u,v)\|_{H^h}\|(u-v)\|_{L^\infty}\}.\end{aligned}$$

When we show (3.12), we let $0\le j\le[\frac{n}{4}]$ and $0\le h\le s+1-\sigma_0(k)-j$. In this case, using (3.5), (3.6) and (3.11), we find that

$$\begin{aligned}\|\partial_x^{k+h}(f(u)-f(v))(t)\|_{L^2} &\le C\|(u,v)\|_{X_2}^{\nu-2}\{\|(u,v)\|_{X_2}\|u-v\|_{X_1}\\&\qquad+\|(u,v)\|_{X_1}\|u-v\|_{X_2}\}(1+t)^{-\frac{j}{2}-\frac{k}{4}-\frac{n}{4}(\nu-1)}\\&\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X(1+t)^{-\frac{j}{2}-\frac{k}{4}-\frac{n}{4}(\nu-1)}.\end{aligned}$$

Substituting this estimate into (3.16), we can estimate J_2 as

$$\begin{aligned}J_2 &\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac12}(1+\tau)^{-\frac{j}{2}-\frac{k}{4}-\frac{n}{4}(\nu-1)}d\tau\\&\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X(1+t)^{-\frac{j}{2}-\frac{k}{4}-\frac{n}{4}(\nu-1)}\int_{\frac{t}{2}}^t(1+\tau)^{-\frac12}d\tau\\&\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho_1(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}},\end{aligned}\tag{3.21}$$

where $0\le j\le[\frac{n}{4}]$, $0\le h\le s+1-\sigma_0(k)-j$, and $\rho_1(t)$ is given by

$$\rho_1(t):=(1+t)^{-\{\frac{n}{4}(\nu-1)-\frac12\}}=\begin{cases}1 & n=1,2\\(1+t)^{-(\frac{n}{4}-\frac12)} & n\ge3.\end{cases}\tag{3.22}$$

On the other hand, when we show (3.13), we let $0\le h\le s-\sigma_1(k,n)$. In this case, by using (3.6) and (3.11), we have

$$\|\partial_x^{k+h}(f(u)-f(v))(t)\|_{L^2}\le C\|(u,v)\|_{X_2}^{\nu-1}\|u-v\|_{X_2}(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)}.$$

Therefore, similarly as in (3.21), we can estimate J_2 as

$$\begin{aligned}J_2 &\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac12}(1+\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)}d\tau\\&\le C\|(u,v)\|_X^{\nu-1}\|u-v\|_X\rho_1(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}\end{aligned}\tag{3.23}$$

for $0\le h\le s-\sigma_1(k,n)$, where $\rho_1(t)$ is given in (3.22).

Now the estimate (3.12) (resp. (3.13)) follows from (3.17), (3.19) and (3.21) (resp. (3.17), (3.20) and (3.23)). Thus the proof of Lemma 3.3 is complete. □

3.3. Proof of Theorem 3.1

We prove Theorem 3.1 by applying the contraction mapping principle. We define the mapping Φ by

$$\Phi[u] := u_L + F[u] \tag{3.24}$$

for $u \in X$, where u_L is the linear solution given in (2.2), $F[u]$ is the nonlinear term defined in (3.3), and X denotes the Banach space defined in (3.3) with the regularity $s \geq \sigma_1(s_0, n)$. To define a subset S of X, we recall the estimates (2.20) and (2.21) for the linear solution u_L. These estimates together with (3.5) and (3.6) give

$$\|u_L\|_X \leq C_0 E_1, \tag{3.25}$$

where C_0 is a positive constant and E_1 is the norm of the initial data given in (2.19). With this constant C_0, we define the subset S of X by

$$S := \{u \in X;\ \|u\|_X \leq 2C_0E_1\}. \tag{3.26}$$

Then we claim that Φ is a contraction mapping of S into itself, provided that E_1 is suitably small.

We verify this claim. Let u, $v \in X$. Then it follows from Lemma 3.3 that

$$\|F[u] - F[v]\|_X \leq C_1\|(u, v)\|_X^{\nu-1}\|u - v\|_X, \tag{3.27}$$

where C_1 is a positive constant. Now we suppose that $u \in S$, *i.e.*, $\|u\|_X \leq 2C_0E_1$. Then we have from (3.25) and (3.27) with $v = 0$ that

$$\|\Phi[u]\|_X \leq \|u_L\|_X + \|F[u]\|_X \leq C_0E_1 + C_1(2C_0E_1)^\nu \leq 2C_0E_1,$$

provided that E_1 is so small as $C_1(2C_0E_1)^{\nu-1} \leq \frac{1}{2}$. This shows that Φ is a mapping of S into itself. Next, letting u, $v \in S$, we have from (3.27) that

$$\|\Phi[u]-\Phi[v]\|_X = \|F[u]-F[v]\|_X \le C_1(2C_0E_1)^{\nu-1}\|u-v\|_X \le \frac{1}{2}\|u-v\|_X$$

provided that $C_1(2C_0E_1)^{\nu-1} \le \frac{1}{2}$. Thus we have shown that Φ is a contraction mapping of S into itself if E_1 is suitably small.

As the consequence of the contraction mapping principle, our mapping Φ has a unique fixed point u in S. This fixed point u satisfies $u = \Phi[u] = u_L + F[u]$ and $\|u\| \le 2C_0E_1$, and hence is the desired solution to our problem (3.2). This complete the proof of Theorem 3.1. □

As the direct consequence of Theorem 3.1 and Lemma 3.3, we have the following linear approximation result for $n \ge 3$.

Corollary 3.4 (Linear approximation)**.** *Assume the same condition in Theorem 3.1. Let u be the global solution to the nonlinear problem* (1.1), (1.2), *which is obtained in Theorem 3.1, and let u_L be the linear solution given in* (2.2). *When $n \ge 3$, the nonlinear solution u is asymptotic to the linear solution u_L as $t \to \infty$. More precisely, we have*

$$\|\partial_x^k(u-u_L)(t)\|_{H^{s+1-\sigma_0(k)-j}} \le CE_1^{\nu}\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \quad (3.28)$$

$$\|\partial_x^k(u-u_L)(t)\|_{H^{s-\sigma_1(k,n)}} \le CE_1^{\nu}\rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \quad (3.29)$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$ and $\sigma_0(k)+j \le s+1$ in (3.7), *$0 \le k \le s_0$ and $\sigma_1(k,n) \le s$ in* (3.8), *and $\rho(t)$ is given in* (3.14).

Proof. In Theorem 3.1 we have shown that $\|u\|_X \le CE_1$. Therefore, applying Lemma 3.3 with $v = 0$ to the expression $u - u_L = F[u]$, we conclude the desired estimates (3.28) and (3.29). This completes the proof. □

4. Linear Approximation for $n \ge 3$

In the last section, we observed that the nonlinear solution u is approximated by the linear solution u_L for $n \ge 3$. The aim of this section is to show the further approximation of the linear solution u_L by the solution v_L of the simpler linear problem (1.10) with

$$v(x,0) = v_0(x) := u_0(x) + u_1(x), \quad (4.1)$$

where u_0 and u_1 are the initial data in (1.2). This solution v_L is given by the formula

$$v_L(t) = G_0(t) * (u_0 + u_1), \tag{4.2}$$

where G_0 is the fundamental solution of (1.10). We note that

$$G_0(x,t) = \mathcal{F}^{-1}[e^{-|\xi|^4 t}](x) = t^{-\frac{n}{4}} G_*(xt^{-\frac{1}{4}}), \tag{4.3}$$

where $G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)$. Since G_* is a rapidly decreasing function, it is not difficult to show the following L^p-L^q estimate

$$\|\partial_x^k G_0(t) * \phi\|_{L^p} \le Ct^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{p})-\frac{k}{4}} \|\phi\|_{L^q}, \tag{4.4}$$

where $k \ge 0$ and $1 \le q \le p \le \infty$. As a simple consequence, we have

$$\|\partial_x^k v_L(t)\|_{H^{s-k}} \le CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}} \tag{4.5}$$

for $0 \le k \le s$. Here we used the fact that $\|u_0 + u_1\|_{H^s \cap L^1} \le E_1$, where E_1 is given in (2.19).

4.1. Simpler linear approximation

For our purpose, we first prepare the decay estimate for $(G - G_0)(t)*$.

Lemma 4.1 ([42]). *Let $1 \le q \le 2$ and $k \ge 0$. Then we have the following decay estimate:*

$$\begin{aligned} &\|\partial_x^k (G - G_0)(t) * \phi\|_{L^2} \\ &\quad \le C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{1}{2}-\frac{k}{4}} \|\phi\|_{L^q} \\ &\qquad + C(1+t)^{-\frac{l+1}{2}} \|\partial_x^{(k+l)_+} \phi\|_{L^2} + Ct^{-\frac{j}{4}} e^{-ct} \|\partial_x^{k-j} \phi\|_{L^2}, \end{aligned} \tag{4.6}$$

where $l + 1 \ge 0$, $(k+l)_+ = \max\{k+l, 0\}$ and $0 \le j \le k$ in (4.6).

To verify this lemma, we consider the difference $\hat{G} - \hat{G}_0$. By using the asymptotic expansion (2.6) in the expression (2.3) for $\hat{G}$ and noting $\hat{G}_0(\xi,t) = e^{-|\xi|^4 t}$, we find that

$$|(\hat{G} - \hat{G}_0)(\xi,t)| \le C|\xi|^2 e^{-c|\xi|^4 t} + Ce^{-ct} \tag{4.7}$$

for $|\xi| \le r_0$ and $t \ge 0$, where r_0 is a small positive constant. In the region $|\xi| \ge r_0$, we have (2.12) and $|\hat{G}_0(\xi,t)| = e^{-|\xi|^4 t}$. These pointwise estimates together with the Plancherel theorem yield the desired estimate (4.6). For the details, we refer to [42].

Next we prepare the decay estimates for $(G - G_0)(t)*$ in the form of Lemma 2.2.

Lemma 4.2. *Let $s \ge 0$. We have the decay estimates*

$$\|\partial_x^k (G-G_0)(t)*\phi\|_{H^{s-\sigma_0(k)-j}} \le C(1+t)^{-\frac{j+1}{2}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}, \quad (4.8)$$

$$\|\partial_x^k (G-G_0)(t)*\phi\|_{H^{s-\sigma_1(k,n)-j}} \le C(1+t)^{-\frac{n}{8}-\frac{j}{2}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}, \quad (4.9)$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$, $\sigma_0(k)+j \le s$ in (4.8), and $k \ge 0$, $0 \le j \le 1$ and $\sigma_1(k,n)+j \le s$ in (4.9).

Proof. Let $k \ge 0$ and $h \ge 0$. We have from (4.6) with k replaced by $k+h$ and with $q=1$ and $j=0$ that

$$\begin{aligned}&\|\partial_x^{k+h}(G-G_0)(t)*\phi\|_{L^2}\\ &\le C(1+t)^{-\frac{n}{8}-\frac{1}{2}-\frac{k+h}{4}}\|\phi\|_{L^1}+C(1+t)^{-\frac{l+1}{2}}\|\partial_x^m\phi\|_{L^2}+Ce^{-ct}\|\partial_x^{k+h}\phi\|_{L^2}, \quad (4.10)\end{aligned}$$

where $l+1 \ge 0$, $m := (k+h+l)_+ \le s$ and $k+h \le s$. To prove (4.8), we set $0 \le j \le [\frac{n}{4}]$. For this j, we choose l in (4.10) as the smallest integer satisfying $\frac{l+1}{2} \ge \frac{j+1}{2} + \frac{k}{4}$, *i.e.*, $l \ge \frac{k}{2}+j$. This gives $l = [\frac{k+1}{2}]+j = \sigma_0(k)-k+j$. For this choice of l, we get from (4.10) that

$$\|\partial_x^{k+h}(G-G_0)(t)*\phi\|_{L^2} \le C(1+t)^{-\frac{j+1}{2}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}$$

for $0 \le h \le s-\sigma_0(k)-j$, which proves (4.8).

To show (4.9), we set $0 \le j \le 1$ and choose l in (4.10) as the smallest integer satisfying $\frac{l+1}{2} \ge \frac{n}{8}+\frac{j}{2}+\frac{k}{4}$, *i.e.*, $l \ge \frac{n+2k}{4}+j-1$. This leads to $l = [\frac{n+2k-1}{4}]+j = \sigma_1(k,n)-k+j$. For this choice of l, we obtain

$$\|\partial_x^{k+h}(G-G_0)(t)*\phi\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{j}{2}-\frac{k}{4}}\|\phi\|_{H^s\cap L^1}$$

for $0 \le h \le s-\sigma_1(k,n)-j$, which proves (4.9). Thus the proof of Lemma 4.2 is complete. □

Now we can show that the linear solution u_L is approximated by the simpler solution v_L.

Proposition 4.3 (Simpler linear approximation)**.** *Let $n \geq 1$ and assume the same conditions as in Proposition* 2.3*. Let u_L be the linear solution given in* (2.2) *and let v_L be the simpler solution in* (4.2)*. Then u_L is asymptotic to v_L as $t \to \infty$ in the following sense:*

$$\|\partial_x^k(u_L - v_L)(t)\|_{H^{s-\sigma_0(k)-j}} \leq CE_1(1+t)^{-\frac{j+1}{2}-\frac{k}{4}}, \tag{4.11}$$

$$\|\partial_x^k(u_L - v_L)(t)\|_{H^{s-\sigma_1(k,n)-1}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{1}{2}-\frac{k}{4}}, \tag{4.12}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s$ in (4.11)*, and $k \geq 0$ and $\sigma_1(k,n) + 1 \leq s$ in* (4.12)*.*

Proof. We have from (2.2) and (4.2) that

$$(u_L - v_L)(t) = (G - G_0)(t) * (u_0 + u_1) + H(t) * u_0.$$

By applying (4.8) and (2.15) with j replaced by $j+1$, we easily obtain (4.11) for $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s$. On the other hand, applying (4.9) and (2.17) both with $j = 1$, we get (4.12) for $k \geq 0$ and $\sigma_1(k,n) + 1 \leq s$ in (4.12). This completes the proof. □

4.2. Approximation by the linear diffusion wave

In the last part of this section, we show that the nonlinear solution u is approximated by the linear diffusion wave

$$v_*(x,t) := MG_0(x, t+1), \qquad M := \int_{\mathbb{R}^n} (u_0 + u_1)(x)dx.$$

We note that since $G_0(x, t+1)$ is a solution of (1.10) with the initial data $G_0(x, 1) = G_*(x)$, we have the expression $G_0(\cdot, t+1) = G_0(t) * G_*$. Therefore

$$v_*(t) = MG_0(t) * G_*.$$

Observing Corollary 3.4 and Proposition 4.3, we recognize that the following decomposition of the difference $(u - v_*)(t)$ is useful to show our purpose:

$$(u - v_*)(t) = (u - u_L)(t) + (u_L - v_L)(t) + (v_L - v_*)(t). \tag{4.13}$$

Then we only need to show the decay estimate for $(v_L - v_*)(t) = G_0(t) * (u_0 + u_1 - MG_*)$. To this aim, we prepare the following lemma.

Lemma 4.4. *Let $n \geq 1$ and $\phi \in L^1_1$ with $\int_{\mathbb{R}^n} \phi(x)dx = 0$. Then we have*

$$\|\partial_x^k G_0(t) * \phi\|_{L^p} \leq Ct^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k+1}{4}} \|\phi\|_{L^1_1}, \tag{4.14}$$

where $1 \leq p \leq \infty$ and $k \geq 0$.

Proof.By $\int_{\mathbb{R}^n} \phi(x)dx = 0$ and the mean value theorem, we see that

$$\begin{aligned} \partial_x^k G_0(t) * \phi &= \partial_x^k G_0(t) * \phi - \partial_x^k G_0(t) \int_{\mathbb{R}^n} \phi(y)dy \\ &= \int_{\mathbb{R}^n} \partial_x^k \big(G_0(x-y,t) - G_0(x,t)\big)\phi(y)dy \\ &= \int_{\mathbb{R}^n} \Big\{ \int_0^1 (-y) \cdot \partial_x^{k+1} G_0(x - \theta y, t) d\theta \Big\} \phi(y)dy. \end{aligned} \tag{4.15}$$

Thus noting that $\|\partial_x^k G_0(t)\|_{L^p} = Ct^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k}{4}}$ with some constant $C > 0$, we obtain that

$$\begin{aligned} \|\partial_x^k G_0(t) * \phi\|_{L^p} &\leq \int_{\mathbb{R}^n} |y| \, \|\partial_x^{k+1} G_0(\cdot, t)\|_{L^p_x} |\phi(y)| \, dy \\ &\leq Ct^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k+1}{4}} \int_{\mathbb{R}^n} |y||\phi(y)| \, dy \\ &\leq Ct^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k+1}{4}} \|\phi\|_{L^1_1}, \end{aligned} \tag{4.16}$$

which is the desired estimate. The proof of Lemma 4.4 is complete. □

As a direct consequence of Lemma 4.4, we have the following approximation results for v_L by the linear diffusion wave v_* corresponding to the regularity of v_0.

Corollary 4.5. *Let $n \geq 1$ and $s \geq 0$. Assume that $v_0 := u_0 + u_1 \in H^s \cap L^1$. Then the difference $v_L - v_*$ satisfies*

$$\|\partial_x^k (v_L - v_*)(t)\|_{H^{s-k}} = o(t^{-\frac{n}{8}-\frac{k}{4}}) \tag{4.17}$$

as $t \to \infty$, where $0 \leq k \leq s$. Also, if $v_0 := u_0 + u_1 \in H^s \cap L^1_1$, then we have

$$\|\partial_x^k (v_L - v_*)(t)\|_{H^{s-k}} \leq C\|v_0\|_{H^s \cap L^1_1} (1+t)^{-\frac{n}{8}-\frac{k+1}{4}}, \tag{4.18}$$

where $0 \le k \le s$.

Proof. We only show the estimate (4.18) and omit the proof of (4.17). Observe that $(v_L - v_*)(t) = G_*(t) * \phi$, where $\phi := v_0 - MG_*$ and $M = \int_{\mathbb{R}^n} v_0(x)dx$. Since $\int_{\mathbb{R}^n} G_*(x)dx = \hat{G}_*(0) = 1$, we see that $\int_{\mathbb{R}^n} \phi(x)dx = 0$. Also, noting $|M| \le \|v\|_{L^1}$, we have $\|\phi\|_{L^1_1} \le C\|v_0\|_{L^1_1}$ and $\|\phi\|_{H^s} \le C\|v_0\|_{H^s \cap L^1}$. Therefore, applying (4.14) with $p = 2$, we obtain

$$\|\partial_x^k (v_L - v_*)(t)\|_{H^{s-k}} \le C\|v_0\|_{L^1_1} t^{-\frac{n}{8}-\frac{k+1}{4}}, \tag{4.19}$$

where $0 \le k \le s$. Also, using (4.4) with $p = q = 2$, we have

$$\|\partial_x^k (v_L - v_*)(t)\|_{H^{s-k}} \le C\|v_0\|_{H^s \cap L^1_1}, \tag{4.20}$$

where $0 \le k \le s$. Combining (4.19) and (4.20), we have the desired estimate (4.18). The proof of Corollary 4.5 is complete. □

As mentioned above, as a consequence of Corollary 3.4, Proposition 4.3, and Corollary 4.5, we can derive the approximation of the nonlinear solution u by the linear diffusion wave v_*:

Proposition 4.6. *Let* $n \ge 3$. *Assume the same condition in Theorem 3.1. Let* u *be the global solution to the nonlinear problem* (1.1), (1.2), *which is obtained in Theorem 3.1, and let* v_* *be the linear diffusion wave. Then the nonlinear solution* u *is approximated by the linear diffusion wave* v_*. *Namely, we have*

$$\|\partial_x^k (u - v_*)(t)\|_{H^{s-\sigma_1(k,n)-1}} = o(t^{-\frac{n}{8}-\frac{k}{4}}) \tag{4.21}$$

as $t \to \infty$. *Furthermore, suppose in addition that* $v_0 = u_0 + u_1 \in L^1_1$ *and put* $\tilde{E}_1 = E_1 + \|u_0 + u_1\|_{L^1_1}$. *Then we have*

$$\|\partial_x^k (u - v_*)(t)\|_{H^{s-\sigma_1(k,n)-1}} \le C\tilde{E}_1(1+t)^{-\frac{n}{8}-\frac{k+1}{4}}, \tag{4.22}$$

where $k \ge 0$ *and* $\sigma_1(k,n) + 1 \le s$ *in* (4.21) *and* (4.22).

Remark 4.7. We note that (4.21) and (4.22) mean that the upper bound (3.8) of $u(t)$ is sharp. Indeed, if $v_0 \in L^1$ and $M = \int_{\mathbb{R}^n} v_0(x)dx \ne 0$, we see that $\|\partial_x^k v_*(t)\|_{L^2} = c_0(1+t)^{-\frac{n}{8}-\frac{k}{4}}$ for some constant $c_0 > 0$. Therefore we observe that

$$\begin{aligned}\|\partial_x^k u(t)\|_{L^2} &\geq \|\partial_x^k v_*(t)\|_{L^2} - \|\partial_x^k (u-v_*)(t)\|_{L^2}\\ &\geq c_0(1+t)^{-\frac{n}{8}-\frac{k}{4}} + o(t^{-\frac{n}{8}-\frac{k}{4}})\end{aligned}$$

as $t \to \infty$ and hence

$$c(1+t)^{-\frac{n}{8}-\frac{k}{4}} \leq \|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \tag{4.23}$$

for large t, where the constants $c > 0$ and $C > 0$ in (4.23) are independent of t.

Proof of Proposition 4.6. We apply the estimates (3.29) and (4.12) to the decomposition (4.13) to have

$$\begin{aligned}&\|\partial_x^k (u-v_*)(t)\|_{H^{s-\sigma_1(k,n)-1}} \leq \|\partial_x^k (u-u_L)(t)\|_{H^{s-\sigma_1(k,n)-1}}\\ &\quad + \|\partial_x^k (u_L-v_L)(t)\|_{H^{s-\sigma_1(k,n)-1}} + \|\partial_x^k (v_L-v_*)(t)\|_{H^{s-\sigma_1(k,n)-1}}\\ &\leq CE_1^\nu \rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} + CE_1(1+t)^{-\frac{n}{8}-\frac{k+2}{4}} + \|\partial_x^k (v_L-v_*)(t)\|_{H^{s-\sigma_1(k,n)-1}}.\end{aligned} \tag{4.24}$$

Therefore (4.17) and (4.18) yield the desired estimates (4.21) and (4.22). The proof is complete. □

We give a final remark on the corresponding estimates for $\|\partial_x^k (u-v_*)(t)\|_{H^{s-\sigma_0(k)-j}}$.

Remark 4.8. If $v_0 \in L^1$, we have

$$\|\partial_x^k (u-v_*)(t)\|_{H^{s-\sigma_0(k)-j}} \leq CE_1\eta_j(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{4.25}$$

where $\eta_j(t) = \max\{\rho(t),\, o(1)(1+t)^{-\frac{1}{2}(\frac{n}{4}-j)}\}$. Also, if $v_0 \in L^1_1$, then we have

$$\|\partial_x^k (u-v_*)(t)\|_{H^{s-\sigma_0(k)-j}} \leq C\tilde{E}_1\tilde{\eta}_j(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{4.26}$$

where $\tilde{\eta}_j(t) = \max\{\rho(t),\, (1+t)^{-\frac{1}{4}-\frac{1}{2}(\frac{n}{4}-j)}\}$. Here $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s$ in (4.25) and (4.26).

Indeed, we apply the estimates (3.28) and (4.11) to the decomposition (4.13) to have

$$\begin{aligned}\|\partial_x^k(u-v_*)(t)\|_{H^{s-\sigma_0(k)-j}} &\le \|\partial_x^k(u-u_L)(t)\|_{H^{s-\sigma_0(k)-j}}\\ &+\|\partial_x^k(u_L-v_L)(t)\|_{H^{s-\sigma_0(k)-j}}+\|\partial_x^k(v_L-v_*)(t)\|_{H^{s-\sigma_0(k)-j}}\\ &\le C(E_1+E_1^\nu)\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}+\|\partial_x^k(v_L-v_*)(t)\|_{H^{s-\sigma_0(k)-j}}. \quad (4.27)\end{aligned}$$

Then, using (4.17) and (4.18), we obtain the desired estimates (4.25) and (4.26).

5. Cahn-Hilliard Type Equation

In this section we consider the parabolic equation

$$v_t+\Delta^2 v=\Delta g(v) \tag{5.1}$$

with the initial data $v_0=u_0+u_1$ as in (4.1), where $g(v)=av^\nu$ for some constant $a\neq 0$. The associated integral equation is

$$v(t)=G_0(t)*v_0+\int_0^t G_0(t-\tau)*\Delta g(v)(\tau)d\tau\,, \tag{5.2}$$

where $G_0(t)$ is given by (4.3), and the solution to (5.1) is always understood as the solution to (5.2). Let us recall that the number ν in the nonlinear term is $1+\frac{2}{n}$ for $n=1,2$, while $\nu=2$ for $n\ge 3$. In view of the global behavior of solutions, there is a significant difference between the case $n=1,2$ and the case $n\ge 3$. To see this briefly, we observe that (5.1) is invariant under the following scaling:

$$v(x,t)\longmapsto v_\lambda(x,t)=\begin{cases}\lambda^{\frac{n}{4}}v(\lambda^{\frac14}x,\lambda t)\,, & n=1,2\,,\\ \lambda^{\frac12}v(\lambda^{\frac14}x,\lambda t)\,, & n\ge 3\,.\end{cases} \tag{5.3}$$

Then, for sufficiently localized and small initial data, the nonlinearity should be negligible in large time if $n\ge 3$, while the nonlinear effect essentially comes into the large time behavior of solutions if $n=1,2$. We will discuss this problem specific to the critical case $n=1,2$ later in Section 7, and in this section we mainly focus on the global existence and the temporal decay estimates of solutions to (5.2). The main result is stated as follows.

Theorem 5.1. *Set* $p_n = 1$ *if* $n = 1, 2$ *and* $p_n = \frac{n}{2}$ *if* $n \geq 3$. *There is a positive constant* $\delta = \delta(n)$ *such that if* $\|v_0\|_{L^{p_n}} \leq \delta$, *then* (5.2) *admits a unique solution* $v \in C^0([0,\infty); L^{p_n})$ *satisfying*

$$\|\partial_x^k v(t)\|_{L^p} \leq C\|v_0\|_{L^{p_n}} t^{-\frac{n}{4}(\frac{1}{p_n}-\frac{1}{p})-\frac{k}{4}}, \qquad t > 0, \tag{5.4}$$

for $k \geq 0$ *and* $p_n \leq p \leq \infty$. *Moreover, if* $v_0 \in H^s \cap L^1$ *for some* $s \geq 0$ *in addition, then*

$$\|v(t) - G_0(t) * v_0\|_{L^1} \leq C\|v_0\|^{\nu}_{L^1\cap L^{p_n}} \rho(t), \tag{5.5}$$

$$\|\partial_x^k\big(v(t) - G_0(t) * v_0\big)\|_{H^{s-k}} \leq C\|v_0\|^{\nu}_{H^s\cap L^1\cap L^{p_n}} \rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{5.6}$$

where $0 \leq k \leq s$ *in* (5.6), *and* $\rho(t)$ *is the function defined by* (3.14).

Remark 5.2. (i) The solution obtained in Theorem 5.1 satisfies

$$\|v(t)\|_{L^1} \leq C\|v_0\|_{L^1\cap L^{p_n}}, \tag{5.7}$$

$$\|\partial_x^k v(t)\|_{H^{s-k}} \leq C\|v_0\|_{H^s\cap L^1\cap L^{p_n}}(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{5.8}$$

$$\|\partial_x^{s+1} v(t)\|_{L^2} \leq C\|v_0\|_{H^s\cap L^1\cap L^{p_n}} t^{-\frac{1}{4}}(1+t)^{-\frac{n}{8}-\frac{s}{4}}, \qquad t > 0, \tag{5.9}$$

where $0 \leq k \leq s$ in (5.8).

(ii) The estimates (5.8) and (5.9) will be used in the next section, where we will show that the solution u to the original problem (1.1), (1.2), constructed in Theorem 3.1, is approximated by the solution v to (5.2) obtained in Theorem 5.1 for large time.

Proof of Theorem 5.1. As usual, we look for the solution to (5.2) as a fixed point of the mapping

$$\Phi_0[v](t) = G_0(t) * v_0 + \int_0^t G_0(t-\tau) * \Delta g(v)(\tau)d\tau \tag{5.10}$$

in the closed ball

$$S_R = \{v \in C^0([0,\infty); L^{p_n})\ ;\ \|v\| \leq R\},$$

$$\|v\| := \sum_{k=0,1} \sup_{t>0} t^{\frac{k}{4}}\big(\|\partial_x^k v(t)\|_{L^{p_n}} + t^{\frac{n}{4p_n}}\|\partial_x^k v(t)\|_{L^\infty}\big),$$

where $R > 0$ will be determined later. Let $k = 0, 1$. For $v \in S_R$ we have from (4.4),

$$\begin{aligned}\|\partial_x^k \Phi_0[v](t)\|_{L^{p_n}} &\leq \|\partial_x^k G_0(t) * v_0\|_{L^{p_n}} + \int_0^t \|\partial_x^k G_0(t-\tau) * \Delta g(v)(\tau)\|_{L^{p_n}} \, d\tau \\ &\leq Ct^{-\frac{k}{4}} \|v_0\|_{L^{p_n}} + C\int_0^t (t-\tau)^{-\frac{k+2}{4}} \|g(v)(\tau)\|_{L^{p_n}} \, d\tau.\end{aligned}$$

Here, since $\|g(v)\|_{L^{p_n}} \leq C\|v\|_{L^\infty}^{\nu-1} \|v\|_{L^{p_n}}$, we see that

$$\|g(v)(t)\|_{L^{p_n}} \leq C\|v\|^\nu t^{-\frac{n}{4p_n}(\nu-1)} = C\|v\|^\nu t^{-\frac{1}{2}}. \tag{5.11}$$

Consequently, we obtain

$$\begin{aligned}\|\partial_x^k \Phi_0[v](t)\|_{L^{p_n}} &\leq Ct^{-\frac{k}{4}} \|v_0\|_{L^{p_n}} + C\|v\|^\nu \int_0^t (t-\tau)^{-\frac{k+2}{4}} \tau^{-\frac{1}{2}} \, d\tau \\ &\leq Ct^{-\frac{k}{4}} \|v_0\|_{L^{p_n}} + Ct^{-\frac{k}{4}} \|v\|^\nu.\end{aligned}$$

Similarly, we have again from (4.4),

$$\begin{aligned}\|\partial_x^k \Phi_0[v](t)\|_{L^\infty} &\leq \|\partial_x^k G_0(t) * v_0\|_{L^\infty} + \int_0^t \|\partial_x^k G_0(t-\tau) * \Delta g(v)(\tau)\|_{L^\infty} \, d\tau \\ &\leq Ct^{-\frac{n}{4p_n}-\frac{k}{4}} \|v_0\|_{L^{p_n}} + C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{4p_n}-\frac{k+2}{4}} \|g(v)(\tau)\|_{L^{p_n}} \, d\tau \\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{k+2}{4}} \|g(v)(\tau)\|_{L^\infty} \, d\tau\end{aligned}$$

Here we have $\|g(v)\|_{L^\infty} \leq C\|v\|_{L^\infty}^\nu$ and hence $\|g(v)(t)\|_{L^\infty} \leq C\|v\|^\nu t^{-\frac{1}{2}-\frac{n}{4p_n}}$. Using this estimate and (5.11), we obtain

$$\begin{aligned}\|\partial_x^k \Phi_0[v](t)\|_{L^\infty} &\leq Ct^{-\frac{n}{4p_n}-\frac{k}{4}} \|v_0\|_{L^{p_n}} + C\|v\|^\nu \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{4p_n}-\frac{k+2}{4}} \tau^{-\frac{1}{2}} \, d\tau \\ &\quad + C\|v\|^\nu \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{k+2}{4}} \tau^{-\frac{1}{2}-\frac{n}{4p_n}} \, d\tau \\ &\leq Ct^{-\frac{n}{4p_n}-\frac{k}{4}} \|v_0\|_{L^{p_n}} + Ct^{-\frac{n}{4p_n}-\frac{k}{4}} \|v\|^\nu.\end{aligned}$$

Hence we have shown that

$$\|\Phi_0[v]\| \le C_0\|v_0\|_{L^{p_n}} + C_1\|v\|^\nu \tag{5.12}$$

for some constants C_0 and C_1 depending only on n. By the same calculation as above, using the estimate $\|g(v)-g(\tilde v)\|_{L^p} \le C\|(v,\tilde v)\|_{L^\infty}^{\nu-1}\|v-\tilde v\|_{L^p}$, we also have

$$\|\Phi_0[v]-\Phi_0[\tilde v]\| \le C_2\|(v,\tilde v)\|^{\nu-1}\|v-\tilde v\| \tag{5.13}$$

for some C_2 depending only on n. Now we take $R = 2C_0\|v_0\|_{L^{p_n}}$ and assume that $\|v_0\|_{L^{p_n}} \le \delta$ with sufficiently small $\delta > 0$ depending only on C_0, C_1, C_2. Then it is easy to see that Φ_0 is a contraction mapping from S_R into S_R. Hence, there is a unique fixed point of Φ_0 in S_R, which is the solution to (5.2). By the above construction of the solution and by the interpolation we have already proved (5.4) for $k = 0, 1$. The same estimate for $k \ge 2$ is obtained by the standard bootstrap argument, and we omit the details here.

Next we assume in addition that $v_0 \in H^s \cap L^1$ for some $s \ge 0$. Then, by the bootstrap argument we observe that $v(t) \in L^1$ for any $t > 0$. Moreover, we have from (4.4),

$$\begin{aligned}\|v(t)\|_{L^1} &\le C\|v_0\|_{L^1} + C\int_0^t (t-\tau)^{-\frac12}\|g(v)(\tau)\|_{L^1}\,d\tau \\ &\le C\|v_0\|_{L^1} + C\|v_0\|_{L^{p_n}}^{\nu-1}\int_0^t (t-\tau)^{-\frac12}\tau^{-\frac12}\|v(\tau)\|_{L^1}\,d\tau\,,\end{aligned} \tag{5.14}$$

where we used $\|g(v)\|_{L^1} \le C\|v\|_{L^\infty}^{\nu-1}\|v\|_{L^1}$ and $\|v(t)\|_{L^\infty}^{\nu-1} \le C\|v\|^{\nu-1}t^{-\frac12} \le C\|v_0\|_{L^{p_n}}^{\nu-1}t^{-\frac12}$. Since $\|v_0\|_{L^{p_n}}$ is sufficiently small, the above inequality implies

$$\sup_{t>0}\|v(t)\|_{L^1} \le C\|v_0\|_{L^1}\,. \tag{5.15}$$

In particular, (5.14) and (5.15) imply (5.5) for $n = 1, 2$. To prove (5.5) for $n \ge 3$, we first show the following L^∞ estimate:

$$\|v(t)\|_{L^\infty} \le Ct^{-\frac{n}{4}}\|v_0\|_{L^1\cap L^{p_n}}\,, \qquad t > 0. \tag{5.16}$$

To veriy this, by applying (4.4) and using $\|v(t)\|_{L^\infty} \le C\|v_0\|_{L^{pn}} t^{-\frac{1}{2}}$ (for $n \ge 3$) and (5.15), we observe that

$$\begin{aligned}\|v(t)\|_{L^\infty} &\le Ct^{-\frac{n}{4}}\|v_0\|_{L^1} + C\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\|v(\tau)\|_{L^\infty}\|v(\tau)\|_{L^1} d\tau \\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|v(\tau)\|_{L^\infty}^2 d\tau \qquad (5.17) \\ &\le Ct^{-\frac{n}{4}}\|v_0\|_{L^1} + C\|v_0\|_{L^1}\|v_0\|_{L^{pn}}\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\tau^{-\frac{1}{2}} d\tau \\ &\quad + C\|v_0\|_{L^{pn}}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\|v(\tau)\|_{L^\infty}\, d\tau \\ &\le Ct^{-\frac{n}{4}}\|v_0\|_{L^1\cap L^{pn}} + C\|v_0\|_{L^{pn}}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\|v(\tau)\|_{L^\infty} d\tau. \qquad (5.18)\end{aligned}$$

Then, the smallness of $\|v_0\|_{L^{pn}}$ yields the estimate (5.16). Now we go back to the similar inequality to (5.14) and apply (5.16) for $\tau > 1$, which gives for $t > 2$,

$$\begin{aligned}&\|v(t) - G_0(t) * v_0\|_{L^1} \\ &\le\ C\|v_0\|_{L^1\cap L^{pn}}^2 \int_0^1 (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\, d\tau + C\|v_0\|_{L^1\cap L^{pn}}^2 \int_1^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{n}{4}}\, d\tau \\ &\le\ Ct^{-\frac{1}{2}}\|v_0\|_{L^1\cap L^{pn}}^2 + C\rho(t)\|v_0\|_{L^1\cap L^{pn}}^2\,. \qquad (5.19)\end{aligned}$$

Thus the estimate (5.5) has been proved.

For later use we also derive the similar estimate in L^∞. We employ the similar inequality to (5.17) and use the estimate (5.16) for $\tau > 1$. This yields for $t > 2$,

$$\begin{aligned}&\|v(t) - G_0(t) * v_0\|_{L^\infty} \\ &\le\ C\|v_0\|_{L^1\cap L^{pn}}^2 \int_0^1 (t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\tau^{-\frac{1}{2}}\, d\tau \\ &\quad + C\|v_0\|_{L^1\cap L^{pn}}^2 \int_1^{\frac{t}{2}} (t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\tau^{-\frac{n}{4}}\, d\tau + C\|v_0\|_{L^1\cap L^{pn}}^2 \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\tau^{-\frac{n}{2}}\, d\tau \\ &\le\ Ct^{-\frac{n}{4}-\frac{1}{2}}\|v_0\|_{L^1\cap L^{pn}}^2 + C\rho(t)t^{-\frac{n}{4}}\|v_0\|_{L^1\cap L^{pn}}^2 + Ct^{-\frac{n}{2}+\frac{1}{2}}\|v_0\|_{L^1\cap L^{pn}}^2\,. \qquad (5.20)\end{aligned}$$

Hence, by interpolating (5.19) and (5.20), we obtain

$$\|v(t)-G_0(t)*v_0\|_{L^p} \le C\rho(t)t^{-\frac{n}{4}(1-\frac{1}{p})}\|v_0\|^2_{L^1\cap L^{p_n}},\ t>2,\ 1\le p\le\infty. \quad (5.21)$$

Finally, let us prove (5.6). We note that (5.6) for $0<t<1$ follows from the assumption $v_0 \in L^1 \cap L^{p_n} \cap H^s$ without difficulty, whose proof is similar to the proof for the existence of v stated above. So we omit the details and focus on the estimate for $t \ge 1$. Since $\rho(t)=1$ for $n=1,2$, thanks to (5.4) with $p=2$, it suffices to consider the case $n\ge 3$. By applying (4.4), we have

$$\begin{aligned}
&\|\partial_x^{k+1}\big(v(t)-G_0(t)*v_0\big)\|_{L^2}\\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{8}-\frac{k+3}{4}}\|g(v)(\tau)\|_{L^1}d\tau + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{4}}\|\partial_x^k g(v)(\tau)\|_{L^2}d\tau\\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{8}-\frac{k+3}{4}}\|v(\tau)\|^2_{L^2}d\tau\\
&\quad +C\|v_0\|_{L^1\cap L^{p_n}}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{3}{4}}\tau^{-\frac{n}{4}}\|\partial_x^k v(\tau)\|_{L^2}d\tau\,, \qquad (5.22)
\end{aligned}$$

where we used $\|g(v)\|_{L^1}\le C\|v\|^2_{L^2}$, $\|\partial_x^k g(v)\|_{L^2}\le C\|v\|_{L^\infty}\|\partial_x^k v\|_{L^2}$ (for $n\ge 3$), and (5.16). The estimate (5.22) shows that the desired estimate (5.6) (for $n\ge 3$) is obtained if we establish the estimate

$$\|\partial_x^j v(t)\|_{L^2}\le C(1+t)^{-\frac{n}{8}-\frac{j}{4}}\|v_0\|_{H^j\cap L^1\cap L^{p_n}}\,, \qquad 0\le j\le k, \qquad (5.23)$$

which will be proved again from (5.22) by the induction on k as follows. Firstly, we note that

$$\|\partial_x^k G_0(t)*v_0\|_{L^2}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|v_0\|_{H^k\cap L^1}\,. \qquad (5.24)$$

Thus, (5.23) with $k=0$ is a direct consequence of (5.21). Assume that (5.23) holds for k. Then, (5.22) yields

$$\begin{aligned}
&\|\partial_x^{k+1}\big(v(t)-G_0(t)*v_0\big)\|_{L^2}\\
&\le C\|v_0\|^2_{L^1\cap L^{p_n}\cap L^2}\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{n}{8}-\frac{k+3}{4}}(1+\tau)^{-\frac{n}{4}}\,d\tau\\
&\quad +C\|v_0\|^2_{H^k\cap L^1\cap L^{p_n}}\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{3}{4}}\tau^{-\frac{n}{4}}(1+\tau)^{-\frac{n}{8}-\frac{k}{4}}\,d\tau
\end{aligned}$$

$$\leq \ C\rho(t)t^{-\frac{n}{8}-\frac{k+1}{4}}\|v_0\|^2_{H^k\cap L^1\cap L^{p_n}}\,, \qquad t>1\,.$$

This estimate together with (5.24) for $k+1$ gives (5.23) for $k+1$. Thus we have proved (5.6) for $n \geq 3$. The proof of Theorem 5.1 is complete. □

6. Nonlinear Approximation

In this section, we show that the nonlinear solution u is approximated by the solution v of Cahn-Hilliard type equation (5.1) with $v(0) = u_0 + u_1$. For our purpose, we need to introduce the quantities $N(t)$, $N_1(t)$ and $N_2(t)$ as follows:

$$N(t) := N_1(t) + N_2(t), \tag{6.1}$$

$$N_1(t) := \sum_{j=0}^{[\frac{n}{4}]} \sum_{\sigma_0(k)+j\leq s} \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{j}{2}+\frac{k}{4}}\tilde\rho(\tau)^{-1}\|\partial_x^k(u-v)(\tau)\|_{H^{s-\sigma_0(k)-j}}, \tag{6.2}$$

$$N_2(t) := \sum_{k=0}^{s_0} \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{n}{8}+\frac{k}{4}}\tilde\rho(\tau)^{-1}\|\partial_x^k(u-v)(\tau)\|_{H^{s-\sigma_1(k,n)-1}}, \tag{6.3}$$

where $s_0 = [\frac{n}{2}]+1$, $s \geq \sigma_1(s_0, n)+1$, and $\tilde\rho(t)$ is given by

$$\tilde\rho(t) := \begin{cases} (1+t)^{-\frac{1}{4}} & n=1 \\ (1+t)^{-\frac{1}{2}+\varepsilon} & n=2 \\ (1+t)^{-\frac{1}{2}} & n\geq 3 \end{cases} \tag{6.4}$$

with small $\varepsilon > 0$. We also introduce the Banach space X':

$$X' = \{u \in C^0([0,\infty); H^s);\ \|u\|_{X'} < \infty\},$$

with the norm

$$\|u\|_{X'} \ = \ \|u\|_{X'_1} + \|u\|_{X'_2}, \tag{6.5}$$

$$\|u\|_{X'_1} \ := \ \sum_{j=0}^{[\frac{n}{4}]} \sum_{\sigma_0(k)+j\leq s} \sup_{t\geq 0} (1+t)^{\frac{j}{2}+\frac{k}{4}}\|\partial_x^k u(t)\|_{H^{s-\sigma_0(k)-j}}, \tag{6.6}$$

$$\|u\|_{X'_2} \ := \ \sum_{k=0}^{s_0} \sup_{t\geq 0} (1+t)^{\frac{n}{8}+\frac{k}{4}}\|\partial_x^k u(t)\|_{H^{s-\sigma_1(k,n)-1}}, \tag{6.7}$$

where $s_0 = [\frac{n}{2}] + 1$ and $s \geq \sigma_1(s_0, n) + 1$. Notice that X' is the space X in (3.4) with s replaced by $s - 1$.

Futhermore, we define the Banach space Y:

$$Y = \{u \in C^0([0,\infty); H^s) \cap C^0((0,\infty); H^{s+1});\ \|u\|_Y < \infty\},$$

with the norms

$$\|u\|_Y := \sum_{k=0}^{s} \sup_{t\geq 0} (1+t)^{\frac{n}{8}+\frac{k}{4}} \|\partial_x^k u(t)\|_{H^{s-k}} + \sup_{t>0} (1+t)^{\frac{n}{8}+\frac{s}{4}} t^{\frac{1}{4}} \|\partial_x^{s+1} u(t)\|_{L^2}. \tag{6.8}$$

This space Y is used for the solution v of (5.2). In fact, it follows from Theorem 5.1 together with the estimates (5.8) and (5.9) that if v_0 is small in $H^s \cap L^1$ with $s \geq s_0$, then we have the solution $v \in Y$ of (5.2) satisfying $\|v\|_Y \leq C\|v_0\|_{H^s\cap L^1}$. For this solution v, we see that

$$\|g(v)(\tau)\|_{L^1} \leq C\|v\|_{L^\infty}^{\nu-2}\|v\|_{L^2}^2 \leq C\|v\|_Y^\nu (1+\tau)^{-\frac{n}{4}(\nu-1)}, \tag{6.9}$$

$$\|\partial_x^m g(v)(\tau)\|_{L^2} \leq C\|v\|_{L^\infty}^{\nu-1}\|\partial_x^m v\|_2 \leq C\|v\|_Y^\nu (1+\tau)^{-\frac{n}{8}-\frac{m}{4}-\frac{n}{4}(\nu-1)}\tau^{-\frac{\theta}{4}}, \tag{6.10}$$

where $\theta = 0$ for $0 \leq m \leq s$ and $\theta = 1$ for $m = s+1$. Moreover, by the similar procedure in the proof of Lemma 3.3, for $u, v \in X'$ (notice that $v \in Y \subset X'$), we have

$$\|(g(u)-g(v))(\tau)\|_{L^1} \leq C\|(u,v)\|_{X'}^{\nu-1} N_2(t)\tilde{\rho}(\tau)(1+\tau)^{-\frac{n}{4}(\nu-1)}, \tag{6.11}$$

$$\|\partial_x^{k+h}(g(u)-g(v))(\tau)\|_{L^2} \leq C\|(u,v)\|_{X'}^{\nu-1} N(t)\tilde{\rho}(\tau)(1+\tau)^{-\frac{j}{2}-\frac{k}{4}-\frac{n}{4}(\nu-1)}, \tag{6.12}$$

$$\|\partial_x^{k+h}(g(u)-g(v))(\tau)\|_{L^2} \leq C\|(u,v)\|_{X'}^{\nu-1} N_2(t)\tilde{\rho}(\tau)(1+\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)}, \tag{6.13}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $0 \leq h \leq s - \sigma_0(k) - j$ in (6.12), $k \geq 0$ and $0 \leq h \leq s - \sigma_1(k,n) - 1$ in (6.13).

In the above situation, we have the following asymptotic behavior of the solution of the problem (1.1), (1.2).

Theorem 6.1. *Let $n \geq 1$ and $s \geq \sigma_1(s_0, n) + 1$. Assume the same condition in Theorem 3.1. Let u be the global solution to the nonlinear problem (1.1), (1.2) and v be the solution of (5.1) with $v(0) = u_0 + u_1$. Then the nonlinear solution u is asymptotic to v as $t \to \infty$ in the following sense:*

$$\|\partial_x^k(u-v)(t)\|_{H^{s-\sigma_0(k)-j}} \leq CE_1\tilde{\rho}(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{6.14}$$

$$\|\partial_x^k(u-v)(t)\|_{H^{s-\sigma_1(k,n)-1}} \leq CE_1\tilde{\rho}(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{6.15}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k)+j \leq s$ in (6.14), $0 \leq k \leq s_0$ *and* $\sigma_1(k,n) \leq s-1$ *in* (6.15), *and $\tilde{\rho}(t)$ is given in* (6.4).

For the proof of Theorem 6.1, we take the difference between the integral equations (3.1) and (5.2) and decompose into five parts:

$$\begin{aligned}(u-v)(t) &= (u_L-v_L)(t) \\ &\quad + \int_0^t G(t-\tau)*(1-\Delta)^{-1}\Delta(f(u)-g(u))(\tau)d\tau \\ &\quad + \int_0^t G(t-\tau)*(1-\Delta)^{-1}\Delta(g(u)-g(v))(\tau)d\tau \\ &\quad + \int_0^t (G-G_0)(t-\tau)*(1-\Delta)^{-1}\Delta g(v)(\tau)d\tau \\ &\quad + \int_0^t G_0(t-\tau)*\{(1-\Delta)^{-1}-1\}\Delta g(v)(\tau)d\tau \\ &= A_0(t)+A_1(t)+A_2(t)+A_3(t)+A_4(t). \end{aligned} \tag{6.16}$$

In what follows, we estimate $A_j(t)$ $(j=1,\cdots,4)$, respectively.

6.1. Preliminaries

First, we show the estimate for the term A_1.

Lemma 6.2. *Let $s \geq \sigma_1(s_0,n)$ and suppose that $u \in X$. Then we have the following decay estimates:*

$$\|\partial_x^k A_1(t)\|_{H^{s+1-\sigma_0(k)-j}} \leq C\|u\|_X^{\nu+1}\tilde{\rho}(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{6.17}$$

$$\|\partial_x^k A_1(t)\|_{H^{s-\sigma_1(k,n)}} \leq C\|u\|_X^{\nu+1}\tilde{\rho}(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{6.18}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k)+j \leq s+1$ in (6.17), $k \geq 0$ *and* $\sigma_1(k,n) \leq s$ *in* (6.18), *and $\tilde{\rho}(t)$ is the function given in* (6.4).

The proof of Lemma 6.2 is given in a similar fashion to the proof of Lemma 3.3. In fact, to show Lemma 6.2, we only follow the proof of Lemma 3.3 with $f(u)-f(v)$ replaced by $f(u)-g(u)=O(u^{\nu+1})$. Here we omit the proof.

Secondly we consider the estimate for the term A_2. In this case, we can obtain the decay estimate with minor change of the proof of Lemma 3.3.

Lemma 6.3. *Let $s \geq \sigma_1(s_0, n) + 1$ and suppose that $u, v \in X'$. Then we have the following decay estimates:*

$$\|\partial_x^k A_2(t)\|_{H^{s-\sigma_0(k)-j}} \leq C\|(u,v)\|_{X'}^{\nu-1} N(t)\tilde{\rho}(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \quad (6.19)$$

$$\|\partial_x^k A_2(t)\|_{H^{s-\sigma_1(k,n)-1}} \leq C\|(u,v)\|_{X'}^{\nu-1} N(t)\tilde{\rho}(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \quad (6.20)$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k) + j \leq s$ in (6.19), $k \geq 0$ and $\sigma_1(k, n) \leq s - 1$ in (6.20), and $\tilde{\rho}(t)$ is the function given in (6.4).

The proof of Lemma 6.3 is also given in a similar way of the proof of Lemma 3.3. In fact, making use of the estimates (6.11), (6.12) and (6.13), we can show the desired estimates (6.19) and (6.20) in the same way as in the proof of Lemma 3.3. We omit the details.

For the term A_3 and A_4, we prepare the linear estimates, which are useful to show the estimates for A_3 and A_4 in our framework.

Lemma 6.4. *Let $n \geq 1$, $s \geq 0$ and $1 \leq q \leq 2$. Then we have:*

$$\begin{aligned}
&\|\partial_x^k (G - G_0)(t) * (1-\Delta)^{-1}\Delta f\|_{L^2} \\
&\quad \leq C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{k+4-j}{4}}\|\partial_x^j f\|_{L^q} \\
&\qquad + C(1+t)^{-\frac{l+1}{2}}\|\partial_x^{(k+l)_+} f\|_{L^2} + Ce^{-ct}t^{-\frac{j'}{4}}\|\partial_x^{k-j'} f\|_{L^2}, \quad (6.21)
\end{aligned}$$

$$\begin{aligned}
&\|\partial_x^k G_0(t) * \{(1-\Delta)^{-1} - 1\}\Delta f\|_{L^2} \\
&\quad \leq C(1+t)^{-\frac{n}{4}(\frac{1}{q}-\frac{1}{2})-\frac{k+4-j}{4}}\|\partial_x^j f\|_{L^q} + Ce^{-ct}t^{-\frac{j'}{4}}\|\partial_x^{k+2-j'} f\|_{L^2}, \quad (6.22)
\end{aligned}$$

where $0 \leq k \leq s$, $0 \leq j \leq k+4$, $0 \leq j' \leq k$, $l+1 \geq 0$, $(k+l)_+ = \max\{k+l, 0\}$ in (6.21) and $0 \leq k \leq s$, $0 \leq j \leq k+4$, $0 \leq j' \leq k+2$ in (6.22).

Proof. First, we prove (6.21). Here we recall (4.7). Then it is easy to see that

$$|(\hat{G} - \hat{G}_0)(\xi, t)|(1+|\xi|^2)^{-1}|\xi|^2 \leq C|\xi|^4 e^{-c|\xi|^4 t} + Ce^{-ct} \quad (6.23)$$

for $|\xi| \leq r_0$ and by (2.12) and the definition of G_0,

$$|(\hat{G} - \hat{G}_0)(\xi, t)|(1+|\xi|^2)^{-1}|\xi|^2 \leq C|\xi|^{-1} e^{-c|\xi|^{-2} t} + Ce^{-c|\xi|^4 t} \quad (6.24)$$

for $|\xi| \geq r_0$, where r_0 is a small positive constant. Therefore the pointwise estimates (6.23) and (6.24) and the Plancherel formula yield the estimate (6.21). Next, we show (6.22). In this case, we use the expression $\hat{G}_0(\xi,t) = e^{-|\xi|^4 t}$ and

$$\left|(1+|\xi|^2)^{-1}-1\right| |\xi|^2 \leq \begin{cases} C|\xi|^4 & \text{for } |\xi| \leq r_0, \\ C|\xi|^2 & \text{for } |\xi| \geq r_0, \end{cases}$$

for small positive constant r_0. Then we again apply the Plancherel formula to have the estimate (6.22). The proof of Lemma 6.4 is complete. □

Based on the estimate (6.21), we have the estimate for A_3 as follows:

Lemma 6.5. *Let $s \geq \sigma_1(s_0,n)+1$ and suppose that $v \in Y$. Then we have the following decay estimates:*

$$\|\partial_x^k A_3(t)\|_{H^{s-\sigma_0(k)-j}} \leq C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}}, \tag{6.25}$$
$$\|\partial_x^k A_3(t)\|_{H^{s-\sigma_1(k,n)-1}} \leq C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{6.26}$$

where $k \geq 0$, $0 \leq j \leq [\frac{n}{4}]$ and $\sigma_0(k)+j \leq s$ in (6.25), $k \geq 0$ and $\sigma_1(k,n) \leq s-1$ in (6.26), and $\eta(t)$ is the function given below:

$$\eta(t) := (1+t)^{-\frac{1}{2}}\rho(t). \tag{6.27}$$

Proof. To show the estimates (6.25) and (6.26), we apply ∂_x^{k+h} to A_3 and decompose into two parts:

$$\begin{aligned}\|\partial_x^{k+h} A_3(t)\|_{L^2} &\leq \int_0^t \|\partial_x^{k+h}(G-G_0)(t-\tau) * (1-\Delta)^{-1}\Delta g(v)(\tau)\|_{L^2} d\tau \\ &= \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t =: A_{31} + A_{32}.\end{aligned}$$

For the term A_{31}, we apply (6.21) with k replaced by $k+h$ and with $j =$

$j'=0$. Then we see that

$$\begin{aligned} A_{31}(t) &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+h+4}{4}}\|g(v)(\tau)\|_{L^1}d\tau \\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l+1}{2}}\|\partial_x^m g(v)(\tau)\|_{L^2}d\tau \\ &\quad + C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^{k+h} g(v)(\tau)\|_{L^2}d\tau \\ &=: A_{311}+A_{312}+A_{313}, \end{aligned} \tag{6.28}$$

where $m=(k+h+l)_+\le s+1$. To obtain the estimate for A_{311}, using (6.9), we have

$$\begin{aligned} A_{311} &\le C\|v\|_Y^\nu \int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+h+4}{4}}(1+\tau)^{-\frac{n}{4}(\nu-1)}d\tau \\ &\le C\|v\|_Y^\nu \rho(t)(1+t)^{-\frac{n}{8}-\frac{k+2}{4}} = C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \end{aligned} \tag{6.29}$$

for $h\ge 0$. Next, we show the estimates for A_{312}. For the proof of (6.25), we choose l in (6.28) as the smallest integer satisfying $\frac{l+1}{2}\ge\frac{j}{2}+\frac{k+4}{4}$, i.e., $l\ge j+\frac{k}{2}+1$. This leads to $l=\sigma_0(k)-k+j+1$. In this case, the regularity assumption $m:=(k+h+l)_+\le s+1$ means that $0\le h\le s-\sigma_0(k)-j$. Then we see that

$$\begin{aligned} A_{312} &\le C\|v\|_Y^\nu \int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{j}{2}-\frac{k+4}{4}}(1+\tau)^{-\frac{n}{8}-\frac{m}{4}-\frac{n}{4}(\nu-1)}\tau^{-\frac{\theta}{4}}d\tau \\ &\le C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}} \end{aligned} \tag{6.30}$$

for $0\le h\le s-\sigma_0(k)-j$, where we used (6.10). To show (6.26), we choose l in (6.28) as the smallest integer satisfying $\frac{l+1}{2}\ge\frac{n}{8}+\frac{k+4}{4}$, i.e., $l\ge\frac{n+2k}{4}+1$. This leads to $l=\sigma_1(k,n)-k+2$. Then the regularity assumption $m:=(k+h+l)_+\le s+1$ means that $0\le h\le s-\sigma_1(k,n)-1$. Therefore we have

$$\begin{aligned} A_{312} &\le C\|v\|_Y^\nu \int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+4}{4}}(1+\tau)^{-\frac{n}{8}-\frac{m}{4}-\frac{n}{4}(\nu-1)}\tau^{-\frac{\theta}{4}}d\tau \\ &\le C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \end{aligned} \tag{6.31}$$

for $0\le h\le s-\sigma_1(k,n)-1$, where we used (6.10). For the term A_{313}, the estimate (6.10) yields that

$$A_{313} \le C\|v\|_Y^{\nu} \int_0^{\frac{t}{2}} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{8}-\frac{k+h}{4}-\frac{n}{4}(\nu-1)} d\tau \le C\|v\|_Y^{\nu} e^{-ct}. \quad (6.32)$$

Finally, we estimate the term A_{32}. We apply (6.21) with k replaced by $k+h$ and $q=2$, $j=k+h+1$, $j'=0$ and $l=1$. This choice of $l=1$ requires $0 \le h \le s-k$ since $(k+h+l)_+ \le s+1$. Then we see

$$\begin{aligned} A_{32}(t) \ &\le \ C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{3}{4}} \|\partial_x^{k+h+1} g(v)(\tau)\|_{L^2} d\tau \\ &\quad + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-1} \|\partial_x^{k+h+1} g(v)(\tau)\|_{L^2} d\tau \\ &\quad + C\int_{\frac{t}{2}}^{t} e^{-c(t-\tau)} \|\partial_x^{k+h} g(v)(\tau)\|_{L^2} d\tau \\ &=: \ A_{321} + A_{322} + A_{323}. \end{aligned}$$

For the terms A_{321} and A_{322}, we use the estimate (6.10) with $m = k+h+1$ to have

$$\begin{aligned} A_{321}+A_{322} \ &\le \ C\|v\|_Y^{\nu} \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{n}{8}-\frac{k+h+1}{4}-\frac{n}{4}(\nu-1)} \tau^{-\frac{\theta}{4}} d\tau \\ &\le \ C\|v\|_Y^{\nu}(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)} \le C\|v\|_Y^{\nu}\eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \quad (6.33) \end{aligned}$$

for $0 \le h \le s-k$, where we used the fact that $(1+t)^{-\frac{n}{4}(\nu-1)} = (1+t)^{-\frac{1}{2}}\rho_1(t) \le \eta(t)$ with $\rho_1(t)$ in (3.22). Similarly, using (6.10) with $m = k+h$ $(\le s)$, we have

$$\begin{aligned} A_{323} \ &\le \ C\|v\|_Y^{\nu} \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{8}-\frac{k+h+2}{4}-\frac{n}{4}(\nu-1)} d\tau \\ &\le \ C\|v\|_Y^{\nu}(1+t)^{-\frac{n}{8}-\frac{k+2}{4}-\frac{n}{4}(\nu-1)} \le C\|v\|_Y^{\nu}\eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}} \quad (6.34) \end{aligned}$$

for $0 \le h \le s-k$. Summing up the above argument, we have the estimate (6.25) (resp. (6.26)) from (6.29), (6.30), (6.32), (6.33) and (6.34) (resp. (6.29), (6.31), (6.32), (6.33) and (6.34)). □

For the last term A_4, we can obtain the desired estimate without regularity-loss.

Lemma 6.6. *Let $s \geq s_0$ and $v \in Y$. Then we have*

$$\|\partial_x^k A_4(t)\|_{H^{s-k}} \leq C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{6.35}$$

where $0 \leq k \leq s$, and $\eta(t)$ is defined by (6.27).

Proof. Let k, $h \geq 0$. We apply ∂_x^{k+h} to A_4 and take the L^2 norm to obtain

$$\begin{aligned}\|\partial_x^{k+h} A_4(t)\|_{L^2} &\leq \int_0^t \|\partial_x^{k+h} G_0(t-\tau) * \{(1-\Delta)^{-1} - 1\}\Delta g(v)(\tau)\|_{L^2} d\tau \\ &= \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t =: A_{41} + A_{42}.\end{aligned} \tag{6.36}$$

For the term A_{41}, we apply (6.22) with k replaced by $k+h$ and $q = 1$, $j = 0$ and $j' = 2$. Then we have

$$\begin{aligned}A_{41}(t) &\leq C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8}-\frac{k+h+4}{4}} \|g(v)(\tau)\|_{L^1} d\tau \\ &\quad + C\int_0^{\frac{t}{2}} e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_x^{k+h} g(v)(\tau)\|_{L^2} d\tau \\ &=: A_{411} + A_{412}.\end{aligned} \tag{6.37}$$

The term A_{411} is just the same as A_{311} and we have $A_{411} = A_{311} \leq C\|v\|_Y^\nu \eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}$ for $h \geq 0$. Also, the term A_{412} is similar to A_{313} and by using (6.10) with $m = k+h$, we have

$$A_{412} \leq C\|v\|_Y^\nu \int_0^{\frac{t}{2}} e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)} d\tau \leq C\|v\|_Y^\nu e^{-ct}$$

for $0 \leq h \leq s-k$. To show the estimate of the term A_{42}, we apply (6.22) with k replaced by $k+h$ and $q = 2$, $j = k+h+1$ and $j' = 2$. Thus we see

$$\begin{aligned}A_{42}(t) &\leq C\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{3}{4}} \|\partial_x^{k+h+1} g(v)(\tau)\|_{L^2} d\tau \\ &\quad + C\int_{\frac{t}{2}}^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_x^{k+h} g(v)(\tau)\|_{L^2} d\tau \\ &=: A_{421} + A_{422}.\end{aligned} \tag{6.38}$$

Here the term A_{421} is just the same as A_{321} and we have $A_{421} = A_{321} \leq$

$C\|v\|_Y^\nu\eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}$ for $0 \le h \le s-k$. Also, the term A_{422} is similar to A_{323} and we have

$$\begin{aligned}A_{422} &\le C\|v\|_Y^\nu \int_{\frac{t}{2}}^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{n}{8}-\frac{k+h}{4}-\frac{n}{4}(\nu-1)}d\tau \\ &\le C\|v\|_Y^\nu(1+\tau)^{-\frac{n}{8}-\frac{k}{4}-\frac{n}{4}(\nu-1)} \le C\|v\|_Y^\nu\eta(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}}\end{aligned}$$

for $0 \le h \le s-k$. Then combining all these estimates, we have (6.35). The proof of Lemma 6.6 is complete. □

6.2. Proof of Theorem 6.1

We show Theorem 6.1 by using Lemmas which we prepared in the previous subsection. Let $s \ge \sigma(s_0, n)+1$. Let $u \in X$ be the global solution to the problem (1.1), (1.2), which was obtained in Theorem 3.1, and let $v \in Y$ be the global solution of (5.1) with $v(0) = v_0 := u_0 + u_1 \in H^s \cap L^1$, which is obtained in Theorem 5.1. Then we apply Lemmas 6.2, 6.3, 6.5 and 6.6 to (6.16). This yields

$$\begin{aligned}&\|\partial_x^k(u-v)(t)\|_{H^{s-\sigma_0(k)-j}} \\ &\le \sum_{l=0}^4 \|\partial_x^k A_l(t)\|_{H^{s-\sigma_0(k)-j}} \\ &\le C(E_1 + \|u\|_X^{\nu+1} + \|(u,v)\|_{X'}^{\nu-1}N(t) + \|v\|_Y^\nu)\tilde\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}} \\ &\le C(E_1 + E_1^{\nu+1} + E_1^{\nu-1}N(t) + E_1^\nu)\tilde\rho(t)(1+t)^{-\frac{j}{2}-\frac{k}{4}},\end{aligned}$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$ and $\sigma_0(k)+j \le s$. Here we used $\|(u,v)\|_{X'} \le \|u\|_X + \|v\|_Y \le CE_1$. Then, multiplying $\tilde\rho(t)^{-1}(1+t)^{\frac{j}{2}+\frac{k}{4}}$ to the both sides and taking summation in j and k, we see

$$N_1(t) \le CE_1 + CE_1^{\nu-1}N(t). \tag{6.39}$$

By the similar way, we have

$$\|\partial_x^k(u-v)(t)\|_{H^{s-\sigma_1(k,n)-1}} \le C(E_1+E_1^{\nu+1}+E_1^{\nu-1}N(t)+E_1^\nu)\tilde\rho(t)(1+t)^{-\frac{n}{8}-\frac{k}{4}},$$

where $0 \le k \le s_0$ and $\sigma_1(k,n) \le s-1$ and then

$$N_2(t) \le CE_1 + CE_1^{\nu-1} N(t). \tag{6.40}$$

It follows from (6.39) and (6.40) that $N(t) \le CE_1 + CE_1^{\nu-1}N(t)$. Therefore, if E_1 is sufficiently small, then we have $N(t) \le CE_1$, which implies the desired estimates (6.14) and (6.15). Thus the proof of Theorem 6.1 is complete.

7. Nonlinear Asymptotic Profile

In this section we consider the parabolic equation

$$v_t + \Delta^2 v = \Delta g_*(v), \tag{7.1}$$

with the initial data $v_0 := u_0 + u_1$ as in (4.1), where the nonlinear term g_* is given as

$$g_*(v) = \begin{cases} a\, v^{1+\frac{2}{n}}, & n = 1, 2, \\ a\, |v|^{\frac{2}{n}} v, & n \ge 3. \end{cases}$$

Here $a \neq 0$ is a fixed constant. In (7.1) one may replace the nonlinear term $g_*(v)$ by $a\,|v|^{2/n}v$ or $a\,|v|^{1+2/n}$, which does not give rise to an essential change of the argument below, except for the statements on the higher-order regularity of solutions. For simplicity we assume $a = 1$.

One of the basic properties of (7.1) is the invariance under the action

$$v(x,t) \longmapsto v_\lambda(x,t) = \lambda^{\frac{n}{4}} v(\lambda^{\frac{1}{4}} x, \lambda t), \qquad \lambda > 0. \tag{7.2}$$

Indeed, if v satisfies(7.1) then v_λ also satisfies (7.1) for any $\lambda > 0$, while the equality $\sup_{t>0} \|v_\lambda(t)\|_{L^1} = \sup_{t>0} \|v(t)\|_{L^1}$ holds. In the presence of such an invariance, even for a sufficiently localized initial data, a simple linearization v_L given in (4.2) does not provide a correct asymptotics in large-time for solutions to (7.1) in general. Instead, one usually needs to consider the self-similar solution, which represents the balance between the linear diffusion by $-\Delta^2$ and the nonlinear reaction by Δg_*.

The purpose of this section is to show the existence of self-similar solutions to (7.1) and to establish their asymptotic stability. For later use we introduce the scaling operator $\{R_\lambda\}_{\lambda>0}$ as

$$(R_\lambda f)(x) = \lambda^{\frac{n}{4}} f(\lambda^{\frac{1}{4}} x)\,, \qquad \lambda > 0\,. \tag{7.3}$$

As in the previous sections, the equation (7.1) is always understood as its integral form

$$v(t) = G_0(t-s) * v(s) + \int_s^t G_0(t-\tau) * \Delta g_*(v)(\tau) d\tau\,, \qquad t > s > 0\,, \tag{7.4}$$

and the solution to the Cauchy problem of (7.1) with initial data v_0 is the solution to (7.4) with $s = 0$ and $v(0) = v_0$ as in Section 5. We say that v is a self-similar solution to (7.1) if v satisfies (7.4) for $t > s > 0$ and also if v is invariant under the scaling (7.2), or equivalently, v is written in the form

$$v(x,t) = (R_{\frac{1}{t}} \Phi)(x)\,, \qquad t > 0\,,$$

for some profile function Φ which is independent of the time variable. Our first result in this section is on the unique existence of small self-similar solutions, stated as follows.

Theorem 7.1. *Fix $m > \frac{n}{2}$. Then there is a positive constant $\delta_2 = \delta_2(m,n)$ such that, for each $|\delta| \in [0, \delta_2]$, the equation (7.1) admits a unique self-similar solution $R_{\frac{1}{t}} \Phi_\delta$ satisfying*

$$\int_{\mathbb{R}^n} \Phi_\delta(x) dx = \delta\,, \qquad \sum_{k=0,1} \|\partial_x^k \Phi_\delta\|_{L^2_m \cap L^\infty_{m+\frac{n}{2}}} \le C|\delta|\,. \tag{7.5}$$

Moreover, if $n = 1, 2$, then $\partial_x^k \Phi_\delta \in L^2_m$ for any $k \ge 0$ with the norms in $L^2_m \cap L^\infty_{m+\frac{n}{2}}$ of the order $O(|\delta|)$.

The next theorem shows the asymptotic stability of the self-similar solution obtained in Theorem 7.1.

Theorem 7.2. *There is a positive constant $\delta_3 = \delta_3(n)$ such that if $\|v_0\|_{L^1_1} \le \delta_3$, then the Cauchy problem (7.1), (4.1) admits a unique solution $v \in C^0([0,\infty); L^1)$ satisfying*

$$\|\partial_x^k (v(t) - R_{\frac{1}{1+t}} \Phi_\delta)\|_{L^p} \le C\|v_0\|_{L^1_1} t^{-\frac{n}{4}(1-\frac{1}{p}) - \frac{k}{4}} (1+t)^{-\frac{1}{4}}\,, \qquad t > 0\,, \tag{7.6}$$

for $k = 0, 1$ and $1 \le p \le \infty$. Here Φ_δ is the profile of the self-similar solution in Theorem 7.1 with $\delta = \int_{\mathbb{R}^n} v_0(x)dx$ and $m > \frac{n}{2} + 1$. Moreover, if in addition $n = 1, 2$, then (7.6) holds for any $k \ge 0$, and in particular, in the case $v_0 \in H^s$ for some $s \ge 0$ we have

$$\|\partial_x^k(v(t) - R_{\frac{1}{1+t}}\Phi_\delta)\|_{H^{s-k}} \le C\|v_0\|_{H^s \cap L^1_1}(1+t)^{-\frac{n}{8}-\frac{k+1}{4}}, \tag{7.7}$$

where $0 \le k \le s$.

As a direct consequence of Theorems 6.1 and 7.2, we can conclude that the nonlinear diffusion wave $R_{\frac{1}{1+t}}\Phi_\delta$ is an asymptotic profile of the solution of (1.1), (1.2).

Corollary 7.3. *Let $n = 1, 2$ and $s \ge \sigma_1(s_0, n) + 1$, where $s_0 = [\frac{n}{2}] + 1$. Suppose that $u_0 \in H^{s+1} \cap L^1$, $u_1 \in H^s \cap L^1$ and $v_0 := u_0 + u_1 \in L^1_1$, and put $\tilde{E}_1 = E_1 + \|u_0 + u_1\|_{L^1_1}$. Let u be the global solution to the nonlinear problem (1.1), (1.2), which is obtained in Theorem 3.1. Then the nonlinear solution u is approximated by the nonlinear diffusion wave $v^* := R_{\frac{1}{1+t}}\Phi_\delta$ with $\delta = M := \int_{\mathbb{R}^n}(u_0 + u_1)(x)dx$. More precisely, the difference $u - v^*$ verifies the decay estimates*

$$\|\partial_x^k(u - v^*)(t)\|_{H^{s-\sigma_0(k)-j}} \le C\tilde{E}_1(1+t)^{-\frac{j}{2}-\frac{k+1}{4}}, \tag{7.8}$$

$$\|\partial_x^k(u - v^*)(t)\|_{H^{s-\sigma_1(k,n)-1}} \le C\tilde{E}_1(1+t)^{-\frac{n}{8}-\frac{k+1}{4}}, \tag{7.9}$$

where $k \ge 0$, $0 \le j \le [\frac{n}{4}]$ and $\sigma_0(k) + j \le s$ in (7.8), $0 \le k \le s_0$ and $\sigma_1(k, n) \le s - 1$ in (7.9).

Remark 7.4. The estimate (7.9) implies the lower bound of the norm of the nonlinear solution u for large t, provided that $M \ne 0$. Indeed, using the self-similar property of $v^* = R_{\frac{1}{1+t}}\Phi_\delta$, we see that $\|\partial_x^k v^*(t)\|_{L^2} = c_0(1+t)^{-\frac{n}{8}-\frac{k}{4}}$ with some constant $c_0 > 0$ depending on M. Therefore we have

$$\begin{aligned}\|\partial_x^k u(t)\|_{L^2} &\ge \|\partial_x^k v^*(t)\|_{L^2} - \|\partial_x^k(u - v^*)(t)\|_{L^2} \\ &= c_0(1+t)^{-\frac{n}{8}-\frac{k}{4}} - \|\partial_x^k(u - v^*)(t)\|_{L^2}.\end{aligned} \tag{7.10}$$

Since the second term on the right hand side of (7.10) decays as in (7.9), the estimate (7.10) means the lower bound of the norm of u.

To prove Theorems 7.1 and 7.2 we follow the argument of Kagei and Maekawa [28], where the abstract theory is developed for the evolution equations in the presence of scaling invariance.

7.1. Similarity transform

As is well known, the existence and the asymptotic stability of self-similar solutions are equivalent with the existence and the stability of stationary solutions to the equation derived through a similarity transform associated with the invariant scaling. Before introducing the similarity transform related to (7.2), we first note that the one parameter family $\{G_0(t)*\}_{t\geq 0}$ in (4.2) defines a C_0-analytic semigroup in L^p_m, $1 \leq p < \infty$, with the generator $-\Delta^2$ whose domain is characterized as

$$D_{L^p_m}(-\Delta^2) = \{f \in L^p_m\,;\ \partial_x^k f \in L^p_m\,,\ 0 \leq k \leq 4\}\,. \tag{7.11}$$

For convenience we use the notation $e^{-t\Delta^2}$ for the operator $G_0(t)*$. By using the scaling operator $\{R_\lambda\}_{\lambda>0}$ in (7.3), the invariance of (7.1) with respect to the scaling (7.2) is represented by the following identities.

$$R_\lambda e^{-\lambda t\Delta^2} = e^{-t\Delta^2} R_\lambda\,, \qquad \lambda R_\lambda \Delta g_*(v) = \Delta g_*(R_\lambda v)\,. \tag{7.12}$$

Then the associated similarity transform is defined as

$$w(x,t) = \big(R_{e^t} v(e^t - 1)\big)(x) = e^{\frac{n}{4}t} v(e^{\frac{t}{4}}x, e^t - 1)\,. \tag{7.13}$$

Since $\{R_\lambda\}_{\lambda>0}$ is a strongly continuous action of the multiplicative group $\{\lambda > 0\}$ on L^p_m for $1 \leq p < \infty$, the first identity of (7.12) implies that the one-parameter family $\{R_{e^t} e^{-(e^t-1)\Delta^2}\}_{t\geq 0}$ defines a C_0-semigroup in L^p_m for $1 \leq p < \infty$; see [28, Lemma 2.1]. We denote by $\mathbf{A}$ the associated generator, i.e.,

$$e^{t\mathbf{A}} = R_{e^t} e^{-(e^t-1)\Delta^2} = e^{-(1-e^{-t})\Delta^2} R_{e^t} \qquad \text{(by (7.12))}\,. \tag{7.14}$$

If v is the solution to the Cauchy problem (7.1), (4.1), then the integral equation for w is given as

$$w(t) = e^{t\mathbf{A}} w_0 + \int_0^t e^{(t-s)\mathbf{A}} \Delta g_*(w(s)) ds\,, \qquad t > 0\,, \tag{7.15}$$

with $w_0 = v_0$. We note that the self-similar solution $R_{\frac{1}{t}}\Phi$ to (7.1) is, in the similarity variables, nothing but the stationary solution to (7.15), i.e.,

$$\Phi = e^{t\mathbf{A}}\Phi + \int_0^t e^{s\mathbf{A}}\Delta g_*(\Phi)ds\,, \qquad t > 0\,. \tag{7.16}$$

As will be seen in Lemma 7.8 (iii) below, the function $e^{t\mathbf{A}}\varphi$ converges to δG_* as $t \to \infty$, where $\delta = \int_{\mathbb{R}^n} \varphi dx$ and $G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)$ is the eigenfunction with $\int_{\mathbb{R}^n} G_*(x)dx = 1$ to the first simple eigenvalue 0 of $\mathbf{A}$. One can also verify from Lemma 7.8 (ii) that the integral $\int_0^\infty e^{s\mathbf{A}}\Delta\varphi ds$ converges in the case $\varphi \in L^2_m$ with $m > \frac{n}{2}$. Thus, by taking $t \to \infty$ in (7.16) we formally obtain the equation for Φ as follows.

$$\Phi = \delta G_* + \int_0^\infty e^{s\mathbf{A}}\Delta g_*(\Phi)ds\,, \qquad \delta \in \mathbb{R}\,. \tag{7.17}$$

The number δ is now a given parameter, and it represents the mass of the stationary solution due to the property $\int_{\mathbb{R}^n}\int_0^\infty (e^{s\mathbf{A}}\Delta\varphi)(x)dxds = 0$, which is justified for $\varphi \in L^2_m$ with $m > \frac{n}{2}$. Clearly (7.16) and (7.17) are formally equivalent.

In Subsection 7.2 we prove the unique existence of small solutions to (7.17) by a standard fixed point argument. In particular, Theorem 7.1 is an immediate consequence of Proposition 7.5. In Subsection 7.3 we show the stability of stationary solutions obtained in Subsection 7.2 with respect to small perturbations. Then Theorem 7.2 follows by returning to the original variables. The estimates for the semigroup $\{e^{t\mathbf{A}}\}_{t\geq 0}$ are the most fundamental tool in Subsections 7.2 and 7.3, which are collected in Subsection 7.4. As is explained in Remark 7.7 below, thanks to the presence of Δ in the nonlinear term, we do not need to analyze the detailed spectral property of the linearization around the stationary solution in order to obtain the optimal convergence rate $(1+t)^{-\frac{1}{4}}$ in Theorem 7.2. Therefore we can skip, to some extent, the detailed spectral analysis of the generator $\mathbf{A}$. However, the study of the spectral property of $\mathbf{A}$ seems to have its own interest and applications, so we go into the details on this topic in Subsection 7.5.

7.2. Existence of stationary solution

In this subsection we prove the existence and the uniqueness of small solutions to (7.17) for a small fixed δ. The next proposition directly leads to Theorem 7.1.

Proposition 7.5. *Fix $m > \frac{n}{2}$. Then there is a positive constant $\delta_2 = \delta_2(m,n)$ such that, for each $|\delta| \in [0,\delta_2]$, the equation (7.17) admits a unique solution Φ_δ satisfying*

$$\int_{\mathbb{R}^n} \Phi_\delta(x)dx = \delta\,, \qquad \sum_{k=0,1} \|\partial_x^k \Phi_\delta\|_{L^2_m \cap L^\infty_{m+\frac{n}{2}}} \le C|\delta|\,. \tag{7.18}$$

Moreover, if $n = 1,2$, then $\partial_x^k \Phi_\delta \in L^2_m$ for any $k \ge 0$ with the norms in $L^2_m \cap L^\infty_{m+\frac{n}{2}}$ of the order $O(|\delta|)$.

Proof. We look for the solution to (7.17) of the form $\Phi = \delta G_* + \phi$, where ϕ is the function in the class

$$X_R = \{f \in L^\infty_{m+\frac{n}{2}}\,;\ \textstyle\sum_{k=0,1} \|\partial_x^k f\|_{L^\infty_{m+\frac{n}{2}}} \le R\,,\ \int_{\mathbb{R}^n} f(x)dx = 0\}\,.$$

The number $R > 0$ will be chosen later, which is of the order $O(|\delta|^{1+\frac{2}{n}})$. Then ϕ should satisfy

$$\phi = F(\phi) := \int_0^\infty e^{s\mathbf{A}} \Delta g_*(\delta G_* + \phi)ds\,. \tag{7.19}$$

Set $a(t) = 1 - e^{-t}$. From Lemma 7.8 (ii) below we have

$$\begin{aligned}\sum_{k=0,1} \|\partial_x^k F(\phi)\|_{L^\infty_{m+\frac{n}{2}}} &\le C \int_0^\infty \frac{e^{-\frac{s}{2}}}{a(s)^{\frac{3}{4}}} \|g_*(\delta G_* + \phi)\|_{L^\infty_{m+\frac{n}{2}}} ds \\ &\le C\|\delta G_* + \phi\|^{1+\frac{2}{n}}_{L^\infty_{m+\frac{n}{2}}} \\ &\le C_1\big(|\delta|^{1+\frac{2}{n}} + \|\phi\|^{1+\frac{2}{n}}_{L^\infty_{m+\frac{n}{2}}}\big)\,, \qquad \phi \in X_R\,.\end{aligned} \tag{7.20}$$

On the other hand, we have for $\phi, \tilde{\phi} \in X_R$,

$$\sum_{k=0,1} \|\partial_x^k (F(\phi) - F(\tilde{\phi}))\|_{L^\infty_{m+\frac{n}{2}}}$$

$$\le C\int_0^\infty \frac{e^{-\frac{s}{2}}}{a(s)^{\frac{3}{4}}}\|g_*(\delta G_*+\phi)-g_*(\delta G_*+\tilde\phi)\|_{L^\infty_{m+\frac{n}{2}}}\,ds$$
$$\le C_2\big(|\delta|^{\frac{2}{n}}+\|\phi\|_{L^\infty}^{\frac{2}{n}}+\|\tilde\phi\|_{L^\infty}^{\frac{2}{n}}\big)\|\phi-\tilde\phi\|_{L^\infty_{m+\frac{n}{2}}}\,. \tag{7.21}$$

Let us take $R=2C_1|\delta|^{1+\frac{2}{n}}$ and $|\delta|\le\delta_3$ with δ_3 small enough depending on C_1 and C_2. Then F is a contraction mapping from X_R into X_R, and hence, there is a unique fixed point of F in X_R, as desired. Let ϕ be the fixed point of F in X_R. Note that $g_*(f)\in L^\infty_{\alpha(1+\frac{2}{n})}$ if $f\in L^\infty_\alpha$. Thus we have again from Lemma 7.8 (ii),

$$\sum_{k=0,1}\|\partial_x^k F(\phi)\|_{L^2_m}\le C\int_0^\infty \frac{e^{-\frac{s}{2}}}{a(s)^{\frac{3}{4}}}\|g_*(\delta G_*+\phi)\|_{L^2_m}\,ds$$
$$\le C\|\delta G_*+\phi\|_{L^\infty_{m+\frac{n}{2}}}^{1+\frac{2}{n}}$$
$$\le C\big(|\delta|^{1+\frac{2}{n}}+\|\phi\|_{L^\infty_{m+\frac{n}{2}}}^{1+\frac{2}{n}}\big)\,. \tag{7.22}$$

This proves the estimate of ϕ in L^2_m. If $n=1,2$ then the nonlinear term g_* is smooth, and one can show the higher order regularity of ϕ by a standard bootstrap argument. The details are omitted here. The proof is complete. □

7.3. Stability of stationary solution

In this subsection we prove the stability of stationary solutions obtained in Proposition 7.5.

Proposition 7.6. *There is a positive constant $\delta_3=\delta_3(n)$ such that if $\|w_0\|_{L^1_1}\le\delta_3$ then the Cauchy problem (7.15) admits a unique solution $w\in C^0([0,\infty);L^1)$ satisfying*

$$\|\partial_x^k(w(t)-\Phi_\delta)\|_{L^p}\le\frac{C\|w_0\|_{L^1_1}e^{-\frac{t}{4}}}{a(t)^{\frac{n}{4}(1-\frac{1}{p})+\frac{k}{4}}}\,,\qquad t>0\,, \tag{7.23}$$

for $k=0,1$ and $1\le p\le\infty$. Here $a(t)=1-e^{-t}$ and Φ_δ is the stationary solution in Proposition 7.5 with $\delta=\int_{\mathbb{R}^n}w_0(x)dx$ and $m>\frac{n}{2}+1$. Moreover, if in addition $n=1,2$, then (7.23) holds for any $k\ge0$, and in particular, *in the case $w_0 \in H^s$ for some $s \geq 0$ we have*

$$\|\partial_x^k (w(t) - \Phi_\delta)\|_{H^{s-k}} \leq C\|w_0\|_{H^s \cap L^1_1} e^{-\frac{t}{4}}\,, \qquad t > 0\,, \tag{7.24}$$

where $0 \leq k \leq s$.

Remark 7.7. (i) The factor $e^{-\frac{t}{4}}$ in (7.24) is considered as an optimal. The reason why one can obtain this optimal rate only from simple estimates in Lemma 7.8 is due to the presence of Δ in the nonlinear term. Indeed, from Lemma 7.8 (i) we have $\|e^{-(t-s)\mathbf{A}}\Delta f\|_{L^p} \leq Ce^{-\frac{t-s}{2}}\|f\|_{L^q}$ for $t - s \geq 1$, and therefore, the rate $e^{-\frac{t}{4}}$ is derived only from the smallness of the stationary solution and the perturbation. If the nonlinear term $\Delta g_*(u)$ is replaced by $\nabla h(u)$ with $h(u) = |u|^{\frac{3}{n}} u$, which also preserves the invariant property with respect to the scaling (7.2), then it is essential to study the spectrum of the linearization around the stationary solution in order to achieve the convergence rate $e^{-\frac{t}{4}}$. In such a case one needs to analyze the spectral properties of $\mathbf{A}$ itself. In particular, the results as in Lemma 7.10 play a central role, which makes us possible to apply the general perturbation theory of linear operators.

(ii) Since (7.1) is invariant under the translation in the x variables, by shifting the stationary solution suitably, one can improve the rate $e^{-\frac{t}{4}}$ in (7.24) to $e^{-\frac{t}{2}}$; see [28] for details on the abstract argument related to this issue.

Proof of Proposition 7.6. From (7.15) and (7.16) we construct w in the form $w = \Phi_\delta + z$, where z is the solution to

$$\begin{aligned} z(t) &= e^{t\mathbf{A}}(w_0 - \Phi_\delta) + \int_0^t e^{(t-s)\mathbf{A}} \Delta\big(g_*(z + \Phi_\delta) - g_*(z)\big)\, ds \\ &=: F[z](t)\,. \end{aligned}$$

To this end we look for the fixed point of F in the closed ball

$$\begin{aligned} X_R &= \{f \in C^0([0,\infty); L^1)\,;\ \|f\| \leq R\}\,, \\ \|f\| &:= \sum_{k=0,1} \sup_{t>0} e^{\frac{t}{4}} \big(a(t)^{\frac{k}{4}} \|\partial_x^k f(t)\|_{L^1} + a(t)^{\frac{n+k}{4}} \|\partial_x^k f(t)\|_{L^\infty}\big)\,, \end{aligned}$$

where $R > 0$ is chosen later. We firstly observe that $\int_{\mathbb{R}^n} (w_0 - \Phi_\delta) dx = 0$ and $w_0 - \Phi_\delta \in L^1_1$ since $\Phi_\delta \in L^2_m$ with $m > \frac{n}{2} + 1$. Thus, Lemma 7.8 (iii) implies

$$\|\partial_x^k F[0](t)\|_{L^p} = \|\partial_x^k e^{t\mathbf{A}}(w_0 - \Phi_\delta)\|_{L^p} \le \frac{Ce^{-\frac{t}{4}}}{a(t)^{\frac{n}{4}(1-\frac{1}{p})+\frac{k}{4}}}\|w_0 - \Phi_\delta\|_{L^1_1}, \ t > 0,$$

for $1 \le p \le \infty$ and $k = 0, 1$. This shows

$$\|F[0]\| \le C_1 \|w_0\|_{L^1_1}. \tag{7.25}$$

Next we estimate $F[z] - F[\tilde{z}]$ for $z, \tilde{z} \in X_R$. By the definition of g_* we have

$$\begin{aligned}
&\|g_*(z(s) + \Phi_\delta) - g_*(\tilde{z}(s) + \Phi_\delta)\|_{L^p} + \|g_*(z(s)) - g(\tilde{z}(s))\|_{L^p} \\
&\quad \le C\big(\|z(s)\|_{L^\infty}^{\frac{2}{n}} + \|\tilde{z}(s)\|_{L^\infty}^{\frac{2}{n}} + \|\Phi_\delta\|_{L^\infty}^{\frac{2}{n}}\big)\|z(s) - \tilde{z}(s)\|_{L^p} \\
&\quad \le Ca(s)^{-\frac{1}{2}}\big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z(s) - \tilde{z}(s)\|_{L^p}.
\end{aligned}$$

Here we have used the fact $0 < a(s) \le 1$. For convenience we set

$$h[z, \tilde{z}](s) = g_*(z(s) + \Phi_\delta) - g_*(\tilde{z}(s) + \Phi_\delta) + g_*(z(s)) - g(\tilde{z}(s)).$$

Then we have proved that

$$\|h[z, \tilde{z}](s)\|_{L^p} \le Ca(s)^{-\frac{1}{2}}\big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z(s) - \tilde{z}(s)\|_{L^p}. \tag{7.26}$$

Now we consider two cases: (i) t is small and (ii) t is large.

(i) The case $0 < t \le 2$: In this case we decompose the integral $\int_0^t$ into $\int_{t/2}^t$ and $\int_0^{t/2}$. By using Lemma 7.8 (i) and (7.26) the first term is estimated as, for $p = 1, \infty$,

$$\begin{aligned}
&\left\|\partial_x^k \int_{\frac{t}{2}}^t e^{(t-s)\mathbf{A}} \Delta h[z, \tilde{z}](s) ds\right\|_{L^p} \\
&\quad \le \ C\int_{\frac{t}{2}}^t a(t-s)^{-\frac{2+k}{4}}\|h[z, \tilde{z}](s)\|_{L^p} ds \\
&\quad \le \ C\int_{\frac{t}{2}}^t a(t-s)^{-\frac{2+k}{4}} a(s)^{-\frac{1}{2}-\frac{n}{4}(1-\frac{1}{p})} ds\big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z - \tilde{z}\|
\end{aligned}$$

$$\leq\ Ca(t)^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k}{4}}\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\,.$$

On the other hand, the integral $\int_0^{t/2}$ is estimated as

$$\begin{aligned}
&\|\partial_x^k\int_0^{\frac{t}{2}}e^{(t-s)\mathbf{A}}\Delta h[z,\tilde z](s)ds\|_{L^p}\\
&\leq\ C\int_0^{\frac{t}{2}}a(t-s)^{-\frac{2+k}{4}-\frac{n}{4}(1-\frac{1}{p})}\|h[z,\tilde z](s)\|_{L^1}ds\\
&\leq\ C\int_{\frac{t}{2}}^{t}a(t-s)^{-\frac{2+k}{4}-\frac{n}{4}(1-\frac{1}{p})}a(s)^{-\frac{1}{2}}ds\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\\
&\leq\ Ca(t)^{-\frac{n}{4}(1-\frac{1}{p})-\frac{k}{4}}\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{k=0,1}\sup_{0<t\leq 2}e^{\frac{t}{4}}\Big(a(t)^{\frac{k}{4}}\|\partial_x^k\big(F[z]-F[\tilde z]\big)(t)\|_{L^1}+a(t)^{\frac{n+k}{4}}\|\partial_x^k\big(F[z]-F[\tilde z]\big)(t)\|_{L^\infty}\Big)\\
&\leq\ C\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\,. \qquad (7.27)
\end{aligned}$$

(ii) The case $t>2$: Note that $1-e^{-1}\leq a(s)\leq 1$ for $s\geq t-1$ in this case. We decompose the integral $\int_0^t$ into $\int_{t-1}^t$ and $\int_0^{t-1}$. The first term is estimated as

$$\begin{aligned}
&\|\partial_x^k\int_{t-1}^{t}e^{(t-s)\mathbf{A}}\Delta h[z,\tilde z](s)ds\|_{L^p}\\
&\leq\ C\int_{t-1}^{t}a(t-s)^{-\frac{2+k}{4}}\|h[z,\tilde z](s)\|_{L^p}ds\\
&\leq\ C\int_{t-1}^{t}a(t-s)^{-\frac{2+k}{4}}e^{-\frac{s}{4}}ds\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\\
&\leq\ Ce^{-\frac{t}{4}}\big(R^{\frac{2}{n}}+\|w_0\|_{L_1^1}^{\frac{2}{n}}\big)\|z-\tilde z\|\,,\quad p=1,\infty\,.
\end{aligned}$$

The second term is estimated as

$$\begin{aligned}
&\|\partial_x^k\int_0^{t-1}e^{(t-s)\mathbf{A}}\Delta h[z,\tilde z](s)ds\|_{L^p}\\
&\leq\ C\int_0^{t-1}e^{-\frac{t-s}{2}}\|h[z,\tilde z](s)\|_{L^p}ds
\end{aligned}$$

$$\le C\int_0^{t-1} e^{-\frac{t-s}{2}} a(s)^{-\frac{1}{2}} e^{-\frac{s}{4}} ds \big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z-\tilde{z}\|$$
$$\le Ce^{-\frac{t}{4}}\big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z-\tilde{z}\|, \quad p=1,\infty.$$

Thus we obtain

$$\sum_{k=0,1}\sup_{t>2} e^{\frac{t}{4}}\Big(a(t)^{\frac{k}{4}}\|\partial_x^k\big(F[z]-F[\tilde{z}]\big)(t)\|_{L^1}+a(t)^{\frac{n+k}{4}}\|\partial_x^k\big(F[z]-F[\tilde{z}]\big)(t)\|_{L^\infty}\Big)$$
$$\le C\big(R^{\frac{2}{n}}+\|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z-\tilde{z}\|. \tag{7.28}$$

Combining (7.27) with (7.28), we have arrived at the estimate

$$\|F[z]-F[\tilde{z}]\| \le C_2\big(R^{\frac{2}{n}} + \|w_0\|_{L^1_1}^{\frac{2}{n}}\big)\|z-\tilde{z}\|. \tag{7.29}$$

Let us take $R = 2C_1\|w_0\|_{L^1_1}$ and take $\|w_0\|_{L^1_1}$ small enough. Then (7.25) and (7.29) imply that F is a contraction mapping from X_R into X_R, and there is unique fixed point of F in X_R, as desired. The estimates of $z(t) = w(t) - \Phi_\delta$ in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ easily follow from the ones for $p = 1, \infty$ by the interpolation. The estimate (7.23) is obtained from the bootstrap argument as in the proof of (5.6) in Theorem 5.1. The proof is complete. □

Proof of Theorem 7.2. We only give a proof of (7.6) for $k = 0$. The estimates for $k = 1$ or (7.7) are proved by the same argument. By the similarity transform (7.13) we have

$$v(x,t)-(R_{\frac{1}{1+t}}\Phi_\delta)(x)=(1+t)^{-\frac{n}{4}}\big\{w\big((1+t)^{-\frac{1}{4}}x,\log(1+t)\big)-\Phi_\delta((1+t)^{-\frac{1}{4}}x)\big\}.$$

Hence, Proposition 7.6 yields

$$\|v(t) - R_{\frac{1}{1+t}}\Phi_\delta\|_{L^p} = (1+t)^{-\frac{n}{4}(1-\frac{1}{p})}\|w(\log(1+t)) - \Phi_\delta\|_{L^p}$$
$$\le C(1+t)^{-\frac{n}{4}(1-\frac{1}{p})}\|v_0\|_{L^1_1} a(\tau)^{-\frac{n}{4}(1-\frac{1}{p})} e^{-\frac{\tau}{4}}, \tau=\log(1+t),$$

which implies

$$\|v(t) - R_{\frac{1}{1+t}}\Phi_\delta\|_{L^p} \le C\|v_0\|_{L^1_1} t^{-\frac{n}{4}(1-\frac{1}{p})}(1+t)^{-\frac{1}{4}},$$

as desired. The proof of Theorem 7.2 is complete. □

7.4. Basic estimates for $\{e^{t\mathbf{A}}\}_{t\ge 0}$

In this section we establish the L^p - L^q estimates for the semigroup $\{e^{t\mathbf{A}}\}_{t\ge 0}$, which are used in the proof of Theorems 7.1 and 7.2.

Lemma 7.8. *Set $a(t) = 1 - e^{-t}$. Let k be a nonnegative integer.*

(i) *Let $1 \le q \le p \le \infty$. Then we have*

$$\|\partial_x^k e^{t\mathbf{A}} f\|_{L^p} \le \frac{Ce^{\frac{n}{4}(1-\frac{1}{q})t}}{a(t)^{\frac{n}{4}(\frac{1}{q}-\frac{1}{p})+\frac{k}{4}}} \|f\|_{L^q}\,, \qquad t > 0\,, \tag{7.30}$$

$$\|e^{t\mathbf{A}} \partial_x^k f\|_{L^p} \le \frac{Ce^{\frac{n-k}{4}(1-\frac{1}{q})t}}{a(t)^{\frac{n}{4}(\frac{1}{q}-\frac{1}{p})+\frac{k}{4}}} \|f\|_{L^q}\,, \qquad t > 0\,. \tag{7.31}$$

(ii) *Let $1 \le q \le p \le \infty$ and $m > n(1 - \frac{1}{q})$. Then we have*

$$\|\partial_x^k e^{t\mathbf{A}} f\|_{L^p_m} \le \frac{C}{a(t)^{\frac{n}{4}(\frac{1}{q}-\frac{1}{p})+\frac{k}{4}}} \|f\|_{L^q_m}\,, \qquad t > 0\,, \tag{7.32}$$

$$\|e^{t\mathbf{A}} \partial_x^k f\|_{L^p_m} \le \frac{Ce^{-\frac{k}{4}t}}{a(t)^{\frac{n}{4}(\frac{1}{q}-\frac{1}{p})+\frac{k}{4}}} \|f\|_{L^q_m}\,, \qquad t > 0\,. \tag{7.33}$$

Moreover, we can take $m = 0$ if $q = 1$.

(iii) *Let $1 \le p \le \infty$ and $f \in L^1_1$. Then*

$$\|\partial_x^k\big(e^{t\mathbf{A}} f - \delta_f G_*\big)\|_{L^p} \le \frac{Ce^{-\frac{t}{4}}}{a(t)^{\frac{n}{4}(1-\frac{1}{p})+\frac{k}{4}}} \|f\|_{L^1_1}\,, \qquad t > 0\,. \tag{7.34}$$

Here $G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)$ and $\delta_f = \int_{\mathbb{R}^n} f(x)dx$.

Proof. We note that $G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)$ is already introduced in (4.3), and we set $G_{*,\alpha}(x) = \partial_x^\alpha G_*(x)$, where α is a multi-index with $|\alpha| = k$. Then the definition of $e^{t\mathbf{A}}$ implies the representation

$$(\partial_x^\alpha e^{t\mathbf{A}} f)(x) = \frac{e^{\frac{n}{4}t}}{a(t)^{\frac{n+k}{4}}} \int_{\mathbb{R}^n} G_{*,\alpha}(a(t)^{-\frac{1}{4}}(x-y)) f(e^{\frac{t}{4}}y)dy\,, \tag{7.35}$$

while by the integration by parts,

$$(e^{t\mathbf{A}}\partial_x^\alpha f)(x) = \frac{e^{\frac{n-k}{4}t}}{a(t)^{\frac{n+k}{4}}}\int_{\mathbb{R}^n} G_{*,\alpha}(a(t)^{-\frac{1}{4}}(x-y))f(e^{\frac{t}{4}}y)dy\,. \tag{7.36}$$

Therefore, (7.30) and (7.31) follow from the Young inequality. For the proof of (7.32) and (7.33) we will use the inequality

$$\langle x\rangle \le \langle x-y\rangle\langle y\rangle\,. \tag{7.37}$$

Since (7.32) and (7.33) for $t \in (0,1]$ are easily obtained from (7.35), (7.37), and the Young inequality, we give a proof only for the case $t > 1$. It suffices to consider the case $k = 0$. Using the embedding $L^q_m \hookrightarrow L^1$ due to the assumption $m > n(1-\frac{1}{q})$, we have again from (7.35) and the Young inequality,

$$\begin{aligned}\|e^{t\mathbf{A}}f\|_{L^p_m} &\le Ce^{\frac{n}{4}t}\|f(e^{\frac{t}{4}}\cdot)\|_{L^1} + Ce^{\frac{n}{4}t}\||x|^m f(e^{\frac{t}{4}}x)\|_{L^q}\\ &\le C\|f\|_{L^1} + Ce^{(\frac{n}{4}-\frac{m}{4}-\frac{n}{4q})t}\|f\|_{L^q_m}\\ &\le C\|f\|_{L^q_m}\,,\end{aligned}$$

as desired. Note that, since $t > 1$, there is no contribution from the factor $a(t) = 1-e^{-t}$ in the above calculations. To show (7.34) it suffices to consider the case $t > 1$ and $k = 0$. We observe that G_* satisfies $\int_{\mathbb{R}^n} G_*(x)dx = \mathcal{F}[G_*](0) = 1$ and $e^{t\mathbf{A}}G_* = G_*$ for all $t \ge 0$ (since $\mathbf{A}G_* = 0$). Therefore, it suffices to estimate $e^{t\mathbf{A}}\tilde{f}$ for $\tilde{f} \in L^1_1$ with $\int_{\mathbb{R}^n}\tilde{f}(x)dx = 1$ by setting $\tilde{f} = f - \delta_f G_*$. Using this zero integral condition, we can write

$$\begin{aligned}(e^{t\mathbf{A}}\tilde{f})(x) &= \frac{e^{\frac{n}{4}t}}{a(t)^{\frac{n}{4}}}\int_{\mathbb{R}^n}\big(G_*(a(t)^{-\frac{1}{4}}(x-y)) - G_*(a(t)^{-\frac{1}{4}}x)\big)\tilde{f}(e^{\frac{t}{4}}y)dy\\ &= -\frac{1}{a(t)^{\frac{n}{4}}}\int_0^1\int_{\mathbb{R}^n}\big(\nabla_x G_*\big)(a(t)^{-\frac{1}{4}}(x-\sigma e^{-\frac{t}{4}}y))\cdot\frac{e^{-\frac{t}{4}}y}{a(t)^{\frac{1}{4}}}\tilde{f}(y)dyd\sigma\,.\end{aligned} \tag{7.38}$$

Taking the L^p norm in (7.38), we obtain the desired estimates by the Minkowski inequality. The proof is complete. □

7.5. Spectral property of A

In this section we study spectral properties of $\mathbf{A}$ in details. As stated in Remark 7.7, the results in this section are not necessarily essential in the proof of Theorems 7.1 and 7.2. However, the spectral analysis of $\mathbf{A}$, which is a fourth order elliptic operator, will have its own interest, and we focus on this issue for reader's convenience. We will mainly work in the space L^2_m, which makes the argument slightly simpler due to the structure of the Hilbert space.

Firstly, let us introduce the operator $\mathcal{B}$,

$$\begin{aligned} D_{L^2_m}(\mathcal{B}) &= \{f \in L^2_m\,;\ x\cdot\nabla f \in L^2_m\}\,,\\ \mathcal{B}f &= \tfrac{x}{4}\cdot\nabla f + \tfrac{n}{4}f\,, \qquad f\in D_{L^2_m}(\mathcal{B})\,, \end{aligned}$$

which is the generator associated with the scaling $\{R_\lambda\}_{\lambda>0}$ (see (7.3)), i.e.,

$$\mathcal{B}f = \lim_{h\to 0}\frac{R_{1+h}f - f}{h} \quad \text{in } \ L^2_m\,, \qquad f\in D_{L^2_m}(\mathcal{B})\,.$$

The next lemma describes the domain of the generator of $\{R_{e^t}e^{-(e^t-1)\Delta^2}\}_{t\ge 0}$.

Lemma 7.9. *The generator* $\mathbf{A}$ *of* $\{R_{e^t}e^{-(e^t-1)\Delta^2}\}_{t\ge0}$ *in* L^2_m *is given as*

$$\begin{aligned} D_{L^2_m}(\mathbf{A}) &= D_{L^2_m}(\Delta^2)\cap D_{L^2_m}(\mathcal{B})\\ &= \{f\in L^2_m\,;\ \partial_x^k f\in L^2_m\,,\ 0\le k\le 4\,,\ x\cdot\nabla f\in L^2_m\}\,,\\ \mathbf{A}f &= -\Delta^2 f + \mathcal{B}f\,, \qquad f\in D_{L^2_m}(\mathbf{A})\,. \end{aligned}$$

Proof. We will use the result in [33]. The direct computations show that the adjoint of $-\Delta^2$ in L^2_m is given by

$$\begin{aligned} D_{L^2_m}\big((-\Delta^2)^*\big) &= D_{L^2_m}(-\Delta^2)\,, \\ (-\Delta^2)^* f &= -\Delta^2 f + \sum_{|\gamma|\le 3} p_\gamma(x)\partial_x^\gamma f\,, \qquad f\in D_{L^2_m}\big((-\Delta^2)^*\big)\,, \end{aligned} \tag{7.39}$$

where each p_γ is a smooth function satisfying

$$|\partial_x^\alpha p_\gamma(x)| \le C\langle x\rangle^{2m+|\gamma|-4-|\alpha|}\,, \qquad x\in\mathbb{R}^n\,.$$

Therefore, by the interpolation inequality we have

$$\begin{aligned}\|(-\Delta^2)f - (-\Delta^2)^* f\|_{L^2_m} &\le \sum_{|\gamma|\le 3} \|p_\gamma \partial_x^\gamma f\|_{L^2_m} \\ &\le \epsilon \|\Delta^2 f\|_{L^2_m} + C_\epsilon \|f\|_{L^2_m} \end{aligned} \tag{7.40}$$

for any small $\epsilon > 0$ and $f \in D_{L^2_m}(-\Delta^2)$. On the other hand, the adjoint of $\mathcal{B}$ in L^2_m is given as

$$D_{L^2_m}(\mathcal{B}^*) = D_{L^2_m}(\mathcal{B}), \qquad \mathcal{B}^* f = -\tfrac{x}{4} \cdot \nabla f - \tfrac{m}{2} f, \quad f \in D_{L^2_m}(\mathcal{B}). \tag{7.41}$$

Thus we see

$$\|\mathcal{B} f + \mathcal{B}^* f\|_{L^2_m} = (\tfrac{n}{4} - \tfrac{m}{2}) \|f\|_{L^2_m}. \tag{7.42}$$

Collecting (7.39), (7.40), (7.41), and (7.42), we can apply [33, Theorem 3.9], and the proof is complete. □

Next we study the spectrum of $\mathbf{A}$ in L^2_m, which is the main object of research in this section.

Lemma 7.10. *Fix $m \ge 0$. The spectrum of $\mathbf{A}$ in L^2_m is given as*

$$\sigma(\mathbf{A}) = \big\{\lambda \in \mathbb{C}\,;\ \mathrm{Re}(\lambda) \le \tfrac{1}{4}(\tfrac{n}{2} - m)\big\} \cup \{-\tfrac{k}{4}\,;\ k = 0, 1, 2, \cdots\}. \tag{7.43}$$

If $m > \frac{n}{2}$ and if $k \in \mathbb{N} \cup \{0\}$ satisfies $k + \frac{n}{2} < m$, then $\lambda_k = -\frac{k}{4}$ is a semisimple eigenvalue of $\mathbf{A}$ with multiplicity $\binom{n+k-1}{k}$. Moreover, we have

$$r_{ess}(e^{t\mathbf{A}}) = e^{-\frac{t}{4}(m - \frac{n}{2})}, \qquad t > 0, \tag{7.44}$$

where $r_{ess}(e^{t\mathbf{A}})$ is the radius of the essential spectrum of $e^{t\mathbf{A}}$.

Remark 7.11. The characterization (7.43) implies that $\{e^{t\mathbf{A}}\}_{t\ge 0}$ is not analytic in L^2_m, and in such case it is known that the general spectral mapping theorem is not applied. Therefore, in order to estimate $\{e^{t\mathbf{A}}\}_{t\ge 0}$ (or its perturbations) in large time, the information on the essential spectrum of $e^{t\mathbf{A}}$ is important, in addition to (7.43). For the definition of the radius of the essential spectrum, we refer to [4, Chapter IV, Section 1].

Proof of Lemma 7.10. We follow the argument of Gallay and Wayne [10, Appendix A], where the spectrum of $\Delta + \frac{x}{2}\cdot\nabla + \frac{n}{2}$ in L^2_m is determined.

Step 1 (Discrete spectrum): As in the proof of Lemma 7.8, we set

$$G_*(x) = \mathcal{F}^{-1}[e^{-|\xi|^4}](x)\,, \qquad G_{*,\alpha}(x) = \partial_x^\alpha G_*(x)\,. \tag{7.45}$$

Note that $\int_{\mathbb{R}^n} G_*(x)dx = 1$. It is straightforward to see $G_{*,\alpha} \in \mathcal{S}(\mathbb{R}^n)$ and $G_{*,\alpha}$ is an eigenfunction of $\mathbf{A}$ with eigenvalue $-\frac{|\alpha|}{4}$. In particular, the multiplicity of the eigenvalue $\lambda_k = -\frac{k}{4}$, $k \in \mathbb{N}\cup\{0\}$, is greater than or equal to

$$\begin{pmatrix} n+k-1 \\ k \end{pmatrix} = \#\{\alpha \in (\mathbb{N}\cup\{0\})^n\,;\ |\alpha| = k\}.$$

Step 2 (Continuous spectrum): Assume that $\lambda \in \mathbb{C}$ satisfies $\mathrm{Re}(\lambda) < \frac{n}{8}$ and $-2\lambda \notin \mathbb{N}\cup\{0\}$. Set

$$\Psi_\lambda(x) = \mathcal{F}^{-1}[|\xi|^{-4\lambda}e^{-|\xi|^4}](x)\,. \tag{7.46}$$

It is clear that $\Psi_\lambda \in C^\infty(\mathbb{R}^n)$. Using the representation of $\mathbf{A}$ in the Fourier variables $-|\xi|^4 - \frac{\xi}{4}\cdot\nabla_\xi$, we can check that Ψ_λ solves the eigenvalue problem $\mathbf{A}\Psi_\lambda = \lambda\Psi_\lambda$. Moreover, from the estimate

$$|\partial_\xi^\beta(\xi^\alpha|\xi|^{-4\lambda}e^{-|\xi|^4})| \le C_{\alpha,\beta,n,\lambda}|\xi|^{-4\,\mathrm{Re}(\lambda)+|\alpha|-|\beta|}e^{-c|\xi|^4}\,, \qquad \xi \in \mathbb{R}^n\,,$$

for some $c > 0$, it is not difficult to see

$$|\partial_x^\alpha\Psi_\lambda(x)| \le C_{\alpha,n,\lambda}|x|^{4\,\mathrm{Re}(\lambda)-n-|\alpha|}\,, \qquad |x| \ge 1\,. \tag{7.47}$$

Therefore, Ψ_λ belongs to $D_{L^2_m}(\mathbf{A})$ if $\mathrm{Re}(\lambda) < \frac{1}{4}(\frac{n}{2} - m)$. Since the spectrum is a closed set, we have

$$\big\{\lambda \in \mathbb{C}\,;\ \mathrm{Re}(\lambda) \le \tfrac{1}{4}(\tfrac{n}{2} - m)\big\} \subset \sigma(\mathbf{A})\,. \tag{7.48}$$

Step 3 (Expansion of $e^{t\mathbf{A}}$): Let $l \in \mathbb{Z}$ be such that $l + \frac{n}{2} < m$. Then we define a bounded operator $\mathcal{P}_l$ in L^2_m by

$$\mathcal{P}_l f = \begin{cases} 0 & \text{if } \ l < 0\,, \\ \displaystyle\sum_{|\alpha| \le l} c_\alpha(f) G_{*,\alpha} & \text{if } \ l \ge 0\,. \end{cases} \tag{7.49}$$

Here $G_{*,\alpha}$ is the function in Step 1 and

$$c_\alpha(f) = \frac{1}{\alpha!} \partial_\xi^\alpha (\hat{f}(\xi) e^{|\xi|^4})|_{\xi=0}\,, \tag{7.50}$$

which is well-defined if $|\alpha| \le l$ and $l + \frac{n}{2} < m$. By (7.50) we have

$$c_\beta(G_{*,\alpha}) = \delta_{\alpha\beta}\,, \qquad \alpha, \beta: \text{ multiindices.} \tag{7.51}$$

Hence, $\mathcal{P}_l^2 = \mathcal{P}_l$ and $\mathcal{P}_l$ is a projection. We also set $\mathcal{Q}_l = I - \mathcal{P}_l$. It is easy to see that $\mathcal{P}_l f = 0$ if and only if

$$\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0 \qquad \text{whenever} \quad |\alpha| \le l\,,$$

which implies

$$\int_{\mathbb{R}^n} x^\alpha \mathcal{Q}_l f(x) dx = 0 \qquad \text{if} \quad |\alpha| \le l\,. \tag{7.52}$$

Now we decompose $e^{t\mathbf{A}}$ in L^2_m as

$$e^{t\mathbf{A}} f = e^{t\mathbf{A}} \mathcal{P}_l f + e^{t\mathbf{A}} \mathcal{Q}_l f\,. \tag{7.53}$$

Take $l \in \mathbb{Z}$ so that $l + \frac{n}{2} < m \le l + \frac{n}{2} + 1$. Then we claim that, for all multi-index α and $\epsilon > 0$, there exists a positive constant C such that

$$\|\partial_x^\alpha e^{t\mathbf{A}} \mathcal{Q}_l f\|_{L^2_m} \le \frac{C e^{-\frac{t}{4}(m - \frac{n}{2} - \epsilon)}}{a(t)^{\frac{|\alpha|}{4}}} \|f\|_{L^2_m}\,, \qquad t > 0\,. \tag{7.54}$$

The proof of (7.54) proceeds exactly as same as [10, Proposition A.2]. Indeed, from the representation (7.35), the estimate (7.54) for $t \in (0, 1]$ is obtained from (7.37) and by the Young inequality. For $t > 1$ there is no contribution from the factor $a(t)$, so the problem is reduced to the estimate of

$$S(t)f(x) = \int_{\mathbb{R}^n} \phi(x-y) f(e^{\frac{t}{4}} y) dy\,,$$

for $\phi \in \mathcal{S}(\mathbb{R}^n)$. Recalling (7.52), we have from [10, Lemma A.4],

$$\|S(t)\mathcal{Q}_l f\|_{L^2_m} \le C_\epsilon e^{-\frac{t}{4}(m-\frac{n}{2}-\epsilon)} \|f\|_{L^2_m}\,, \qquad t>0\,,$$

when $l+\frac{n}{2} < m \le l+\frac{n}{2}+1$, which implies (7.54).

Step 4 (Proof of (7.43) and (7.44)): By Step 1 and Step 2 we have already proved that

$$\big\{\lambda \in \mathbb{C}\,;\ \mathrm{Re}(\lambda) \le \tfrac{1}{4}(\tfrac{n}{2}-m)\big\} \cup \{-\tfrac{k}{2}\,;\ k=0,1,2,\cdots\} \subset \sigma(\mathbf{A})\,. \qquad (7.55)$$

Take $l \in \mathbb{Z}$ so that $l+\frac{n}{2} < m \le l+\frac{n}{2}+1$. Since the closed subspace

$$L^2_{m,(l)} = \{f \in L^2_m\,;\ \mathcal{P}_l f = 0\} = \{f \in L^2_m\,;\ \textstyle\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0\,,\ \ |\alpha| \le l\}$$

is invariant uder the action $\{e^{t\mathbf{A}}\}_{t\ge 0}$, which can be checked by computing the evolution of each moment $\int_{\mathbb{R}^n} x^\alpha e^{t\mathbf{A}} f(x) dx$ with $|\alpha| \le l$ for $f \in D_{L^2_m}(\mathbf{A})$ based on the integration by parts, we have $\mathcal{P}_l e^{t\mathbf{A}} \mathcal{Q}_l = 0$ for each $t \ge 0$. Hence $\{e^{t\mathbf{A}}\mathcal{Q}_l\}_{t\ge 0}$ defines a C_0-semigroup in $L^2_{m,(l)}$. Then, since $e^{t\mathbf{A}}\mathcal{P}_l$ is a finite rank operator, we have from (7.53) and (7.54),

$$r_{ess}(e^{t\mathbf{A}}) = r_{ess}(e^{t\mathbf{A}}\mathcal{Q}_l) \le e^{-\frac{t}{4}(m-\frac{n}{2})}\,.$$

If $r_{ess}(e^{t\mathbf{A}}) \le e^{-\frac{t}{4}(m-\frac{n}{2}+\epsilon)}$ for some $\epsilon > 0$ then the set $\{\lambda \in \sigma(\mathbf{A})\,;\ \mathrm{Re}(\lambda) > -\frac{1}{4}(m-\frac{n}{2}+\epsilon)\}$ consists of isolated eigenvalues; see [4, Corollary IV-2.11]. This contradicts with (7.55), and thus, $r_{ess}(e^{t\mathbf{A}}) = e^{-\frac{t}{4}(m-\frac{n}{2})}$. Assume that $\lambda_0 \in \sigma(\mathbf{A})$ and $\mathrm{Re}(\lambda_0) > \frac{1}{4}(\frac{n}{2}-m)$. Then, since $r_{ess}(e^{t\mathbf{A}}) = e^{-\frac{t}{4}(m-\frac{n}{2})}$, again from [4, Corollary IV-2.11], the spectrum λ_0 must be an isolated eigenvalue with finite algebraic multiplicities. Let f_0 be an associated eigenfunction. Then we have

$$e^{\lambda_0 t} f_0 = e^{t\mathbf{A}} f_0 = e^{t\mathbf{A}}\mathcal{P}_l f_0 + e^{t\mathbf{A}}\mathcal{Q}_l f_0\,,$$

where $l \in \mathbb{Z}$ is taken as $l+\frac{n}{2} < m \le l+\frac{n}{2}+1$. From (7.54), $\mathrm{Re}(\lambda_0) > \frac{1}{4}(\frac{n}{2}-m)$, and $f_0 \neq 0$, the number l must be nonnegative and we see

$$f_0 = \lim_{t\to\infty} e^{-\lambda_0 t} e^{t\mathbf{A}} \mathcal{P}_l f_0 = \lim_{t\to\infty} \sum_{|\alpha|\le l} e^{-(\lambda_0+\frac{|\alpha|}{4})t} c_\alpha(f_0) G_{*,\alpha}\,.$$

This implies $\lambda_0 = -\frac{k}{4}$ for some integer $k \le l$, and $f_0 = \sum_{|\alpha|=k} c_\alpha(f_0) G_{*,\alpha}$. Thus we have proved that $\big\{\lambda \in \sigma(\mathbf{A})\,;\ \mathrm{Re}(\lambda) > \frac{1}{4}(\frac{n}{2} - m)\big\}$ is contained in $\{-\frac{k}{4}\,;\ k = 0, 1, 2, \cdots\}$, that is, (7.43) is proved. Let $m > \frac{n}{2}$ and let $k \in \mathbb{N}\cup\{0\}$ be such that $k + \frac{n}{2} < m$. Then the above proof shows that $\mathrm{Ker}\,(\mathbf{A} + \frac{k}{4})$ is spanned by $\{G_{*,\alpha}\,;\ |\alpha| = k\}$, and hence, the geometrical multiplicity of the eigenvalue $-\frac{k}{4}$ is $\binom{n+k-1}{k}$. To show the semisimple property it suffices to prove $\mathrm{Ker}\,\big((\mathbf{A} + \frac{k}{4})^2\big) = \mathrm{Ker}\,(\mathbf{A} + \frac{k}{4})$. Since $\mathcal{P}_l e^{t\mathbf{A}} \mathcal{Q}_l = 0$ for all $t > 0$, we have

$$\mathbf{A}\mathcal{P}_l f = \mathcal{P}_l \mathbf{A} f\,, \qquad \mathbf{A}\mathcal{Q}_l = \mathcal{Q}_l \mathbf{A} f\,, \tag{7.56}$$

for $f \in D_{L^2_m}(\mathbf{A})$. Assume that $f \in \mathrm{Ker}\,\big((\mathbf{A} + \frac{k}{4})^2\big)$. Then

$$(\mathbf{A} + \tfrac{k}{4}) f = \textstyle\sum_{|\alpha|=k} a_\alpha G_{*,\alpha}\,, \qquad a_\alpha \in \mathbb{C}\,,$$

and thus,

$$(\mathbf{A} + \tfrac{k}{4}) \mathcal{Q}_k f = -(\mathbf{A} + \tfrac{k}{4}) \mathcal{P}_k f + \textstyle\sum_{|\alpha|=k} a_\alpha G_{*,\alpha}\,.$$

From (7.56) and $\mathcal{Q}_k^2 = \mathcal{Q}_k$ we conclude that $(\mathbf{A} + \frac{k}{4})\mathcal{Q}_k f = 0$, that is, $\mathcal{Q}_k f = \sum_{|\alpha|=k} b_\alpha G_{*,\alpha} = \mathcal{P}_k \sum_{|\alpha|=k} b_\alpha G_{*,\alpha} = \mathcal{P}_k \mathcal{Q}_k f = 0$. Hence, it follows that $f = \mathcal{P}_k f = \sum_{|\alpha|\le k} c_\alpha(f) G_{*,\alpha}$, which implies $f \in \mathrm{Ker}\,\big((\mathbf{A} + \frac{k}{4})^2\big)$ if and only if $f \in \mathrm{Ker}\,\big(\mathbf{A} + \frac{k}{4}\big)$. The proof of Lemma 7.10 is complete. □

Acknowledgments

This work is supported in part by JSPS Grant-in-Aid for Scientific Research (A) No. 22244009. The first author is partially supported by JSPS Grant-in-Aid for Young Scientists (B), No. 23740116. The second author is partially supported by JSPS Grant-in-Aid for Young Scientists (B), No. 25800079.

References

1. J. M. Cahn and J. E. Hilliard, Free energy of a non-uniform system I. Interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258-367.

2. R. Duan, Global smooth flows for the compressible Euler-Maxwell system: I The relaxation case, *J. Hyperbolic Differential Equations*, **8** (2011), 375-413.

3. R. Duan, Dissipative property of the Vlasov-Maxwell-Boltzmann system with a uniform ionic background, *SIAM J. Math. Anal.*, **43** (2011), 2732-2757.

4. K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math, Springer-Verlag, New York, 2000.

5. J.-D. Evans, V.-A. Galaktionov and J.-F. Williams, Blow-up and global asymptotics of the limit unstable Cahn-Hilliard equation, *SIAM J. Math. Anal.*, **38** (2006), 64-102.

6. P. Galenko and D. Jou, Diffuse-interface model for rapid phase transformations in nonequilibrium systems, *Phys. Rev. E*, **71** (2005), 046125.

7. P. Galenko and V. Lebedev, Analysis of the dispersion relation in spinodal decomposition of a binary system, *Philos. Mag. Let.*, **87** (2007), 821-827.

8. P. Galenko and V. Lebedev, Local nonequilibrium effect on spinodal decomposition in a binary system, *Int. J. of Thermodynamics*, **11** (2008), 21-28.

9. P. Galenko and V. Lebedev, Non-equilibrium effects in spinodal decomposition of a binary system, *Physics Letters A*, **372** (2008), 985-989.

10. Th. Gallay and C. E. Wayne, Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbf{R}^2$, *Arch. Ration. Mech. Anal.*, **163** (2002), 209-258.

11. S. Gatti, M. Grasselli, A. Miranville and V. Pata, On the hyperbolic relaxation of the one dimensional Cahn-Hilliard equation, *J. Math. Anal. Appl.*, **312** (2005), 230-247.

12. S. Gatti, M. Grasselli, V. Pata and A. Miranville, Hyperbolic relaxation of the viscous Cahn-Hilliard equation in 3-D, *Math. Models Methods Appl. Sci.*, **15** (2005), 165-198.

13. M. Grasselli, H. Petzeltová and G. Schimperna, Asymptotic behavior of a nonisothermal viscous Cahn-Hilliard equation with inertial term, *J. Differential Equations*, **239** (2007), 38-60.

14. M. Grasselli, G. Schimperna, A. Segatti and S. Zelik, On the 3D Cahn-Hilliard equation with inertial term, *J. Evol. Equ.*, **9** (2009), 371-404.

15. M. Grasselli, G. Schimperna and S. Zelik, On the 2D Cahn-Hilliard equation with inertial term, *Comm. Partial Differential Equations*, **34** (2009), 137-170.

16. M. Grasselli, G. Schimperna and S Zelik, Trajectory Trajectory and smooth attractors for Cahn-Hilliard equations with inertial term, *Nonlinearity*, **23** (2010), 707-737.

17. T. Hosono and S. Kawashima, Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system, *Math. Models Meth. Appl. Sci.*, **16** (2006), 1839-1859.

18. K. Ide, K. Haramoto and S. Kawashima, Decay property of regularity-loss type for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008), 647-667.

19. K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008), 1001-1025.

20. T. Iguchi and S. Kawashima, On space-time decay properties of solutions to hyperbolic-elliptic coupled systems, *Hiroshima Math. J.*, **32** (2002), 229-308.

21. K. Ishige, T. Kawakami, and K. Kobayashi, Asymptotics for a nonlinear integral equation with a generalized heat kernel, *J. Evol. Equ.*, **14** (2014), 749-777.

22. M. Kato, Large time behavior of solutions to the generalized Burgers equations, *Osaka J. Math.*, **44** (2007), 923-943.

23. M. Kato, Y.-Z. Wang and S. Kawashima, Asymptotic behavior of solutions to the generalized cubic double dispersion equation in one space dimension, *Kinetic and Related Models*, **6** (2013), 969-987.

24. S. Kawashima, Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications, *Proc. Roy. Soc. Edinburgh*, **106A** (1987), 169-194.

25. S. Kawashima, Large-time behavior of solutions of the discrete Boltzmann equation, *Commun. Math. Phys.*, **109** (1987), 563-589.

26. S. Kawashima, Y. Nikkuni and S. Nishibata, The initial value problem for hyperbolic-elliptic coupled systems and application to radiation hydrodynamics, *Analysis of Systems of Conservation Laws* (H. Freistühler, ed.), Chapman & Hall/CRC, 1998, 87-127.

27. T. Kubo and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for some hyperbolic-elliptic system, *Kyushu J. Math.*, **63** (2009), 1-21.

28. Y. Kagei and Y. Maekawa, Asymptotic behaviors of solutions to evolution equations in the presence of translation and scaling invariance, *J. Funct. Anal.*, **260** (2011), 3036-3096.

29. Y. Liu and S. Kawashima, Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation, *Discrete Contin. Dyn. Syst.*, **29** (2011), 1113-1139.

30. Y. Liu and S. Kawashima, Global existence and decay of solutions for a quasi-linear dissipative plate equation, *J. Hyperbolic Differential Equations*, **8** (2011), 591-614.

31. T.-P. Liu and Y. Zeng, *Large time behavior of solutions to general quasilinear hyperbolic-parabolic systems of conservation laws*, Memoirs Amer. Math. Soc., 1997.

32. C. R. da Luz and R. C. Charão, Asymptotic properties for a semilinear plate equation in unbounded domains, *J. Hyperbolic Differential Equations*, **6** (2009), 269-294.

33. Y. Maekawa, Remark on C_0-semigroups with scaling invariance, *Nonlinear Dynamics in Partial Differential Equations, Advanced Studies in Pure Mathematics*, **64** (2015), 337-345.

34. A. Matsumura, On the asymptotic behaviour of solutions of semi-linear wave equations, *Publ. Res. Inst. Math. Sci.*, **12** (1976), 169-189.

35. N. Mori and S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, *J. Hyperbolic Differential Equations*, **11** (2014), 135-157.

36. N. Mori and S. Kawashima, Decay property of the Timoshenko-Cattaneo system, *Analysis and Applications* (accepted).

37. K. Nishihara, L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their applications, *Math. Z.*, **244** (2003), 631-649.

38. M. Nakao and K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, *Math. Z.*, **214** (1993), 325-342.

39. J.E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, *Discrete and Continuous Dynamical Systems*, **9** (2003), 1625-1639.

40. A. Savostianov and S. Zelik, Finite dimensionality of the attractor for the hyperbolic Cahn-Hilliard-Oono equation in $\mathbb{R}^3$, arXiv:1502.02156.

41. M. L. Santos, D. S. Almeida Júnior and J. E. Muñoz Rivera, The stability number of the Timoshenko system with second sound, *J. Differential Equations*, **253** (2012), 2715-2733.

42. Y. Sugitani and S. Kawashima, Decay estimates of solutions to a semi-linear dissipative plate equation, *J. Hyperbolic Differential Equation*, **7** (2010), 471-501.

43. H. Takeda and S. Yoshikawa, Asymptotic profile of solutions for the limit unstable Cahn-Hilliard equation with inertial term, *Differential Integral Equations*, **25** (2012), 341-362.

44. H. Takeda and S. Yoshikawa, On the initial value problem of the semilinear beam equation with weak damping I: Smoothing effect, *J. Math. Anal. Appl.*, **401** (2013), 244-258.

45. H. Takeda and S. Yoshikawa, On the initial value problem of the semilinear beam equation with weak damping II: Asymptotic profles, *J. Differential Equations*, **253** (2012), 3061-3080.

46. G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, *J. Differential Equations*, **174** (2000), 464-489.

47. Y. Ueda and S. Kawashima, Large time behavior of solutions to a semilinear hyperbolic system with relaxation, *J. Hyperbolic Differential Equations*, **4** (2007), 147-179.

48. Y. Ueda and S. Kawashima, Decay property of regularity-loss type for the Euler-Maxwell system, *Methods and Applications of Analysis*, **18** (2011), 245–268.

49. Y.-Z. Wang and Y.-X. Wang, *Global existence and asymptotic behavior of solutions to a nonlinear wave equation of fourth-order*, *J. Math. Phys.*, **53** (2012), 013512-01-013512-13.

50. S. Wang and H. Xu, On the asymptotic behavior of solution for the generalized IBq equation with hydrodynamical damped term, *J. Differential Equations*, **252** (2012), 4243-4258.