Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **10** (2015), No. 3, pp. 393-421

SELF-SIMILAR SOLUTIONS OF 2-D COMPRESSIBLE EULER EQUATIONS AND MIXED-TYPE PROBLEMS

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Dedicated to Professor Tai-Ping Liu for his 70th birthday

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Abstract

The Riemann problem has been proved to play the role of building blocks in various aspects of theory, numerics and applications of one-dimensional conservation laws. In contrast, the solution structures of two-dimensional Riemann problems are much poorly understood due to the instantaneous space-time interaction of nonlinear waves which leads to complex but fascinating wave structures. These structures have the universal self-similarity feature that reflects the invariant property under dilation. With the self-similarity reduction, the underlying problems change from the purely hyperbolic type to the hyperbolic-elliptic mixed type. In this paper we will formulate and review precisely some mathematical problems with plausible explicit structures in the construction of 2-D Riemann problems and propose some doable problems.

1. Introduction

Hyperbolic conservation laws take the form

$$\mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad t > 0, \tag{1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_m)^\top$ is a conservative vector and $\mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}), \ldots, \mathbf{f}_d(\mathbf{u}))$ is the flux function, t is the time variable. The gradient operator ∇ is taken with respect to the spatial variable $\mathbf{x} = (x_1, \ldots, x_d)$ that is x

Received March 30, 2015 and in revised form May 30, 2015.
AMS Subject Classification: 35L02, 35M10, 35M30, 76H05.
Key words and phrases: Hyperbolic conservation laws, compressible Euler equations, 2-D Riemann problems, self-similar solutions, mixed-type problems.

in one-dimension and $(x_1, x_2) = (x, y)$ in two-dimensions. We are particularly interested in the compressible Euler equations in gas dynamics: The conservative variable $\mathbf{u}$ and the flux function $\mathbf{f}(\mathbf{u})$ are

$$\mathbf{u} = (\rho, \mathbf{m}, E)^\top, \quad \mathbf{f}(\mathbf{u}) = (\mathbf{m}, \mathbf{m} \otimes \mathbf{m}/\rho + P\mathbf{I}, \mathbf{m}(E + P)/\rho)^\top, \tag{1.2}$$

where ρ, $\mathbf{m}$ and P are the density, momentum and pressure, respectively; E is the total energy which is defined as the sum of kinematic energy $|\mathbf{m}|^2/(2\rho)$ and the internal energy, $E = |\mathbf{m}|^2/(2\rho) + \rho e$, e is the specific internal energy, $\mathbf{I}$ is the standard identity matrix. The velocity function $\mathbf{v}$ is defined as $\mathbf{v} = \mathbf{m}/\rho$. For polytropic gases, we use the equation of state,

$$P = (\gamma - 1)\rho e, \quad \gamma > 1. \tag{1.3}$$

The solution we are considering for (1.1) is self-similar in the sense that the solution is invariant under dilation $(t, \mathbf{x}) \to (\alpha t, \alpha \mathbf{x})$ for any $\alpha > 0$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\zeta}, 1), \quad \boldsymbol{\zeta} = \mathbf{x}/t, \tag{1.4}$$

provided that local structures have such a property (Galilean invariance) or the initial data just take the form,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\boldsymbol{\theta}), \tag{1.5}$$

where $\boldsymbol{\theta}$ is the polar angle in the $\mathbf{x}$-space. Then (1.1) becomes

$$-\boldsymbol{\zeta} \cdot \nabla_{\boldsymbol{\zeta}} \mathbf{u} + \nabla_{\boldsymbol{\zeta}} \cdot \mathbf{f}(\mathbf{u}) = 0, \tag{1.6}$$

or

$$(-\boldsymbol{\zeta} \mathbf{I} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}) \cdot \nabla_{\boldsymbol{\zeta}} \mathbf{u} = 0, \tag{1.7}$$

where $\nabla_{\boldsymbol{\zeta}}$ is the gradient operator with respect to $\boldsymbol{\zeta}$. We call a flow governed by (1.7) *the pseudo-steady flow* when the compressible fluid flow is considered and the variable $\boldsymbol{\zeta}$ *a self-similarity variable.*

The system (1.7) is unusual because topological structures of solutions are fundamentally changed: (1.1) may change from the purely hyperbolic type to the hyperbolic-elliptic mixed type under the self-similarity reduction (1.4), in addition that the coefficient of (1.7) depends on the self-similarity variable $\boldsymbol{\zeta}$. This fundamental change naturally arises issues how to formulate and propose doable problems of mixed-type and establish inherently related mathematical theory.

In this paper we will discuss general properties of the pseudo-steady flows and formulate some doable problems. In Section 2, we show that the self-similar reduction of (1.1) leads to the study of mixed-type problems inevitably following Canic and Keyfitz. In Section 3, we review three basic problems of mixed-type that may help the study of self-similar solutions to (1.7). In Section 4, we formulate some doable problems based on our understanding of current progresses in this field.

2. General Theory on Pseudo-steady Flows

In this section, we will discuss the general property of pseudo-steady flows and answer a basic question that why a purely hyperbolic problem becomes a mixed-type one in terms of self-similarity variable $\boldsymbol{\zeta}$.

2.1. Hyperbolicity in the $(t, \mathbf{x})$-space

We write (1.1) as

$$[\mathbf{I}\partial_t + \mathbf{A}(\mathbf{u}) \cdot \nabla]\, \mathbf{u} = 0, \quad \mathbf{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \tag{2.1}$$

where $\mathbf{I}$ is the identity matrix. For any fixed value $\mathbf{u}_0$, the linearized symbol matrix of (2.1) is

$$\mathbf{Q}(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0) := -\lambda \mathbf{I} + \mathbf{A}(\mathbf{u}_0) \cdot \boldsymbol{\alpha}, \quad |\boldsymbol{\alpha}| = 1, \tag{2.2}$$

and the linearized symbol is

$$q(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0) := \det(-\lambda \mathbf{I} + \mathbf{A}(\mathbf{u}_0) \cdot \boldsymbol{\alpha}), \tag{2.3}$$

where $(\lambda, \boldsymbol{\alpha})$ is a co-vector lying in the space dual to $(t, \mathbf{x})$. We say (1.1) *is hyperbolic if for any direction* $\boldsymbol{\alpha} \in \mathbb{R}^d$ *and value* $\mathbf{u}_0$, *the symbol matrix* $\mathbf{Q}(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0)$ *has all real eigenvalues* λ *and a complete set of eigenvectors [15].* We assume that the eigenvalues can be found through the fundamental theorem in algebra for $q(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0)$,

$$q(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0) = \prod_{\ell=1}^{K}(\lambda - \lambda_\ell(\boldsymbol{\alpha}, \mathbf{u}_0)) \prod_{\ell=1}^{J} q_\ell(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0), \qquad K + 2J = m, \tag{2.4}$$

where $\lambda_\ell(\boldsymbol{\alpha}, \mathbf{u}_0)$, $\ell = 1, \ldots, K$, have the linear relation with $\boldsymbol{\alpha}$,

$$\lambda_\ell(\boldsymbol{\alpha}, \mathbf{u}_0) = \boldsymbol{\alpha} \cdot \boldsymbol{\kappa}_\ell(\mathbf{u}), \tag{2.5}$$

and q_ℓ, $\ell = 1, \ldots, J$, are the quadratic form of $(\lambda, \boldsymbol{\alpha})$,

$$q_\ell(\lambda, \boldsymbol{\alpha}, \mathbf{u}) = (-\lambda, \boldsymbol{\alpha})\boldsymbol{\theta}_\ell(\mathbf{u})(-\lambda, \boldsymbol{\alpha})^\top, \tag{2.6}$$

where $\boldsymbol{\kappa}_\ell(\mathbf{u})$ is a vector and $\boldsymbol{\theta}_\ell(\mathbf{u})$ is a $(d + 1) \times (d + 1)$ matrix. This assumption is reasonable because there are many physical examples such as the compressible Euler equations (1.2) and MHD sharing such a property.

Example. For the Euler equations (1.2), $m = d + 2$, $K = d$, and $J = 1$. The factorization (2.4) is

$$\begin{aligned} \lambda_\ell(\boldsymbol{\alpha}, \mathbf{u}_0) &= \mathbf{v} \cdot \boldsymbol{\alpha}, \qquad \ell = 1, \ldots, K, \\ q_\ell(\lambda, \boldsymbol{\alpha}, \mathbf{u}_0) &= (\lambda - \mathbf{v} \cdot \boldsymbol{\alpha})^2 - (c\boldsymbol{\alpha}) \cdot (c\boldsymbol{\alpha}), \quad \ell = 1, \end{aligned} \tag{2.7}$$

where c is the sound speed, given with $c^2 = \gamma P/\rho$. Since the wave equation can be regarded as the linearized version of the Euler equations, it shares the same property at this point too.

We define ℓ-nondegenerate normal cone for any value $\mathbf{u}_0$ (see [15]),

$$\mathcal{E}_N(\mathbf{u}_0) = \{(\lambda, \boldsymbol{\alpha}); (-\lambda, \boldsymbol{\alpha})\boldsymbol{\theta}_\ell(\mathbf{u}_0)(-\lambda, \boldsymbol{\alpha})^\top = 0\}. \tag{2.8}$$

Its dual is ℓ-nondegenerate wave cone

$$\mathcal{E}_D(\mathbf{u}_0) = \{(t, \mathbf{x}); (t, \mathbf{x})\boldsymbol{\theta}_\ell^{-1}(\mathbf{u}_0)(t, \mathbf{x})^\top = 0\}, \tag{2.9}$$

where $\boldsymbol{\theta}_\ell^{-1}(\mathbf{u}_0)$ is the inverse of $\boldsymbol{\theta}_\ell(\mathbf{u}_0)$. For any $(t, \mathbf{x}) \in \mathcal{E}_D(\mathbf{u}_0)$, there exists a $(\lambda, \boldsymbol{\alpha}) \in \mathcal{E}_N(\mathbf{u}_0)$ such that

$$(-\lambda, \boldsymbol{\alpha}) \cdot (t, \mathbf{x}) = 0. \tag{2.10}$$

Such a plane is a characteristic plane of (1.1) with normal $(-\lambda, \boldsymbol{\alpha})$. Indeed, $\mathcal{E}_D(\mathbf{u}_0)$ is the envelope of planes of the form (2.10) and therefore it is convex.

Next we discuss the geometry of discontinuities. Let $\Gamma : \sigma(t, \mathbf{x}) = 0$ be a discontinuity separating two states $\mathbf{u}_0$ and $\mathbf{u}$. The Rankine-Hugoniot relation holds stating the continuity of $(\mathbf{u}, \mathbf{f}(\mathbf{u}))$ in the normal direction

$$([\mathbf{u}], [\mathbf{f}(\mathbf{u})]) \cdot (\sigma_t, \sigma_{\mathbf{x}}) = 0, \tag{2.11}$$

where $(\sigma_t, \sigma_{\mathbf{x}})$ is the normal vector of Γ, $[\mathbf{u}] = \mathbf{u} - \mathbf{u}_0$ and similarly for $[\mathbf{f}(\mathbf{u})]$. Certainly, some additional physical criteria should be introduced to select admissible solutions and they includes entropy criteria, small viscosity limit and some others.

We write (2.11) as

$$(\sigma_t I + \overline{\mathbf{A}}(\mathbf{u}, \mathbf{u}_0) \cdot \sigma_{\mathbf{x}})[\mathbf{u}] = 0, \tag{2.12}$$

where $\overline{\mathbf{A}}(\mathbf{u}, \mathbf{u}_0)$ is the Roe matrix of $\mathbf{f}(\mathbf{u})$,

$$\overline{\mathbf{A}}(\mathbf{u}, \mathbf{u}_0) = \int_0^1 \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_0 + \pi(\mathbf{u} - \mathbf{u}_0))d\pi. \tag{2.13}$$

Then we denote the matrix

$$\overline{\mathbf{Q}}(\sigma_t, \sigma_{\mathbf{x}}, \mathbf{u}_0, \mathbf{u}) := \sigma_t I + \overline{\mathbf{A}}(\mathbf{u}, \mathbf{u}_0) \cdot \sigma_{\mathbf{x}}, \tag{2.14}$$

and the Rankine-Hugoniot (abbr. R-H) symbol

$$\bar{q}(\sigma_t, \sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0) := \det(\sigma_t I + \overline{\mathbf{A}}(\mathbf{u}, \mathbf{u}_0) \cdot \sigma_{\mathbf{x}}). \tag{2.15}$$

With the assumption (2.4), we have the decomposition, similar to (2.4),

$$\bar{q}(\sigma_t, \sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0) = \prod_{\ell=1}^{K}(\sigma_t + \bar{\lambda}_\ell(\sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0)) \prod_{\ell=1}^{J} \bar{q}_\ell(\sigma_t, \sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0). \tag{2.16}$$

It turns out that such a discontinuity should satisfy the condition of the form,

$$\sigma_t + \bar{\lambda}_\ell(\sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0) \;=\; \sigma_t + \bar{\boldsymbol{\kappa}}_\ell(\mathbf{u}, \mathbf{u}_0) = 0, \tag{2.17}$$

or

$$\bar{q}_\ell(\sigma_t, \sigma_{\mathbf{x}}, \mathbf{u}, \mathbf{u}_0) \;=\; (\sigma_t, \sigma_{\mathbf{x}})\; \bar{\boldsymbol{\theta}}_\ell(\mathbf{u}, \mathbf{u}_0)\; (\sigma_t, \sigma_{\mathbf{x}})^\top = 0. \tag{2.18}$$

The former corresponds to contact discontinuities and the latter corresponds to shocks. Therefore, we can define a *Rankine-Hugoniot cone*,

$$(t, \mathbf{x})\; \bar{\boldsymbol{\theta}}_\ell^{-1}(\mathbf{u}, \mathbf{u}_0)\; (t, \mathbf{x})^\top = 0, \tag{2.19}$$

which is the envelope of all plane of the form $(\sigma_t, \sigma_{\mathbf{x}}) \cdot (t, \mathbf{x}) = 0$. As $\mathbf{u}$ tends to $\mathbf{u}_0$, this cone becomes the ℓ-nondegenerate wave cone (2.9).

2.2. Pseudo-steady flows

As far as the self-similar solutions of (1.1) is considered, (1.1) is reduced as (1.7). Then the linearized symbol matrix for (1.7) with a fixed value $\mathbf{u}_0$ is

$$\mathbf{Q}_{\boldsymbol{\zeta}}(\boldsymbol{\beta}, \mathbf{u}_0) := \left(-\boldsymbol{\zeta} I + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right) \cdot \boldsymbol{\beta} = -\boldsymbol{\zeta} \cdot \boldsymbol{\beta} \mathbf{I} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \cdot \boldsymbol{\beta}, \tag{2.20}$$

and the corresponding symbol writes

$$q_{\boldsymbol{\zeta}}(\boldsymbol{\beta}, \mathbf{u}_0) = \det\left(-\boldsymbol{\zeta} \cdot \boldsymbol{\beta} I + \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \cdot \beta\right), \tag{2.21}$$

where $\boldsymbol{\beta} \in \mathbb{R}^n$ is a unit vector. We say that (1.7) is hyperbolic at $\boldsymbol{\zeta}$ if $q_{\boldsymbol{\zeta}}(\boldsymbol{\beta}, \mathbf{u}_0)$ has real roots $\boldsymbol{\beta}$. Note that every component of $\boldsymbol{\beta}$ plays the same role and no preference is given to a fixed direction. It is evident that the hyperbolicity depends on the self-similarity variable $\boldsymbol{\zeta}$, besides the fixed value $\mathbf{u}_0$. In [5], Canic and Keyfitz showed that for the case of two spatial variables, the self-similar perturbation about a fixed state $\mathbf{u}_0$ must be of mixed-type. This is true in general; and the proposition is stated as follows.

Proposition 2.1 (Theorem 2.1, Page 134, [5]). *For any fixed* $\mathbf{u}_0 \in \mathbb{R}^m$, (1.7) *is hyperbolic if and only if* $(1, \boldsymbol{\zeta})$ *is outside of the wave cone* $\mathcal{E}_D(\mathbf{u}_0)$.

Proof. The proof is not difficult and we would like to include it for completeness. For this purpose, we write

$$q_{\boldsymbol{\zeta}}(\boldsymbol{\beta},\mathbf{u}_0) \;=\; \prod_{\ell=1}^{K}(-\boldsymbol{\zeta}\cdot\boldsymbol{\beta}+\lambda_\ell(\boldsymbol{\beta},\mathbf{u}_0))\prod_{\ell=1}^{J}q_\ell(-\boldsymbol{\zeta},\boldsymbol{\beta},\mathbf{u}_0), \tag{2.22}$$

where

$$-\boldsymbol{\zeta}\cdot\boldsymbol{\beta}-\lambda_\ell(\boldsymbol{\beta},\mathbf{u}_0)=(-\boldsymbol{\zeta}+\mathbf{k}_\ell(\mathbf{u}_0))\cdot\boldsymbol{\beta}, \tag{2.23}$$

and

$$q_\ell(\boldsymbol{\zeta},\boldsymbol{\beta},\mathbf{u}_0)=(-\boldsymbol{\zeta}\cdot\boldsymbol{\beta},\boldsymbol{\beta})\boldsymbol{\theta}_\ell(\mathbf{u}_0)(-\boldsymbol{\zeta}\cdot\boldsymbol{\beta},\boldsymbol{\beta})^\top. \tag{2.24}$$

We assume that (1.7) is hyperbolic at $\boldsymbol{\zeta}_0$. Then for some real vector $\boldsymbol{\beta}\in\mathbb{R}^n$, $q_{\boldsymbol{\zeta}_0}(\boldsymbol{\beta},\mathbf{u}_0)=0$. This means that $\boldsymbol{\sigma}_0=(-\boldsymbol{\zeta}_0\cdot\boldsymbol{\beta},\boldsymbol{\beta})\in\mathcal{E}_N(\mathbf{u}_0)$ and the plane $\sigma_0\cdot(t,\mathbf{x})=0$ is tangent to $\mathcal{E}_D(\mathbf{u}_0)$. Due to convexity of $\mathcal{E}_D(\mathbf{u}_0)$ and the fact that $\boldsymbol{\sigma}_0\cdot(1,\boldsymbol{\zeta}_0)=0$, we conclude that $(1,\boldsymbol{\zeta}_0)$ is outside of $\mathcal{E}_D(\mathbf{u}_0)$.

Conversely, if $(1,\boldsymbol{\zeta}_0)$ is outside of $\mathcal{E}_D(\mathbf{u}_0)$, then we can make a tangent plane to $\mathcal{E}_D(\mathbf{u}_0)$ through $(1,\boldsymbol{\zeta}_0)$. Since $\mathcal{E}_D(\mathbf{u}_0)$ is convex, such a plane is outside $\mathcal{E}_D(\mathbf{u}_0)$, and there holds for some $\boldsymbol{\sigma}\in\mathcal{E}_N(\mathbf{u}_0)$,

$$\boldsymbol{\sigma}\cdot(1,\boldsymbol{\zeta}_0)=0. \tag{2.25}$$

Denote $\boldsymbol{\sigma}=(-\lambda,\boldsymbol{\alpha})$. Then we have $\lambda=\boldsymbol{\alpha}\cdot\boldsymbol{\zeta}_0$ and $(-\boldsymbol{\alpha}\cdot\boldsymbol{\zeta}_0,\boldsymbol{\zeta}_0)\in\mathcal{E}_N(\mathbf{u}_0)$. This implies that

$$q_{\boldsymbol{\zeta}_0}(\boldsymbol{\alpha},\mathbf{u}_0)=0, \tag{2.26}$$

has a non-degenerate real solution $\boldsymbol{\alpha}$; and therefore (1.7) is hyperbolic at $\boldsymbol{\zeta}_0$. □

This proposition shows that the self-similarity reduction of (1.1) into (1.7) leads inevitably to mixed-type (elliptic-hyperbolic coupled) problems. We specialize to the two-dimensional case in the next subsection to see more technical details.

We emphasize that, in terms of the self-similar variable $\boldsymbol{\zeta}$, the wave cone (2.9) becomes

$$(1,\boldsymbol{\zeta})\;\boldsymbol{\theta}_\ell^{-1}(\mathbf{u}_0)\;(1,\boldsymbol{\zeta})^\top=0; \tag{2.27}$$

and the Rankine-Hugoniot cone (2.19) becomes,

$$(1, \boldsymbol{\zeta})\ \bar{\boldsymbol{\theta}}_\ell^{-1}(\mathbf{u}, \mathbf{u}_0)\ (1, \boldsymbol{\zeta})^\top = 0. \tag{2.28}$$

They are the projection of (2.9) and (2.19) onto the plane $\{t = 1\}$, respectively.

2.3. Two-dimensional Riemann problems and pseudo-steady flows

We specialize to the two-dimensional case for which the number of spatial variables are two. We write (1.1) in a special form

$$\mathbf{u}_t + \mathbf{g}(\mathbf{u})_x + \mathbf{h}(\mathbf{u})_y = 0, \tag{2.29}$$

and the Riemann initial data for (1.1) is sectorial constant and radially invariant. That is, the data just depend on the polar angle,

$$\mathbf{u}(x, y, 0) = \mathbf{u}_0(\theta), \quad \tan\theta = y/x, \tag{2.30}$$

and $\mathbf{u}_0(\theta)$ is piecewise constant,

$$\mathbf{u}_0(\theta) = \mathbf{u}_i, \quad \theta_i < \theta < \theta_{i+1}, \quad i = 1, \dots, J, \quad \theta_{J+1} = \theta_1 + 2\pi. \tag{2.31}$$

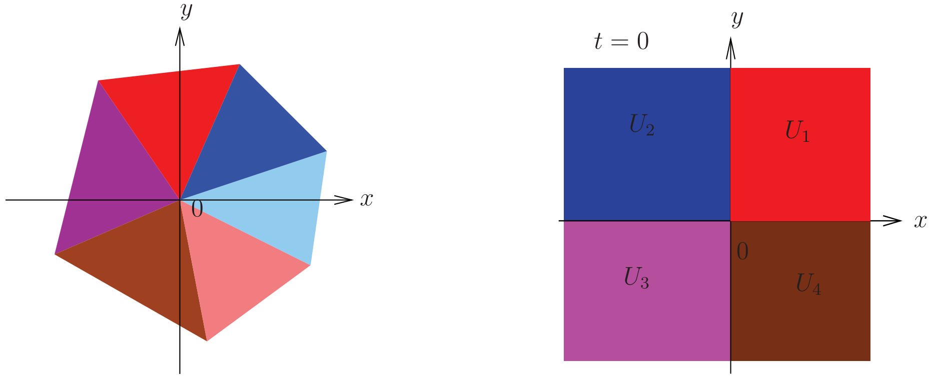
We display the distribution of the data (2.31) in Figure 2.1 (a). In [41], the data is restricted to be constant in each quadrant and assumed that only one planar elementary wave emanates from each initial discontinuity (half of each coordinate) for the reason of simplicity and symmetry, see Figure 2.1 (b). The number of pieces of the initial data is not essential. Instead, it is important to disclose some fundamental phenomena through a simple and reasonable setting.

Denote $\boldsymbol{\zeta} = (\xi, \eta) = (x/t, y/t)$. Then (2.29) becomes

$$[-\xi I + A(\mathbf{u})]\mathbf{u}_\xi + [-\eta I + B(\mathbf{u})]\mathbf{u}_\eta = 0, \tag{2.32}$$

where $A(\mathbf{u}) = \dfrac{\partial \mathbf{g}}{\partial \mathbf{u}}$ and $B(\mathbf{u}) = \dfrac{\partial \mathbf{h}}{\partial \mathbf{u}}$. The initial data (2.31) are now transformed into boundary values imposed in the far field,

$$\lim_{\xi^2+\eta^2\to\infty} \mathbf{u}(\xi, \eta) = \mathbf{u}_0(\theta), \quad \theta = \eta/\xi, \tag{2.33}$$



(a) 2-D Riemann initial data: $\mathbf{u}(x, y, 0) = \mathbf{u}_0(\theta)$, $\theta = \arctan(y/x)$

(b) Four-wave Riemann data by Zhang & Zheng (1990): SIAM J. Math. Anal.

Figure 2.1: The display of the Riemann initial data (2.31)

where the limit is taken by keeping $\eta/\xi = \tan^{-1}(\theta)$. Thus the purely initial value problem (1.1) and (2.31) is formulated as the boundary value problem. Please note that this boundary value problem is unusual. Parts of reasons are the following:

(1) The boundary data is imposed at infinity, but not on any boundary of bounded domains.

(2) The coefficient matrices $-\xi I + A(\mathbf{u})$ and $-\eta I + B(\mathbf{u})$ may be singular even though the Jacobian $\dfrac{\partial \mathbf{g}}{\partial \mathbf{u}}$ and $\dfrac{\partial \mathbf{h}}{\partial \mathbf{u}}$ are not on their own. This is because we assume that the system (2.29) is hyperbolic in every direction (of course in the x or y-direction). Also they depend on independent variables $\boldsymbol{\zeta}$ so that (2.32) is not autonomous.

(3) Characteristics are the projection of corresponding characteristic surfaces of (1.1). This projection is made from three-dimensional sets onto two-dimensional sets and therefore the topology changes fundamentally. For example, the system of 2-D Euler equations possibly change the hyperbolic type from the elliptic type after the dimension reduction, which will be specified later.

Nevertheless, this dimension reduction still supplies great benefits so that we are able to apply current existing (elliptic and hyperbolic) theories to resolve some physics-based problems.

The eigenvalues for (2.32) are computed through the equation for $\boldsymbol{\beta} = (\alpha, \beta)$,

$$\det(\alpha(-\xi I + A(\mathbf{u})) + \beta(-\eta I + B(\mathbf{u}))) = 0, \tag{2.34}$$

and characteristic curves are defined using the equation

$$\alpha d\xi + \beta d\eta = 0, \tag{2.35}$$

Due to the distinct feature of (2.34), the characteristic curves defined by (2.35) have unique properties different from those in the $(t, \mathbf{x})$-space.

(i) Singularity of characteristic curves. With the assumption of hyperbolicity of (2.29) in any direction, the matrix $-\xi\mathbf{u} + A(\mathbf{u})$ and $-\eta\mathbf{u} + B(\mathbf{u})$ are singular. This means that each characteristic curve has a singularity point that can be regarded as the end point, which corresponds to the bi-characteristic curve. There are two subcases:

The first one is defined through the equation

$$-\boldsymbol{\zeta} \cdot \boldsymbol{\beta} + \lambda_\ell(\boldsymbol{\beta}, \mathbf{u}_0) = (-\boldsymbol{\zeta} + \boldsymbol{\kappa}_\ell(\mathbf{u}_0)) \cdot \boldsymbol{\beta} = 0, \quad \boldsymbol{\beta} = (-d\eta, d\xi). \tag{2.36}$$

Each of this family of characteristics has a singularity point $\boldsymbol{\zeta}_0 = (\xi_0, \eta_0)$ if it carries the value $\mathbf{u}_0$. We can orient this characteristic line from infinite to this singularity point. This family of characteristics are usually degenerate and correspond to particle trajectories (flow characteristics).

The second one is defined as

$$(-\boldsymbol{\zeta} \cdot \boldsymbol{\beta}, \boldsymbol{\beta})\ \boldsymbol{\theta}_\ell(\mathbf{u}_0)\ (-\boldsymbol{\zeta} \cdot \boldsymbol{\beta}, \boldsymbol{\beta})^\top = 0, \quad \boldsymbol{\beta} = (-d\eta, d\xi). \tag{2.37}$$

Each of them are tangential to the sonic ellipse

$$C_\ell : (1, \boldsymbol{\zeta})\boldsymbol{\theta}_\ell^{-1}(\mathbf{u}_0)(1, \boldsymbol{\zeta})^\top = 0, \tag{2.38}$$

which is just the wave cone in the $(t, \mathbf{x})$-space following (2.9) or the projection of wave cone (2.9) onto the $\boldsymbol{\zeta}$-plane. All characteristic lines defined by (2.35) with such $\boldsymbol{\beta}$ are straight and tangent to C_ℓ. We illustrate these two sub-cases in Figure 2.2.

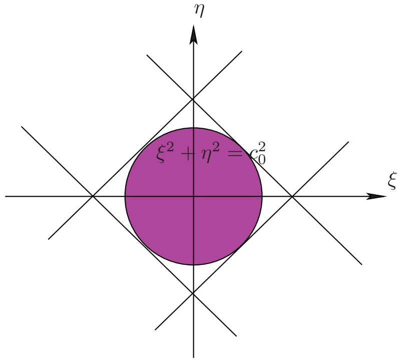


Figure 2.2: Characteristics for the pseudo-steady wave equation or linearized Euler equations.

(ii) Mixed-type of systems. Once there are wave-characteristics defined by (2.37), the system (1.7) must be of mixed-type due to Proposition 2.1. Then we have to solve mixed-type problems subject to boundary dada imposed at infinity in general. In the far field, the system (2.32) is purely hyperbolic, thanks again to Proposition 2.1.

The typical example is the linearized Euler equations in two dimensions,

$$\begin{aligned} &\rho_t + \rho_0 \nabla \cdot \mathbf{v} = 0, \\ &\mathbf{v}_t + \tfrac{c_0^2}{\rho_0} \nabla \rho = 0, \end{aligned} \tag{2.39}$$

where $\mathbf{v} = (u, v)$, $c_0^2 = P'(\rho_0)$. This system is obtained by linearizing the Euler equations around the background state $\mathbf{u}_0 = (\rho_0, \mathbf{0})$. In terms of the self-similarity variable $\boldsymbol{\zeta} = (\xi, \eta)$, (2.39) is written as

$$\begin{aligned} &-\xi\rho_\xi - \eta\rho_\eta + \rho_0(u_\xi + v_\eta) = 0, \\ &-\xi u_\xi - \eta u_\eta + \tfrac{c_0^2}{\rho_0}\rho_\xi = 0, \\ &-\xi v_\xi - \eta v_\eta + \tfrac{c_0^2}{\rho_0}\rho_\eta = 0. \end{aligned} \tag{2.40}$$

The symbol is

$$q_{\boldsymbol{\zeta}}(\boldsymbol{\beta}, \mathbf{u}_0) = (-\boldsymbol{\zeta} \cdot \boldsymbol{\beta})((\boldsymbol{\beta} \cdot \boldsymbol{\zeta})^2 - c_0^2|\boldsymbol{\beta}|^2), \tag{2.41}$$

which defines characteristics

$$\begin{aligned} &\xi d\eta - \eta d\xi = 0, \\ &(\xi^2 - c_0^2)(d\eta)^2 - 2\xi\eta d\xi d\eta + (\eta^2 - c_0^2)(d\xi)^2 = 0. \end{aligned} \tag{2.42}$$

The sonic curve is $C : \{(\xi,\eta); \xi^2 + \eta^2 = c_0^2\}$. (2.40) is hyperbolic for (ξ,η) is outside C and of mixed-type inside C. This distribution of characteristics is displayed in Figure (2.2).

3. Elliptic-Hyperbolic Mixed-type Problems

Proposition 2.1 tells that we have to deal with elliptic-hyperbolic mixed-type problems when pseudo-steady flows (1.7) are considered. As we all know, the study of elliptic-hyperbolic problems has a long history and the pioneering work may be attributed to Tricomi [35]. There are two fundamental families of mixed-type problems: Tricomi-type and Keldysh-type problems. In between, the Lavrentiev-Bitsatze equation may be useful in the treatment of transonic shocks. We would like to review them below. Useful books highly recommended are [14] and [22], which summarize many interesting pictures reflecting mixed-type problems.

3.1. The Tricomi equation

The Tricomi equation was named after Tricomi [35] and reads

$$yu_{xx} + u_{yy} = 0. \tag{3.1}$$

It can be derived from the isentropic irrotational Euler equations using the holograph transformation [22]. Obviously, this equation is elliptic in the upper-plane $y > 0$; hyperbolic in the lower plane $y < 0$ and parabolically degenerate (sonic) on the line $y = 0$. See Figure 3.1.

In the hyperbolic region $y < 0$, the characteristics are defined as

$$y(dy)^2 + (dx)^2 = 0. \tag{3.2}$$

Integrating it yields

$$C_{\pm} : x \pm \frac{2}{3}(-y)^{3/2} = C, \tag{3.3}$$

where C is an arbitrary constant.

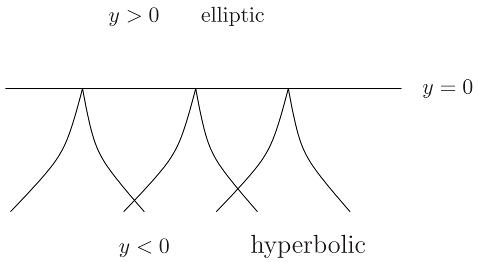


Figure 3.1: Characteristics in the hyperbolic region for the Tricomi equation.

The two families of characteristics coincide and are vertical to $y = 0$. The formulation of boundary value problem for (3.1) can be found in [24]. A common approach is to transform a boundary value problem into the problem finding a solution of a singular integral equation. The study about its fundamental solution can be found in [1].

A typical example of self-similar solutions whose characteristics are tangential to sonic curves in the type of Tricomi is the propagation of planar rarefaction waves. In Figure 4.8 below, we see the Tricomi structure in the propagation of planar rarefaction waves.

3.2. The Keldysh equation

There is another type of mixed-type equation named after Keldysh, which was proposed in [20] to study a class of degenerate elliptic equations on the boundary of a domain and it reads

$$u_{xx} + yu_{yy} = 0. \tag{3.4}$$

This equation is still elliptic in the upper plane $y > 0$ and hyperbolic in the lower plane $y < 0$, with the transition line $y = 0$. The characteristic equations are

$$(dy)^2 + y(dx)^2 = 0, \tag{3.5}$$

which gives

$$C_\pm : 2\sqrt{-y} \pm x = C. \tag{3.6}$$

Substantially different from those in (3.3) for the Tricomi equation (3.1), the characteristic curves in (3.6) are all tangent to $y = 0$. Therefore the solution behavior is substantially different too. It is pointed out in [11] by investigating the fundamental solution that the behavior of solution near $y = 0$ is more singular than that for the Tricomi equation.

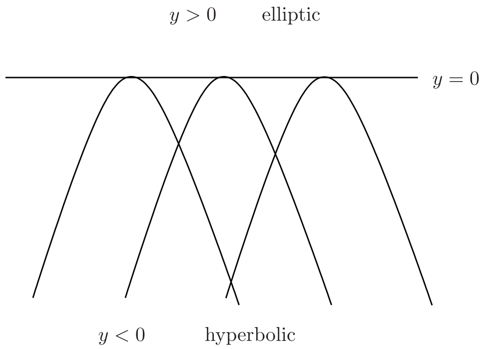


Figure 3.2: Characteristics in the hyperbolic region for the Keldysh equation.

In Figure 2.2, the characteristic lines are tangential to the sonic circle in the type of Keldysh.

3.3. The Lavrentiev-Bitsatze equation

The Lavrentiev-Bitsatze equation was proposed in 1950s [3, 4, 23] and takes the form

$$u_{yy} + \operatorname{sgn}(y) u_{xx} = 0. \tag{3.7}$$

In the upper plane it is the Laplace equation and in the lower plane it is the wave equation. The characteristic lines are

$$x \pm y = C. \tag{3.8}$$

We can mimic the approach for the Tricomi equation (3.1) to study the boundary value problem of (3.7). In order to make such a kind of

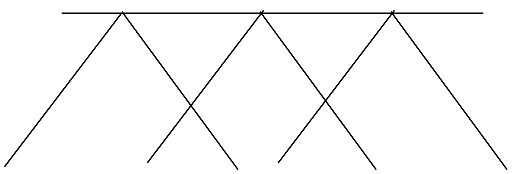


Figure 3.3: Characteristics in the hyperbolic region for the Lavrentiev-Bitsatze equation.

study possibly more applicable for nonlinear problems, Chen proposed the following nonlinear equation in [12] and studied similar problems,

$$u_{yy} + \operatorname{sgn}(u) u_{xx} = 0. \tag{3.9}$$

This type of problems may be often present in the study of a transonic shock problem in which a transonic shock separates a supersonic state from a subsonic state.

The above three examples belong to linear theory. Just as pointed out in [14, Page 6]: *For technical applications it would be useful if linear theory gave a good approximation...... Unfortunately, linearized theory cannot give the correct answer in the transonic range.* Hence we have to develop nonlinear theory for nonlinear mixed-type problems (transonic flows).

4. Pseudo-steady Euler Equations

Undoubtedly, the most important model in the context of conservation laws is the Euler system (1.2). For the two-dimensional case, the velocity $\mathbf{v} = (u, v)$. Denote the pseudo-velocity $(U, V) = (u-\xi, v-\eta)$, the velocity relative to the spatial location at a fixed time. Then we compute the eigenvalues,

$$\lambda_{01} = \lambda_{02} = \frac{V}{U}, \quad \lambda_{\pm} = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}, \quad \lambda = \frac{d\eta}{d\xi}. \tag{4.1}$$

Hence this pseudo-steady Euler equations are:

(1) **purely hyperbolic or supersonic** if $U^2 + V^2 - c^2 > 0$;

(2) **of mixed-type or subsonic** if $U^2 + V^2 - c^2 < 0$.

The curve $\{(\xi,\eta); U^2 + V^2 - c^2 = 0\}$ is a sonic curve. Generally speaking, the pseudo-steady Euler equations are of mixed-type but they are purely hyperbolic in the far field since the pseudo-velocity is very large there. Indeed, only planar elementary waves are present in the far field. One of main missions of 2-D Riemann problems is to investigate how the planar waves coming from far field interact to produce new flow patterns and establish corresponding mathematical theories.

Due to extremely difficulties of the pseudo-steady Euler equations, it is plausible to proceed our study from simplified models. Two examples we choose are the pressure-gradient equations and the potential equation.

4.1. Pressure-gradient flows

As a ladder step, we may think of a flow by ignoring the influence of inertial effect, which is achieved through some asymptotic approximation as follows; or through the first moment closure of the Boltzmann equation by ignoring the transport effect. Assume that the flow is almost stationary

$$(u,v) \sim (0,0). \tag{4.2}$$

Then we formally have

$$\rho\mathbf{v} \ll \rho; \quad \rho\mathbf{v}\otimes\mathbf{v} \ll \rho\mathbf{v}; \quad \mathbf{v}\rho E \ll \rho E. \tag{4.3}$$

Thus the system of Euler equations reduces to

$$\begin{cases} \rho_t = 0, \\ (\rho\mathbf{v})_t + \nabla p = 0, \\ (\rho E)_t + \nabla\cdot(\mathbf{v}p) = 0. \end{cases} \tag{4.4}$$

We can make a further assumption that $\rho \equiv 1$. Then we derive the pressure-gradient system

$$\begin{cases} \mathbf{v}_t + \nabla p = 0, \\ E_t + \nabla\cdot(\mathbf{v}p) = 0. \end{cases} \tag{4.5}$$

where $E = \frac{u^2+v^2}{2} + p$. This system does not produce new swirls. In other words, if the flow is vorticity-free initially, so is it always. Then we can focus on the study of nonlinear shock waves or rarefaction waves. More interestingly, the pressure variable p can be decoupled from the system (4.5) to satisfy a quasi-linear wave equation

$$\left(\frac{p_t}{p}\right)_t - (p_{xx} + p_{yy}) = 0. \tag{4.6}$$

This equation is very analogous to the 2-D wave equation, just adding a slight nonlinearity. Using this equation, we can understand the flow patterns related to shocks or rarefaction waves quite thoroughly.

The self-similar form of (4.6) is

$$(p-\xi)^2 p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p-\eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{4.7}$$

The eigenvalues are

$$\lambda_\pm = \frac{\xi\eta \pm \sqrt{p(\xi^2+\eta^2-p)}}{\xi^2 - p}, \quad \lambda = \frac{d\eta}{d\xi}. \tag{4.8}$$

Even though (4.7) is still a nonlinear mixed-type equation, it is much easier to handle. Hence the pressure-gradient system can be regarded as a ladder model to investigate compressible fluid flows (of course it is on its own an important model).

In recent years, a lot of progresses were made using the pressure-gradient model (4.5). In [38], the authors mimicked [41] to analyze the two-dimensional Riemann problem for (4.5). The solution structures are strikingly analogous to those of Euler equations except those involving contact discontinuities. Then the analysis was followed by numerical simulations. It is observed that there are numerous problems involving the study of (4.7) in a domain with degeneracy on the boundary. Then Zheng in [43] proved the following theorem concerning the existence of solutions in subsonic-sonic regions.

Theorem 4.2 (Existence of subsonic solutions,[43])**.** *Consider* (4.7) *inside the domain* Ω *with the boundary data* $p|_{\partial\Omega} = \xi^2 + \eta^2$. *Then there exists a positive weak solution* $p \in H^1_{loc}(\Omega)$ *with* $p \in C^{0,\alpha}_{loc}(\Omega)$. *It takes on the boundary value in the sense* $[p - (\xi^2+\eta^2)]^{3/2} \in H^1_0(\Omega)$. *Furthermore, it has*

(i) *maximum principle:* $min_{\partial\Omega}(\xi^2+\eta^2) \le p(\xi,\eta) \le max_{\partial\Omega}(\xi^2+\eta^2)$*;*

(ii) *interior ellipticity:* $p(\xi,\eta) - (\xi^2+\eta^2) > 0$ *in* Ω*.*

In hyperbolic regions $\xi^2+\eta^2 > p$, the equation (4.7) has characteristic decompositions. In fact, we let $r = \sqrt{\xi^2+\eta^2}$, $\theta = \arctan(\eta/\xi)$, $\lambda = \sqrt{\frac{p}{r^2(r^2-p}}$, and $\partial_\pm = \partial_\theta \pm \frac{1}{\lambda}\partial_r$. Then we have

$$\begin{cases} \partial_+\partial_- p = mp_r\partial_- p, \\ \partial_-\partial_+ p = -mp_r\partial_+ p, \end{cases} \tag{4.9}$$

where $m = \frac{\lambda r^4}{2p^2}$. We can use the coordinates (ξ,η) to derive similar characteristic decompositions. Using these decompositions, we are able to discuss the solution to (4.7) in hyperbolic regions. Dai and Zhang solved the problem of planar rarefaction wave interaction in [16] with a vacuum bubble near the origin, and then Lei and Zheng in [25] showed that the vacuum bubble is imaginary and does not exist in reality.

Theorem 4.3 (Existence, [16, 25])**.** *There is a classical smooth solution to the interaction of planar rarefaction waves for the pressure gradient equations* (4.5)*. See Figure* 4.1*. This also applies to the case* $0 < \theta \le \pi/2$*.*

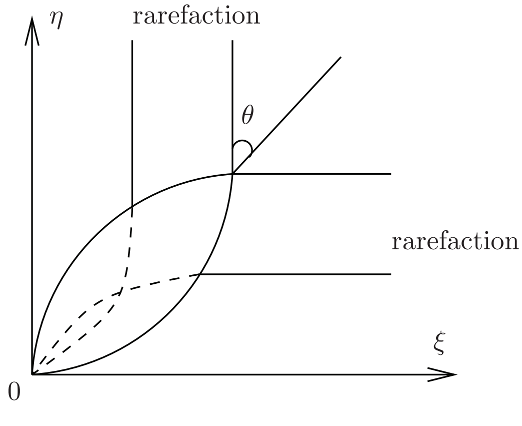


Figure 4.1: Interaction of binary planar rarefaction waves for pressure gradient equations

These two theorems are interesting on their own, but they work separately in a purely elliptic region and a purely hyperbolic region. It is interesting to see how to combine them to construct a global structure in elliptic-hyperbolic mixed regions.

For the Riemann problem involving shocks, we have to solve transonic problem with shocks as free boundaries. Along this direction Zheng in [45] provided a global solution to the case that has initially two contact discontinuities plus two shocks. That is just Configuration H in [29, Page 206]. Then Zheng in [46] proved the existence of solutions to the regular reflection problem of shocks. As for the solution behavior near sonic curves, Tianyou Zhang and Yuxi Zheng recently made an interesting study on the solution behavior near sonic curves via the pressure gradient equation (4.7) in [39].

4.2. Unsteady potential flows

As the flow is vorticity-free, we consider the unsteady potential equation,

$$\phi_{tt} + 2\phi_x\phi_{tx} + 2\phi_y\phi_{ty} - (c^2 - \phi_x^2)\phi_{xx} + 2\phi_x\phi_y\phi_{xy} - (c^2 - \phi_y^2)\phi_{yy} = 0, \quad (4.10)$$

where ϕ is the potential function, such that

$$\phi_x = u, \quad \phi_y = v. \quad (4.11)$$

As usual, (4.10) is closed with the Bernoulli law

$$\phi_t + \frac{\phi_x^2 + \phi_y^2}{2} + \frac{c^2}{\gamma - 1} = 0. \quad (4.12)$$

Introduce the pseudo-potential function,

$$\Phi(\xi, \eta) = \frac{1}{t}\phi(t, x, y) - \frac{\xi^2 + \eta^2}{2}, \quad (4.13)$$

which satisfies

$$\Phi_\xi = u - \xi, \quad \Phi_\eta = v - \eta. \quad (4.14)$$

Then in terms of Φ we arrive at the pseudo–potential flow equation

$$(c^2 - \Phi_\xi^2)\Phi_{\xi\xi} - 2\Phi_\xi\Phi_\eta\Phi_{\xi\eta} + (c^2 - \Phi_\eta^2)\Phi_{\eta\eta} = \Phi_\xi^2 + \Phi_\eta^2 - 2c^2, \quad (4.15)$$

and the Bernoulli law becomes

$$\Phi + \frac{\Phi_\xi^2 + \Phi_\eta^2}{2} + \frac{c^2}{\gamma - 1} = 0. \tag{4.16}$$

It is clear that (4.15) is of mixed-type,

$$(c^2 - \Phi_\xi^2)(d\eta)^2 + 2\Phi_\xi\Phi_\eta d\xi d\eta + (c^2 - \Phi_\eta^2)(d\xi)^2 = 0. \tag{4.17}$$

Using the pseudo-potential equation (4.15), many interesting works have been done. For example, Elling and Liu use it for proving the stability of weak shock when a strong supersonic upstream flow moves against a sharp wedge [18]; Chen and Feldman adopt it for the study of the regular reflection of shock [9]; Kim proves the existence of solutions in a subsonic domain [21]. Morawetz had done the pioneering works on the steady counterpart of (4.10) using the compensated compactness framework and constructed a non-smooth transonic solution around a blunt airfoil in [34]. Let's go to next section for more details.

4.3. Some doable problems

There are lots of progresses in recent years on mixed-type problems although this field is still pre-mature compared to purely elliptic or one-dimensional hyperbolic problems. The existing results are sporadic mostly on specific physics-based problems. It seems that we can only work in this way at the present stage. We will propose some doable problems and mainly concern about the following points regarding existence of solutions and solution structures:

(1) Supersonic-sonic problems;

(2) Subsonic-sonic problems;

(3) Full transonic (mixed-type) problems;

(4) Regularity and solution details near sonic curves.

The useful governing equations may be one of the Euler system (1.2), the pressure-gradient equations (4.5) and the potential equation (4.15).

4.3.1. Transonic nozzles or airfoil flows

There are three typical (steady) flow patterns for the Laval nozzle. We denote by M the local Mach number; M' some critical Mach number less than 0.8 experimentally; M_{in} the Mach number of upstream flows. We refer to [22] for more structures.

1. $M_{in} < M' \ll 1$. For this case, the upstream flow is slow and it will flow through the whole nozzle smoothly without shocks. The existence of smooth solution was established in 1950s by Bers [2] and others using 2-D steady isentropic irrotational Euler equations or 2-D steady potential flow equations. See Figure 4.2.

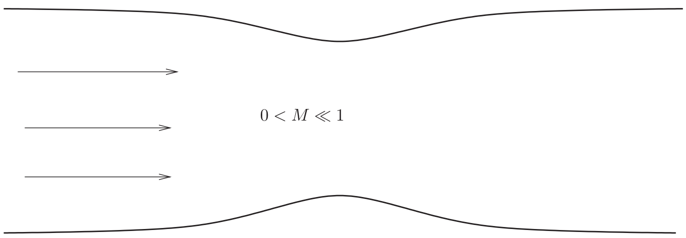


Figure 4.2: Subsonic smooth solutions for Laval nozzle.

2. $0 < M' < M_{in} < M'' < 1$. As the Mach number of subsonic upstream flow is larger than a critical number M', the situation is extremely complicated. Usually, a supersonic bubble is observed to separate from a downstream flow by a shock. See Figure 4.3.

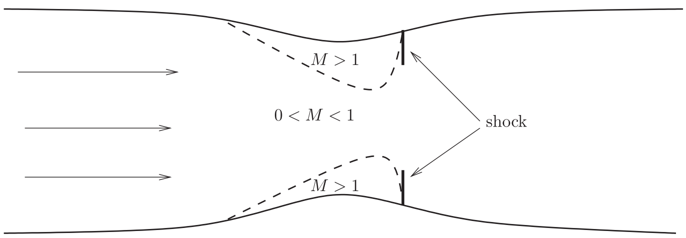


Figure 4.3: Transonic nozzle flows.

3. $M_{in} > 1$. As the upstream flow is supersonic, shocks are produced in the process of the flow passing through the nozzle. A simplest case is that only a shock is produced to separate the upstream flow from a downstream flow. See Figure 4.4.

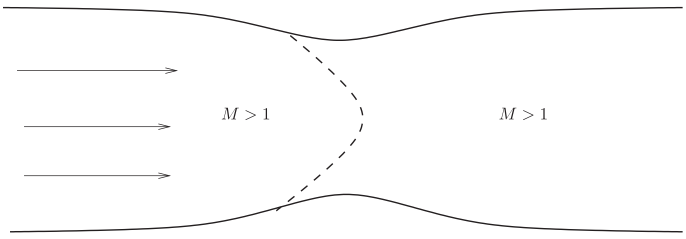


Figure 4.4: A shock separates the upstream flow from a downstream flow.

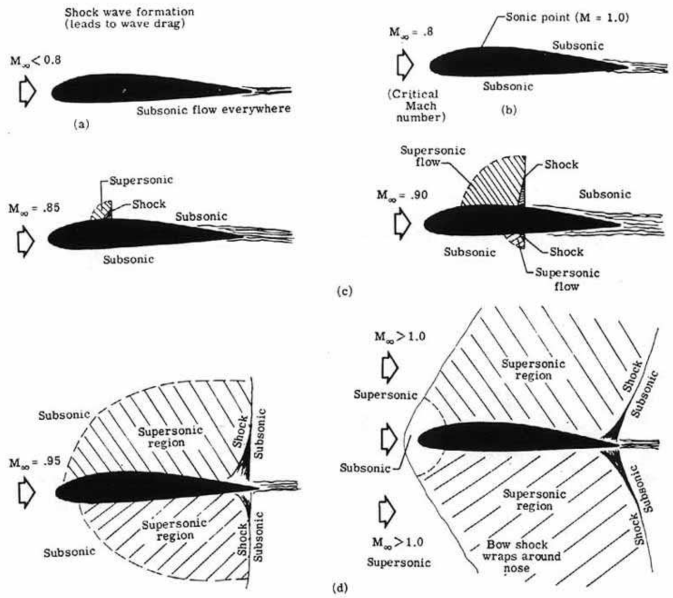


Figure 4.5: Airfoil problem (Downloaded from Google).

Many contributions were made for the nozzle flow. For example, Morawetz uses the framework of compensated-compactness to show the nonexistence of smooth solutions [34]; Chen *et al.* show (also use the framework of compensated-compactness) the existence of weak solutions [7, 8]; Xie and Xin adopt a stream function formulation to prove the existence of solutions in a subsonic-sonic part of the nozzle [37]. Recently, Tianyou Zhang and Yuxi Zheng give more insight into detailed structures of the solution near the sonic curves [40]. A question is how to construct a global weak solution involving shocks. The results by Chen *et al.* did not resolve this issue because their solutions are constructed, respectively, in a supersonic region and in a subsonic region. In [15] there are some explicit examples of transonic solutions , e.g, the Ringleb flow. However, it is quite difficult to apply them for practical uses.

Similar situations occur as a flow passes over an airfoil. We sketch several cases in Figure 4.5. We can use the steady Euler equations, or the potential flow equation to study this class of problems.

4.3.2. A supersonic flow over a blunt body

As a supersonic upstream flow goes over a blunt body, a transonic shock usually forms, as shown in Figure 4.6.

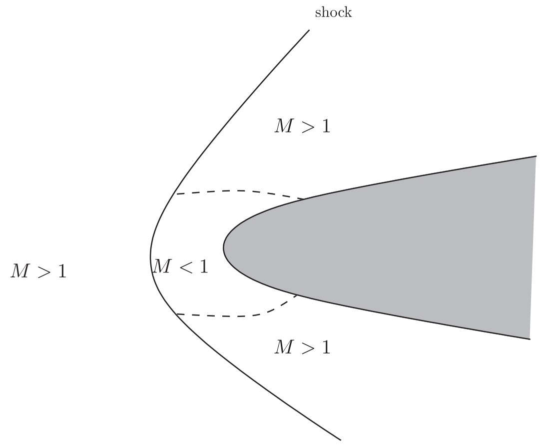


Figure 4.6: A supersonic flow over a blunt body.

There is an extreme case that the blunt body becomes a sharp wedge. Then as a strong supersonic flow goes over it, two symmetric shocks attach at the tip of the wedge. We can use a shock polar to find the existence of such shocks: a strong shock or a weak shock. The downstream flow is supersonic for the weak shock; while it may be supersonic or subsonic for the strong shock. A simple algebraic entropy criterion cannot select out which shock is physically admissible. Prandtl and Meyer conjectured that the weak shock is stable under some perturbations. However, this issue has not been settled down yet although there are various beautiful arguments available. For example, Chen, Xin and Yin show the local structural stability of weak shock near the tip of wedge [13] with many extensions afterwards; Elling and Liu study the asymptotic stability of weak solution [18]. The strong shock may be transonic and is much harder to be investigated. See Elling's recent works (eg. [17]).

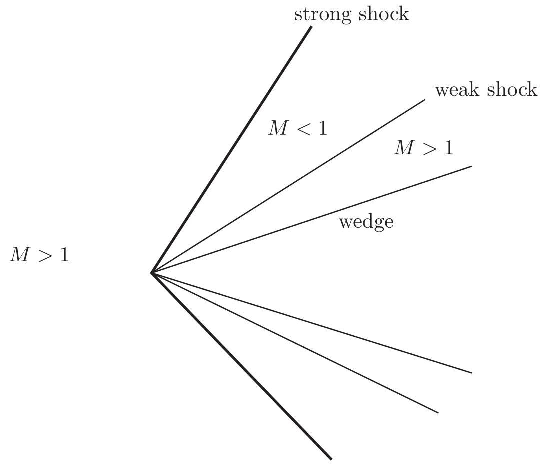


Figure 4.7: A supersonic flow over a solid wedge.

4.3.3. A wedge-shaped gas expansion problem

The expansion problem of a wedge-shaped gas into a vacuum has a long history and it can be hydraulically interpreted as a dam collapse problem. As the half angle of the wedge is less than $\pi/2$, it is completely solved recently. In [16, 25], the pressure-gradient model (4.5) is used; and in [26, 27, 30, 31, 32, 28, 42] the system of Euler equations is used. In this context, an approach of characteristic decomposition is developed to handle the interaction of simple waves. This approach can be even used to extend the supersonic solution to sonic boundaries and construct a so-called *semi-hyperbolic patch* that exists extensively in gas dynamics [36, 33],.

An obvious problem remains: What happens as the half wedge angle is larger than $\pi/2$? We can formulate this problem for the Euler or pressure gradient equations. For example, we consider the latter. The initial data is set to be

$$p(x,y,0) = \begin{cases} 0, & \text{for } x < 0, y > 0, \\ 1, & \text{otherwise.} \end{cases} \tag{4.18}$$

The initial velocity could be imposed properly. The solution structure is roughly described in Figure 4.8. In order to depict a global picture, we may have to use all ingredients including semi-hyperbolic patches, the theory of degenerate elliptic equations and something else.

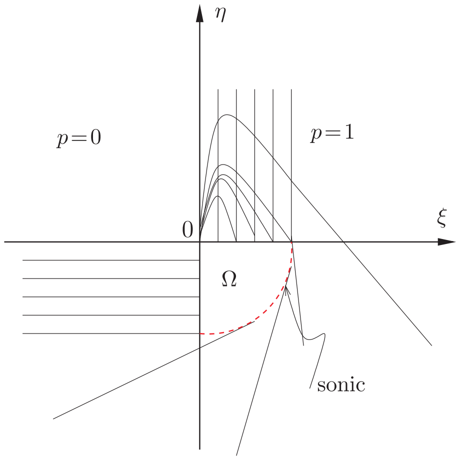


Figure 4.8: A wedge-shaped gas expansion into a vacuum with obtuse angle.

4.3.4. Symmetric shock interaction

If one checks all cases of 2-D Riemann problems simulated by our very delicate scheme in [19], he finds that one case that could be promisingly worked out rigorously is the regular interaction of symmetric shocks. See Figure 4.9. The picture is very clean. Related to this pattern is the regular reflection of oblique shock against a rigid wall, which is solved in [9], and the stability of Mach reflection of shocks in [10].

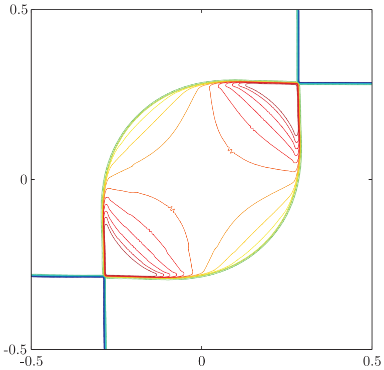


Figure 4.9: Regular reflection resulting from 2–D Riemann problem for Euler system by the GRP scheme.

For this problem, we may start from the pressure-gradient model and adjust the interaction angle of shocks, as shown in Figure 4.10. There are already some ladder results for this problem [45, 46].

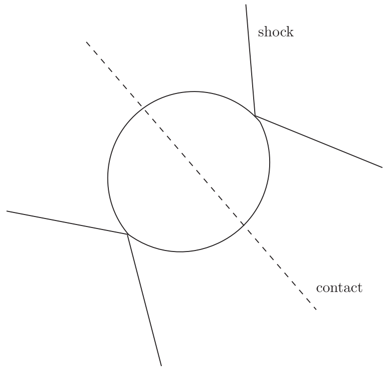


Figure 4.10: Regular interaction of shocks sketched from 2–D Riemann problem for pressure-gradient system

Acknowledgment

With admire and respect, the author thanks Professor Tai-Ping Liu for his constant guidance as an inspiring mentor. This paper is supported by NSFC (No. 11371063), the doctoral program from the Education Ministry of China (No. 20130003110004) and the Innovation Program from Beijing Normal University (No. 2012LZD08).

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