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# THE ONE WAY LINEARIZED WATER WAVE EQUATIONS

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**Dedication**

*With best wishes to Tai-Ping on his seventieth birthday. In recognition and appreciation for years of comradeship in the search to understand the propagation of waves in the natural world and the structure of solutions of hyperbolic partial differential equations.*

**Abstract**

In analogy with D'Alembert's analysis of the one dimensional wave equation, the solutions of the linearized equations describing the perturbations of the $1d$ horizontal interface between $2d$ water and air are sums of rightward and leftward moving waves. This paper introduces and analyses the qualitative behavior of the corresponding one way equations. The one way equations are the core elements in an efficient numerical algorithm.

## 1. Introduction

Our papers [6], [7], and [8] introduce a numerical method for constructing approximate solutions of the linearized water wave equation on the entire real line

$$u_{tt} + \big|D\big|\, u = 0, \qquad D := \frac{1}{i}\,\partial_x. \tag{1.1}$$

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The key difficulty addressed is the infinite domain of $x \in \mathbb{R}$. Most approximate methods, including ours, truncate the domain to be finite and compute an approximate solution on the finite domain. The goal is to obtain accurate values on a subdomain that is not too much smaller.

For differential equations such truncations are challenging. It is more difficult for (1.1) since the the operator $|D|$ is nonlocal. The time derivative $u_{tt}$ at each point $(t,x)$ depends on the values of $u(t,\cdot)$ on the whole real line. With truncation many values of $u(t,\cdot)$ are not available. Another peculiarity of the water wave equation is that the group velocities tend to infinity as $\xi \to 0$. Waves reach artificial boundaries instantly. There is no short reprieve as there would be for problems with finite propagation speed.

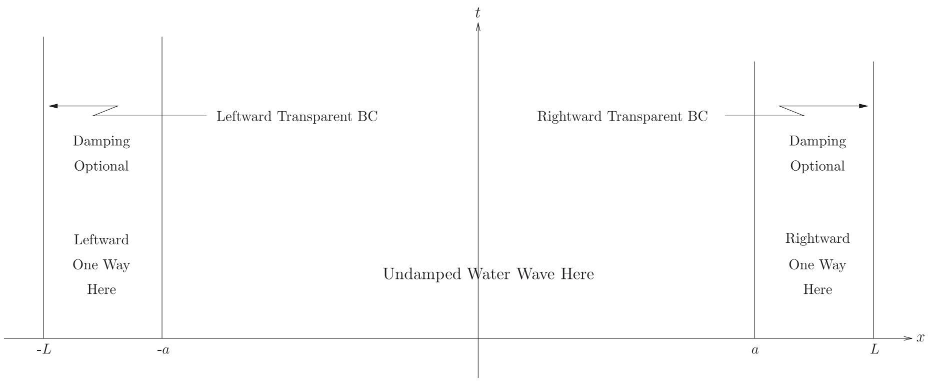


Figure 1: Computational domain $[-L,L]$ and interest domain $[-a,a]$.

The method involves a computational domain $-L \le x \le L$ containing the domain of interest $[-a,a]$. The domain of interest contains the support of initial data. The strategy is based on the observation that solutions of (1.1) are sums of essentially leftward and rightward going waves that are solutions of one way equations. A wave initially supported in $[-a,a]$ that reaches the right layer $[a,L]$ is essentially rightward moving. The numerical algorithm replaces the water wave equation by the rightward equation in the layer $a < x < L$. The values in the layer are advanced using the rightward equation rather than the water wave equation. An advantage is that errors committed in the layer tend to be swept rightward away from the domain of interest. A second advantage is that the rightward equation is easily damped without generating backward moving waves. Damping reduces the size of waves reaching the external boundary $x = L$. It is as if the layers are in wind tunnels with the wind driving the flow away from the domain of interest. The rightward boundary is treated so as to be essentially transparent to rightward waves. Errors from the boundary tend not to reenter the domain thanks to the same wind tunnel effect. The left hand layer is treated similarly. The leftward and rightward one way equations are the key elements in this strategy. Their analysis is the subject of this article. The numerical method and the details of high order implementation are discussed in the pair of articles [7], and [8].

Plane waves $e^{i(\xi x-\omega t)}$ satisfy the linearized water wave equation (1.1) if and only if they satisfy the *dispersion relation*

$$\omega^2 = |\xi|. \tag{1.2}$$

**Definition 1.1.** Denote by $\omega(\xi)$ the unique non decreasing function that satisfies (1.2), that is

$$\omega(\xi) = \mathrm{sgn}(\xi)\sqrt{|\xi|}. \tag{1.3}$$

The function $\omega(\xi)$ is called the *smart square root.*

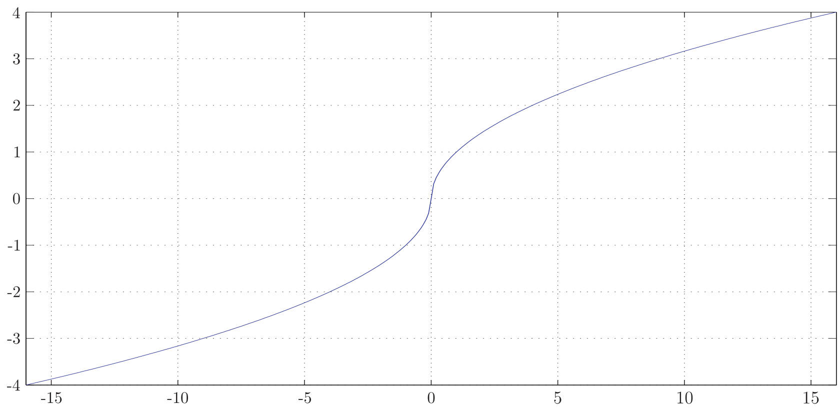


Figure 2: The smart square root $\omega(\xi)$ plotted against $\xi$.

The group velocities defined by

$$\mathbf{v}_{\mathrm{group}}(\xi) := \omega'(\xi) = |\xi|^{-1/2}$$

are positive exactly because $\omega$ is nondecreasing. The operator $\omega(D)$ is defined by $\mathcal{F}^{-1}\omega(\xi)\mathcal{F}$ where $\mathcal{F}$ is the Fourier Transform. Such multiplication operators are discussed in §2. The plane waves

$$e^{i(\xi x-\omega(\xi)t)} \tag{1.4}$$

generate the solutions of $v_t+i\omega(D)v=0$ so this equation describes essentially rightward motion. Similarly, $v_t-i\omega(D)v=0$ describes leftward motion. These equations are called the *one way water wave equations.*

§4 shows that the one way Cauchy problems are well set. §5 proves an analogue of D'Alembert's analysis of the one dimensional wave equation showing that solutions the water wave equation are sums of solutions of the leftward and rightward one way equations.

§6, and §7, contain precise asymptotic descriptions as $t\to\infty$, as $x\to\infty$, and, also along lines in space time. These estimates are important since the tails at infinity in $x$ represent data that is not available to the truncated numerical algorithm.

In §8 the well posedness of the spatially truncated problem that corresponds to our numerical method is proved for smooth truncations.

**Acknowledgments**

Peter Miller gave us an important shove in the right direction when considering the asymptotics as $x\to\infty$, and, Sijue Wu helped us in many ways. A counterexample of Mike Taylor steered us away from dangerous shoals. The thesis of Jennifer Beichman [2] stimulated several refinements. We thank the referee for calling [1] to our attention.

## 2. Multiplication Operators

In this paper, in contrast to its siblings [7] and [8], the Fourier transform and its inverse are defined by

$$\widehat{f}(\xi) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\; e^{-ix\xi}\; dx, \quad \text{so,} \quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\xi)\; e^{ix\xi}\; d\xi.$$

The Fourier transform is unitary on $L^2(\mathbb{R})$, and the usual partial derivative $\partial_x$ and the operator $D := \frac{1}{i}\partial_x$ are given by

$$\widehat{\partial_x f}(\xi) = i\xi \widehat{f}(\xi), \qquad \text{and,} \qquad \widehat{Df}(\xi) = \xi \widehat{f}(\xi).$$

Next define Fourier multiplication operators.

**Definition 2.1.** The Weiner algebra $\mathbb{A}^0$ consists of the tempered distributions on $\mathbb{R}$ whose Fourier transform belongs to $L^1(\mathbb{R})$. More generally, for any $s \in \mathbb{R}$ define

$$\mathbb{A}^s := \Big\{ \phi \in \mathcal{S}'(\mathbb{R}) : \langle \xi \rangle^s \widehat{\phi} \in L^1(\mathbb{R}) \Big\}.$$

$\mathbb{A}^\infty(\mathbb{R}) := \cap_{s\in\mathbb{R}} \mathbb{A}^s(\mathbb{R})$, and $\mathbb{A}^{-\infty}(\mathbb{R}) := \cup_{s\in\mathbb{R}} \mathbb{A}^s(\mathbb{R})$.

**Definition 2.2.** For a measurable $G(\xi) \in L^\infty_{loc}(\mathbb{R})$ growing at most polynomially as $|\xi| \to \infty$, define an operator $G(D)$ from $\mathbb{A}^\infty(\mathbb{R})$ to itself and also $\mathbb{A}^{-\infty}(\mathbb{R})$ to itself by

$$\widehat{G(D)f} := G(\xi)\widehat{f}(\xi).$$

**Example 2.3.** The operators $|D|$, $|D|^{1/2}$, and the Hilbert Transform $H := -i\,\mathrm{sgn}(D)$ are given by

$$\widehat{|D|^{1/2}f}(\xi) = |\xi|^{1/2}\widehat{f}(\xi), \qquad \widehat{|D|f}(\xi) = |\xi|\widehat{f}(\xi),$$

and,

$$\widehat{(Hf)}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi). \tag{2.1}$$

The Hilbert transform is anti self adjoint on $L^2(\mathbb{R})$. The formulas

$$\mathcal{F}\Big(\mathrm{P.V.}\,\frac{1}{x}\Big) = i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,\xi \qquad \text{and,} \qquad \widehat{f*g} = \sqrt{2\pi}\;\widehat{f}\,\widehat{g}$$

imply that the Hilbert transform is given by

$$(Hf)(x) := \frac{1}{\pi}\,\mathrm{P.V.}\int \frac{f(y)}{x-y}dy. \tag{2.2}$$

Multiplication operators commute regardless of their symbols. They are translation invariant. Each is a convolution operator. When there is a $c$ so that $|G(\xi)| \leq c\langle\xi\rangle^m$, then $G(D)$ is continuous from $H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ and from $\mathbb{A}^s(\mathbb{R}) \to \mathbb{A}^{s-m}(\mathbb{R})$ for all $s \in \mathbb{R}$.

## 3. The One Way Water Wave Equations

Consider the non steady incompressible inviscid irrotational infinitely deep two dimensional Euler equation in a time dependent domain $\Omega(t)$. The boundary of the domain is an interface with fluid below and a massless medium of constant pressure above. Linearizing these equations at the flat interface, small perturbations are described by solutions of the scalar equation (1.1).

It follows that for *any* measurable solution $\omega(\xi)$ of (1.2),

$$\begin{aligned}\partial_t^2 + |D| = \partial_t^2 + \omega(D)^2 &= (\partial_t - i\omega(D))(\partial_t + i\omega(D)) \\ &= (\partial_t + i\omega(D))(\partial_t - i\omega(D)). \qquad (3.1)\end{aligned}$$

*Make the choice $\omega(\xi)$ that is the unique non decreasing solution from Definition 1.1.* From here on, $\omega(\xi)$ denotes this solution. One has

$$i\omega = i\,\mathrm{sgn}(\xi)\sqrt{|\xi|}, \qquad \text{so,} \qquad \omega(D) = -\,H\,|D|^{1/2}. \qquad (3.2)$$

The function $\omega(\xi)$ is odd and strictly monotone increasing.

The superpositions in $\xi$ of the plane waves (1.4) satisfy $\partial_t v + i\omega(D)v = 0$. The opposite choice of sign $e^{i(\xi x + \omega(\xi)t)}$ satisfies $\partial_t v - i\omega(D)v = 0$.

Thanks to (3.1) it follows that solutions of the one way equations are solutions of the linearized water wave equation. Like D'Alembert's analysis of $u_{tt} - u_{xx} = 0$, Proposition 5.2 below shows that the general solution of the linearized equation is equal to a sum $v^+ + v^-$ of solutions of the rightward and leftward one way equations.

## 4. The Cauchy Problem for the One Way Equations

The formula $v(t) = e^{-it\omega(D)}v(0)$ involving the Fourier multiplier $e^{-it\omega(D)}$ is the natural candidate solution of $v_t + i\omega(D)v = 0$ with initial value $v(0)$. The next proposition shows that under suitable assumptions of regularity in time and decay at infinity it is the only solution.

Begin by showing that the Cauchy problem for the differential equation

$$v_t + i\,\omega(D)v = 0 \tag{4.1}$$

makes sense when $v \in L^1_{\rm loc}(\mathbb{R}\,;\,\mathbb{A}^s(\mathbb{R}))$ for some $s \in \mathbb{R}$, possibly very negative. In this case $v$ is a well defined distribution and $v_t = -i\omega(D)v \in L^1_{\rm loc}(\mathbb{R}\,;\,\mathbb{A}^{s-1/2}(\mathbb{R}))$. Thus the left hand side of the differential equation makes sense as a distribution. The definition of a solution is that this distribution is equal to zero.

For such a solution, both $v$ and $v_t$ belong to $L^1_{\rm loc}(\mathbb{R}\,;\,\mathbb{A}^{s-1/2}(\mathbb{R}))$. Therefore $v \in C(\mathbb{R}\,;\,\mathbb{A}^{s-1/2}(\mathbb{R}))$. In particular, the value $v(t)$ is well defined for all $t$.

**Proposition 4.1. i.** *If $s \in \mathbb{R}$ and $v \in L^1_{\rm loc}(\mathbb{R}\,;\,\mathbb{A}^s(\mathbb{R}))$ satisfies* (4.1) *then for all $t$, $v(t) = e^{-it\omega(D)}v(0)$.*

**ii.** *Conversely, if $f \in \mathbb{A}^s(\mathbb{R})$ then $v(t) := e^{-it\omega(D)}f$ belongs to $(L^\infty \cap C)(\mathbb{R}\,;\,A^s(\mathbb{R}))$ and satisfies* (4.1) *with initial value $v(0) = f$. In addition for all $0 \le k \in \mathbb{N}$, $\partial_t^k v \in (L^\infty \cap C)(\mathbb{R}\,;\,A^{s-k/2}(\mathbb{R}))$.*

***Proof.*** The remarks before the Proposition show that $v \in C(\mathbb{R}\,;\,\mathbb{A}^{s-1/2}(\mathbb{R}))$. An induction shows that $\partial_t^k v = (-i\omega(D))^k v \in C(\mathbb{R}\,;\,\mathbb{A}^{s-k/2}(\mathbb{R}))$.

Take the Fourier transform of the equation to find

$$\partial_t \widehat{v} + i\omega(\xi)\widehat{v} = 0.$$

Therefore

$$\partial_t\big(e^{i\,t\,\omega(\xi)}\;\widehat{v}\big) = 0, \tag{4.2}$$

proving that $v$ must be given by formula in **i**. Part **ii** is immediate. □

**Example 4.2. i.** Because $e^{-it\omega(D)}$ is $L^2$ unitary, the theory of the water wave equation is dominated by $L^2$ methods. The results above in the spaces $\mathbb{A}^s$ suffice to get the ball rolling. For example, if $v \in L^1_{loc}(\mathbb{R}; H^s(\mathbb{R}))$ then $v \in L^1_{loc}(\mathbb{R}; \mathbb{A}^\sigma(\mathbb{R}))$ for all $\sigma < s - 1/2$ so the one way equation makes sense and solutions are given by the formula $e^{-it\omega(D)}v(0)$.

**ii.** In contrast, if

$$\widehat{f} \in L^1_{comp} \text{ is compactly supported with } \int |\widehat{f}|^2\, d\xi = \infty,$$

then $f \in \mathbb{A}^\infty$ is smooth and $v = e^{-it\omega(D)} f$ is a smooth solution covered by Proposition 4.1. However, $f \notin H^{-\infty}$ so this solution is not amenable to an $L^2$ Sobolev analysis. This greater generality explains the choice of the $\mathbb{A}^s$ setting over the $H^s$ setting.

**Remark 4.3. i.** The solution formula shows that for all $\xi \in \mathbb{R}$, $|\widehat{v}(t,\xi)|^2$ is independent of $t$. Therefore for all $\sigma \le s$, the Sobolev norms $\|v(t)\|^2_{H^\sigma(\mathbb{R})}$ are independent of time.

**ii.** The evolution is a unitary group on $H^s(\mathbb{R})$ whose generator is the anti self adjoint operator $i\omega(D)$ with domain equal to the set of functions $f \in H^s(\mathbb{R})$ such that $\omega(D) f \in H^s(\mathbb{R})$.

**iii.** The Fourier transform intertwines this operator with multiplicaiton by $\omega(\xi)$ whence the spectrum is absolutely continuous and equal to $\mathbb{R}$.

A pair of cousin equations with $\underline{\omega}$ positive homogeneous of degree 1/2 are

$$\partial_t u + e^{i\pi/4}\, \underline{\omega}(D\,) u = 0, \qquad \text{with} \qquad \underline{\omega}^2 = \xi/i. \tag{4.3}$$

They occur in the construction of absorbing boundary conditions for the one dimensional Schrödinger equation (see [1]). With $t$ and $x$ interchanged it is the Dirichlet to Neumann operator of Schrödinger equation. The operator $\underline{\omega}(D)$ is an Abel fractional integral operator with kernel supported in a half line. Because of the $\xi$ in place of $|\xi|$, the values of $\underline{\omega}(\xi)$ for $\xi \in \mathbb{R}$ lie on a pair of orthogonal rays. The dispersion relation $\underline{\omega}$ is not real valued. The initial value problem for (4.3) is ill posed either toward the future or the past. The analysis of the absorbing boundary condition has elements in common with this paper.

## 5. A D'Alembert Formula for The Linearized Water Wave Equation

The analysis of the initial value problem for the linearized water wave equation is analogous. The proof of the next proposition is omitted.

**Proposition 5.1.** *If $s \in \mathbb{R}$, $f \in H^s(\mathbb{R})$ and $g \in H^{s-1/2}(\mathbb{R})$ then there is a unique solution*

$$u \in C(\mathbb{R}\,;\, H^s(\mathbb{R})) \qquad \text{with} \qquad u_t \in C(\mathbb{R}\,;\, H^{s-1/2}(\mathbb{R}))$$

*to the initial value problem*

$$u_{tt} + |D|u = 0, \qquad u\big|_{t=0} = f, \quad u_t\big|_{t=0} = g.$$

There is an analogue of D'Alembert's formula for the one dimensional wave equation.

**Proposition 5.2.** *If $f, g, s, u$ are as in the preceding proposition with $g \in |D|^{1/2}\, H^s(\mathbb{R})$ then there are solutions $v^+$ and $v^-$ in $C\big(\mathbb{R}\,;\, H^s(\mathbb{R})\big)$ satisfying the rightward and leftward one way equations respectively and so that $u = v^+ + v^-$. The $v^+$ and $v^-$ are uniquely determined by $u$.*

***Proof.*** It is sufficient to find the initial values of $v^\pm$. In order that $v^+ + v^-$ and $\partial_t(v^+ + v^-)$ match the corresponding initial values for $u$ it is necessary and sufficient that

$$v^+(0) + v^-(0) = f, \quad \text{and,} \quad -i\omega(D)v^+(0) + i\omega(D)v^-(0) = g. \tag{5.1}$$

Multiplying the first relation by $i\omega(D)$ then adding and subtracting yields

$$2i\omega(D)v^-(0) = \omega(D)f + g, \qquad 2i\omega(D)v^+(0) = \omega(D)f - g.$$

These formulas uniquely determine $v^\pm(0) \in H^s(\mathbb{R})$ thanks to the hypothesis $g \in |D|^{1/2}\, H^s(\mathbb{R})$. Defining $v^+$ and $v^-$ to be the solutions of the rightward and leftward one way equations with these initial values proves the proposition. □

## 6. Asymptotics of the One Way Equations as $x \to \pm\infty$

**Proposition 6.1. i.** *Suppose that $v = e^{-it\omega(D)} f$ with $f$ satisfying*

$$\widehat{f}(\xi) \in C^2(\mathbb{R}) \qquad \text{and} \qquad \frac{\partial^j \widehat{f}(\xi)}{\partial \xi^j} \in L^1(\mathbb{R}) \quad \text{for} \quad 0 \le j \le 2.$$

*Then* $v(t,x) = O(x^{-3/2})$ *as* $x \to +\infty$ *uniformly on compact time intervals. More precisely there is a* $c \neq 0$ *computed in the proof so that uniformly on bounded time intervals as* $x \to +\infty$

$$v(t,x) = c\,t\,\frac{\widehat{f}(0)}{x^{3/2}} + O(|x|^{-2}).$$

**ii.** *An analogous result with a different* $c$ *holds for* $x \to -\infty$. *Analogous results hold for the leftward equation.*

**Remark 6.2. i.** When all the derivatives of $\widehat{f}$ belong to $L^1(\mathbb{R})$, there is a complete asymptotic expansion $\sum_{j=3}^{\infty} c_j\, x^{-j/2}$ as $x \to \infty$ whose leading term is given in **i**.

**ii.** When $\widehat{f}$ vanishes to order $k$ at the origin, the decay rate is $x^{-(3+k)/2}$.

***Proof.*** The solution is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-i\omega(\xi)t} \widehat{f}(\xi) d\xi. \tag{6.1}$$

Choose $\chi(\xi) \in C_0^\infty(\mathbb{R})$ with $\chi$ identically equal to 1 on a neighborhood of $\xi = 0$. The integral is equal to

$$\int_{-\infty}^{\infty} e^{ix\xi} e^{-i\omega(\xi)t} (1 - \chi(\xi)) \widehat{f}(\xi) d\xi + \int_{-\infty}^{\infty} e^{ix\xi} e^{-i\omega(\xi)t} \chi(\xi) \widehat{f}(\xi) d\xi := J_1 + J_2. \tag{6.2}$$

The integral $J_1$ is estimated to be $O(1/x^2)$ by the method of non stationary phase as follows. Use

$$e^{ix\xi} = \Big(\frac{1}{ix}\,\partial_\xi\Big)^2 e^{ix\xi}$$

to find

$$\Big(\frac{1}{ix}\Big)^2 \int e^{ix\xi}\, \partial_\xi^2 \Big(e^{-i\omega(\xi)t}\ (1 - \chi(\xi))\ \widehat{f}(\xi)\Big)\ d\xi.$$

Using the fact that $\omega', \omega''$ are bounded on the domain of integration together with the integrablility hypothesis on $\partial_\xi^j \widehat{f}$ shows that the integral is bounded uniformly for bounded $t$ and all $x$.

It remains to analyze $J_2$. Thanks to the factor $\chi$, it suffices to treat the case of $f \in C_0^2(\mathbb{R})$. Introduce the bijective change of variables on $\mathbb{R}$,

$y = \omega(\xi)$. Then,

$$y^2 = |\xi|, \qquad \mathrm{sgn}(y) = \mathrm{sgn}(\xi), \qquad 2ydy = \mathrm{sgn}(\xi)\, d\xi, \qquad \xi = y^2\,\mathrm{sgn}(y).$$

The integral $J_2$ is equal to

$$\int_0^\infty \widehat{f}(y^2)\; e^{-iyt}\; e^{ixy^2}\; 2ydy + \int_{-\infty}^0 \widehat{f}(-y^2)\; e^{-iyt}\; e^{-ixy^2}\; (-2y)dy := I_1 + I_2$$

In this integral $t$ and $y$ are bounded. The interest is $|x| \to \infty$. Each integral has a point of stationary phase at the endpoint $y = 0$. The behavior is determined by the Taylor expansions of $\widehat{f}(y^2)e^{-iyt}$ and $\widehat{f}(-y^2)e^{-iyt}$ at $y = 0$.

The formal computation is straightforward. See [4] for the justification. Expand about $y = 0$ to find

$$\widehat{f}(y^2)\, e^{-iyt} = \widehat{f}(0)\big(1-iyt\big)+O(y^2), \qquad \widehat{f}(-y^2)e^{-iyt} = \widehat{f}(0)\big(1-iyt\big)+O(y^2).$$

The leading term in $I_1 + I_2$ comes from the two constant terms in the Taylor expansions to give $\widehat{f}(0)$ times the sum of oscillatory integrals in the sense of Hörmander,

$$\int_0^\infty e^{ixy^2}\; 2ydy + \int_{-\infty}^0 e^{-ixy^2}\; (-2y)dy. \tag{6.3}$$

Next show that the sum (6.3) vanishes. Perform the computation for $x > 0$. Make the change of variable

$$\rho := \sqrt{x}y, \quad y = \rho/\sqrt{x}, \quad d\rho = \sqrt{x}dy \tag{6.4}$$

to find

$$\int_0^\infty e^{i\rho^2} 2\frac{\rho}{\sqrt{x}}\frac{d\rho}{\sqrt{x}} + \int_{-\infty}^0 e^{-\rho^2}(-2\frac{\rho}{\sqrt{x}})\frac{d\rho}{\sqrt{x}} = \frac{2}{x}\left(\int_0^\infty e^{i\rho^2}\rho d\rho - \int_{-\infty}^0 e^{-i\rho^2}\rho d\rho\right).$$

Write $e^{\pm i\rho^2}$ using DeMoivre's idententity the $\sin(\pm\rho^2)$ contributions of the last two integrals cancel by parity. This leaves

$$\int_0^\infty \cos(\rho^2)\rho d\rho - \int_{-\infty}^0 \cos(-\rho^2)\rho d\rho.$$

Each of these oscillatory integrals vanish. For example with $\psi \in C_0^\infty(\mathbb{R})$ with $\psi(0) = 1$,

$$\int_0^\infty \cos(\rho^2)\rho d\rho := \lim_{\varepsilon\to 0} \int_0^\infty \psi(\varepsilon\rho)\cos(\rho^2)\rho d\rho.$$

Compute using an integration by parts that

$$\begin{aligned} 2\int_0^\infty \psi(\varepsilon\rho)\cos(\rho^2)\rho d\rho &= \int_0^\infty \psi(\varepsilon\rho)\frac{d}{d\rho}\sin(\rho^2)d\rho \\ &= -\varepsilon\int_0^\infty \psi'(\varepsilon\rho)\sin(\rho^2)d\rho \end{aligned}$$

with boundary contributions that vanish. The integral on the right is $O(1)$ showing that the desired limit as $\varepsilon \to 0$ is equal to zero.

The next term in the asymptotics is $it\widehat{f}(0)$ times

$$\int_0^\infty e^{ixy^2}\, 2y^2 dy + \int_{-\infty}^0 e^{-ixy^2}\, (-2y^2)dy. \tag{6.5}$$

The change of variables (6.4) writes the first term for $x > 0$ as

$$\int_0^\infty e^{i\rho^2}\, \frac{\rho^2}{x}\, \frac{d\rho}{\sqrt{x}} = x^{-3/2}\int_0^\infty e^{i\rho^2}\, \rho^2\, d\rho.$$

Continuing, write

$$\int_0^\infty e^{i\rho^2}\, \rho^2\, d\rho = \frac{1}{i}\frac{d}{dx}\int_0^\infty e^{ixy^2}\, dy\Big|_{x=1} = \frac{1}{i}\frac{d}{dx}\Big(\frac{1}{\sqrt{x}}\int_0^\infty e^{i\rho^2}\, d\rho\Big)\Big|_{x=1}.$$

The last term is $i/2$ times the nonzero Fresnel integral $\int_0^\infty e^{i\rho^2}\, d\rho$. Treating the second term in (6.5) the same way verifies that (6.5) is a nonzero constant times $x^{-3/2}$ as $x \to +\infty$. □

**Example 6.3.** The rightward equation is not strictly rightward moving. The solution of the initial value problem with $\operatorname{supp} f(x) \subset \{x \geq 0\}$ is not usually supported in $\{x \geq 0\}$ for $t > 0$. In fact, for $t > 0$ and $x \to -\infty$ one has $v(t,x) \approx c\,\widehat{f}(0)\,x^{-3/2}$ with $c \neq 0$. This shows that the solution has a tail that extends all the way to $-\infty$ whenever $\widehat{f}(0) \neq 0$. There is an analogous non vanishing tail whenever $\widehat{f}$ is smooth in a neighborhood of $\xi = 0$ and vanishes there at most to finite order.

## 7. Asymptotics of the One Way Equations as $t \to \infty$

### 7.1. Decay uniform in $x$

The next proposition yields $t^{-1/2}$ decay. There are two $L^1 \to L^\infty$ decay estimates. The first when $\operatorname{supp} \widehat{f} \subset \{|\xi| \le 2\}$ the second when $\operatorname{supp} \widehat{f} \subset \{\Lambda/2 \le |\xi| \le 2\Lambda\}$. The first result concerns the small frequency contribution.

**Proposition 7.1.** *There is a constant $C$ so that if $f \in L^1$ with $\widehat{f}$ supported in $|\xi| \le 2$, then for $|t| \ge 1$,*

$$\big\| e^{\pm i\omega(D)t} f \big\|_{L^\infty(\mathbb{R})} \le \frac{C}{|t|^{1/2}} \|f\|_{L^1}.$$

***Proof.*** **Step I.** The plus and minus signs are nearly identical. Treat only the sign plus. Denote $w(t) := e^{i\omega(D)t} f$. The first step is to prove that if $\widehat{f}$ is in addition absolutely continuous, then

$$\big\| w(t) \big\|_{L^\infty(\mathbb{R})} \le \frac{C}{|t|^{1/2}} \int_{-2}^{2} \Big| \frac{\partial \widehat{f}(\xi)}{\partial \xi} \Big| d\xi.$$

Treat only $t \to +\infty$. Define $f^\varepsilon$ to be the function whose Fourier Transform is equal to $\widehat{f}$ on $|\xi| \ge \varepsilon$ and equal to zero on $|\xi| \le \varepsilon$. Denote by $w^\varepsilon := e^{i\omega(D)t} f^\varepsilon$. It is sufficient to prove that $\|w^\varepsilon(t)\|_{L^\infty} \le C t^{-1/2} \int_{|\xi| \ge \varepsilon} |\partial \widehat{f^\varepsilon} / \partial \xi| d\xi$ with a constant independent of $\varepsilon \in ]0,1]$.

The support of $\widehat{f^\varepsilon}$ is contained in the union of two closed intervals, one in $]0,\infty[$ and the other in $]-\infty, 0[$. Treat the first. The second is nearly identical.

For each $t > 0$,

$$\max_{x \in \mathbb{R}} |w^\varepsilon(t,x)| = \max_{c \in \mathbb{R}} |w^\varepsilon(t,ct)|, \quad w^\varepsilon(t,ct) = \int_\varepsilon^2 e^{ict\xi} e^{i\omega(\xi)t} \widehat{f}(\xi) d\xi. \quad (7.1)$$

Van der Corput's lemma (see for example [10]), asserts for phase functions $\phi$ that are real valued and $C^2$ in $[\varepsilon, 2]$ with $|\phi''(x)| \ge \lambda$ on $[\varepsilon, 2]$ one has

$$\left| \int_a^b e^{i\phi(x)} \psi(x) dx \right| \le \frac{8}{\lambda^{1/2}} \left( |\psi(b)| + \int_a^b |\psi'(x)| dx \right). \quad (7.2)$$

Take

$$\phi(\xi, c) := t\big(\omega(\xi) + c\xi\big).$$

On $[\varepsilon, 2]$, $\phi''$ is bounded below by a strictly positive constant times $t$. Estimate (7.2) with $b = 2$ $\psi = \widehat{f}$, and, $\psi(2) = 0$, completes the proof.

**Step II.** Next suppose that $f \in L^1$ with $\widehat{f}$ supported in $[-2, 2]$. Choose $\chi$ with $\widehat{\chi} \in C_0^\infty(\mathbb{R})$ and $\widehat{\chi} = 1$ on $-2 \le \xi \le 2$ so $\widehat{\chi}\widehat{f} = \widehat{f}$. Then

$$\widehat{w}(t) = e^{i\omega(\xi)t}\widehat{f} = e^{i\omega(\xi)t}\widehat{f}\widehat{\chi} = \Big(e^{i\omega(\xi)t}\ \widehat{\chi}\Big)\widehat{f}$$

Taking the inverse Fourier transform yields a convolution representation

$$w = c\Big(e^{i\omega(D)t}\chi\Big) * f.$$

The proof of Step **I** shows that

$$\|e^{i\omega(D)t}\chi\|_{L^\infty} \lesssim t^{-1/2}.$$

Young's inequality yields

$$\|w(t)\|_{L^\infty} \lesssim \|e^{i\omega(D)t}\chi\|_{L^\infty}\|f\|_{L^1}$$

completing the proof. □

**Proposition 7.2.** *There is a constant $C$ so that for all $\Lambda \ge 1$ and $f \in L^1$ with*

$$\operatorname{supp}\widehat{f} \subset \left\{\frac{\Lambda}{2} \le |\xi| \le 2\Lambda\right\}$$

*one has*

$$\big\|e^{\pm i\omega(D)t}f\big\|_{L^\infty(\mathbb{R})} \le \frac{C\,\Lambda^{1/2}}{|t|^{1/2}}\big\|f\big\|_{L^1(\mathbb{R})}. \tag{7.3}$$

***Proof.*** Denote $w(t) := e^{i\omega(D)t}f$. Define $z(t,x) := v(t/(\Lambda)^{1/2}, x/\Lambda)$. Then

$$z = e^{i\omega(D)t}g, \quad \text{with} \quad g(x) := f(x/\Lambda).$$

The function $g \in L^1$ with Fourier transform supported in $1/2 \le |\xi| \le 2$ so Proposition 7.1 applies. Using that proposition to estimate $z$ yields for

$t \geq 1$,

$$\|w(t)\|_{L^\infty} = \big\|z\big((\Lambda)^{1/2}t\big)\big\|_{L^\infty} \lesssim (\Lambda t)^{-1/2}\|g\|_{L^1} = \frac{\Lambda}{(\Lambda t)^{1/2}}\|f\|_{L^1} = \frac{\Lambda^{1/2}}{t^{1/2}}\|f\|_{L^1}.$$

□

**Example 7.3.** Denote by $f = \sum f_n$ a standard Littlewood-Paley decomposition with $f_0$ supported in $|\xi| \leq 2$, and, for $n \geq 1$ $\widehat{f_n}$ supported in $\Lambda/2 < |\xi| < 2\Lambda$ with $\Lambda = 2^n$. Then

$$\|D^s f_n\|_{L^1} \lesssim 2^{ns}\|f_n\|_{L^1} \lesssim 2^{ns}\|f\|_{L^1}.$$

Applying (7.3) with $\Lambda = 2^n$ shows for $s > 1/2$, one has for $|t| \geq 1$,

$$\|e^{\pm it\omega(D)} f\|_{L^\infty} \lesssim |t|^{-1/2}\Big(\|f\|_{L^1} + \|D^s f\|_{L^1}\Big).$$

**Corollary 7.4.** *If $f \in H^s(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$ then, $\phi(x)\, e^{\pm it\omega(D)} f$ tends strongly to zero in $H^s(\mathbb{R})$ as $|t| \to \infty$.*

***Proof.*** The family of maps indexed by $t$, $f \mapsto \phi(x)e^{\pm it\omega(D)} f$, are uniformly bounded from $H^s(\mathbb{R})$ to itself. Therefore it suffices to prove the conclusion for $f$ belonging to a dense subset of $H^s(\mathbb{R})$. Take the dense set consisting of $f$ so that $\widehat{f} \in C_0^\infty(\mathbb{R} \setminus 0)$. For these data the solution $v := e^{\pm it\omega(D)} f$ is smooth and for all $j$, $\lim_{|t| \to ]\infty} \partial_x^j v \to 0$ uniformly on compact subsets of $x$ from the previous propositions applied to $\partial_x^j v$. □

**Remark 7.5.** The estimates of Propositions 7.2 and 7.1 are sufficient as inputs to yield Strichartz estimates following, for example, [5].

### 7.2. Large time asymptotics on lines $x = \underline{x} + \mathbf{v}t$

The results of the preceding section do not reveal the one way character of the equation. This is remedied by computing the large time behavior on lines $x = \underline{x} + \mathbf{v}t$ moving with velocity $\mathbf{v}$. For the rightward equation there is faster decay for $\mathbf{v} < \mathbf{0}$. The analysis is straight forward when $\widehat{f}$ is compactly supported and vanishes in a neighborhood of $\xi = 0$.

**Proposition 7.6.** *Suppose that $2 \leq k \in \mathbb{N}$ and $\widehat{f} \in C_0^k(\mathbb{R} \setminus 0)$ and that $v = e^{-it\omega(D)} f$. Define the set of active group velocities as*

$$\Gamma(f) := \big\{\omega'(\xi)\ :\ \xi \in \operatorname{supp} \widehat{f}\big\} \ \subset\subset\ ]0, \infty[. \tag{7.4}$$

**i.** *If* $\mathbf{v} \notin \Gamma(f)$ *then as* $t \to +\infty$

$$v(t, \underline{x} + \mathbf{v}t) = O(t^{-k}).$$

*The estimate is uniform for* $(\underline{x}, \mathbf{v})$ *in compact subsets of* $\mathbb{R} \times (\mathbb{R} \setminus \Gamma(f))$. *The hypothesis* $\mathbf{v} \notin \Gamma(f)$ *holds for all* $\mathbf{v} \leq 0$.

**ii.** *If* $0 < \mathbf{v} \in \Gamma(f)$ *then there is a unique* $\underline{\xi} \in \operatorname{supp} f$ *so that* $\omega'(\underline{\xi}) = \mathbf{v}$. *As* $t \to +\infty$

$$v(t, \underline{x} + \mathbf{v}t) = \frac{e^{i(\underline{\xi}\underline{x} - \omega(\underline{\xi})t)}}{(\omega''(\underline{\xi})t)^{1/2}} \, \widehat{f}(\underline{\xi}) + O(t^{-1}).$$

*The estimate is uniform for* $\underline{x}$ *in compact sets and* $\mathbf{v} \in \Gamma$.

**iii.** *Analogous results hold for* $t \to -\infty$ *and also for the leftward equation.*

**Remark 7.7. i.** Following part **ii** of Proposition 7.1 one can find a slightly slower decay applicable for $f$ that are not compactly supported but decay at infinity.

**ii.** The solution is a familiar dispersive wave packet with all group velocities strictly positive because of the compact support of $f$.

**iii.** There is a complete asymptotic expansion $\sum_{j=1}^{\infty} c_j \, t^{-j/2}$. If $\widehat{f}$ vanishes to order $k$ at $\underline{\xi}$ the decay in **ii** becomes $O(t^{-(1+k)/2})$.

***Proof.*** The solution is equal to $1/\sqrt{2\pi}$ times (6.1).

**i.** Follows from the principal of non stationary phase. The exponential term in the integral is equal to

$$e^{i(\xi(\underline{x}+\mathbf{v}t)-\omega(\xi)\mathbf{t})} = e^{i\xi\underline{x}} e^{it(\xi\mathbf{v}-\omega(\xi))}.$$

The first factor has derivatives uniformly bounded and $\mathbf{v} - \omega(\xi)$ is smooth with non vanishing derivative on the compact region of integration. The lower bound on the derivative is uniform for $x$ in compact sets and $\mathbf{v}$ bounded away from $\Gamma(f)$. The only subtlety is when $\mathbf{v} = \mathbf{0}$ where $\omega'$ approaches zero as $\xi \to \infty$. But $\xi$ is limited to $\operatorname{supp} \widehat{f}$ so the region near $\xi = \infty$ does not appear in the integral.

**ii.** Follows upon applying the stationary phase formula. $\square$

There are three singular points. At $\xi = 0$ the function $\omega(\xi)$ is Hölder 1/2 and not better. As $\xi \to \pm\infty$ the integration domain becomes non compact. The next two propositions estimate the decay of solutions when the support of $\widehat{f}$ reaches these singular points. When applying the results of this section the function $\widehat{f}$ is typically split into pieces corresponding to a neighborhood of the stationary point $\underline{\xi}$ to which part **ii** of Proposition 7.6 applies, a neighborhood of $\infty$ to which Proposition 7.10 applies, a neighborhood of 0 where the next proposition applies, and, a function to which the result of part **i** of Proposition 7.6 applies. The next result concerns the neighborhood of the origin.

**Proposition 7.8.** *Suppose that* $\widehat{f} \in C_0^2(\mathbb{R})$ *and that* $0 \in \operatorname{supp} \widehat{f}$. *Then* $\Gamma(f)$ *defined by (7.4) is of the form* $[\alpha, \infty[$ *for some* $\alpha > 0$. *If* $\mathbf{v} < \alpha$ *then* $v = e^{-it\omega(D)} f$ *satisfies*

$$v(t, \underline{x} + \mathbf{v}t) = \frac{4\widehat{f}(0)}{\sqrt{2\pi}\, t^2} + \frac{o(1)}{t^2}.$$

*The estimate is uniform for* $(\underline{x}, \mathbf{v})$ *belonging to compact subsets of* $\mathbb{R}\times]-\infty, \alpha[$.

***Proof.*** The solution is a constant multiple of (6.1). The proof is by the method of non stationary phase. The subtlety is that the phase is singular at $\xi = 0$. Write the integral as $\int_{-\infty}^0 + \int_0^\infty$ and treat first $\int_0^\infty$.

Must estimate

$$\int_0^\infty e^{i\underline{x}\xi}\ \widehat{f}(\xi)\ e^{it\phi(\xi)}\ d\xi, \qquad \phi(\xi) := \xi\mathbf{v} - \omega(\xi).$$

The hypothesis on $\mathbf{v}$ guarantees that the phase is non stationary in $\xi > 0$ so the asymptotic behavior is dominated by contributions from $\xi = 0$ where the phase is singular. Make the change of variables $\xi = y^2$. For $\xi < 0$ the appropriate change would have been $\xi = -y^2$. Use $d\xi = 2ydy$ to find,

$$\int_0^\infty a(y)\ e^{it\psi(y)} dy, \quad a(y) := e^{i\underline{x}y^2}\,\widehat{f}(y^2) 2y, \quad \psi(y) := y^2\mathbf{v} - y.$$

Note that $a(0) = 0$ and $a'(0) = 2\widehat{f}(0)$. Introduce the differential operator $L$

and its transpose $L^\dagger$

$$L := \frac{1}{it\psi'(y)}\frac{\partial}{\partial y}, \quad \text{so}, \quad Le^{it\psi} = e^{it\psi}, \quad L^\dagger v := -\frac{\partial}{\partial y}\Big(\frac{1}{it\psi'}v\Big). \tag{7.5}$$

The hypothesis on $\mathbf{v}$ implies that $\psi' \neq 0$ on the domain of integration so $L$ has smooth coefficients.

An integration by parts without boundary terms because $a(0) = 0$ yields

$$\begin{aligned}\int_0^\infty a(y)e^{it\psi(y)}\,dy &= \int_0^\infty a(y)Le^{it\psi(y)}\,dy = \int_0^\infty L^\dagger a(y)e^{it\psi}\,dy\\ &= \int_0^\infty L^\dagger a(y)Le^{it\psi}\;dy = -\int_0^\infty \frac{1}{it\psi'}L^\dagger a(y)\frac{\partial}{\partial y}e^{it\psi}\,dy.\end{aligned} \tag{7.6}$$

Integrate by parts with a nonvanishing boundary contribution to find

$$\int_0^\infty a(y)\; e^{it\psi(y)}\,dy = \Big[\frac{-e^{it\psi}}{it\psi'}L^\dagger a(y)\Big]_{y=0} + \int_0^\infty (L^\dagger)^2 a(y)e^{it\psi}\,dy. \tag{7.7}$$

The integral on the right of (7.7) is of the form

$$\frac{1}{t^2}\int_0^\infty g(y)e^{it\psi(y)}\,dy, \quad g \in L^1(]0,\infty][.$$

This is $o(1)/t^2$. Indeed, for any $\varepsilon > 0$ choose $g_\varepsilon \in C_0^\infty(]0,\infty[)$ with $\|g - g_\varepsilon\|_{L^1} < \varepsilon$. The corresponding integral with $g_\varepsilon$ is $O(t^{-\infty})$ and the difference is $\leq \varepsilon$.

Therefore

$$\int_0^\infty e^{i\underline{x}\xi}\widehat{f}(\xi)e^{it\phi(\xi)}\;d\xi = \int_0^\infty a(y)e^{it\psi(y)}\,dy = \frac{-a'(0)}{(it)^2} + \frac{o(1)}{t^2} = \frac{2\widehat{f}(0)}{t^2} + \frac{o(1)}{t^2}. \tag{7.8}$$

An entirely analogous computation shows that

$$\int_{-\infty}^0 e^{i\underline{x}\xi}\widehat{f}(\xi)e^{it\phi(\xi)}\,d\xi = \frac{2\widehat{f}(0)}{t^2} + \frac{o(1)}{t^2}. \tag{7.9}$$

In fact, changing variable to $\xi = -y^2$ with $d\xi = -2y$ yields

$$\int_{-\infty}^0 e^{i\underline{x}\xi}\,\widehat{f}(\xi)\,e^{it\phi(\xi)}\,d\xi = \int_{-\infty}^0 b(y)\,e^{it\gamma(y)}\,dy, \quad b(y) := e^{-i\underline{x}y^2} f(-y^2)(-2y),$$
$$\gamma(y) := \xi\underline{x} - \omega(\xi) = -y^2\underline{x} - y.$$

Introduce $M := (it\gamma')^{-1}\partial_y$ and its transpose $M^\dagger$ with $Me^{it\gamma} = e^{it\gamma}$. An integration by parts with vanishing boundary term yields

$$\int_{-\infty}^0 b(y)\; M^2 e^{it\gamma(y)} dy = \int_{-\infty}^0 M^\dagger b(y)\; M e^{it\gamma(y)} dy$$
$$= \int_{-\infty}^0 \frac{1}{it\gamma'}\, M^\dagger b(y)\partial_y e^{it\gamma(y)} dy.$$

A final integration by parts yields

$$\int_{-\infty}^0 e^{i\underline{x}\xi}\widehat{f}(\xi)\; e^{it\phi(\xi)} d\xi = \left[\frac{1}{it\gamma'}\; M^\dagger b(y) e^{it\gamma(y)}\right]_{y=0} + \int_{-\infty}^0 (M^\dagger)^2 b(y) e^{it\gamma(y)} dy.$$

The second term is $o(1)/t^2$ proving (7.9).

Adding (7.8) and (7.9) proves the Proposition. □

**Remark 7.9. i.** There is a complete asymptotic expansion in negative powers of $t$ if $\widehat{f} \in C_0^\infty(\mathbb{R})$.

**ii.** If $\widehat{f}$ vanishes to order $k \geq 0$ at the origin, the decay rate of $v(t, \underline{x} + \mathbf{v}t)$ is $O(1/t^{2+k})$.

**Proposition 7.10. i.** *Suppose that* $\widehat{f} \in C^k(\mathbb{R})$, $0 \notin \operatorname{supp} \widehat{f}$, *and that for all* $0 \leq j \leq k$, $\partial_\xi^j \widehat{f} \in L^1(\mathbb{R})$. *Then* $\Gamma(f) = ]0, \alpha]$ *for some* $\alpha$. *If* $\mathbf{v} \notin [0, \alpha]$, *Then as* $t \to \infty$, $v = e^{-it\omega(D)} f$ *satisfies*

$$v(t, \underline{x} + \mathbf{v}t) = O(t^{-k}).$$

*The estimate is uniform for* $\underline{x}, \mathbf{v}$ *belonging to compact subsets of* $\mathbb{R} \times (\mathbb{R} \setminus [0, \alpha])$.

**ii.** *The analogous result is valid for* $t \to -\infty$ *and for the leftward equation.*

***Proof.*** With $L$ from (7.5), the function $\sqrt{2\pi}\, v(t, \underline{x} + \mathbf{v}t)$ is equal to

$$\int e^{i\underline{x}\xi} \widehat{f}(\xi) e^{it\phi(\xi)} d\xi = \int e^{i\underline{x}\xi} \widehat{f}(\xi) L^k \Big(e^{it\phi(\xi)}\Big) d\xi$$
$$= \frac{1}{(it)^k} \int e^{i\underline{x}\xi} \widehat{f}(\xi) \Big(\frac{1}{\phi'(\xi)} \frac{\partial}{\partial \xi}\Big)^k \Big(e^{it\phi(\xi)}\Big) d\xi.$$

The coefficient $1/\phi'$ is smooth on the support of the integrand and has an expansion

$$\frac{1}{\phi'(\xi)} = \sum_{j=0}^{\infty} c_j \; \xi^{-j/2} \tag{7.10}$$

near $\xi = \infty$. There is an analogous expansion at $-\infty$. It follows that each derivative of the coefficient $1/\phi'$ is uniformly bounded on the support of the integrand. The integrability hypothesis on $\widehat{f}$ implies that one can integrate by parts to find

$$ct^k v(t, \underline{x} + \mathbf{v}t) = \int \Big( - \frac{\partial}{\partial \xi} \frac{1}{\phi'(\xi)}\Big)^k \Big[e^{i\underline{x}\xi} \; \widehat{f}(\xi)\Big] \Big(e^{it\phi(\xi)}\Big) d\xi,$$
$$\Big( - \frac{\partial}{\partial \xi} \frac{1}{\phi'(\xi)}\Big)^k \Big[e^{i\underline{x}\xi} \; \widehat{f}(\xi)\Big] \in L^1(\mathbb{R})$$

implying the desired estimate. □

## 8. Spatially Truncated One Way Equations

In our numerical algorithms, we compute approximate values of $u$ in an interval $I := [-L, L] \subset \mathbb{R}_x$. Denote by $\mathbb{1}_I(x)$ the indicator function of $I$. The algorithms march forward by computing the restriction to $I$ of $\omega(D)$ applied to the restriction of $u$ to $I$. This is equivalent to marching forward in discrete time steps approximating the equation

$$u_t + \mathbb{1}_I(x) i\omega(D) \mathbb{1}_I(x) u = 0. \tag{8.1}$$

If the initial data are supported in $I$ then the solution remains supported in that interval for all $t > 0$. The evolution equation with the discontinuous cutoff $\mathbb{1}_I$ is singular. We can prove a good existence result if this cutoff is replaced by a cutoff function $a(x)$ that is Hölder $1/2$. Rather than give that slightly harder result we prove existence for a smooth cutoff.

**Proposition 8.1.** *Suppose that* $a \in C^\infty(\mathbb{R})$ *and for each* $j$, $\partial_x^j a \in L^\infty(\mathbb{R})$. *If* $s \in \mathbb{R}$ *and* $g \in H^s(\mathbb{R})$ *then there is exactly one solution* $u \in C(\mathbb{R}\,;\, H^s(\mathbb{R}))$ *of the initial value problem*

$$u_t \;+\; a(x)\; i\,\omega(D)\; a(x)\, u = 0, \qquad u(0) = g. \tag{8.2}$$

*In addition for all* $j \in \mathbb{N}$,

$$\partial_x^j u \in C\big(\mathbb{R}\;:\; H^{s-j/2}(\mathbb{R})\big).$$

*If* $s \geq 0$, *then* $\|u(t)\|_{L^2(\mathbb{R})}$ *is independent of* $t$.

**Sketch of proof.** The essential step is to derive $H^s(\mathbb{R}^d)$ *a priori* estimates. Treat only $0 \leq s \in \mathbb{N}$. The $L^2$ conservation for a solution $u \in C(\mathbb{R}; H^{1/2}(\mathbb{R})) \cap C^1(\mathbb{R}; L^2(\mathbb{R}))$ is proved by computing

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= \frac{d}{dt}(u,u) = (u_t,u) + (u,u_t) = (-ia\omega(D)au, u) + (u, -ia\omega(Da)u) \\ &= -i\Big((\omega(D)au, au) - (au, \omega(D)au)\Big) = 0.\end{aligned}$$

The key is derivative estimates. Estimate the $L^2(\mathbb{R})$ norm of $\partial_x u$. The equation satisfied by $\partial u$ is

$$(\partial_t + i\,a\,\omega(D)\,a)\partial u = [\partial, a\,i\omega(D)a]\,u.$$

The key is that $[\partial, a\omega(D)a]$ is a bounded operator from $H^1$ to $L^2$. To prove the boundedness, write the commutator as

$$[\partial, a]\omega(D)a + a\omega(D)[\partial, a]$$

Since $[\partial, a]$ is multiplication by $a'$ it is bounded. This is the step that fails when $a = \mathbb{1}_I$.

Using the $L^2$ estimate in Duhamel's formula yields

$$\|\partial u(t)\|_{L^2} \leq \|\partial u(0)\|_{L^2} + C\int_0^t \|[\partial, a\omega(D)a]u(\sigma)\|_{L^2}\, d\sigma.$$

Insert

$$\|[\partial, a\omega(D)a]u(\sigma)\|_{L^2} \leq C\|u(\sigma)\|_{H^1}$$

to find an integral inequality

$$\|u(t)\|_{H^1} \leq \|u(0)\|_{H^1} + C \int_0^t \|u(\sigma)\|_{H^1} d\sigma.$$

Gronwall's inequality implies that

$$\|u(t)\|_{H^1} \leq e^{C\,|t|} \|u(0)\|_{H^1}.$$

One gets an $H^s(\mathbb{R})$ estimate for integer $s \geq 0$ in the same way.

Denote by $\delta^h$ is the centered finite difference approximation to $\partial_x$. Introduce the family of operators

$$\omega^h(D) := \frac{\delta^h}{|D|^{1/2}}, \quad \omega^h(\xi) = \frac{\sin(h\xi)}{h\,|\xi|^{1/2}}.$$

For $h$ fixed, the operator $\omega^h(D)$ is bounded from $H^s(\mathbb{R})$ to itself. As $h \to 0$ the operator converges to $\omega(D)$. Choose $g^h \in \mathcal{S}(\mathbb{R})$ with $g^h \to g$ in $H^s(\mathbb{R})$. A sequence of approximate solutions is defined as the solutions of

$$\partial_t u^h + a(x)\ i\,\omega^h(D)\ a(x)\,u^h = 0, \quad u^h(0) = -g^h.$$

$H^s$ estimates for $u^h$ uniform on compact time intervals and $0 < h < 1$ are proved as for the case $h = 0$. As in §2.2 of [9], passage to the limit $h \to 0$ proves existence.

For uniqueness observe that a solution in $C(\mathbb{R}\,;\, H^s(\mathbb{R}))$ is automatically $C^1(\mathbb{R}\,;\, H^{s-1/2}(\mathbb{R}))$. This regularity is sufficient to prove the $H^{s-1/2}$ *a priori* estimate and therefore uniqueness. This completes the sketch of proof. □

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