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THE FIELD OF DEFINITION FOR DYNAMICAL SYSTEMS ON $\mathbb{P}^N$

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Abstract

Let Hom_d^N be the set of morphisms $\phi : \mathbb{P}^N \to \mathbb{P}^N$ of degree d. For $f \in \mathrm{PGL}_{N+1}$, let $\phi^f = f^{-1} \circ \phi \circ f$ be the conjugation action and let $M_d^N = \mathrm{Hom}_d^N/\mathrm{PGL}_{N+1}$ be the moduli space of degree d morphisms of $\mathbb{P}^N$. A *field of definition* for $\xi \in M_d^N$ is a field over which at least one representative $\phi \in \xi$ is defined. The *field of moduli* for ξ is the fixed field of $G_\xi = \{\sigma \in \mathrm{Gal}(\bar{K}/K) : \xi^\sigma = \xi\}$. Every field of definition contains the field of moduli. In this article, we give a sufficient condition for the field of moduli to be a field of definition for morphisms with trivial stabilizer group.

1. Introduction

We begin by fixing some notation: K is a field with fixed separable closure $\overline{K}$, and $G_K = \mathrm{Gal}(\overline{K}/K)$. We write PGL_{N+1} for the projective linear group over $\overline{K}$, and $\mathrm{PGL}_{N+1}(K)$ for the G_K-stable elements of this group.

Let Hom_d^N be the set of morphisms $\mathbb{P}^N \to \mathbb{P}^N$ of (algebraic) degree d defined over $\overline{K}$, meaning that ϕ is given in coordinates by homogeneous polynomials of degree d with no common zeros. We write $\mathrm{Hom}_d^N(K)$ to indicate the subset of morphisms that can be written in each coordinate as homogeneous polynomials with coefficients in K.

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Let $\phi \in \mathrm{Hom}_d^N$. For $f \in \mathrm{PGL}_{N+1}$, let $\phi^f = f^{-1} \circ \phi \circ f$ be the conjugation action and let $M_d^N = \mathrm{Hom}_d^N / \mathrm{PGL}_{N+1}$ be the quotient space for this action. We know that M_d^N exists as a geometric quotient [5]; hence, we call M_d^N the moduli space of dynamical systems of degree d morphisms of $\mathbb{P}^N$. We will denote the conjugacy class of $\phi \in \mathrm{Hom}_d^N$ by $[\phi]$ and points in the moduli space as $\xi \in M_d^N$. We use $\phi \in \xi$ to denote a representative ϕ of a point in the moduli space ξ. In this case, we have a natural identification $\xi = [\phi]$. However, it is possible that ξ is defined over K while no representative $\phi \in \xi$ has this property, so we maintain the distinct notation.

Definition 1. Let $\phi \in \mathrm{Hom}_d^N$. A field K'/K is a *field of definition* for ϕ if $\phi^f \in \mathrm{Hom}_d^N(K')$ for some $f \in \mathrm{PGL}_{N+1}$.

Of course, a given map ϕ has many fields of definition, but we are generally interested in finding a minimal such field if it exists. However, this is not always possible. For example, in [8, Section 6] Silverman shows that a number field K is a field of definition for the map

$$\phi(z) = i\left(\frac{z-1}{z+1}\right)^3$$

if and only if -1 is a sum of two squares in K. We conclude that every field of definition is at least a quadratic extension of $\mathbb{Q}$, and there are many such fields.

We can also define a minimal field for ϕ in a Galois-theoretic sense. First note that an element $\sigma \in G_K$ acts on a map $\phi \in \mathrm{Hom}_d^N$ by acting on its coefficients. We denote this left action by ϕ^σ. Using Hilbert's Theorem 90, we see that ϕ is defined over K if and only if ϕ is fixed by every element in G_K. Furthermore, if $\psi = \phi^f$, then $\psi^\sigma = (\phi^\sigma)^{f^\sigma}$. So the action of G_K on Hom_d^N descends to an action on M_d^N. We make the following definition.

Definition 2. Let $\phi \in \mathrm{Hom}_d^N$, and define

$$G_\phi = \{\sigma \in G_K \mid \phi^\sigma \text{ is } \overline{K} \text{ equivalent to } \phi\}.$$

The *field of moduli* of ϕ is the fixed field $\overline{K}^{G_\phi}$.

In other words, the field of moduli of ϕ is the smallest field L with the property that for every $\sigma \in \mathrm{Gal}(\overline{K}/L)$ there is some $f_\sigma \in \mathrm{PGL}_{N+1}$ such that $\phi^\sigma = \phi^{f_\sigma}$. It is not hard to see that the field of moduli of ϕ must be contained in any field of definition of ϕ. A fundamental question is:

When is the field of moduli also a field of definition?

For $[\phi] = \xi \in M_d^1$, Silverman [8] proved that when the degree of ϕ is even, or when ϕ is a polynomial map of any degree, the field of moduli is also a field of definition. In the present work, we solve the higher dimensional problem for morphisms with a trivial stabilizer group.

Definition 3. For any $\phi \in \mathrm{Hom}_d^N$ define $\mathcal{A}_\phi$ to be the *stabilizer group* of ϕ, i.e.,

$$\mathcal{A}_\phi = \{f \in \mathrm{PGL}_{N+1} \mid \phi^f = \phi\}.$$

From [6], $\mathcal{A}_\phi$ is well-defined as a finite subgroup of PGL_{N+1}. For a conjugacy class $[\phi] \in M_d^N$, however, the stabilizer group is only defined up to conjugacy class in PGL_{N+1}. For most $\phi \in \mathrm{Hom}_d^N$ (all but a Zariski closed subset), $\mathcal{A}_\phi$ is trivial.

Theorem 4. *Let* $[\phi] = \xi \in M_d^N(K)$, *with* $\mathcal{A}_\phi = \{\mathrm{id}\}$. *Let* $D_n = \sum_{i=0}^N d^{ni}$. *If there is a positive integer* n *such that* $\gcd(D_n, N+1) = 1$, *then* K *is a field of definition for* ξ.

Remark. It may appear that to apply Theorem 4 we must test infinitely many cases. Reducing modulo $N+1$, however, makes clear that we need only check $\gcd(D_n, N+1)$ for $n \in [1, \varphi(N+1))$, where φ is the Euler totient function. So if we let $N = 1$, for example, then the theorem says that the field of moduli is a field of definition when there is some n such that $\gcd(1 + d^n, 2) = 1$, i.e., whenever d is even. This partially recovers Silverman's result in one dimension. In general, if $N+1$ is prime, the condition reduces to $d \not\equiv 1 \pmod{N+1}$.

While the proof of the one-dimensional result in [8] was different in the cases of trivial versus nontrivial stabilizer group, the end result was independent of $\mathcal{A}_\phi$. In higher dimensions, this is not the case. The map

$$\phi = [i(x-y)^3, (x+y)^3, z^3] \in \mathrm{Hom}_3^2$$

has field of moduli strictly smaller than any field of definition. Specifically, the field of moduli for $[\phi]$ is $\mathbb{Q}$, but $\mathbb{Q}$ is not a field of definition. The map seems to satisfy the hypotheses of Theorem 4 since $\gcd(D_1, N+1) = 1$. However, it is a simple matter to check that $\mathcal{A}_\phi \supseteq C_2$, where C_2 is the cyclic group of order two. So the hypothesis on the stabilizer group in Theorem 4 is, in fact, essential. See Section 4 for more on this and other examples.

Silverman [8] also shows that his result is the best possible by giving an example in every odd degree of a morphism with field of moduli strictly smaller than any field of definition, as follows.

Example 5. Let $\alpha(z) = i\left(\frac{z-1}{z+1}\right)^d$ with d odd. Clearly $\mathbb{Q}(i)$ is a field of definition for α. Furthermore, if we let σ represent complex conjugation, then we can check that

$$\alpha^\sigma = \alpha^f \qquad \text{for} \qquad f(z) = -\frac{1}{z}. \tag{1}$$

Hence, $\mathbb{Q}$ is the field of moduli for α. But $\mathbb{Q}$ cannot be a field of definition for α because any such field must have -1 as a sum of two squares. (See [8] for details.) Note that the computation in equation (1) depends on the fact that we chose d odd. When d is even, α and α^σ are not conjugate. In that case the field of definition and field of moduli are equal (as we know they must be); both are $\mathbb{Q}(i)$.

Outline. In Section 2, we generalize to $\mathbb{P}^N$ some results from [8] regarding morphisms on $\mathbb{P}^1$. The methods are essentially the same as in [8], with the details adjusted for arbitrary dimension. The proof of Theorem 4 occupies Section 3. In Section 4, we provide examples illustrating the necessity of both conditions in Theorem 4. The first example gives a morhpism with trivial stabilizer for which the field of moduli is not a field of definition because the morphism does not satisfy the gcd condition. The other example is a morphism which satisfies the gcd condition but for which the field of moduli is not a field of definition because the morphism has nontrivial stabilizer group.

2. First Constructions

We begin this section by briefly developing the idea of K-twists of rational maps.

Definition 6. Two maps $\phi, \psi \in \mathrm{Hom}_d^N(K)$ are *K-equivalent* if $\psi = \phi^f$ for some $f \in \mathrm{PGL}_{N+1}(K)$.

Maps that are K-equivalent have the same arithmetic dynamical behavior over K. In particular, f maps ψ-orbits of K-rational points to ϕ-orbits of K-rational points. Further, the field extension of K generated by the period n points of ϕ and ψ must agree for every $n \geq 1$. However, maps that are $\overline{K}$-equivalent but not K-equivalent may exhibit very different dynamics on $\mathbb{P}^N(K)$.

Definition 7. Let $\phi \in \mathrm{Hom}_d^N(K)$. The map $\psi \in \mathrm{Hom}_d^N(K)$ is a *K-twist* of ϕ if ψ and ϕ are $\overline{K}$-equivalent. Further, we define

$$\begin{aligned}\mathrm{Twist}_K(\phi) &= \frac{\{K\text{-twists of } \phi\}}{K\text{-equivalence}} \\ &= \{K\text{-equivalence classes of maps defined over } K \text{ and} \\ &\qquad\qquad \overline{K}\text{-equivalent to } \phi\}.\end{aligned}$$

Remark. One checks easily (or see [9, Proposition 4.73]) that if $\mathcal{A}_\phi = \{\mathrm{id}\}$, then $\#\mathrm{Twist}_K(\phi) = 1$.

Example 8. Let

$$\phi(z) = 2z + \frac{5}{z} \quad \text{and} \quad \psi(z) = \frac{z^2 - 3z}{3z - 1}.$$

One can check that

$$\phi^f(z) = \psi(z) \quad \text{where} \quad f(z) = \frac{i\sqrt{5}\,(z-1)}{1+z},$$

so these maps are $\mathbb{Q}$-twists of each other. However, they cannot be $\mathbb{Q}$-equivalent because the finite fixed points of $\psi(z)$ are rational (they are $z = -1$ and $z = 0$), but the finite fixed points of $\phi(z)$ are $\pm i\sqrt{5}$.

For $\phi, \psi \in \mathrm{Hom}_d^N(K)$, ψ is a K-twist of ϕ if and only if $[\phi] = [\psi]$ in M_d^N. Thus, the study of twists may be viewed as the study of the fibers of

$$[\cdot] : \mathrm{Hom}_d^N(K)/\mathrm{PGL}_{N+1}(K) \to M_d^N(K),$$

where we define

$$M_d^N(K) = \left\{\xi \in M_d^N : \forall \sigma \in G_K, \xi^\sigma = \xi\right\}. \tag{2}$$

In particular, the existence of a K-twist for $\phi \in \mathrm{Hom}_D^N(K)$ is equivalent to $[\cdot]$ not being injective at $[\phi]$. Additionally, the field of moduli K of $\xi = [\phi]$ is also a field of definition precisely when ξ is in the image of $[\cdot]$. So the field of moduli fails to be a field of definition where $[\cdot]$ is not surjective.

Proposition 9. *Let $\xi \in M_d^N(K)$ be a dynamical system whose field of moduli is contained in K, and let $\phi \in \xi$ be any representative of ξ.*

(a) *For every $\sigma \in G_K$, there exists an $f_\sigma \in \mathrm{PGL}_{N+1}$ such that*

$$\phi^\sigma = \phi^{f_\sigma}.$$

The map f_σ is determined by ϕ up to (left) multiplication by an element of $\mathcal{A}_\phi$.

(b) *Having chosen f_σ's as in (a), the resulting map $f : G_K \to \mathrm{PGL}_{N+1}$ satisfies*

$$f_\sigma f_\tau^\sigma f_{\sigma\tau}^{-1} \in \mathcal{A}_\phi \quad \text{for all } \sigma, \tau \in G_K.$$

We say that f is a G_K-to-PGL_{N+1} cocycle relative to $\mathcal{A}_\phi$.

(c) *Let $\Phi \in \xi$ be any other representative of ξ, and for each $\sigma \in G_K$ choose an automorphism $F_\sigma \in \mathrm{PGL}_{N+1}$ as in (a) so that $\Phi^\sigma = \Phi^{F_\sigma}$. Then there exists a $g \in \mathrm{PGL}_{N+1}$ such that*

$$g^{-1} F_\sigma g^\sigma f_\sigma^{-1} \in \mathcal{A}_\phi \quad \text{for all } \sigma \in G_K. \tag{3}$$

We say that f and F are G_K-to-PGL_{N+1} cohomologous relative to $\mathcal{A}_\phi$.

(d) *The field K is a field of definition for ξ if and only if there exists a $g \in \mathrm{PGL}_{N+1}$ such that*

$$g^{-1} g^\sigma f_\sigma^{-1} \in \mathcal{A}_\phi \quad \text{for all } \sigma \in G_K.$$

Remark. The conclusions of Proposition 9 hold for any $\mathcal{A}_\xi$. However, if $\mathcal{A}_\xi = \{\mathrm{id}\}$, the proposition says that the map $f : G_K \to \mathrm{PGL}_{N+1}$ is a one-cocycle whose cohomology class in the set $H^1(G_K, \mathrm{PGL}_{N+1})$ depends only on ξ. On the other hand, if $\mathcal{A}_\xi \neq \{\mathrm{id}\}$, then the criterion in equation (3) is not an equivalence relation, so we cannot even define a "cohomology set relative to $\mathcal{A}_\phi$."

Proof.

(a) Fix $\sigma \in G_K$. By definition of $M_d^N(K)$ in (2), there exists an f_σ such that

$$\phi^\sigma = \phi^{f_\sigma}.$$

Suppose that $g_\sigma \in \mathrm{PGL}_{N+1}$ has the same property. Then

$$\phi^{g_\sigma} = \phi^\sigma = \phi^{f_\sigma},$$

and consequently

$$g_\sigma f_\sigma^{-1} \in \mathcal{A}_\phi.$$

(b) Let $\sigma, \tau \in G_K$. We compute

$$\phi^{f_{\sigma\tau}} = \phi^{\sigma\tau} = (\phi^\tau)^\sigma = (\phi^{f_\tau})^\sigma = (\phi^\sigma)^{f_\tau^\sigma} = \phi^{f_\sigma f_\tau^\sigma}.$$

Hence, $f_\sigma f_\tau^\sigma f_{\sigma\tau}^{-1} \in \mathcal{A}_\phi$.

(c) Since $[\phi] = \xi = [\Phi]$, we can find some $g \in \mathrm{PGL}_{N+1}$ so that $\phi = \Phi^g$. Then for any $\sigma \in G_K$ we compute

$$\phi^{f_\sigma} = \phi^\sigma = (\Phi^g)^\sigma = (\Phi^\sigma)^{g^\sigma} = (\Phi^{F_\sigma})^{g^\sigma} = \phi^{g^{-1}F_\sigma g^\sigma}.$$

Hence, $g^{-1}F_\sigma g^\sigma f_\sigma^{-1} \in \mathcal{A}_\phi$.

(d) Suppose first that K is a field of definition for ξ, so there is a map $\Phi \in \xi$ which is defined over K. Then $\Phi^\sigma = \Phi$ for all $\sigma \in G_K$, so in (c) we let F_σ be 1, which gives the desired result.

Now suppose there is a $g \in \mathrm{PGL}_{N+1}$ such that $g^{-1}g^\sigma f_\sigma^{-1} \in \mathcal{A}_\phi$ for all $\sigma \in G_K$. Let $h = g^{-1}g^\sigma f_\sigma^{-1} \in \mathcal{A}_\phi$ and let $\Phi = \phi^{g^{-1}} \in \xi$. Note that $h^{-1} \in \mathcal{A}_\phi$ and $h^{-1}g^{-1} = f_\sigma(g^{-1})^\sigma$. Then

$$\Phi^\sigma = \left(\phi^{g^{-1}}\right)^\sigma = (\phi^\sigma)^{(g^{-1})^\sigma} = \phi^{f_\sigma(g^{-1})^\sigma} = \phi^{h^{-1}g^{-1}} = \phi^{g^{-1}} = \Phi.$$

Hence, Φ is defined over K, so K is a field of definition for ξ. □

Definition 10. A scheme X/K is called a *Brauer-Severi variety* of dimension N if there exists a finite, separable field extension L/K such that $X \otimes_K \mathrm{Spec} L$ is isomorphic to $\mathbb{P}_L^N$. Two Brauer-Severi varieties are considered *equivalent* if they are isomorphic over K.

Proposition 11.

(a) *There is a one-to-one correspondence between the set of Brauer-Severi varieties of dimension N up to equivalence and the cohomology set $H^1(G_K, \mathrm{PGL}_{N+1})$. This correspondence is defined as follows: Let X/K be a Brauer-Severi variety of dimension N and choose a $\overline{K}$-isomorphism $j : \mathbb{P}^N \to X$. Then the associated cohomology class $c_X \in H^1(G_k, \mathrm{PGL}_{N+1})$ is given by the cocycle*

$$G_K \to \mathrm{PGL}_{N+1}, \quad \sigma \mapsto j^{-1} \circ j^{\sigma}.$$

(b) *The following conditions are equivalent:*
   (i) *X is K-isomorphic to $\mathbb{P}^N$.*
   (ii) *$X(K) \neq \emptyset$.*
   (iii) *$c_X = 1$.*

Proof. Brauer-Severi varieties are discussed by Serre [7] and in more detail by Jahnel [4]. In particular, (a) is discussed in [7, X.6] and the equivalence of (i) and (iii) follow from (a). The equivalence of (i) and (ii) is [7, Exercise X.6.1] and also [4, Proposition 4.8]. □

Definition 12. The *Brauer group* of a field K is given by

$$\mathrm{Br}(K) \cong H^2(G_K, \overline{K}^*).$$

Proposition 13. *Let X/K be a Brauer-Severi variety of dimension N. If there is a zero-cycle D on X such that D is defined over K and* $\deg(D)$ *is relatively prime to $N+1$, then $X(K) \neq \emptyset$.*

Proof. We consider the two exact sequences

$$1 \to \mu_{N+1} \to \mathrm{SL}_{N+1}(\overline{K}) \to \mathrm{PGL}_{N+1}(\overline{K}) \to 1$$

and

$$1 \to \mu_{N+1} \to \overline{K}^* \xrightarrow{z \to z^{N+1}} \overline{K}^* \to 1.$$

We take their cohomology to determine maps

$$0 \to H^1(G_K, \mathrm{PGL}_{N+1}) \to H^2(G_K, \mu_{N+1}), \text{ and}$$
$$0 \to H^2(G_K, \mu_{N+1}) \to H^2(G_K, \overline{K}^*) \xrightarrow{z \to z^{N+1}} H^2(G_K, \overline{K}^*) \to \cdots$$

Note: We use the facts that $H^1(G_K, SL_{N+1})$ is trivial by [7, Chapter X, Corollary to Proposition 3] and $H^1(G_K, \overline{K}^*)$ is trivial by Hilbert's Theorem 90. Hence,

$$\begin{aligned} H^1(G_K, \mathrm{PGL}_{N+1}) &\hookrightarrow H^2(G_K, \mu_{N+1}) \hookrightarrow H^2(G_K, \overline{K}^*) \\ &\cong \mathrm{Br}(K)[N+1] \subset \mathrm{Br}(K). \end{aligned}$$

Let c_X be the cohomology class associated to X as in Proposition 11. Choose a Galois extension M/K such that $X(M) \neq \emptyset$. Then by Proposition 11(b), the restriction of c_X to $H^1(G_M, \mathrm{PGL}_{N+1})$ is trivial, so c_X comes from an element in $H^1(G_{M/K}, \mathrm{PGL}_{N+1}(M))$, which we will also denote by c_X.

For any prime p relatively prime to $N+1$, let G_p denote the p-Sylow subgroup of $G_{M/K}$, and let $M_p = M^{G_p}$ be the fixed field of G_p. Taking appropriate inflation and restriction maps gives the following commutative diagram:

$$\begin{array}{ccccc} H^1(G_K, \mathrm{PGL}_{N+1}) & \xhookrightarrow{1-1} & H^2(G_K, \mu_{N+1}) & \xrightarrow{\sim} & \mathrm{Br}(K)[N+1] \subset \mathrm{Br}(K)\,. \\ \big\uparrow{\scriptstyle \mathrm{Inf}} & & & & {\scriptstyle \mathrm{Res}}\big\downarrow \\ H^1(G_{M/K}, \mathrm{PGL}_{N+1}(M)) & & & & \mathrm{Br}(M_p) \\ \big\downarrow{\scriptstyle \mathrm{Res}} & & & & {\scriptstyle \mathrm{Inf}}\big\uparrow{\scriptstyle 1-1} \\ H^1(G_p, \mathrm{PGL}_{N+1}(M)) & & \xhookrightarrow{\qquad 1-1 \qquad} & & H^2(G_p, M^*) \end{array}$$

The bottom row comes from taking G_p cohomology of the exact sequence

$$1 \to M^* \to \mathrm{GL}_{N+1} \to \mathrm{PGL}_{N+1}(M) \to 1$$

and using the fact that $H^1(G_p, \mathrm{GL}_{N+1}) = 1$ from [7, X.Proposition 3].

If we start with $c_X \in H^1(G_{M/K}, \mathrm{PGL}_{N+1}(M))$ and trace it around the diagram to $\mathrm{Br}(M_p)$, we find that it has order dividing $N+1$ since it maps through $\mathrm{Br}(K)[N+1]$, and it has order a power of p since it maps through $H^1(G_p, M^*)$. Hence, the image of c_X in $\mathrm{Br}(M_p)$ is 0. The injectivity of the maps along the bottom and up the right-hand side shows that $\mathrm{Res}(c_X) = 0$ in $H^1(G_p, \mathrm{PGL}_{M_p})$. So by Proposition 11(b), we have $X(M_p) \neq \emptyset$.

Let $P \in X(M_p)$ and let $P_1, \ldots, P_r$ be the complete set of M_p/K conjugates of P. Then r is prime to p because it divides $[M_p : K]$, and the zero-cycle $(P_1) + \cdots + (P_r)$ is defined over K. In other words, there is a zero-cycle D_p on X defined over K whose degree is prime to p.

So for all primes p relatively prime to $(N+1)$, we have a zero-cycle D_p with degree prime to p. Hence, the greatest common divisor of the set

$$\{\deg(D_p) \mid (p, N+1) = 1\}$$

is a product of divisors of $(N+1)$.

Given a zero-cycle D defined over K and with $\deg(D)$ relatively prime to $N+1$, we can find a finite linear combination

$$E = nD + \sum_{(p,N+1)=1} n_p D_p \qquad \text{with } \deg(E) = 1.$$

Since E is defined over K, we have found a zero-cycle of degree 1 that is defined over K. Applying classical results on homogeneous spaces (or [1, Theorem 0.3]), we have $X(K) \neq \emptyset$. □

3. Proof of the Main Theorem

In this section, we focus on dynamical systems ξ with trivial stabilizer; that is, we assume that $\mathcal{A}_\phi = \{\text{id}\}$ for all $\phi \in \xi$.

Proposition 14. *Let $\xi \in M_d^N(K)$ be a dynamical system with trivial stabilizer.*

(a) *There is a cohomology class $c_\xi \in H^1(G_K, \mathrm{PGL}_{N+1})$ such that for any $\phi \in \xi$ there is a one-cycle $f : G_K \to \mathrm{PGL}_{N+1}$ in the class of c_ξ so that*

$$\phi^\sigma = \phi^{f_\sigma} \quad \text{for all } \sigma \in G_K.$$

(b) *Let X_ξ/K be the Brauer-Severi variety associated to the cohomology class c_ξ. Then for any $\phi \in \xi$ there exists an isomorphism $i : \mathbb{P}^N \to X_\xi$ defined over $\overline{K}$ and a morphism $\Phi : X_\xi \to X_\xi$ defined over K so that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{P}^N & \xrightarrow{\phi_{/\overline{K}}} & \mathbb{P}^N \\ \wr\big\downarrow i_{/\overline{K}} & & \wr\big\downarrow i_{/\overline{K}} \\ X_\xi & \xrightarrow{\Phi_{/K}} & X_\xi. \end{array} \tag{4}$$

(c) *The following are equivalent.*

(i) *K is a field of definition for ξ.*

(ii) *$X_\xi(K) \neq \emptyset$.*

(iii) *$c_\xi = 1$.*

Proof. (a) Since $\mathcal{A}_\phi = \{\text{id}\}$, the result follows immediately from the following:

- Proposition 9(a) says that ϕ determines $f : G_K \to \text{PGL}_{N+1}$.
- By Proposition 9(b) f is a one cocycle.
- By Proposition 9(c), any other choice of $\phi \in \xi$ gives a cohomologous cocycle.

(b) Let $j : \mathbb{P}^N \to X_\xi$ be a $\overline{K}$-isomorphism, so c_ξ is the cohomology class associated to the cocycle $\sigma \mapsto j^{-1}j^\sigma$. However, from (a) we know that c_ξ is associated to the cocycle $\sigma \mapsto f_\sigma$. Thus, these two cocycles must be cohomologous, meaning there is an element $g \in \text{PGL}_{N+1}$ so that

$$f_\sigma = g^{-1}(j^{-1}j^\sigma)g^\sigma \quad \text{for all } \sigma \in G_K.$$

Define $i = jg$, and $\Phi = i\phi i^{-1}$. Then diagram (4) commutes, and it only remains to check that Φ is defined over K. For any $\sigma \in G_K$, we compute

$$\begin{aligned} \Phi^\sigma &= i^\sigma \phi^\sigma (i^{-1})^\sigma = (j^\sigma g^\sigma)(f_\sigma^{-1}\phi f_\sigma)(j^\sigma g^\sigma)^{-1} \\ &= (j^\sigma g^\sigma)(g^{-1}j^{-1}j^\sigma g^\sigma)^{-1}\phi(g^{-1}j^{-1}j^\sigma g^\sigma)(j^\sigma g^\sigma)^{-1} \\ &= jg\phi g^{-1}j^{-1} = i\phi i^{-1} \\ &= \Phi. \end{aligned}$$

(c) The equivalence of (ii) and (iii) was already proven in Proposition 11(b). The equivalence of (i) and (iii) follows from Proposition 9(d) when $\mathcal{A}_\phi = \{\text{id}\}$. ☐

We now prove Theorem 4.

***Proof of Theorem* 4.** Take any $\phi \in \xi$ and choose $i : \mathbb{P}^N \to X_\xi$ and $\Phi : X_\xi \to X_\xi$ as in Proposition 14(b) so that the diagram in (4) commutes.

Let $\Gamma_n \subseteq \mathbb{P}^N \times \mathbb{P}^N$ be the graph of Φ^n and let Δ be the diagonal map on X_ξ. Define

$$\mathrm{Per}_n(\Phi) = \Delta^*(\Gamma_n),$$

which denotes the set of periodic points of period n of Φ taken with multiplicity.

The zero-cycle $\mathrm{Per}_n(\Phi)$ has degree D_n (see [3, Proposition 4.17]) and is defined over K. Hence, X_ξ has a zero-cycle of degree relatively prime to $N+1$. Thus, from Proposition 11, $X_\xi(K) \neq \emptyset$ and by Proposition 14(c) K is a field of definition for ξ. ☐

4. Examples

In this section, we demonstrate that both hypotheses in Theorem 4 are necessary. First, we give a map where $\gcd(D_n, N+1) > 1$ for all positive integers n, and we show that the field of moduli is not a field of definition. Then, we give a map where $\gcd(D_1, N+1) = 1$ but the stabilizer group is nontrivial, and we show that the field of moduli is not a field of definition.

The second example is particularly interesting because it demonstrates a striking difference between Hom_d^1 and Hom_d^N for $N > 1$. In dimension 1, the *condition* for the field of moduli to be a field of definition is independent of the stabilizer group, but the proof is more complicated in the case of nontrivial stabilizer. Our example shows that in higher dimensions, the condition (not just the proof) must, in fact, be different for maps with nontrivial stabilizer.

4.1. Example A

We construct a map $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ with trivial stabilizer group whose field of moduli is $\mathbb{Q}$ but for which $\mathbb{Q}$ is not a field of definition. This is explained by the fact that $\gcd(D_n, N+1) = \gcd(D_n, 3) > 1$ for all n.

Consider the map

$$\phi = [(x - iz)^4, (y + iz)^4, z^4] : \mathbb{P}^2 \to \mathbb{P}^2.$$

Claim 1: ϕ has trivial stabilizer.

Proof. It is clear that any element $f \in \mathcal{A}_\phi$ must be of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then by direct computation we can see there are no solutions to

$$\phi^f = \phi. \qquad \square$$

Claim 2: ϕ has field of moduli $\mathbb{Q}$.

Proof. Notice that ϕ is defined over $\mathbb{Q}(i)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(i))$ represent complex conjugation. If we can exhibit an f such that $\phi^f = \phi^\sigma$, then we have that the field of moduli is $\mathbb{Q}$. The element

$$f = \begin{pmatrix} 0 & \zeta_6 & 0 \\ \zeta_6 & 0 & 0 \\ 0 & 0 & \zeta_6 \end{pmatrix}, \tag{5}$$

where ζ_6 is a primitive 6^{th} root of unity, has the necessary property. $\square$

Claim 3: $\mathbb{Q}$ is not a field of definition for $[\phi]$.

Proof. From Proposition 9(d) and the fact that $\mathcal{A}_\phi = \{\mathrm{id}\}$, we know that $\mathbb{Q}$ is a field of definition for $[\phi]$ if and only if there exists a $g \in \mathrm{PGL}_{N+1}$ such that

$$g^\sigma f_\sigma^{-1} = g \qquad \forall \sigma \in \mathrm{Gal}(\overline{K}/K).$$

In that case, we have $\phi^{g^{-1}}$ defined over $\mathbb{Q}$.

Since $\mathbb{Q}(i)$ is a field of definition for ϕ, we only need to consider σ to be complex conjugation, so the associated f_σ is given by (5). We now show that if g satisfies $g^\sigma f_\sigma^{-1} = g$, then necessarily $g \in \mathrm{PGL}_3(\mathbb{Q}(i)) \setminus \mathrm{PGL}_3(\mathbb{Q})$.

Recall we are assuming that g^{-1} conjugates ϕ to some $\psi \in \mathrm{Hom}_4^2(\mathbb{Q})$. Let K be the Galois closure of the field of definition of g, and let $G = \mathrm{Gal}(K/\mathbb{Q})$. Write $H = \mathrm{Gal}(K/\mathbb{Q}(i)) \trianglelefteq G$. For $\sigma \in H$, we have

$$\psi = g \circ \phi \circ g^{-1}, \text{ so}$$
$$\psi^\sigma = g^\sigma \circ \phi^\sigma \circ (g^{-1})^\sigma = g^\sigma \circ \phi \circ (g^{-1})^\sigma, \text{ since } \phi \in \mathrm{Hom}_4^2(\mathbb{Q}(i)).$$

Since we assume ψ is defined over $\mathbb{Q}$, $\psi^\sigma = \psi$ and we have $\{(g^{-1})^\sigma : \sigma \in H\}$ is contained in the conjugating set

$$\mathrm{Conj}_{\phi,\psi} = \{f \in \mathrm{PGL}_3 \colon \phi^f = \psi\}.$$

But when $\mathrm{Conj}_{\phi,\psi}$ is non-empty, it is a principle homogeneous space for $\mathcal{A}_\phi$ (see [2, Section 3]). Since $\mathcal{A}_\phi$ is trivial, we conclude that $\#\mathrm{Conj}_{\phi,\psi} \leq 1$. In other words, $(g^{-1})^\sigma = g^{-1}$ for all $\sigma \in H$, meaning $g^{-1} \in \mathrm{PGL}_3(\mathbb{Q}(i))$. But clearly $g^{-1} \notin \mathrm{PGL}_3(\mathbb{Q})$ since that would imply $\phi = \psi^g$ is defined over $\mathbb{Q}$ as well.

Now that we know that $g \in \mathrm{PGL}_3(\mathbb{Q}(i))$, we can simply set up a system of equations with rational coefficients from

$$g^\sigma f_\sigma^{-1} = g$$

by equating each entry in the matrices. The resulting system has solutions only of the form

$$\begin{pmatrix} 0 & 0 & a - ai \\ 0 & 0 & b - bi \\ 0 & 0 & c - ci \end{pmatrix}$$

which are clearly not elements of PGL_3. Thus, the necessary g does not exist and $\mathbb{Q}$ is not a field of definition. □

Claim 4: $\gcd(D_n, N+1) > 1$ for all n.

Proof. We have $N = 2$, so the only possible divisors of $N+1$ are $1, 3$. With $d = 4$ we compute

$$D_n = 1 + 4^n + 4^{2n} \equiv 1 + 1 + 1 \equiv 0 \pmod 3.$$ □

We see that ϕ is a morphism of $\mathbb{P}^2$ with trivial stabilizer; but since $\gcd(D_n, N+1) > 1$ for all n, the conclusions of Theorem 4 do not apply.

4.2. Example B

We construct a map $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ with $\gcd(D_1, N+1) = \gcd(D_1, 3) = 1$, whose field of moduli is $\mathbb{Q}$ but for which $\mathbb{Q}$ is not a field of definition. This is explained by the fact that $\mathcal{A}_\phi$ is nontrivial.

Consider the map

$$\phi = [i(x-y)^3, (x+y)^3, z^3] : \mathbb{P}^2 \to \mathbb{P}^2.$$

Note that $D_1 = 1 + 3 + 3^2 = 13$, so indeed $\gcd(D_1, N+1) = 1$.

Claim 1: ϕ has field of moduli $\mathbb{Q}$.

Proof. Notice that ϕ is defined over $\mathbb{Q}(i)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(i))$ represent complex conjugation. If we can exhibit an f such that $\phi^f = \phi^\sigma$, then we have that the field of moduli is $\mathbb{Q}$. A calculation shows that

$$f = \begin{pmatrix} 0 & -\zeta_8 & 0 \\ \zeta_8 & 0 & 0 \\ 0 & 0 & -\zeta_8^2 \end{pmatrix}$$

has this property. Here ζ_8 is a primitive eighth root of unity. □

Claim 2: $\mathbb{Q}$ is not a field of definition.

Proof. If ψ is conjugate to this map by $g \in \mathrm{PGL}_3$, then we may assume that g fixes the line $L : \{z = 0\}$. Moreover, the image $g(L)$ must be totally ramified and fixed by ψ, and it is the unique line with these properties. Hence, $g(L)$ must be stable under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (i.e., defined over $\mathbb{Q}$). So, conjugating by an element of $\mathrm{PGL}_3(\mathbb{Q})$ and renaming g appropriately, we may assume that $g(L) = L$.

Now, restricting ϕ to the first two coordinates gives a morphism of $\mathbb{P}^1$ defined by

$$\alpha(z) = i\frac{(z-1)^3}{(z+1)^3}.$$

If ϕ has field of definition $\mathbb{Q}$ and we take the conjugating map g to fix L, then restricting $\psi = \phi^g$ to the first two coordinates yields a map conjugate to α and defined over $\mathbb{Q}$. But Silverman shows [8] that $\mathbb{Q}$ is not a field of definition for α. So $\mathbb{Q}$ is not a field of definition for ϕ. $\square$

Claim 3: ϕ has nontrivial stabilizer.

Proof. Let

$$g = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is a simple matter to check that $\phi^g = \phi$, so $g \in \mathcal{A}_\phi$. $\square$

In fact, this argument shows that

$$\phi = [i(x-y)^d, (x+y)^d, z^d] : \mathbb{P}^2 \to \mathbb{P}^2$$

has field of moduli $\mathbb{Q}$ but field of definition at least quadratic over $\mathbb{Q}$ whenever d is odd.

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