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CONNECTING ORBITS FOR SUBHARMONIC SOLUTIONS IN TIME REVERSIBLE HAMILTONIAN SYSTEMS

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Abstract

This note deals with subharmonic solutions of time reversible Hamiltonian systems. Based on variational approach, a large number of subharmonic solutions can be found out by using penalization arguments. As a further consequence, there exist connecting orbits joining with pairs of subharmonic solutions which have different primitive periods.

1. Introduction

The equation

$$\ddot{q} + \sin q = 0 \tag{1.1}$$

describes the motion of an unforced pendulum in ideal situation. If the pendulum is forced via a support which is moving vertically, then the motion is governed by

$$\ddot{q} + (1 + \ddot{H}(t)) \sin q = 0, \tag{1.2}$$

where $H(t)$ is the vertical displacement of the support at time t.

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With (1.2) as a model equation, we investigate the subharmonic solutions and their connecting orbits for the second order periodic Hamiltonian system

$$\ddot{q} - V'(t,q) = 0, \tag{1.3}$$

where $q : \mathbb{R} \to \mathbb{R}^n$, $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $V'(t,y) = D_y V(t,y)$. For $n > 1$, (1.3) can be viewed as a simple model for the n-pendulum problem with appropriate forcing.

The potential function V is assumed to satisfy the following conditions:

(V1) There is a set $\mathcal{K}_e \subset \mathbb{R}^n$ such that if $\eta \in \mathcal{K}_e$ then $V(t,\eta) = \inf_{y \in \mathbb{R}^n} V(t,y) = V_0$ for all $t \in \mathbb{R}$.

(V2) There are positive numbers μ_1, μ_2 and ρ_0 such that if $|y - \eta| \le \rho_0$ for some $\eta \in \mathcal{K}_e$ then $\mu_2 |y-\eta|^2 \ge V(t,y) - V_0 \ge \mu_1 |y - \eta|^2$ for all $t \in \mathbb{R}$. Moreover, if $\eta_i, \eta_j \in \mathcal{K}_e$ and $i \ne j$, then $|\eta_i - \eta_j| > 8\rho_0$.

(V3) There is a $\mu_0 > 0$ such that if $V(t,y) \le V_0 + \mu_0$ for some $t \in \mathbb{R}$ then $|y - \eta| \le \rho_0$ for some $\eta \in \mathcal{K}_e$.

(V4) V is T-periodic in t.

(V5) $V(t,y) = V(-t,y)$ for all $t \in \mathbb{R}$, $y \in \mathbb{R}^n$.

(V6) V is T_i-periodic in y_i, $1 \le i \le n$, or $\mathcal{K}_e$ contains at least two elements.

In particular (1.3) is a time reversible Hamiltonian system if (V5) is satisfied.

By (V1), $\mathcal{K}_e$ contains a set of equilibria of (1.3). Phase plane analysis shows that (1.1) has a heteroclinic orbit joining the adjacent minima of the potential. Equation (1.1) also possesses a large number of non-constant periodic solutions. However, there is no connecting orbits for such periodic solutions.

In [39], Strobel used variational methods to study the connecting orbits of (1.3). He showed that, for any $\eta_i \in \mathcal{K}_e$, there is a heteroclinic orbit q of (1.3) which satisfies

$$q(t) \to \eta_1 \text{ as } t \to -\infty$$

and

$$q(t) \to \mathcal{K}_e \backslash \{\eta_1\} \text{ as } t \to \infty.$$

Such a solution is referred to as a basic heteroclinic orbit. Moreover for any pair of $\eta_i, \eta_j \in \mathcal{K}_e$, they can be joined by a chain of basic heteroclinics. If additional nondegeneracy conditions are satisfied, Strobel showed that there exist infinitely many multibump heteroclinic orbits originating at η_i and terminating at η_j.

To study the existence of non-constant periodic solutions of (1.3), we may assume $V_0 = 0$, since the potential V is only determined up to an additive constant. Let $E' = W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^n)$. For $m \in \mathbb{N}$, set $E_m = \{z \in E' | z(t + mT) = z(t)\}$ and

$$\hat{I}_m(z) = \int_0^{mT} \mathcal{L}(z)dt, \tag{1.4}$$

where $\mathcal{L}(q) = \frac{1}{2}|\dot{q}|^2 + V(t, q)$, the Lagrangian associated with (1.3). It is known that a critical point of $\hat{I}_m$ is a periodic solution of (1.3). A periodic solution with minimal (i.e., primitive) period mT, $m > 1$, is called a subharmonic solution. Clearly $\mathcal{K}_e$ consists of all the global minimizers of $\hat{I}_m$. For $m = 1$, it has been shown [11] that there exist non-constant periodic solutions of (1.3). Our aim is to extend this idea to study subharmonic solutions. A simple example is $V(t, y) = F_\varepsilon(t)W(y)$, where F is a positive non-constant periodic function and $F_\varepsilon(t) = F(\varepsilon t)$ with $0 < \varepsilon << 1$ so that $F_\varepsilon(t)$ oscillates slowly between a maximum and a minimum.

Consider adding a penalty function ψ_m to $\hat{I}_m$; that is, set

$$I_m(z) = \hat{I}_m(z) + \int_0^{mT} \psi_m(t, z(t))dt \tag{1.5}$$

for $z \in E_m$. If ψ_m is suitably chosen, a global minimizer of I_m actually coincides with a local minimizer of $\hat{I}_m$. Moreover by means of adding different constraints through minimization yields a large number of subharmonic solutions.

Theorem 1. *Suppose that $V(t, y) = F_\varepsilon(t)W(y)$ and F is a positive non-constant periodic function. If ε is sufficiently small then* (1.3) *possesses infinitely many subharmonic solutions.*

The existence of connecting orbits of Hamiltonian systems has been extensively investigated [2, 9, 10, 13, 14, 15, 17, 23, 24, 28, 29, 31, 34, 35, 36, 37, 38, 39]. Rabinowitz [34] considered a class of periodic Hamiltonian systems, where a family $\mathcal{K}$ of $T-$periodic solutions were obtained as the global minimizers of $\hat{I}_1$. He showed that, if $\mathcal{K}$ consists of isolated points, then for any periodic solution $p \in \mathcal{K}$, there is a heteroclinic orbit connecting p to an element of $\mathcal{K}\backslash\{p\}$. For a time reversible periodically forced pendulum equation, Calanchi and Serra [12] studied the heteroclinic orbit connecting two consecutive periodic solutions at minimal energy level. Under a gap condition between such two periodic solutions, they showed that there exists a two-bump homoclinic solution.

Let $\mathcal{K}_m = \{p \in E_m | I_m(p) = \inf_{z \in E_m} I_m(z)\}$. The next theorem gives a heteroclinic orbit joining with two periodic solutions which have different primitive periods. For simplicity in statement, we deal with the same hypotheses as in Theorem 1. More general situation will be treated in Theorem 3.

Theorem 2. *If $\mathcal{K}_m$ and $\mathcal{K}_j$ consist of isolated points, then there is a solution q of* (1.3) *which satisfies*

$$q(t) \ \to \ p_1(t) \ \textit{uniformly as } t \to \infty \tag{1.6}$$

and

$$q(t) \ \to \ p_2(t) \ \textit{uniformly as } t \to -\infty \tag{1.7}$$

for some $p_1 \in \mathcal{K}_m$ and $p_2 \in \mathcal{K}_j$.

Let us remark that, for $n = 1$, Theorem 2 still hods even if $\mathcal{K}_m$ and $\mathcal{K}_j$ are not isolated sets. In fact all the periodic solutions belonging to $\mathcal{K}_m$ live in a region where the penalization vanishes.

There are considerable works [5, 7, 8, 14, 15, 16, 18, 23, 24, 29, 37, 39] devoted to the investigation of multibump solutions of differential equations. Roughly speaking, a multibump solution consists of a number of one bump solutions nicely concatenated. The argument that Strobel used to prove multibump solutions is in the spirit of delicate variational deformation methods developed by Séré [37] and the others (see e.g. [14, 36]). A key requirement for the construction of multibump solutions is that the one bump solutions satisfy certain nondegeneracy conditions. This hypothesis plays the role in variational settings of the classical transversality conditions used in the study of analogous questions for dynamical systems. Namely the standard condition there is that the stable and unstable manifolds through an equilibrium point for the Poincaré map associated with a dynamical system intersect transversally at a homoclinic point. For a given potential V, it is no easy matter to verify if such a nondegeneracy condition or the classical transversality hypothesis holds. With the aid of penalization method, we obtain multibump connecting orbits joining with two subharmonic solutions.

The study of chaoticity of dynamical systems goes back to Poincaré. A classical result of Smale and Birkhoff gives a precise description of the chaotic behavior of dynamics generated by a map having a transversal homoclinic point to a hyperbolic equilibrium (see e.g. [20]). A tool to apply the Smale-Birkhoff theory to continuous flows generated by differential equations is Melnikov method [30]. Recent advance in calculus of variations provides another way to detect chaotic dynamics of Hamiltonian systems [8, 14, 15, 23, 24, 37, 38, 39]. For the slowly perturbed pendulum equation, if we identify $\eta = (2n+1)\pi$, $n \in \mathbb{Z}$, as one point, then a result corresponding to Theorem 1.3 of [5] is that there exists a trajectory $q(t)$ homoclinic to η, and "near" the bumps of $q(t)$ there exists a periodic solution. Here a generalized version for the chaotic behavior of dynamics is that the basic role played by a equilibrium point can be further extended to that of a periodic or subharmonic solution. By taking a suitable penalty function, we find a subharmonic solution p whose trajectory can be close to each one of p_i for a certain amount of time. Furthermore, for a given p_i, the time at which the trajectory of p is close to p_i can be larger than any prescribed number. As been known that the restriction of a Poincaré map to a compact invariant set is semiconjugate to the Bernoulli shift with two symbols, which results in positivity of the topological entropy [25].

In addition to Theorem 1, a number of results for the subharmonic solutions are given in Section 2-3. Section 4 aims at the connecting orbits for subharmonic solutions. As the ides of penalization method has been illustrated in [11], we only sketch the proofs to demonstrate further applications.

2. Subharmonic Solutions

In this section, we study the existence of subharmonic solutions of (1.3). For $\eta_1, \eta_2 \in \mathcal{K}_e$ and $j_1 < j_2$, let

$$\hat{E}(j_1, j_2) = \{z \in W^{1,2}([j_1, j_2], \mathbb{R}^n) | z(j_1) = \eta_1 \text{ and } z(j_2) = \eta_2\}$$

and

$$\hat{\alpha}(j_1, j_2) = \inf_{z \in \hat{E}(j_1, j_2)} I(z),$$

where

$$I(z) = \int_{j_1}^{j_2} \mathcal{L}(z) dt.$$

A cross from one equilibrium to another will be called a transition layer. The periodic solutions obtained here possess at least two transition layers between two equilibria. A minimizer of I would prefer to have its transition layers stay where cost relatively less. To seek the locations where the transition layers prefer to stay, the quantity $\hat{\alpha}(j_1, j_2)$ will be used to measure the cost of a layer. Set $\theta(\rho) = \min(\mu_1 \rho^2, \mu_0)$ and

$$\Lambda = \sup\{\|V'(t, y)\| + 1/2 | t \in \mathbb{R} \text{ and } y \in \bigcup_{\eta \in \mathcal{K}_e} \overline{B_{\rho_0}(\eta)}\}.$$

Assume that the minimal period of V in t is T. Theorem 1 is a consequence of the following result.

Theorem 3. *Assume that* (V1)-(V6) *are satisfied. Suppose that there are* $k_0 < k_1 < k_2 < 0$ *such that* $k_1 + T = -k_0$,

$$\hat{\alpha}(k_1, k_2) \ < \ \min(\hat{\alpha}(k_0, k_1), \hat{\alpha}(k_2, -k_2)) \tag{2.1}$$

and

$$\min(-2k_2, k_1 - k_0) \ > \ 6\rho_0 + 2(2\hat{\alpha}(k_1, k_2) + \rho_0\sqrt{2\theta(\rho_0)})/\theta(r), \tag{2.2}$$

where

$$r = \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{2\Lambda}, \frac{\bar{\theta}}{4\Lambda}\right) \tag{2.3}$$

and $\bar{\theta} = \min(\hat{\alpha}(k_0, k_1) - \hat{\alpha}(k_1, k_2), \hat{\alpha}(k_2, -k_2) - \hat{\alpha}(k_1, k_2))$. *Then, for each* $m \in \mathbb{N}$, (1.3) *possesses a periodic solution with minimal period* mT.

Remark 1. Note that $\hat{\alpha}$ has a monotonicity property depending on the choice of the boundary points j_1 and j_2. For instance, $\hat{\alpha}(j_1, j_2) < \hat{\alpha}(j_3, j_2)$ if $j_1 < j_3$.

We start with two preliminary lemmas.

Lemma 1. *Suppose* $z(t) = \dfrac{t - t_1}{t_2 - t_1} z(t_2) + \dfrac{t_2 - t}{t_2 - t_1} z(t_1)$ *for* $t \in (t_1, t_2)$. *If* $z(t_1) \in \mathcal{K}_e$ *and* $|z(t_2) - z(t_1)| = t_2 - t_1 \leq \rho_0$, *then*

$$\int_{t_1}^{t_2} \mathcal{L}(z) dt \leq \Lambda |z(t_2) - z(t_1)|. \tag{2.4}$$

Proof. It immediately follows from the mean value theorem.

Lemma 2. *Let* $\eta_i \in \mathcal{K}_e$ *and* $\hat{A} = \{z \in E' | z(t_1), z(t_2) \in \partial B_r(\eta_i)\}$. *If* $q \in \hat{A}$ *and* $\int_{t_1}^{t_2} \mathcal{L}(q) dt = \min_{z \in \hat{A}} \int_{t_1}^{t_2} \mathcal{L}(z) dt$, *then* $q(t) \in B_{2\rho_0}(\eta_i)$ *for all* $t \in [t_1, t_2]$.

The above lemma is a property for the flow near an equilibrium $\eta \in \mathcal{K}_e$. Its proof can be found in [10]. We refer to [7, 5, 14, 15, 16, 18, 23, 24] for some analogous results which have been used to study multibump solutions of various equations.

Proof of Theorem 3. Without loss of generality, we may assume $\eta_1 = 0$. To seek a non-constant periodic solution of (1.3), we use penalization method. Let $k_3 = -k_2$, $k_4 = -k_1$,

$$\begin{aligned}
&\hat{t}_3 = k_3 - 3\rho_0 - \Big(\hat{\alpha}(k_1, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}\Big)/\theta(r), \\
&\hat{t}_4 = k_4 + 3\rho_0 + \Big(\hat{\alpha}(k_1, k_2) + \rho_0 \sqrt{2\theta(\rho_0)}\Big)/\theta(r), \\
&\hat{t}_1 = -\hat{t}_4, \hat{t}_2 = -\hat{t}_3, \\
&\hat{t}_{j+4\ell} = \hat{t}_j + \ell T, \\
&t^* = \min(\hat{t}_3 - \hat{t}_2, \hat{t}_1 - \hat{t}_0), \bar{\rho} = 5\rho_0/2
\end{aligned}$$

and

$$M_1 = \theta(r) + 2\hat{\alpha}(k_1, k_2)/t^*.$$

For each $m \in \mathbb{N}$, a penalty function ψ_m will be constructed as follows. Let $\psi_m \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ such that $0 \le \psi_m \le M_1$, $\psi_m(t,y) = \psi_m(-t,y)$, $\psi_m(t+mT,y) = \psi_m(t,y)$ and

$$\psi_m(t,y) = \begin{cases} 0 & \text{if } \ t \in [\hat{t}_1+\rho_0, \hat{t}_2-\rho_0] \cup [\hat{t}_3+\rho_0, \hat{t}_4-\rho_0] \\ & \qquad \cup [\hat{t}_5+\rho_0, \hat{t}_0+mT-\rho_0] \\ M_1 & \text{if } \ y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in [\hat{t}_2, \hat{t}_3] \\ 0 & \text{if } \ y \in B_{\bar{\rho}}(\eta_2) \text{ and } t \in (\hat{t}_2-\rho_0, \hat{t}_3+\rho_0) \\ M_1 & \text{if } \ y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in [\hat{t}_0, \hat{t}_1] \cup [\hat{t}_4, \hat{t}_5], \\ 0 & \text{if } \ y \in B_{\bar{\rho}}(\eta_1) \text{ and } t \in (\hat{t}_0-\rho_0, \hat{t}_1+\rho_0) \cup (\hat{t}_4-\rho_0, \hat{t}_5+\rho_0) \end{cases}$$

where $[s,t] = \phi$ if $s > t$. Set

$$I_m(z) = \int_0^{mT} [\mathcal{L}(z) + \psi_m(t,z)]dt$$

and

$$\alpha_m = \inf_{z \in E_m} I_m(z).$$

Then $\alpha_m < 2\hat{\alpha}(k_1,k_2)$ and there is a $p_m \in E_m$ such that $I_m(p_m) = \alpha_m$. It is easy to see from the construction of ψ_m that $\alpha_m > 0$. Hence $p_m \notin \mathcal{K}_e$.

To prove that p_m is a solution of (1.3), we are going to show that the global minimizers of I_m live in a region where the penalization vanishes. Observe that

> there exist a $t_1 \in (\hat{t}_0, \hat{t}_1)$ and a $t_2 \in (\hat{t}_2, \hat{t}_3)$ such that $p_m(t_1) \in B_r(\eta_1)$ and $p_m(t_2) \in B_r(\eta_2)$;

for otherwise, we would have $I_m(p_m) > \alpha_m$. Let

$$\tau_3 = \tau_3(p_m) = \inf\{t | t \in (t_1, t_2] \text{ and } p_m(t) \in \overline{B_r(\eta_2)}\}, \tag{2.5}$$

$$\tau_2 = \tau_2(p_m) = \sup\{t | t \in [t_1, \tau_3) \text{ and } p_m(t) \in \overline{B_r(\eta_1)}\}, \tag{2.6}$$

$$\tau_4 = \tau_4(p_m) = \sup\{t | t \in [t_2, t_1+T) \text{ and } p_m(t) \in \overline{B_r(\eta_2)}\} \tag{2.7}$$

and

$$\tau_5 = \tau_5(p_m) = \inf\{t | t \in (\tau_4, t_1+T] \text{ and } p_m(t) \in \overline{B_r(\eta_1)}\}. \tag{2.8}$$

By utilizing the penalty function ψ_m, a comparison argument used in [10] shows that

$$\tau_2 > \hat{t}_1 + 2\rho_0, \tau_3 < \hat{t}_2 - 2\rho_0, \tau_4 > \hat{t}_3 + 2\rho_0 \text{ and } \tau_5 < \hat{t}_4 - 2\rho_0. \tag{2.9}$$

Invoking Lemma 2 yields

$$p_m(t) \in B_{2\rho_0}(\eta_2) \text{ if } t \in [\tau_3, \tau_4] \tag{2.10}$$

and

$$p_m(t) \in B_{2\rho_0}(\eta_1) \text{ if } t \in [\tau_5, \tau_2 + mT]. \tag{2.11}$$

It is clear from (2.9)-(2.11) that the minimal period of p_m is mT. The proof is complete. □

Theorem 3 can be extended to the case where (V5) is not assumed. Nevertheless, as to be seen in the following proposition, the additional hypothesis (V5) ensures that the periodic solutions obtained in Theorem 3 are local minimizers among a larger family of functions than the periodic ones. Let $E'_m = W^{1,2}([0, mT], \mathbb{R}^n)$ and recall that $E_m = \{z \in E' | z(t + mT) = z(t)\}$.

Proposition 1. *Assume that the hypotheses of Theorem* 3 *are satisfied. Then*

$$\inf_{z \in E'_m} I_m(z) = \alpha_m \tag{2.12}$$

and $p_m(t) = p_m(-t)$ *if* $I_m(p_m) = \alpha_m$.

We refer to [11] for a proof in the case $m = 1$. The proof of $m > 1$ is similar.

With modification on penalty functions, we establish the following multiplicity result for the subharmonic solutions of (1.3).

Theorem 4. *If the hypothesis of Theorem* 3 *is satisfied,* (1.3) *possesses* $2m - 1$ *periodic solutions with minimal period* mT.

Proof. Let $i, \ell \in \mathbb{Z}$, $m \in \mathbb{N}$ and $\hat{t}_i$ be the same as in the proof of Theorem 3. For $m > 1$ and $0 \le \ell \le 2m - 2$, let $\hat{\psi}_{m,\ell} \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be such that

$0 \le \hat{\psi}_{m,\ell} \le M_1$, $\hat{\psi}_{m,\ell}(t+mT,y) = \hat{\psi}_{m,\ell}(t,y)$ and

$$\hat{\psi}_{m,\ell}(t,y) = \begin{cases} M_1 & \text{if} \quad y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in [\hat{t}_0, \hat{t}_1] \cup [\hat{t}_{2\ell+4}, \hat{t}_{2\ell+5}] \\ M_1 & \text{if} \quad y \notin B_{3\rho_0}(\eta_2) \text{ and } t \in \bigcup_{i \in F_\ell} [\hat{t}_{2i}, \hat{t}_{2i+1}] \\ 0 & \text{if} \quad t \in (\hat{t}_{2i-1} + \rho_0, \hat{t}_{2i} - \rho_0) \\ 0 & \text{if} \quad y \in B_{\bar{\rho}}(\eta_1) \text{ and } t \in [\hat{t}_0 - \rho_0, \hat{t}_1 + \rho_0] \\ & \qquad\qquad \cup [\hat{t}_{2\ell+4} - \rho_0, \hat{t}_{2\ell+5} + \rho_0] \\ 0 & \text{if} \quad y \in B_{\bar{\rho}}(\eta_2) \text{ and } t \in \bigcup_{i \in F_\ell} [\hat{t}_{2i} - \rho_0, \hat{t}_{2i+1} + \rho_0], \end{cases} \tag{2.13}$$

where $F_\ell = \{i \in \mathbb{N} | i \le 2m \text{ and } i \ne \ell + 2\}$. Adding a penalty function $\hat{\psi}_{m,\ell}$ to $\hat{I}_m$, we may proceed as Theorem 3 to get a periodic solution $\hat{p}_{m,\ell}$ of (1.3). By the construction of $\hat{\psi}_{m,\ell}$, the minimal period of $\hat{p}_{m,\ell}$ is mT. Moreover, if $\ell \ne i$ and $\ell, i \in \{0, 1, 2, \cdots, 2m-2\}$ then there is no $j \in \mathbb{Z}$ such that $\hat{p}_{m,\ell}(t) = \hat{p}_{m,i}(t + jT)$. This completes the proof.

Remark 2. Following the terminology used in [15], $\hat{p}_{m,\ell}$ is called a two-bump periodic solution, because roughly speaking it consists of two basic heteroclinics nicely concatenated.

3. Constrained Minimization

It has been shown that, for each $m \in \mathbb{N}$, there are at least $m-1$ subharmonic solutions of (1.3) with minimal period mT. Each of such solutions consists of two transitions in minimal period. We next consider the subharmonic solutions which consist of more than two transitions in minimal period. Let $\mathcal{P}_m = \{p | p \text{ is a periodic solution of (1.3) with minimal period } mT\}$. For $1 \le i \le m$, let $e_i \in \{0, 1\}$. An element p of $\mathcal{P}_m$ is of type $(e_1, e_2, \cdots, e_m)$ if p has a transition on $[(i-1)T, iT)$ when $e_i = 1$ and no transition when $e_i = 0$. In addition to penalization, an elementary constrained minimization will be employed to obtain the following result.

Theorem 5. *Assume that* (V1)–(V5) *are satisfied. Let* $k_1 = -2T - k_0$ *and* $k_2 = k_0 + T$. *Suppose* $k_0 < -T$,

$$\hat{\alpha}(k_1, k_2) < \hat{\alpha}(k_0, k_1) \tag{3.1}$$

and

$$k_1 - k_0 > 6\rho_0 + 2(2\hat{\alpha}(k_1, k_2) + \rho_0\sqrt{2\theta(\rho_0)})/\theta(r), \tag{3.2}$$

where

$$r = \min\left(1, \frac{\rho_0}{2}, \sqrt[4]{\frac{\rho_0^2}{8\mu_2}}, \frac{\rho_0\sqrt{2\theta(\rho_0)}}{2\Lambda}, \frac{\hat{\alpha}(k_0,k_1)-\hat{\alpha}(k_1,k_2)}{4\Lambda}\right). \tag{3.3}$$

For $m \geq 5$, (1.3) *possesses a subharmonic solution of type* $(e_1, e_2, \cdots, e_m)$ *with* $e_i = 1$ *if* $i = 1, 2, 3, m$ *and* $e_i = 0$ *if* $4 \leq i \leq m-1$.

Proof of Theorem 5. With the same $\hat{t}_i$ and $\hat{\psi}_{m,0}$ as in (2.13), we consider $m \geq 5$ and let $\psi_m^* \in C^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ be such that $0 \leq \psi_m^* \leq M_1$, $\psi_m^*(t + mT, y) = \psi_m^*(t, y)$ and

$$\psi_m^*(t,y) = \begin{cases} \hat{\psi}_{m,0}(t,y) & \text{if } \ t \in [\hat{t}_0 - \rho_0, \hat{t}_1 + 2T + \rho_0] \\ \hat{\psi}_{m,0}(t-2T,y) & \text{if } \ t \in [\hat{t}_1 + 2T + \rho_0, \hat{t}_1 + 4T + \rho_0] \\ 0 & \text{if } \ t \in (\hat{t}_1 + 4T + \rho_0, \hat{t}_0 + mT - \rho_0). \end{cases}$$

Set

$$\mathcal{J}_m(z,s,t) = \int_s^t [\mathcal{L}(z) + \psi_m^*(t,z)]dt.$$

Let $s_i = \hat{t}_i + (-1)^{i+1}2\rho_0$, $E_m^* = \{z \in E_m | \mathcal{J}_m(z, s_{2i}, s_{2i+1}) < 2\rho_0\sqrt{2\theta(\rho_0)}, i \in \mathbb{Z}\}$ and

$$\alpha_m^* = \inf_{z\in E_m^*} \mathcal{J}_m(z, 0, mT).$$

Then there is a $p_m^* \in E_m^*$ such that $\mathcal{J}_m(p_m^*, 0, mT) = \alpha_m^*$. Furthermore, the construction of ψ_m^* induces that $\alpha_m^* > 0$ and thus $p_m^* \notin \mathcal{K}_e$.

Next we are going to show that p_m^* lives in a region where the penalization vanishes. Note that

$$\text{there is a } t_1 \in (\hat{t}_0, \hat{t}_1) \text{ such that } p_m^*(t_1) \in B_r(\eta_1);$$

for otherwise,

$$\mathcal{J}_m(p_m^*, s_0, s_1) > \mathcal{J}_m(p_m, \hat{t}_0, \hat{t}_1) \geq \theta(r)(2\rho_0\sqrt{2\theta(\rho_0)}/\theta(r)) = 2\rho_0\sqrt{2\theta(\rho_0)}.$$

Likewise, there is a $t_2 \in (\hat{t}_2, \hat{t}_3)$ such that $p_m^*(t_2) \in B_r(\eta_2)$. Let $\tau_i = \tau_i(p_m^*)$, $2 \leq i \leq 5$, be defined as (2.5)-(2.8). As in Theorem 3, (2.9) must hold. This indicates that p_m^* possesses two transition layers in $(-T, T)$. Owing to the construction of ψ_m^*, the same lines of reasoning shows that p_m^* possesses two transitions in $(T, 3T)$.

Next, we show that $\mathcal{J}_m(z, s_{2i}, s_{2i+1}) < 2\rho_0\sqrt{2\theta(\rho_0)}$ for all $i \in \mathbb{Z}$. We only carry out the proof of

$$\mathcal{J}_m(p_m^*, s_2, s_3) < 2\rho_0\sqrt{2\theta(\rho_0)};$$

the others are the same. In view of (3.3), it suffices to prove

$$\mathcal{J}_m(p_m^*, \tau_3, \tau_4) < 2\Lambda r. \tag{3.4}$$

Let

$$Z_3(t) = \begin{cases} \frac{\tau_3+r-t}{r}p_m^*(\tau_3) + \frac{t-\tau_3}{r}\eta_2 & \text{if} \quad t \in [\tau_3, \tau_3 + r] \\ \eta_2 & \text{if} \quad t \in (\tau_3 + r, \tau_4 - r) \\ \frac{t-\tau_4+r}{r}p_m^*(\tau_4) + \frac{\tau_4-t}{r}\eta_2 & \text{if} \quad t \in [\tau_4 - r, \tau_4]. \end{cases}$$

Invoking Lemma 1 gives

$$\mathcal{J}_m(Z_3, \tau_3, \tau_4) \le 2\Lambda r. \tag{3.5}$$

Set $A_2 = \{z \in W^{1,2}([\tau_3, \tau_4], \mathbb{R}^n) | z(\tau_3) = p_m^*(\tau_3) \text{ and } z(\tau_4) = p_m^*(\tau_4)\}$. Since $Z_3 \in A_2$, (3.4) easily follows from (3.5) and the fact that

$$\mathcal{J}_m(p_m^*, \tau_3, \tau_4) = \inf_{z \in A_2} \mathcal{J}_m(z, \tau_3, \tau_4).$$

With the aid of Lemma 2, we now conclude that p_m^* lives in a region where the penalization is vanished. Thus p_m^* is a periodic solution of (1.3). The locations of the transitions in p_m^* indicate that the minimal period of p_m^* is mT.

Remark 3.

(a) Shifting the time of the solution by one period, we get a solution of type $(e_1, e_2, \cdots, e_m)$ with $e_i = 1$ if $1 \le i \le 4$ and $e_i = 0$ if $5 \le i \le m$.

(b) Using different penalty functions, we obtain subharmonic solutions of type $(e_1, e_2, \cdots, e_m)$ with a given combination of $e_i \in \{0, 1\}$, $1 \le i \le m$.

4. Connecting Orbits

With a large number of subharmonic solutions of (1.3), we next investigate the connecting orbits joining with a pair of periodic solutions. For $z \in E'_m$ there is a constant $c = c(m)$ such that

$$\|z\|_{L^\infty} \le \frac{c}{2}\|z\|. \tag{4.1}$$

Let $\mathcal{B}_\rho(z)$ denote an open ball centered at z with radius ρ and $\mathcal{B}_\rho(\Omega) = \cup_{z\in\Omega}\mathcal{B}_\rho(z)$.

Proposition 2. *There is a positive function* d_m *which satisfies* $\lim_{\rho\to 0^+} d_m(\rho) = 0$, *and*

$$I_m(z) \ge I_m(p_m) + d_m(\rho) \tag{4.2}$$

if $p_m \in \mathcal{K}_m$ *and* $z \in E'_m \backslash \mathcal{B}_\rho(\mathcal{K}_m)$. *Suppose* $\mathcal{K}_m$ *consists of isolated points, then there are only a finite number of elements in* $\mathcal{K}_m$.

We refer to [34] for a proof of Proposition 2. A modification here is to add penalization in the proof.

To formulate a variational framework for the connecting orbits of (1.3), we take $p_1 \in \mathcal{K}_m$, $p_2 \in \mathcal{K}_j$ and set

$$\begin{aligned}\Gamma(p_2,p_1) = \{z \in E' | z(t) \to p_1(t) \text{ uniformly } \text{ as } t \to \infty \\ \text{and } z(t) \to p_2(t) \text{ uniformly as } t \to -\infty\}.\end{aligned} \tag{4.3}$$

Consider a penalty function as follows. Let $\bar\rho, \hat t_0, \hat t_1, M_1$ and ψ_m be defined as in the proof of Theorem 3. For fixed $N \in \mathbb{N}$, let $\Psi \in C^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ be such that $0 \le \Psi \le M_1$ and

$$\Psi(t,y) = \begin{cases} \psi_j(t+NT,y) & \text{if } \; t \in (-\infty, \hat t_0 - NT - \rho_0] \\ M_1 & \text{if } \; y \notin B_{3\rho_0}(\eta_1) \text{ and } t \in (\hat t_1 - NT, \hat t_0) \\ 0 & \text{if } \; y \in B_{\bar\rho}(\eta_1) \text{ and } t \in (\hat t_1 - NT, \hat t_0) \\ \psi_m(t,y) & \text{if } \; t \in [\hat t_0 - \rho_0, \infty). \end{cases}$$

Set $A = \{(t,y)|\Psi(t,y) = 0\}$. As in the proof of Theorem 5, $s_i = \hat{t}_i + (-1)^{i+1}2\rho_0$. Let $E = \{z \in E'|\mathcal{J}(z, s_{2i}, s_{2i+1}) < 2\rho_0\sqrt{2\theta(\rho_0)}, i \in \mathbb{Z}\}$, where

$$\mathcal{J}(z, s, \hat{s}) = \int_s^{\hat{s}} \mathcal{L}(z)dt.$$

For $z \in E'$, define $G(z) = \{(t, z(t))|t \in \mathbb{R}\}$ and

$$a_{-\ell}(z) = \int_{-\ell jT-NT}^{-(\ell-1)jT-NT} [\mathcal{L}(z) + \Psi(t,z) - \mathcal{L}(p_2)]dt,$$
$$a_{\ell}(z) = \int_{(\ell-1)mT}^{\ell mT} [\mathcal{L}(z) + \Psi(t,z) - \mathcal{L}(p_1)]dt$$

for $\ell \in \mathbb{N}$, and

$$a_0(z) = \int_{-NT}^{0} [\mathcal{L}(z) + \Psi(t,z)]dt.$$

It is clear that $a_\ell(z) \geq 0$ for all ℓ. Set

$$J(z) = \sum_{\ell=-\infty}^{\infty} a_\ell(z) \tag{4.4}$$

and

$$\beta(p_2, p_1) = \inf_{z\in\Gamma(p_2,p_1)} J(z). \tag{4.5}$$

It will be seen that if p_1 and p_2 are isolated points of $\mathcal{K}_m$ and $\mathcal{K}_j$ respectively, and $\beta(p_2,p_1) = \inf_{p\in\mathcal{K}_m, p'\in\mathcal{K}_j} \beta(p',p)$, then there is a connecting orbit joining p_2 and p_1.

For $z \in E'$, let $\sigma_\ell(z)$ denote the restriction of z on $[\ell mT, (\ell+1)mT]$. The next lemma will be used to prove Theorem 2. Its proof can be found in [11].

Lemma 3. *Given $\gamma > 0$ and $\bar{\ell} > \ell$, there exist positive numbers $\hat{\rho}$ and $C = C(\gamma, \bar{\ell} - \ell)$ such that, for any $p, p' \in \mathcal{K}_m$ and $z \in E'$, if $\|p - p'\| \geq \gamma$,*

$$\|\sigma_\ell(z - p')\|_{L^\infty} < \hat{\rho} \tag{4.6}$$

and

$$\|\sigma_{\bar{\ell}}(z - p)\|_{L^\infty} < \hat{\rho}, \tag{4.7}$$

then

$$\int_{\ell mT}^{(\bar{\ell}+1)mT} (\mathcal{L}(z) + \psi_m(t,z) - \mathcal{L}(p))dt \geq C. \tag{4.8}$$

Moreover, C is independent of the length of $\bar{\ell} - \ell$, if $\mathcal{K}_m$ consists of isolated points.

***Proof of Theorem* 2.** By Proposition 2 there are $p_1 \in \mathcal{K}_m$ and $p_2 \in \mathcal{K}_j$ such that

$$\beta(p_2, p_1) = \inf_{p\in\mathcal{K}_m, p'\in\mathcal{K}_j} \beta(p', p). \tag{4.9}$$

Let $\{z_k\} \subset \Gamma(p_2, p_1)$ be a minimizing sequence for J. Along a subsequence, $z_k \to q$ weakly in E' and strongly in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^n)$. It follows that

$$J(q) \leq \beta(p_2, p_1). \tag{4.10}$$

Arguing like Theorem 3 yields $G(q) \subset \mathring{A}$. Thus q is a solution of (1.3).

Next, we prove (1.6). By (4.10) and Proposition 2, for any $\rho > 0$, there is an $\hat{\ell} = \hat{\ell}(\rho)$ such that if $\ell \geq \hat{\ell}$ then

$$\sigma_\ell(q) \in \mathcal{B}_\rho(\bar{p}) \text{ for some } \bar{p} \in \mathcal{K}_m.$$

Furthermore, there is a $p \in \mathcal{K}_m$ such that $\sup\{\ell | \sigma_\ell(q) \in \mathcal{B}_\rho(p)\} = \infty$. Suppose (1.6) is false, then for any $\rho > 0$, there is an $\ell \in \mathbb{N}$ such that

$$\sigma_\ell(q) \in \mathcal{B}_\rho(p).$$

If k is large enough then

$$J(z_k) < \beta(p_2, p_1) + \rho$$

and

$$\|\sigma_\ell(z_k - p)\|_{L^\infty} < \rho.$$

Moreover, $\|\sigma_{\bar{\ell}}(z_k - p_1)\|_{L^\infty} < \rho$ if $\bar{\ell}$ is sufficiently large. Define a continuous function v_k by

$$v_k(t) = \begin{cases} z_k(t) & \text{if} \quad t \in (-\infty, (\ell+1)mT - \rho] \\ p(t) & \text{if} \quad t \in [(\ell+1)mT, \infty) \\ \text{a linear function} & \text{if} \quad t \in ((\ell+1)mT - \rho, (\ell+1)mT). \end{cases}$$

A straightforward calculation yields

$$\int_{(\ell+1)mT-\rho}^{(\bar{\ell}+1)mT} (\mathcal{L}(v_k) + \Psi(t, v_k) - \mathcal{L}(p_1))dt \leq b\rho,$$

where b is a constant independent of k and ρ. This together with Lemma 3 gives

$$\int_{\ell mT}^{(\bar{\ell}+1)mT} (\mathcal{L}(z_k) + \Psi(t, z_k) - \mathcal{L}(p_1))dt \geq C.$$

Consequently

$$J(v_k) \leq J(z_k) - C + b\rho < \beta(p_2, p_1) + \rho - C + b\rho < \beta(p_2, p_1),$$

if we pick $\rho < \min(\hat{\rho}, C/(b+1))$. Then

$$\beta(p_2, p) < \beta(p_2, p_1),$$

which is contrary to (4.10). Thus (1.6) holds, so does (1.7), following the same argument.

Remark 4.

(a) If $n = 1$, Theorem 2 still holds without assuming that $\mathcal{K}_m$ and $\mathcal{K}_j$ consist of isolated points. This follows from an argument used in the proof of Poincaré-Bendixson Theorem.

(b) If $p_2 = p_1$ this solution is a homoclinic orbit.

(c) Taking different N actually yields infinitely many connecting orbits of (1.3).

(d) Choosing a suitable penalty function leads to the existence of a connecting orbit joining a periodic solution of type $(e_1, e_2, \cdots, e_m)$ with an equilibrium or a periodic solution of type $(e_1, e_2, \cdots, e_j)$.

The penalization method can also be empolyed to study multibump connecting orbits for periodic solutions of (1.3). Let $\hat{\Psi} \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be such that $0 \le \hat{\Psi} \le M_1$ and

$$\hat{\Psi}(t,y) = \begin{cases} \Psi(t,y) & \text{if} \quad t \in (-\infty, \hat{t}_0 - NT - \rho_0] \cup [\hat{t}_0 - \rho_0, \infty) \\ \psi_i(t+NT, y) & \text{if} \quad t \in (\hat{t}_0 - NT - \rho_0, \hat{t}_0 - \rho_0), \end{cases}$$

where $N = ki$ and k, i are positive integers. For $p_1 \in \mathcal{K}_m$, $p_2 \in \mathcal{K}_j$, $p_3 \in \mathcal{K}_i$ and $z \in E'$, define

$$\begin{aligned} \hat{a}_{-\ell}(z) &= \int_{-\ell jT - NT}^{-(\ell-1)jT - NT} [\mathcal{L}(z) + \hat{\Psi}(t,z) - \mathcal{L}(p_2)]dt, \\ \hat{a}_\ell(z) &= \int_{(\ell-1)mT}^{\ell mT} [\mathcal{L}(z) + \hat{\Psi}(t,z) - \mathcal{L}(p_1)]dt \end{aligned}$$

for $\ell \in \mathbb{N}$, and

$$\hat{a}_0(z) = \int_{-NT}^{0} [\mathcal{L}(z) + \hat{\Psi}(t,z) - \mathcal{L}(p_3)]dt.$$

Set

$$\hat{J}(z) = \sum_{\ell=-\infty}^{\infty} \hat{a}_\ell(z)$$

and

$$\hat{\beta}(p_2, p_1) = \inf_{z \in \Gamma(p_2, p_1)} \hat{J}(z).$$

Theorem 6. *Assume that the hypothesis of Theorem 3 is satisfied. Suppose that $\mathcal{K}_m, \mathcal{K}_j$ and $\mathcal{K}_i$ consist of isolated points. If $\hat{\beta}(p_2, p_1) = \inf_{p \in \mathcal{K}_m, p' \in \mathcal{K}_j} \hat{\beta}(p', p)$, then there is a solution q of (1.3) which satisfies (1.6) and (1.7). Moreover, for any sufficiently small positive number $\hat{\rho}$, if N is chosen large enough, then there exist $N_1, N_2 \in (-N, 0)$ and a $\hat{p} \in \mathcal{K}_i$ such that*

$$\tau_\ell(q) \in \mathcal{B}_{\hat{\rho}}(\hat{p})$$

for some $\ell \in [N_1, N_2]$, where $\tau_\ell(q)$ denotes the restriction of q on $[\ell iT, (\ell+1)iT]$.

The proof follows from the same lines of reasoning based on adding penalty function.

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References

1. V. S. Afraimovich, A. V. Babin and S. N. Chow, Spatial chaotic structure of attractors of reaction-diffusion systems, *Trans. Amer. Math. Soc.*, **348** (1996), 5031-5063.
2. A. Ambrosetti and M. Badiale, Homoclinics: Poincaré-Melnikov type results via a variational approach, *C. R. Acad. Sci. Sér. I Math.*, **323** (1996), 753-758.
3. V. Bangert, Mather sets for twist maps and geodesics on tori, Dynamics reported, vol. 1 (eds. U. Kirchgraber and H. O. Walter), John Wiley, 1988.
4. U. Bessi, Multiple homoclinic orbits for autonomous singular potentials, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), 785-802.
5. S. Bolotin and R. Mackay, Multibump orbits near the anti-integrable limit for Lagrangian systems, *Nonlinearity*, **10** (1997), 1015-1029.
6. B. Buffoni, A. R. Champneys, and J. F. Toland, Bifurcation and coalescence of a plethora of homoclinic orbits, *J. Dynam. Differential Equations*, **8** (1996), 221-279.
7. B. Buffoni and E. Séré, A global condition for quasi-random behavior in a class of conservative systems, *Comm. Pure Appl. Math.*, **49** (1996), 285-305.
8. P. Caldiroli and L. Jeanjean, Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems, *J. Differential Equations*, **136** (1997), 76-114.
9. P. Caldiroli and P. Montecchiari, Homoclinic orbits for second order Hamiltonian systems with potential changing sign. *Comm. on Appl. Nonlinear Anal.*, **1** (1994), 97-129.
10. C. -N. Chen and S. -Y. Tzeng, Existence and multiplicity results for heteroclinic orbits of second order Hamiltonian systems, *J. Differential Equations*, **158** (1999), 211-250.
11. C. -N. Chen and S. -Y. Tzeng, Periodic solutions and their connecting orbits of Hamiltonian systems, *J. Differential Equations*, **177** (2001), 121-145.
12. M. Calanchi and E. Serra, Homoclinic solutions to periodic motions in a class of reversible equations, *Calc. Var. Partial Differential Equations*, **9** (1999), no. 2, 157-184.
13. V. Coti Zelati, I. Ekeland and E. Séré, A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann.*, **288** (1990), 133-160.
14. V. Coti Zelati and P. H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.*, **4** (1991), 693-727.
15. V. Coti Zelati and P. H. Rabinowitz, Multibump periodic solutions of a family of Hamiltonian systems, *Topol. Methods Nonlinear Anal.*, **4** (1994), 31-57.

16. V. Coti Zelati and P. H. Rabinowitz, Heteroclinic solutions between stationary points at different energy levels, *Topol. Methods Nonlinear Anal.*, **17** (2001), no. 1, 1-21.

17. R. L. Devaney, Homoclinic orbits in Hamiltonian systems, *J. Differential Equations*, **21** (1976), 431-438.

18. M. del Pino and P. Felmer, Multi-peak bound states for nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **15** (1998), 127-149.

19. G. Gallavotti, Arnold's diffusion in isochronous systems, *Math. Phys. Anal. Geom.*, **1** (1999), 295-312.

20. J. Guckenheimer and P. J. Holmes, "*Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*", Springer-Verlag, Berlin, 1983.

21. V. V. Kozlov, Calculus of variations in the large and classical mechanics, *Russian Math. Surveys*, **40** (1985), 37-71.

22. U. Kirchgraber and D. Stoffer, Chaotic behavior in simple dynamical systems, *SIAM Rev.*, **32** (1990), 424-452.

23. W. D. Kalies and R. C. A. M. VanderVorst, Multitransition homoclinic and heteroclinic solutions of the extended Fisher-Kolmogorov equation, *J. Differential Equations*, **131** (1996), 209–228.

24. W. D. Kalies, J. Kwapisz, and R. C. A. M. VanderVorst, Homotopy classes for stable connections between Hamiltonian saddle-focus equilibria, *Comm. Math. Phys.*, **193** (1998), 337-371.

25. A. Katok and B. Hasselblatt, "*Introduction to the Modern Theory of Dynamical Systems*", Cambridge University Press, Cambridge, 1995.

26. J. N. Mather, Variational construction of connecting orbits, *Ann. Inst. Fourier* **43** (1993), 1349-1386.

27. J. Mawhin and M. Willem, "*Critical Point Theory and Hamiltonian Systems*", Appl. Math. Sci., Vol. 74, Springer-Verlag, New York/Berlin, 1989.

28. T. O. Maxwell, Heteroclinic chains for a reversible Hamiltonian system, *Nonlinear Anal.*, **28** (1997), 871-887.

29. T. O. Maxwell, Multibump heteroclinics for a reversible Hamiltonian system, preprint.

30. V. K. Melnikov, On the stability of the center for periodic perturbations, *Trans. Moscow Math. Soc.*, **12** (1963), 1-57.

31. J. Moser, Minimal solutions of variational problems on a torus, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **3** (1986), 229-272.

32. H. Poincaré, "*Les Méthodes Nouvelles de la Mécanique Céleste*", Gauthier-Villars, Paris, 1897-1899.

33. L. A. Peletier and W. C. Troy, Chaotic spatial patterns described by the extended Fisher-Kolmogorov equation, *J. Differential Equations*, **129** (1996), 458-508.

34. P. H. Rabinowitz, Heteroclinics for a reversible Hamiltonian system, *Ergodic Theory Dynam. Systems*, **14** (1994), 817-829.

35. P. H. Rabinowitz, Periodic and heteroclinic orbits for a periodic Hamiltonian system, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **6** (1989), 331-346.

36. P. H. Rabinowitz, Homoclinic and heteroclinic orbits for a class of Hamiltonian systems, *Calc. Var.*, **1** (1993), 1-36.

37. E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.*, **209** (1992), 27-42.

38. E. Séré, Looking for the Bernoulli shift, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **10** (1993), 561-590.

39. K. Strobel, Multibump solutions for a class of periodic Hamiltonian systems, University of Wisconsin, Thesis, 1994.

40. K. Tanaka, Periodic solutions for singular Hamiltonian systems and closed geodesics on non-compact Riemannian manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **17** (2000), 1-33.