

## BALL CONCEPTS IN PARTIAL DIFFERENTIAL EQUATIONS

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### Abstract

In this article, we present that ball concepts could be applied to solve existence and non-existence of solutions in semilinear elliptic equations.

### 1. (PS)-values and Index of a Domain

Let  $\Omega$  be a domain in  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $2^* = \infty$  if  $N = 1, 2$ ,  $2^* = \frac{2N}{N-2}$  if  $N > 2$ , and  $2 < p < 2^*$ . Let  $H_0^1(\Omega)$  be the Sobolev space in  $\Omega$ . Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega. \\ u \in H_0^1(\Omega). \end{cases} \quad (1)$$

Associated with Equation (1), we consider the energy functionals  $a$ ,  $b$  and  $J$ , for  $u \in H_0^1(\Omega)$ , as follows

$$\begin{cases} a(u) = \int_{\Omega} (|\nabla u|^2 + u^2). \\ b(u) = \int_{\Omega} |u|^p. \\ J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u). \end{cases}$$

The solutions of Equation (1) and the critical points of the energy functional  $J$  are the same. The existence and the nonexistence of solutions of

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Equation (1) in a domain  $\Omega$  have been the focus of a great deal of research in recent years. They are affected by the geometry and the topology of the domain  $\Omega$ . To characterize, in what kind of domains, the existence and the nonexistence of solutions of Equation (1) is an open question. In this article, we try to obtain new compactness results to answer partially the question.

We give the definition of a Palais-Smale (briefly (PS)) sequence.

**Definition 1.**

- (i) For  $\beta \in \mathbb{R}$ , a sequence  $\{u_n\}$  is a  $(PS)_\beta$ -sequence in  $H_0^1(\Omega)$  for  $J$  if  $J(u_n) = \beta + o(1)$  and  $J'(u_n) = o(1)$  strongly in  $H^{-1}(\Omega)$  as  $n \rightarrow \infty$ .
- (ii)  $\beta \in \mathbb{R}$  is a  $(PS)$ -value in  $H_0^1(\Omega)$  for  $J$  if there is a  $(PS)_\beta$ -sequence in  $H_0^1(\Omega)$  for  $J$ .
- (iii)  $J$  satisfies the  $(PS)_\beta$ -condition in  $H_0^1(\Omega)$  if every  $(PS)_\beta$ -sequence in  $H_0^1(\Omega)$  for  $J$  contains a convergent subsequence.
- (iv)  $J$  satisfies the  $(PS)$ -condition in  $H_0^1(\Omega)$  if for every  $\beta \in \mathbb{R}$ ,  $J$  satisfies the  $(PS)_\beta$ -condition in  $H_0^1(\Omega)$ .

A  $(PS)_\beta$ -sequence is bounded.

**Lemma 2.** *Let  $\beta \in \mathbb{R}$  and let  $\{u_n\}$  be a  $(PS)_\beta$ -sequence in  $H_0^1(\Omega)$  for  $J$ , then*

- (i)  $\|u_n\|_{H^1} \leq c$  for each  $n$ .
- (ii)  $a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1)$ .
- (iii)  $\beta \geq 0$ .

**Proof.**

- (i) Since  $\{u_n\}$  is a  $(PS)_\beta$ -sequence in  $H_0^1(\Omega)$  for  $J$ , we have

$$|\beta| + \delta_n + \frac{\varepsilon_n \|u_n\|_{H^1}}{p} \geq J(u_n) - \frac{1}{p} \langle J'(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|_{H^1}^2,$$

where  $\delta_n = o(1)$  and  $\varepsilon_n = o(1)$ . Take

$$c_n(\beta) = \frac{1}{p-2} \left( \varepsilon_n + \sqrt{\varepsilon_n^2 + 2p(p-2)(|\beta| + \delta_n)} \right),$$

then  $c_n(\beta) = o(1)$  as  $n \rightarrow \infty$  and  $\beta \rightarrow 0$ . There is a constant  $c$  such that  $\|u_n\|_{H^1} \leq c_n(\beta) \leq c$  for each  $n$ .

(ii) Since  $\{u_n\}$  is bounded, we have  $o(1) = \langle J'(u_n), u_n \rangle = a(u_n) - b(u_n)$ , or

$$\beta + o(1) = J(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \frac{p-2}{2p}a(u_n) + o(1).$$

Therefore,

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\beta + o(1).$$

(iii) By (ii), we have  $\beta \geq 0$ . □

Consider the Nehari manifold  $\mathbf{M}(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u)\}$ , clearly, the Nehari manifold  $\mathbf{M}(\Omega)$  contains every non-zero solutions of equation (1). Let

$$\alpha(\Omega) = \inf_{v \in \mathbf{M}(\Omega)} J(v).$$

**Theorem 3.** *There is a constant  $c > 0$  such that  $\alpha(\Omega) \geq c$ .*

**Proof.** By the Sobolev embedding theorem, there is a constant  $d > 0$  such that, for each  $u \in \mathbf{M}(\Omega)$ ,  $a(u) = b(u) \leq d^p a(u)^{\frac{p}{2}}$ , so  $a(u) \geq d^{-\frac{2p}{p-2}}$ , we have then

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) a(u) \geq c,$$

where  $c = \left(\frac{1}{2} - \frac{1}{p}\right) d^{-\frac{2p}{p-2}}$ . Thus  $\alpha(\Omega) \geq c$ . □

Consider the unit ball  $\mathbf{U}(\Omega) = \{u \in H_0^1(\Omega) \mid \|u\|_{H^1} = 1\}$  in the space  $H_0^1(\Omega)$ , we have

**Lemma 4.** *There is a bijective  $C^{1,1}$  map from  $\mathbf{U}(\Omega)$  to  $\mathbf{M}(\Omega)$ .*

**Proof.** By Calculus. □

A minimizing sequence  $\{u_n\}$  in  $\mathbf{M}(\Omega)$  of  $\alpha(\Omega)$  is a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ .

**Theorem 5.** *Let  $\{u_n\}$  be in  $H_0^1(\Omega)$ . Then  $\{u_n\}$  is a  $(PS)_{\alpha(\Omega)}$ -sequence for  $J$  if and only if  $J(u_n) = \alpha(\Omega) + o(1)$  and  $a(u_n) = b(u_n) + o(1)$ . In particular, every minimizing sequence  $\{u_n\}$  in  $\mathbf{M}(\Omega)$  of  $\alpha(\Omega)$  is a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ . Therefore,  $\alpha(\Omega)$  is a  $(PS)$ -value in  $H_0^1(\Omega)$  for  $J$ .*

**Proof.** (i) Suppose  $\{u_n\}$  is a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ . By Lemma 2, we have  $a(u_n) = b(u_n) + o(1)$ .

(ii) Let  $\{u_n\}$  satisfy  $J(u_n) = \alpha(\Omega) + o(1)$  and  $a(u_n) = b(u_n) + o(1)$ . We have

$$\alpha(\Omega) + o(1) = J(u_n) = \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \frac{p-2}{2p}a(u_n) + o(1).$$

Therefore,

$$a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2}\alpha(\Omega) + o(1). \quad (2)$$

For  $n = 1, 2, \dots$ , denote

$$l_n(\varphi) = \int_{\Omega} |u_n|^{p-2} u_n \varphi \quad \text{for } \varphi \in H_0^1(\Omega). \quad (3)$$

Let  $\phi \in H_0^1(\Omega)$ ,  $\|\phi\|_{H^1} = 1$ ,  $t > 0$  exists such that  $t\phi \in \mathbf{M}(\Omega)$ :  $\|t\phi\|_{H^1}^2 = \|t\phi\|_{L^p}^p$ ; we conclude that  $t = \|\phi\|_{L^p}^{\frac{-p}{p-2}}$  and

$$\alpha(\Omega) \leq \left(\frac{1}{2} - \frac{1}{p}\right) \|t\phi\|_{H^1}^2 = \frac{p-2}{2p} t^2 = \frac{p-2}{2p} \|\phi\|_{L^p}^{\frac{-2p}{p-2}}.$$

Therefore,  $\|\phi\|_{L^p} \leq \left(\frac{2p}{p-2}\alpha(\Omega)\right)^{\frac{2-p}{2p}}$ . For each  $n$ ,

$$\begin{aligned} |l_n(\phi)| &= \left| \int_{\Omega} |u_n|^{p-2} u_n \phi \right| \leq \left( \int_{\Omega} |u_n|^p \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\phi|^p \right)^{1/p} \\ &\leq \left( \frac{2p}{p-2}\alpha(\Omega) \right)^{\frac{p-1}{p}} \left( \frac{2p}{p-2}\alpha(\Omega) \right)^{\frac{2-p}{2p}} + o(1) \\ &= \left( \frac{2p}{p-2}\alpha(\Omega) \right)^{1/2} + o(1) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

we have

$$\|l_n\|_{H^{-1}} \leq \left( \frac{2p}{p-2}\alpha(\Omega) \right)^{1/2} + o(1) \quad \text{as } n \rightarrow \infty. \quad (4)$$

Furthermore, by (3), we have

$$l_n \left( \frac{u_n}{\|u_n\|_{H^1}} \right) = \frac{\int_{\Omega} |u_n|^p}{\|u_n\|_{H^1}} = (b(u_n))^{1/2} + o(1)$$

$$= \left( \frac{2p}{p-2} \alpha(\Omega) \right)^{1/2} + o(1) \quad \text{as } n \rightarrow \infty. \quad (5)$$

By (4) and (5), we conclude that

$$\|l_n\|_{H^{-1}} = \left( \frac{2p}{p-2} \alpha(\Omega) \right)^{1/2} + o(1) \quad \text{as } n \rightarrow \infty.$$

By the Riesz representation theorem, for each  $n$ ,  $w_n \in H_0^1(\Omega)$  exists such that, for each  $\varphi \in H_0^1(\Omega)$ ,

$$l_n(\varphi) = \langle w_n, \varphi \rangle_{H^1} = \int_{\Omega} (\nabla w_n \cdot \nabla \varphi + w_n \varphi),$$

and

$$\|w_n\|_{H^1} = \|l_n\|_{H^{-1}} = \left( \frac{2p}{p-2} \alpha(\Omega) \right)^{1/2} + o(1). \quad (6)$$

Consequently,

$$\langle w_n, u_n \rangle_{H^1} = l_n(u_n) = \int_{\Omega} |u_n|^p = \frac{2p}{p-2} \alpha(\Omega) + o(1). \quad (7)$$

By (2), (6), and (7), we obtain

$$\begin{aligned} \|u_n - w_n\|_{H^1}^2 &= \langle u_n, u_n \rangle_{H^1} - 2\langle u_n, w_n \rangle_{H^1} + \langle w_n, w_n \rangle_{H^1} \\ &= \|u_n\|_{H^1}^2 - 2\langle u_n, w_n \rangle_{H^1} + \|w_n\|_{H^1}^2 \\ &= \frac{2p}{p-2} \alpha(\Omega) - 2\frac{2p}{p-2} \alpha_{\mathbf{M}}(\Omega) + \frac{2p}{p-2} \alpha(\Omega) + o(1) \\ &= o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

For  $\varphi \in H_0^1(\Omega)$ ,  $\|\varphi\|_{H^1} = 1$ , we have

$$\begin{aligned} \langle J'(u_n), \varphi \rangle &= \int_{\Omega} (\nabla u_n \cdot \nabla \varphi + u_n \varphi) - \int_{\Omega} |u_n|^{p-2} u_n \varphi \\ &= \langle u_n, \varphi \rangle_{H^1} - \langle w_n, \varphi \rangle_{H^1} = \langle u_n - w_n, \varphi \rangle_{H^1}, \end{aligned}$$

so  $\|J'(u_n)\|_{H^{-1}} \leq \|u_n - w_n\|_{H^1} = o(1)$ . We conclude that  $J'(u_n) = o(1)$  strongly in  $H^{-1}(\Omega)$  as  $n \rightarrow \infty$ .  $\square$

$\alpha(\Omega)$  is the minimal positive (PS)-value in  $H_0^1(\Omega)$  for  $J$ .

**Theorem 6.** *Let  $\beta > 0$  be a (PS)-value in  $H_0^1(\Omega)$  for  $J$ . Then  $\beta \geq \alpha(\Omega)$ . In particular,  $\alpha(\Omega)$  is the minimal positive (PS)-value in  $H_0^1(\Omega)$  for  $J$ .*

**Proof.** Let  $\{u_n\}$  in  $H_0^1(\Omega)$  is a  $(PS)_\beta$ -sequence for  $J$ , by Theorem 2, we have that  $J(u_n) = \beta + o(1)$  and  $a(u_n) = b(u_n) + o(1)$ , therefore we may assume that  $u_n \neq 0$ , for each  $n$ . By Lemma 4, there is a sequence  $\{s_n\}$  in  $\mathbb{R}^+$  such that  $\{s_n u_n\}$  is in  $\mathbf{M}(\Omega)$ :  $s_n^2 a(u_n) = s_n^p b(u_n)$  for each  $n$ . Since  $a(u_n) = b(u_n) + o(1)$  and  $J(u_n) = \beta + o(1)$ , we have  $s_n = 1 + o(1)$ . Therefore,  $J(s_n u_n) = \beta + o(1)$ . Since for each  $n$ , we have  $\alpha(\Omega) \leq J(s_n u_n)$ , so  $\beta \geq \alpha(\Omega)$ . Moreover, by Theorem 5,  $\alpha(\Omega)$  is the minimal positive (PS)-value in  $H_0^1(\Omega)$  for  $J$ .  $\square$

**Definition 7.**  $\alpha(\Omega)$  is called the index of  $J$  in  $\Omega$ . If  $u$  is a nonzero solution of Equation (1), then  $J(u)$  is a positive (PS)-value in  $H_0^1(\Omega)$  for  $J$ . Thus  $J(u) \geq \alpha(\Omega)$ . Let  $u$  be a nonzero solution of Equation (1) in  $H_0^1(\Omega)$ .

- (i)  $u$  is a ground state solution (or a least energy solution) if  $J(u) = \alpha(\Omega)$ .
- (ii)  $u$  is a higher energy solution if  $J(u) > \alpha(\Omega)$ .

**Theorem 8.** *Let  $u \in \mathbf{M}(\Omega)$  be such that  $J(u) = \alpha(\Omega)$ . Then  $u$  is a nonzero solution of (1).*

**Proof.** Set  $g(w) = a(w) - b(w)$  for  $w \in H_0^1(\Omega)$ . Let  $v \in \mathbf{M}(\Omega)$ , then

$$\begin{aligned} g(v) &= a(v) - b(v) = 0 \\ \langle g'(v), v \rangle &= (2 - p)a(v) \neq 0. \end{aligned}$$

Since the minimum of  $J$  is achieved at  $u \in \mathbf{M}(\Omega)$  and is constrained in the manifold  $\mathbf{M}(\Omega)$ , by the Lagrange multiplier theorem,  $\lambda \in \mathbb{R}$  exists such that  $J'(u) = \lambda g'(u)$  in  $H_0^1(\Omega)$ . Thus,

$$0 = \langle J'(u), u \rangle = \lambda \langle g'(u), u \rangle.$$

Since  $\langle g'(u), u \rangle \neq 0$ , we have  $\lambda = 0$ . Thus,  $J'(u) = 0$ . Hence,  $u$  is a weak solution of (1) such that  $J(u) = \alpha(\Omega)$ .  $\square$

**Lemma 9.** *Let  $u$  in  $H_0^1(\Omega)$  be a change sign solution of (1). Then  $J(u) > 2\alpha(\Omega)$ .*

**Proof.** Let  $u^- = \max\{-u, 0\}$ . Then  $u^-$  is nonzero. Multiply (1) by  $u^-$  and integrate to obtain

$$\int_{\Omega} \nabla u \nabla u^- + \int_{\Omega} uu^- = \int_{\Omega} |u|^{p-2} uu^-.$$

Consequently,

$$\int_{\Omega} |\nabla u^-|^2 + \int_{\Omega} |u^-|^2 = \int_{\Omega} |u^-|^p.$$

Thus,  $u^- \in \mathbf{M}(\Omega)$  and hence  $J(u^-) \geq \alpha(\Omega)$ . Suppose that  $J(u^-) = \alpha(\Omega)$ . By Theorem 8,  $u^-$  is a nonzero solution of (1). By the maximum principle,  $u = u^-$ , which contradicts the sign assumption on  $u$ . Thus  $J(u^-) > \alpha(\Omega)$ . Similarly,  $J(u^+) > \alpha(\Omega)$ , where  $u^+ = \max\{u, 0\}$ . Thus,  $J(u) = J(u^+) + J(u^-) > 2\alpha(\Omega)$ .  $\square$

**Remark 10.** A ground state solution admits rich nice properties: by Lemma 9, a ground state solution in  $H_0^1(\Omega)$  is of constant sign. Note that if  $u$  is a solution of Equation (1), then  $-u$  is also a solution of Equation (1). By the maximum principle, if  $u$  is a nonzero and nonnegative solution of Equation (1), then  $u$  is positive. From now on, by a ground state solution in  $H_0^1(\Omega)$ , we mean a positive solution of Equation (1). A ground state solution in  $H_0^1(\Omega)$  is symmetric if  $\Omega$  is symmetric.

## 2. (PS)-condition

We first present the following useful result.

### Theorem 11.

- (i) For each  $(PS)_{\beta}$ -sequence  $\{u_n\}$  in  $H_0^1(\Omega)$  for  $J$ , there is a subsequence  $\{u_n\}$  and a  $u$  in  $H_0^1(\Omega)$  such that  $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$ .
- (ii) Let  $\{u_n\}$  be a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$  and  $u$  in  $H_0^1(\Omega)$  satisfying  $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$ . Then  $u$  is a solution of Equation (1).
- (iii) Let  $\{u_n\}$  be a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$  and  $u$  in  $H_0^1(\Omega)$  such that  $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$  and  $u$  is nonzero. Then  $u$  is a ground state solution of Equation (1) and  $u_n \rightarrow u$  strongly in  $H_0^1(\Omega)$ .

(iv) The  $(PS)_{\alpha(\Omega)}$ -condition holds for  $J$  if and only if for each  $(PS)_{\alpha(\Omega)}$ -sequence  $\{u_n\}$  in  $H_0^1(\Omega)$  for  $J$ , there is a subsequence  $\{u_n\}$  and a nonzero  $u$  in  $H_0^1(\Omega)$  such that  $u_n$  converges to  $u$  weakly and strongly in  $H_0^1(\Omega)$ .

**Proof.** (i) By Lemmas 2.

(ii) By (i), there is a subsequence  $\{u_n\}$  such that  $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$ , a.e. in  $\Omega$ , and strongly in  $L_{loc}^q(\Omega)$  where  $1 \leq q < 2^*$ . We obtain, for  $\phi \in H_0^1(\Omega) \cap C_c^\infty(\Omega)$ ,  $\text{supp } \phi = K$ ,

$$\int_{\Omega} \nabla u_n \cdot \nabla \phi \rightarrow \int_{\Omega} \nabla u \cdot \nabla \phi, \quad \int_{\Omega} u_n \phi \rightarrow \int_{\Omega} u \phi.$$

Let  $g_n = |u_n|^{p-1}|\phi|$  and  $g = |u|^{p-1}|\phi|$ . Then  $||u_n|^{p-2}u_n\phi| \leq g_n$  for each  $n$ ,  $g_n \rightarrow g$  a.e. Since  $1 < p-1 < 2^*$ , by the Rellich-Kondrakov theorem 13,  $g_n \rightarrow g$  in  $L^1(K)$ . By Generalized Lebesgue Dominated Convergence Theorem,

$$\int_{\Omega} |u_n|^{p-2}u_n\phi \rightarrow \int_{\Omega} |u|^{p-2}u\phi.$$

Thus,  $\langle J'(u_n), \phi \rangle \rightarrow \langle J'(u), \phi \rangle$  for each  $\phi \in H_0^1(\Omega) \cap C_c^\infty(\Omega)$ . Since  $\langle J'(u_n), \phi \rangle = o(1)$ , for each  $\phi \in H_0^1(\Omega) \cap C_c^\infty(\Omega)$ , we have  $J'(u) = 0$  in  $H^{-1}(\Omega)$ . Therefore,  $u$  is a solution of Equation (1).

(iii) By (ii),  $u$  is a nonzero solution of Equation (1), hence  $u \in \mathbf{M}(\Omega)$  and

$$\begin{aligned} J(u_n) &= \frac{1}{2}a(u_n) - \frac{1}{p}b(u_n) = \alpha(\Omega) + o(1), \\ \langle J'(u_n), u_n \rangle &= a(u_n) - b(u_n) = o(1). \end{aligned}$$

Thus,

$$a(u_n) = \frac{2p}{p-2}\alpha(\Omega) + o(1).$$

Since  $a$  is weakly lower semicontinuous, we have

$$\alpha(\Omega) \leq J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)a(u) \leq \left(\frac{1}{2} - \frac{1}{p}\right) \liminf_{n \rightarrow \infty} a(u_n) = \alpha(\Omega),$$

or  $J(u) = \alpha(\Omega)$ . By Remark 10, we may assume that  $u$  is positive. Let  $p_n = u_n - u$ , by Brézis-Lieb Lemma, we have

$$J(p_n) = J(u_n) - J(u) + o(1) = o(1).$$



We have that  $\{p_n\}$  is a (PS)-sequence for  $J$ , thus  $\langle J'(p_n), p_n \rangle = o(1)$ . We have

$$a(p_n) = \frac{2p}{p-2} J(p_n) + o(1) = o(1).$$

Thus,  $u_n \rightarrow u$  strongly in  $H_0^1(\Omega)$ .

(iv) This follows by (iii).  $\square$

**Theorem 12.** *Let  $\Omega$  be a domain in  $\mathbb{R}^N$ . If the  $(PS)_{\alpha(\Omega)}$ -condition holds for  $J$ , then there is a ground state solution of Equation (1) in  $\Omega$ .*

**Proof.** By Theorem 11, let  $\{u_n\}$  be a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ , there is a subsequence  $\{u_n\}$  and a nonzero  $u$  in  $H_0^1(\Omega)$  such that  $u_n \rightarrow u$  strongly in  $H_0^1(\Omega)$ , so  $a(u) = b(u)$  and  $J(u_n) = J(u) + o(1)$ , we have that  $u$  is a solution of equation (1) and  $J(u) = \alpha(\Omega)$ .  $\square$

Recall the well known theorem.

**Theorem 13.** *(Rellich-Kondrakov Theorem) Let  $\Omega$  be a domain of finite measure in  $\mathbb{R}^N$ . Then the embedding  $H_0^1(\Omega)$  into  $L^q(\Omega)$  is compact, where  $q \in [1, 2^*)$ .*

**Theorem 14.** *The  $(PS)_{\alpha(\Omega)}$ -condition for  $J$  holds in a bounded domain  $\Omega$ . In particular, there is a ground state solution of (1) in a bounded domain  $\Omega$ .*

**Proof.** Let  $\{u_n\}$  be a  $(PS)_{\alpha(\Omega)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ , by Lemma 2,  $\{u_n\}$  is bounded and

$$J(u_n) = \alpha(\Omega) + o(1), \quad a(u_n) = b(u_n) + o(1).$$

Take a subsequence  $\{u_n\}$  and  $u \in H_0^1(\Omega)$  such that  $u_n \rightharpoonup u$  weakly in  $H_0^1(\Omega)$ . By the Rellich-Kondrakov Theorem 13,  $u_n \rightarrow u$  strongly in  $L^p(\Omega)$ . Suppose  $u = 0$ , then  $b(u_n) = o(1)$ . Thus,  $a(u_n) = o(1)$  and  $J(u_n) = o(1)$ , contradicting that  $\alpha(\Omega) > 0$ . By Theorem 11,  $u$  is a ground state solution in  $H_0^1(\Omega)$  for  $J$  and  $u_n \rightarrow u$  strongly in  $H_0^1(\Omega)$ .  $\square$

Therefore, only interesting domains for studying existence of non-trivial solutions of equation (1) are unbounded domains. We establish the following very useful characterization of (PS)-conditions.

**Theorem 15.** *Let  $\Omega_1 \subsetneq \Omega_2$  and  $J : H_0^1(\Omega_2) \rightarrow \mathbb{R}$  be the energy functional. Let  $\alpha_1 = \alpha(\Omega_1)$  and  $\alpha_2 = \alpha(\Omega_2)$ . Suppose that  $\alpha_2 = \alpha_1$ . Then*

- (i) Equation (1) does not admit any ground state solution in  $\Omega_1$ .
- (ii)  $J$  does not satisfy the  $(PS)_{\alpha_1}$ -condition.
- (iii)  $J$  does not satisfy the  $(PS)_{\alpha_2}$ -condition.

**Proof.** (i) Suppose that Equation (1) admits a ground state solution  $u \in \mathbf{M}(\Omega_1) \subset \mathbf{M}(\Omega_2)$  such that  $J(u) = \alpha_1$ . Then we have  $J(u) = \alpha_1 = \alpha_2 = \min_{v \in \mathbf{M}(\Omega_2)} J(v)$ . By Lemma 8,  $u$  is a ground state solution of (1) in  $\Omega_2$ . By Remark 10,  $u > 0$  in  $\Omega_2$ , which contradicts the fact that  $u \in H_0^1(\Omega_1)$ .

(ii) By part (i) and Theorem 12.

(iii) Let  $\{u_n\}$  in  $H_0^1(\Omega_1)$  satisfy  $J(u_n) = \alpha_1 + o(1)$  and  $J'(u_n) = o(1)$  strongly in  $H^{-1}(\Omega_1)$ . By Lemma 4,  $\{s_n\}$  in  $\mathbb{R}^+$  exists such that  $s_n = 1 + o(1)$ ,  $w_n = s_n u_n \in \mathbf{M}(\Omega_1)$  and  $J(w_n) = \alpha_1 + o(1)$  and  $J'(w_n) = o(1)$  strongly in  $H^{-1}(\Omega_1)$ . Since  $\mathbf{M}(\Omega_1) \subset \mathbf{M}(\Omega_2)$ , we have  $\{w_n\} \subset \mathbf{M}(\Omega_2)$  and  $J(w_n) = \alpha_2 + o(1)$ . By Theorem 5, we have

$$\begin{aligned} J(w_n) &= \alpha_2 + o(1), \\ J'(w_n) &= o(1) \quad \text{strongly in } H^{-1}(\Omega_2). \end{aligned}$$

Suppose that  $J$  satisfies the  $(PS)_{\alpha_2}$ -condition. Then there is a subsequence  $\{w_n\}$  and a  $w \in H_0(\Omega_2)$  satisfying  $w_n \rightarrow w$  strongly in  $H_0^1(\Omega_2)$  and  $J(w) = \alpha_2$ . By Theorem 8,  $w$  is a ground state solution of (1) in  $\Omega_2$ , by Remark 10,  $w > 0$  in  $\Omega_2$ . Since  $\{w_n\} \subset \mathbf{M}(\Omega_1)$  and  $w_n \rightarrow w$  strongly in  $H_0^1(\Omega_2)$ , we have  $w = 0$  in  $(\Omega_1)^c$ . This contradicts the fact that  $w$  is a positive solution of Equation (1) in  $\Omega_2$ . Thus,  $J$  does not satisfy the  $(PS)_{\alpha_2}$ -condition.  $\square$

Let  $z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ . Consider the infinite strip  $\mathbf{A}^r$ , the section strip  $\mathbf{A}_{-2,2}^r$ , the ball  $B^N(z_0; s)$ , the upper semi-strip  $\mathbf{A}_s^r$ , and the interior flask domain  $\mathbf{F}_s^r$  as follows.

$$\begin{aligned} \mathbf{A}^r &= \{(x, y) \in \mathbb{R}^N \mid \|x\| < r\}. \\ \mathbf{A}_{s,t}^r &= \{(x, y) \in \mathbb{R}^N \mid s \leq |y| \leq t\}. \\ B^N(z_0; s) &= \{z \in \mathbb{R}^N : \|z - z_0\| < s\}. \\ \mathbf{A}_s^r &= \{(x, y) \in \mathbf{A}^r \mid s < y\}. \\ \mathbf{F}_s^r &= \mathbf{A}_0^r \cup B^N(0; s). \end{aligned}$$

**Definition 16.** We call  $\Omega$  a large domain in  $\mathbb{R}^N$  if for any  $r > 0$ ,  $z \in \Omega$  exists such that  $B^N(z; r) \subset \Omega$ . We call  $\Omega$  a large domain in  $\mathbf{A}^r$  if for any positive number  $m, a, b$  exist such that  $b - a = m$  and  $\mathbf{A}_{a,b}^r \subset \Omega$ .

By Theorem 15, we have

**Theorem 17.** Let  $\Omega$  be a proper large domain in  $\mathbb{R}^N$ , then

- (i)  $\alpha(\Omega) = \alpha(\mathbb{R}^N)$ .
- (ii)  $\lim_{r \rightarrow \infty} \alpha(B^N(0; r)) = \alpha(\mathbb{R}^N)$ .
- (iii) Equation (1) does not admit any ground state solution in  $\Omega$ .
- (iv)  $J$  does not satisfy the (PS)-condition in  $\Omega$ .
- (v)  $J$  does not satisfy the (PS)-condition in  $\mathbb{R}^N$ .

**Proof.** It suffices to prove part (i). By Theorem 22 below, there is a ground state solution  $w \in H_0^1(\mathbb{R}^N)$  of Equation (1) satisfying

$$a(w) = \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) = b(w) = \int_{\mathbb{R}^N} |w|^p = \left( \frac{2p}{p-2} \right) \alpha(\mathbb{R}^N).$$

For  $r_n \rightarrow \infty$ , take  $\{z_n\} \subset \Omega$  such that  $B^N(z_n; r_n) \subset \Omega$ . Consider the cut-off function  $\eta \in C_c^\infty([0, \infty))$  as follows

$$\eta(t) = \begin{cases} 1 & \text{for } t \in [0, 1]; \\ 0 & \text{for } t \in [2, \infty), \end{cases}$$

For each  $n$ , let  $\eta_n(z) = \eta\left(\frac{2|z|}{n}\right)$  and  $w_n(z) = \eta\left(\frac{2|z-z_n|}{r_n}\right) w(z - z_n)$ . Then  $w_n \in H_0^1(\Omega)$  and

$$\begin{aligned} a(w_n) &= \int_{\Omega} (|\nabla w_n|^2 + w_n^2) = \left( \frac{2p}{p-2} \right) \alpha(\mathbb{R}^N) + o(1), \\ b(w_n) &= \int_{\Omega} |w_n|^p = \left( \frac{2p}{p-2} \right) \alpha(\mathbb{R}^N) + o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus,

$$\begin{aligned} J(w_n) &= \alpha(\mathbb{R}^N) + o(1), \\ a(w_n) &= b(w_n) + o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By Theorem 5,  $\{w_n\}$  is a  $(\text{PS})_{\alpha(\mathbb{R}^N)}$ -sequence in  $H_0^1(\Omega)$  for  $J$ . Therefore,  $\alpha(\Omega) \leq \alpha(\mathbb{R}^N)$ . Clearly,  $\alpha(\mathbb{R}^N) \leq \alpha(\Omega)$ , thus we have  $\alpha(\Omega) = \alpha(\mathbb{R}^N)$ .  $\square$

Similarly, we have

**Theorem 18.** *Let  $\Omega$  be a proper large domain in  $\mathbf{A}^r$ , then*

- (i)  $\alpha(\Omega) = \alpha(\mathbf{A}^r)$ .
- (ii) Equation (1) does not admit any ground state solution in  $\Omega$ .
- (iii)  $J$  does not satisfy the (PS)-condition in  $\Omega$ .
- (iv)  $J$  does not satisfy the (PS)-condition in  $\mathbf{A}^r$ .

For  $k \geq 2$ , let  $\Omega$  be a domain and let  $\Omega_i$  be a proper subdomain in  $\Omega$  for  $i = 1, 2, \dots, k$ , such that  $\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_k$  and  $\Omega_i \cap \Omega_j$  is bounded, for each  $i \neq j$ . Let  $\alpha = \alpha(\Omega)$  and  $\alpha_i = \alpha(\Omega_i)$ , then we have  $\alpha \leq \min\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ .

**Theorem 19.** *The following properties are equivalent:*

- (i)  $J$  satisfies the  $(PS)_\alpha$ -condition.
- (ii)  $\alpha < \min\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ .

**Proof.** See Wang [5] for the proof. □

### 3. Periodic Domains

Let  $\Omega$  be a domain in  $\mathbb{R}^N$ . By Theorem 12, if the  $(PS)_{\alpha(\Omega)}$ -condition holds for  $J$ , then there is a ground state solution of Equation (1) in  $\Omega$ . By Theorem 17,  $J$  does not satisfy the  $(PS)_{\alpha(\mathbb{R}^N)}$ -condition in  $\mathbb{R}^N$ . However, we may use the concentration theorem of P. L. Lions to assert that there is a ground state solution of Equation (1) in  $\mathbb{R}^N$ .

Balls with various centers and a fixed radius was used to the concepts in the concentration theorem. Define the concentration function of  $|u_n|^2$  in  $\mathbb{R}^N$  by

$$Q_n(t) = \sup_{z \in \mathbb{R}^N} \int_{z+B^N(0;t)} |u_n|^2,$$

where  $t > 0$ . Then we have the following concentration theorem.

**Theorem 20** (Lions' Concentration Theorem). *Let  $\{u_n\}$  be bounded in  $H_0^1(\mathbb{R}^N)$  and for some  $t_0 > 0$ , we have  $Q_n(t_0) = o(1)$ . Then*

- (i)  $u_n = o(1)$  strongly in  $L^q(\mathbb{R}^N)$  for  $2 < q < 2^*$ ;

(ii) in addition, if  $u_n$  satisfies

$$-\Delta u_n + u_n - |u_n|^{p-2}u_n = o(1) \quad \text{in } H^{-1}(\mathbb{R}^N),$$

then  $u_n = o(1)$  strongly in  $H_0^1(\mathbb{R}^N)$ .

**Proof.** (i) Decompose  $\mathbb{R}^N$  into the family  $\mathcal{F}_0 = \{P_i^0\}_{i=1}^\infty$  of unit cubes  $P_i^0$  of edge 1. Continue to bisect the cubes to obtain the family  $\mathcal{F}_m = \{P_i^m\}_{i=1}^\infty$  of unit cubes  $P_i^m$  of edge  $\frac{1}{2^m}$ . Let  $m_0$  satisfy  $\sqrt{N}\frac{1}{2^{m_0}} < t_0$ . For each  $i$ , let  $B_i^{m_0}$  be a ball in  $\mathbb{R}^N$  with radius  $t_0$  such that the centers of  $B_i^{m_0}$  and  $P_i^{m_0}$  are the same. Then  $P_i^{m_0} \subset B_i^{m_0}$ ,  $\mathbb{R}^N = \cup_{i=1}^\infty P_i^{m_0}$  and  $\{P_i^{m_0}\}_{i=1}^\infty$  are nonoverlapping. Write  $P_i = P_i^{m_0}$ ,  $2 < q < r < 2^*$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^q &= \sum_{i=1}^\infty \int_{P_i} |u_n|^q = \sum_{i=1}^\infty \int_{P_i} |u_n|^{2(1-t)} |u_n|^{rt} \\ &\leq \sum_{i=1}^\infty \left( \int_{P_i} |u_n|^2 \right)^{1-t} \left( \int_{P_i} |u_n|^r \right)^t \\ &\leq (Q_n(t_0))^{(1-t)} \sum_{i=1}^\infty \left( \int_{P_i} |u_n|^r \right)^t \\ &\leq c(Q_n(t_0))^{(1-t)} \sum_{i=1}^\infty \left( \int_{P_i} |\nabla u_n|^2 + u_n^2 \right)^{rt/2}, \end{aligned}$$

where  $0 < t < 1$ . Since  $\frac{rt}{2} \rightarrow \frac{q}{2} > 1$  as  $r \rightarrow q$ , we may choose  $r$  satisfying  $2 < q < r < 2^*$  and  $s = \frac{rt}{2} > 1$ . Recall that

$$\|\{a_n\}\|_{\ell^s} = \left( \sum_{n=1}^\infty |a_n|^s \right)^{1/s} \leq \sum_{n=1}^\infty |a_n| = \|\{a_n\}\|_{\ell^1}, \quad \ell^1 \subset \ell^2 \subset \dots \subset \ell^\infty.$$

Thus,

$$\begin{aligned} \sum_{i=1}^\infty \left( \int_{P_i} |\nabla u_n|^2 + |u_n|^2 \right)^{rt/2} &\leq \left( \sum_{i=1}^\infty \int_{P_i} (|\nabla u_n|^2 + |u_n|^2) \right)^s \\ &= \left( \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \right)^s = \|u_n\|_{H^1(\mathbb{R}^N)}^{2s} \leq c \quad \text{for } n = 1, 2, \dots \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} |u_n|^q \leq c(Q_n(t_0))^{(1-t)}, \quad \text{or} \quad \int_{\mathbb{R}^N} |u_n|^q = o(1) \quad \text{as } n \rightarrow \infty.$$

(ii) In addition, if  $u_n$  satisfies

$$-\Delta u_n + u_n - |u_n|^{p-2}u_n = o(1) \quad \text{in } H^{-1}(\mathbb{R}^N), \tag{8}$$

Multiply Equation (8) by  $u_n$  and integrate it to obtain

$$a(u_n) = b(u_n) + o(1).$$

By part (i),  $b(u_n) = o(1)$ . Thus,  $a(u_n) = o(1)$ , or  $\|u_n\|_{H^1} = o(1)$  strongly in  $H^1_0(\mathbb{R}^N)$ . □

**Definition 21.** A domain  $\Theta$  in  $\mathbb{R}^N$  is a periodic domain if a partition  $\{Q_m\}_{m=0}^\infty$  of  $\Theta$  and points  $\{z_m\}_{m=1}^\infty$  in  $\mathbb{R}^N$  exist, satisfying the following conditions:

- (i)  $\{z_m\}_{m=1}^\infty$  forms a subgroup of  $\mathbb{R}^N$ .
- (ii)  $Q_0$  is bounded;
- (iii)  $Q_m = z_m + Q_0$  for each  $m$ .

Typical examples of periodic domains are the infinite strip  $\mathbf{A}^r$  and the whole space  $\mathbb{R}^N$ . In Theorem 11, we proved that if a  $(\text{PS})_\alpha(\Omega)$ -sequence for  $J$  admits a nonzero weak limit  $u$ , then  $u$  is a ground state solution for  $J$ . However, even though the weak limit is zero we can still obtain a ground state solution for  $J$  if the domain is periodic.

**Theorem 22.** *Let  $\Theta$  be a periodic domain in  $\mathbb{R}^N$ . Then there is a ground state solution of Equation (1) in  $\Theta$ . In particular, there is a ground state solution of Equation (1) in the infinite strip  $\mathbf{A}^r$  and in the whole space  $\mathbb{R}^N$ .*

**Proof.** It suffices to prove the case  $\Omega = \mathbf{A}^r$ . Let  $\{u_n\}$  be a  $(\text{PS})_{\alpha(\mathbf{A}^r)}$ -sequence such that

$$J(u_n) = \alpha(\mathbf{A}^r) + o(1), \quad J'(u_n) = o(1).$$

By Lemma 2, there are a subsequence  $\{u_n\}$  and a  $u \in H^1_0(\mathbf{A}^r)$  such that

$$u_n \rightharpoonup u \quad \text{weakly in } H^1_0(\mathbf{A}^r).$$

Suppose that  $u$  is nonzero, then by Theorem 11, we are done. Suppose that  $u_n \rightharpoonup 0$  weakly in  $H_0^1(\mathbf{A}^r)$ . Since  $\alpha(\Omega)$  is positive, we have  $u_n \not\rightarrow 0$  strongly in  $H_0^1(\Omega)$ . By Lions' Concentration Theorem 20, there is a subsequence  $\{u_n\}$ , and a constant  $\alpha > 0$  such that for  $n = 1, 2, \dots$ ,

$$Q_n = \sup_{y \in \mathbb{R}} \int_{(0,y) + \mathbf{A}_{-2,2}^r} |u_n(z)|^2 dz > \alpha > 0.$$

Take  $\{z_n\}$  in  $\mathbf{A}^r$ , where  $z_n = (0, y_n)$  such that  $\int_{z_n + \mathbf{A}_{-2,2}^r} |u_n(z)|^2 dz \geq \alpha/2$ , and let  $w_n(z) = u_n(z + z_n)$ . Then for  $n = 1, 2, \dots$ ,

$$\begin{aligned} \int_{\mathbf{A}_{-2,2}^r} |w_n(z)|^2 dz &= \int_{z_n + \mathbf{A}_{-2,2}^r} |u_n(z)|^2 dz \geq \alpha/2, \\ \|w_n\|_{H^1(\mathbf{A}^r)} &= \|u_n\|_{H^1(\mathbf{A}^r)} \leq c, \end{aligned}$$

so  $w \in H_0^1(\mathbf{A}^r)$  exists such that  $w_n \rightharpoonup w$  weakly in  $H_0^1(\mathbf{A}^r)$ . Clearly,  $\{w_n\}$  is a (PS)-sequence in  $H_0^1(\mathbf{A}^r)$  for  $J$ . By Rellich-Kondrakov Theorem 13,

$$\int_{\mathbf{A}_{-2,2}^r} |w|^2 = \lim_{n \rightarrow \infty} \int_{\mathbf{A}_{-2,2}^r} |w_n|^2 \geq \alpha/2,$$

so  $w \not\equiv 0$ . By Theorem 11, there is a ground state solution of Equation (1) in  $H_0^1(\Omega)$ .  $\square$

#### 4. Esteban-Lions Domains

**Definition 23.** A proper smooth unbounded domain  $\Omega$  in  $\mathbb{R}^N$  is an Esteban-Lions domain if  $\chi \in \mathbb{R}^N$  exists with  $\|\chi\| = 1$  such that  $n(z) \cdot \chi \geq 0$ , and  $n(z) \cdot \chi \not\equiv 0$  on  $\partial\Omega$ , where  $n(z)$  is the unit outward normal vector to  $\partial\Omega$  at the point  $z$ .

**Example 24.** The epigraph  $\Pi = \{(x, y) \in \mathbb{R}^N : f(x) < y\}$  and the degenerate interior flask domain  $\mathbf{F}_r^r$  are Esteban-Lions domain, where  $f : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  is a smooth function.

Since the epigraph  $\Pi$  is a proper large domain in  $\mathbb{R}^N$ , by Theorem 17, Equation (1) does not admit any ground state solution in  $\Pi$ . Moreover, Esteban-Lions [1, Theorem I.1] proved that Equation (1) does not admit any higher energy solution in  $\Pi$  as follows.

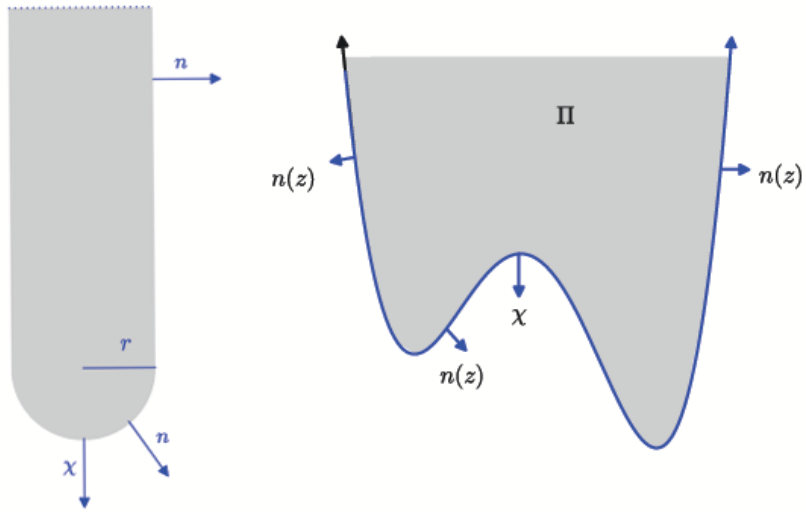


Figure 1: Degenerate interior flask domain (left) and Epigraph  $\Pi$  (right).

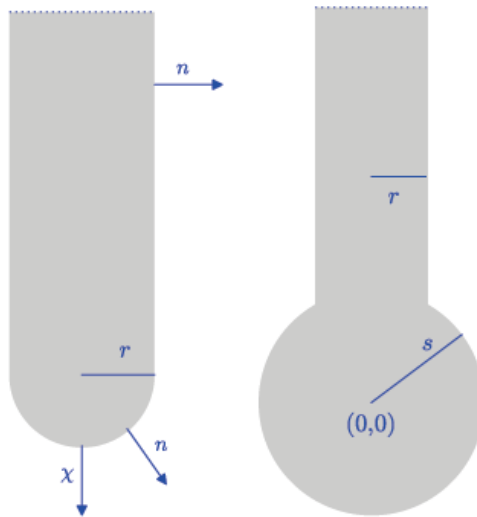


Figure 2: Degenerate interior flask domain (left) and interior flask domain (right).

**Theorem 25.** *Equation (1) in an Esteban-Lions domain  $\Omega$  does not admit any nontrivial solution. In particular, the epigraph  $\Pi$  or the degenerate interior flask domain  $\mathbf{F}_r^r$  admits no any non-trivial solution.*



## 5. Interior Flask Domains

By Theorem 25, Equation (1) does not admit any ground state solution in the degenerate interior flask domain  $\mathbf{F}_r^r$ . In this section, we try to study the existence and non-existence of solutions of general interior flask domains  $\mathbf{F}_s^r$ .

**Theorem 26.** *There is  $s_0 > 0$  such that*

- (i) *the interior flask domain  $\mathbf{F}_s^r$  admits (PS)-condition domain if  $s > s_0$ , in particular, Equation (1) admits a ground state solution in  $\mathbf{F}_s^r$ , for  $s > s_0$ .*
- (ii) *Equation (1) does not have any ground state solution in  $\mathbf{F}_s^r$  if  $s < s_0$ .*

**Proof.** By Theorem 22, the infinite strip  $\mathbf{A}^r$  admits a ground state solution. Then by Theorem 15 (ii),  $\alpha(\mathbf{A}^r) > \alpha(\mathbb{R}^N)$ . By Theorem 18 (ii), we have  $\alpha(\mathbf{A}^r) = \alpha(\mathbf{A}_0^r)$  and by Theorem 17,  $\lim_{s \rightarrow \infty} \alpha(B^N(0; s)) = \alpha(\mathbb{R}^N)$ . Take  $s$  large enough so that

$$\alpha(B^N(0; s)) < \alpha(\mathbf{A}^r) = \alpha(\mathbf{A}_0^r).$$

By Theorem 14, there is a ground state solution of Equation (1) in  $B^N(0; s)$ . Then by Theorem 15 (ii), we have

$$\alpha(\mathbf{F}_s^r) < \alpha(B^N(0; s)).$$

We conclude that

$$\alpha(\mathbf{F}_s^r) < \alpha(B^N(0; s)) < \alpha(\mathbf{A}_0^r),$$

or

$$\alpha(\mathbf{F}_s^r) < \min\{\alpha(B^N(0; s)), \alpha(\mathbf{A}_0^r)\}.$$

By the equivalence of (i) and (ii) in Theorem 19, Equation (1) has a ground state solution in  $\mathbf{F}_s^r$  for large  $s$ . If Equation (1) has a ground state solution in  $\mathbf{F}_{s_1}^r$  and  $s_1 < s_2$ , then  $\mathbf{F}_{s_2}^r = \mathbf{F}_{s_1}^r \cup B^N(0; s_2)$ . By Theorem 14 and Theorem 15 (ii),  $\alpha(\mathbf{F}_{s_2}^r) < \alpha(B^N(0; s_2))$  and  $\alpha(\mathbf{F}_{s_2}^r) < \alpha(\mathbf{F}_{s_1}^r)$ . By the equivalence of (i) and (ii) in Theorem 19, Equation (1) has a ground state solution in  $\mathbf{F}_{s_2}^r$ . Let

$$s_0 = \inf\{s > r : \text{Equation (1) has a ground state solution in } \mathbf{F}_s^r\}.$$

We then conclude that Equation (1) has a ground state solution in  $\mathbf{F}_s^r$  if  $s > s_0$ , and Equation (1) does not have any ground state solution in  $\mathbf{F}_s^r$  if  $s < s_0$ .  $\square$

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