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HIGHER ORDER ANALOGUES OF EXTERIOR DERIVATIVE

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Abstract

We give new examples of linear differential operators of order $k = 2m + 1$ (any given odd integer) that are invariant under the isometries of $\mathbb{R}^n$ and satisfy so-called L^1-duality estimates and div/curl inequalities.

1. Introduction

The purpose of this note is to exhibit (elementary) examples of kth-order linear differential operators $\{\mathcal{S}_{(k)}\}_k$ acting on $\mathbb{R}^n$ that can be regarded as higher order analogues of the exterior derivative complex

$$d : C_q^{\infty,c}(\mathbb{R}^n) \;\to\; C_{q+1}^{\infty,c}(\mathbb{R}^n), \qquad 0 \le q \le n$$

(Here $C_q^{\infty,c}(\mathbb{R}^n)$ and $C_{q+1}^{\infty,c}(\mathbb{R}^n)$ stand for the q-forms and $(q+1)$-forms on $\mathbb{R}^n$ whose coefficients are smooth and compactly supported.) More precisely we require that, for each k, $\mathcal{S}_{(k)}$ map q-forms to $(q+1)$-forms and $\mathcal{S}_{(k)} \circ \mathcal{S}_{(k)} = 0$; that the Hodge Laplacian for $\mathcal{S}_{(k)}$, namely the operator $\mathcal{S}_{(k)}\mathcal{S}_{(k)}^* + \mathcal{S}_{(k)}^*\mathcal{S}_{(k)}$, be elliptic, and that the first-order operator in this family be the exterior derivative (that is, $\mathcal{S}_1 = d$). We also require that $\mathcal{S}_{(k)}$ and $\mathcal{S}_{(k)}^*$ have non-trivial invariance properties and satisfy so-called L^1-duality estimates as well as div-curl inequalities (more on these below). While various operators satisfying one or more of these conditions were recently constructed for any order

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$k = 1, 2, 3, \ldots$, see [6], [12] and [26]-[28], those operators fail to be invariant under pullback by the rotations of $\mathbb{R}^n$ as soon as $k \geq 2$. By contrast, here we define linear differential operators $\mathcal{S}_{(k)}$ of *odd* order

$$k = 2m + 1, \quad m = 0, 1, 2, \ldots,$$

that have the same invariance properties as the codifferential d^* (the L^2-adjoint of exterior derivative) as soon as $k \geq 3$ (i.e. $m \geq 1$); that is

$$\mathcal{S}_{(k)} \circ \psi^* = \psi^* \circ \mathcal{S}_{(k)} \qquad \text{and} \qquad \mathcal{S}^*_{(k)} \circ \psi^* = \psi^* \circ \mathcal{S}^*_{(k)}$$

for any isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ (as customary, ψ^* denotes the pullback of ψ acting on q-forms). While such invariance is non-trivial, it is far weaker than the invariance of d, which indeed is what should be expected of any linear differential operator of order greater than 1, see [19, Note 4] and [23].

Specifically, given $m = 0, 1, 2, 3, \ldots$, we define

$$\mathcal{S}_{(2m+1)} := d\,(d^* d)^m \quad \text{and, consequently,} \quad \mathcal{S}^*_{(2m+1)} = (d^* d)^m d^* \tag{1}$$

It is clear that $\mathcal{S}_{(1)} = d$ and, more generally, that $\mathcal{S}_{(2m+1)}$ takes q-forms to $(q+1)$-forms and $\mathcal{S}_{(2m+1)} \circ \mathcal{S}_{(2m+1)} = 0$. It is also clear that the Hodge Laplacian for $\mathcal{S}_{(2m+1)}$ is

$$\Box_{(2m+1)} = \Box^{2m+1} = \Box \circ \Box \circ \cdots \circ \Box$$

where the composition above is performed $(2m+1)$-many times and

$$\Box = dd^* + d^* d$$

is the Hodge Laplacian for the exterior derivative, so in particular $\Box_{(2m+1)}$ is elliptic because it is the composition of elliptic operators [30].

Note, however, that

$$d \circ \mathcal{S}_{(2m+1)} = 0 \quad \text{and} \quad d^* \circ \mathcal{S}^*_{(2m+1)} = 0$$

see (1), and so the natural compatibility conditions for the data of the Hodge system for $\mathcal{S}_{(2m+1)}$ and $\mathcal{S}^*_{(2m+1)}$ are the same as for the system for d and d^*. As a consequence, the L^1-duality inequalities that are relevant to the Hodge system for $\mathcal{S}_{(2m+1)}$ and $\mathcal{S}^*_{(2m+1)}$ are the same as in [13, page 61] and [24], namely

Proposition 1.1 ([13])**.** *There is $C = C(n)$ such that for any $0 \le q \le n-2$ and for any $f \in C^{\infty,c}_{q+1}(\mathbb{R}^n)$*

$$df = 0 \quad \Rightarrow \quad |\langle f, h\rangle| \le C\|f\|_{L^1_{q+1}(\mathbb{R}^n)}\|\nabla h\|_{L^n_{q+1}(\mathbb{R}^n)} \tag{2}$$

for any $h \in L^\infty_{q+1}(\mathbb{R}^n)$ such that $\nabla h \in L^n_{q+1}(\mathbb{R}^n)$.

There is $C = C(n)$ such that for any $2 \le q \le n$ and for any $g \in C^{\infty,c}_{q-1}(\mathbb{R}^n)$

$$d^*g = 0 \quad \Rightarrow \quad |\langle g, h\rangle| \le C\|g\|_{L^1_{q-1}(\mathbb{R}^n)}\|\nabla h\|_{L^n_{q-1}(\mathbb{R}^n)} \tag{3}$$

for any $h \in L^\infty_{q-1}(\mathbb{R}^n)$ such that $\nabla h \in L^n_{q-1}(\mathbb{R}^n)$.

Here $L^p_{q\pm1}(\mathbb{R}^n)$ denote the spaces of $(q\pm1)$-forms whose coefficients are in the Lebesgue class $L^p(\mathbb{R}^n)$, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2_{q\pm1}(\mathbb{R}^n)$:

$$\langle f, h\rangle = \int_{\mathbb{R}^n} f \wedge *h$$

where $*$ denotes the Hodge-star operator for $\mathbb{R}^n$.

We take this opportunity to point out that these inequalities can be restated in the seemingly more invariant, in fact equivalent, fashion (see also [6, Theorem 1″])

Proposition 1.2. *There is $C = C(n)$ such that for any $0 \le q \le n-2$ and for any $f \in C^{\infty,c}_{q+1}(\mathbb{R}^n)$*

$$df = 0 \quad \Rightarrow \quad |\langle f, h\rangle| \le C\|f\|_{L^1_{q+1}(\mathbb{R}^n)}\|d^*h\|_{L^n_q(\mathbb{R}^n)} \tag{4}$$

for any $h \in L^\infty_{q+1}(\mathbb{R}^n)$ such that $d^*h \in L^n_q(\mathbb{R}^n)$.

There is $C = C(n)$ such that for any $2 \le q \le n$ and for any $g \in C^{\infty,c}_{q-1}(\mathbb{R}^n)$

$$d^*g = 0 \quad \Rightarrow \quad |\langle g, h\rangle| \le C\|g\|_{L^1_{q-1}(\mathbb{R}^n)}\|dh\|_{L^n_q(\mathbb{R}^n)} \tag{5}$$

for any $h \in L^\infty_{q-1}(\mathbb{R}^n)$ such that $dh \in L^n_q(\mathbb{R}^n)$.

We show below that this result is equivalent to each of the following div/curl-type inequalities (one for any choice of $m = 0, 1, 2, \dots$) which are proved with the methods of [13]:

Theorem 1.3. *Fix* $0 \le q \le n$ *and let* $f \in L^1_{q+1}(\mathbb{R}^n)$ *with* $df = 0$, *and* $g \in L^1_{q-1}(\mathbb{R}^n)$ *with* $d^*g = 0$ *be given. Then, for any* $m = 0, 1, 2, 3, \dots,$ *the (unique) q-form* $v_{(m)}$ *that solves the system*

$$\begin{cases} \mathcal{S}_{(2m+1)} v_{(m)} &= f \\ \mathcal{S}^*_{(2m+1)} v_{(m)} &= g \end{cases} \tag{6}$$

belongs to the Sobolev space $W_q^{2m,r}(\mathbb{R}^n)$ *with* $r = n/(n-1)$ *whenever q is neither* 1 *(unless* $g = 0$*) nor* $n-1$ *(unless* $f = 0$*), and we have*

$$\|v_{(m)}\|_{W_q^{2m,r}(\mathbb{R}^n)} \le C\big(\|f\|_{L^1_{q+1}(\mathbb{R}^n)} + \|g\|_{L^1_{q-1}(\mathbb{R}^n)}\big). \tag{7}$$

Here $W_q^{2m,r}(\mathbb{R}^n)$ denotes the space of q-forms whose coefficients belong to the Sobolev space $W^{2m,r}(\mathbb{R}^n)$ of functions that are $2m$-many times differentiable in the sense of distributions and whose derivatives of any order order α $(0 \le |\alpha| \le 2m)$ are in the Lebesgue class $L^r(\mathbb{R}^n)$.

Proposition 1.4. *With same hypotheses as Theorem* 1.3, *if* $q = 1$ *and* $g \ne 0$ *a substitute of* (7) *holds with* $\|g\|_{L^1(\mathbb{R}^n)}$ *replaced by* $\|g\|_{H^1(\mathbb{R}^n)}$, *where* $H^1(\mathbb{R}^n)$ *is the real Hardy space. If* $q = n-1$ *and* $f \ne 0$, *then* (7) *holds with* $\|f\|_{H^1_n(\mathbb{R}^n)}$ *in place of* $\|f\|_{L^1_n(\mathbb{R}^n)}$, *where* $H^1_n(\mathbb{R}^n)$ *is the space of n-forms whose coefficients are in* $H^1(\mathbb{R}^n)$.

In the case when $m = 0$, Theorem 1.3 and Proposition 1.4 were proved in [13], as in such case we have $\mathcal{S}_{(1)} = d$ and $W_q^{0,r}(\mathbb{R}^n) = L^r_q(\mathbb{R}^n)$, and so Theorem 1.3 and Proposition 1.4 can be viewed as a generalization (actually, as we will see, a consequence) of those earlier results.

We remark in closing that one could also consider the operators

$$\mathcal{S}_{(2m)} := (dd^*)^m \quad \text{and} \quad \widetilde{\mathcal{S}}_{(2m)} := (d^*d)^m$$

but these fail to map q-forms to $(q+1)$-forms and do not form a complex and as such are not pertinent to this note.

2. Proofs

We begin by recalling the elliptic estimates for $\Box^s = \Box \circ \cdots \circ \Box$, see [8] and e.g., [30], [20].

Theorem 2.1. *Given any $s \in \mathbb{Z}^+$, we have that*

$$\Box^s : C_q^{\infty,c}(\mathbb{R}^n) \to C_q^{\infty,c}(\mathbb{R}^n)$$

is invertible, and

$$\|(\Box^s)^{-1} u\|_{W_q^{2s,r}(\mathbb{R}^n)} \lesssim \|u\|_{L_q^r(\mathbb{R}^n)} \tag{8}$$

for any $1 < r < \infty$.

***Proof of Theorem* 1.3.** The case $m = 0$ was proved in [13] and here we will show that the estimates in the case when $m \in \mathbb{Z}^+$ follow from the inequalities for $m = 0$. Without loss of generality we may assume: $f \in C_{q+1}^{\infty,c}(\mathbb{R}^n)$ and $g \in C_{q-1}^{\infty,c}(\mathbb{R}^n)$, so that each of d^*f and dg has smooth and compactly supported coefficients.

Applying the codifferential d^* to the first equation in (6) and the exterior derivative d to the second equation, and then adding the two equations, see (1), we find that

$$\Box^{m+1} v_{(m)} = d^* f + dg \tag{9}$$

Comparing $v_{(m)}$ with the solution u of the Hodge system for d and d^* with *same* data as (6), namely

$$\begin{cases} du &= f \\ d^* u &= g \end{cases} \tag{10}$$

we find

$$\Box^m v_{(m)} = u$$

and so the elliptic estimate (8) (with $s := m$) grants

$$\|v_{(m)}\|_{W_q^{2m,r}(\mathbb{R}^n)} \lesssim \|u\|_{L_q^r(\mathbb{R}^n)} \tag{11}$$

for any $1 < r < \infty$. On the other hand, by [13] we have that $u \in L^r_q(\mathbb{R}^n)$ with $r := n/(n-1)$ and

$$\|u\|_{L^r_q(\mathbb{R}^n)} \leq C(n)\big(\|f\|_{L^1_{q+1}(\mathbb{R}^n)} + \|g\|_{L^1_{q-1}(\mathbb{R}^n)}\big). \tag{12}$$

The desired conclusion (7) now follows by combining (11) and (12). □

***Proof of Proposition* 1.4.** The case $m = 0$ was proved in [13] and here we will again only consider $m \in \mathbb{Z}^+$. As before, we may assume: $f \in C^{\infty,c}_{q+1}(\mathbb{R}^n)$ and $g \in C^{\infty,c}_{q-1}(\mathbb{R}^n)$. Now (11) holds as before, and if $q = 1$ and $g \neq 0$ it was proved in [13] that a substitute of (12) holds with $\|g\|_{L^1(\mathbb{R}^n)}$ replaced by $\|g\|_{H^1(\mathbb{R}^n)}$, so the proof of Proposition 1.4 in the case $q = 1$ follows by combining (11) and the H^1-substitute for (12). (The case $q = n-1$ and $f \neq 0$ is proved in a similar fashion.) □

Next we show that Theorem 1.3 (for any choice of $m = 0, 1, 2, \ldots$) is equivalent to Proposition 1.2.

Theorem 1.3 ⇒ *Proposition* 1.2. To prove (4), it again suffices to consider the case when f and h have smooth and compactly supported coefficients; given f as in (4) we consider the solution $v_{(m)}$ (for m fixed arbitrarily) of the system (6) with $g := 0$, namely

$$\begin{cases} d\,(d^*d)^m\, v_{(m)} &= f \\ (d^*d)^m d\, v_{(m)} &= 0 \end{cases}$$

see (1), so that

$$\langle f, h\rangle = \langle d\,(d^*d)^m\, v_{(m)}, h\rangle$$

Integrating by parts the right-hand side of this identity we obtain

$$\langle f, h\rangle = \langle v_{(m)}, (d^*d)^m d^* h\rangle$$

Hölder inequality for $W_q^{2m,\,n/(n-1)}(\mathbb{R}^n)$ and its conjugate space $W_q^{-2m,\,n}(\mathbb{R}^n)$ now grants

$$|\langle f, h\rangle| \leq \|v_{(m)}\|_{W_q^{2m,\,n/(n-1)}(\mathbb{R}^n)} \|(d^*d)^m d^* h\|_{W_q^{-2m,\,n}(\mathbb{R}^n)}$$

and by Theorem 1.3 it thus follows that

$$|\langle f,h\rangle| \le \|f\|_{L^1_{q+1}(\mathbb{R}^n)}\|(d^*d)^m d^*h\|_{W_q^{-2m,\,n}(\mathbb{R}^n)}$$

On the other hand, we have

$$\|(d^*d)^m d^*h\|_{W_q^{-2m,\,n}(\mathbb{R}^n)} = \sup_{\|\zeta\|_{W_q^{2m,\,n/(n-1)}}\le 1} |\langle (d^*d)^m d^*h,\zeta\rangle|$$

Integrating the latter by parts $2m$-many times and applying Hölder inequality for $L^n_q(\mathbb{R}^n)$ and its dual space $L_q^{n/(n-1)}(\mathbb{R}^n)$ we find

$$|\langle (d^*d)^m d^*h,\zeta\rangle| \le \|d^*h\|_{L^n_q}\,\|(d^*d)^m\zeta\|_{L_q^{n/(n-1)}}$$

but

$$\|(d^*d)^m\zeta\|_{L_q^{n/(n-1)}} \le \|\zeta\|_{W_q^{2m,\,n/(n-1)}}$$

which concludes the proof of (4). To prove (5) it suffices to apply (4) to $f := *h \in C^{\infty,c}_{\widetilde{q}+1}(\mathbb{R}^n)$ with $\widetilde{q} := n-q$ (recall that $d^* \approx *d*$ and that $* : L^1_q(\mathbb{R}^n) \to L^1_{n-q}(\mathbb{R}^n)$ is an isometry). □

Proposition 1.2 $\Rightarrow$ *Theorem* 1.3 *for any* $m = 0,1,2,\ldots$ Without loss of generality we may assume, as before, that $f \in C^{\infty,c}_{q+1}(\mathbb{R}^n)$ and $g \in C^{\infty,c}_{q-1}(\mathbb{R}^n)$. Fix $m \in \{0,1,2,3,\ldots\}$ arbitrarily and write

$$v_{(m)} = X_{(m)} + Y_{(m)}$$

where

$$\begin{cases} d(d^*d)^m X_{(m)} &=& f \\ (d^*d)^m d^*\, X_{(m)} &=& 0 \end{cases} \tag{13}$$

and

$$\begin{cases} d(d^*d)^m Y_{(m)} &=& 0 \\ (d^*d)^m d^*\, Y_{(m)} &=& g \end{cases} \tag{14}$$

see (1). We claim that

$$\|X_{(m)}\|_{W_q^{2m,\,n/(n-1)}} \le C\|f\|_{L^1_{q+1}}, \tag{15}$$

and

$$\|Y_{(m)}\|_{W_q^{2m,n/(n-1)}} \leq C\|g\|_{L^1_{q-1}} \tag{16}$$

Note that if $Y_{(m)}$ solves (14) then $X_{(m)} := *Y_{(m)}$ solves (13) with $f := *g \in C^{\infty,c}_{\widetilde{q}+1}(\mathbb{R}^n)$ and $\widetilde{q} := n - q$, and so it suffices to prove (15) for f and $X_{(m)}$ as in (13). (Note that the proof of (15) is non-trivial only for $q \neq n$, and the hypotheses of Theorem 1.3 require $q \neq n-1$, so all together we may assume $0 \leq q \leq n-2$.) By duality, proving (15) is equivalent to showing

$$\left|\langle D^\beta X_{(m)}, \varphi\rangle\right| \leq C\|f\|_{L^1_{q+1}}\|\varphi\|_{L^n_q} \tag{17}$$

for any $\varphi \in C^{\infty,c}_q(\mathbb{R}^n)$ and for any multi-index β of length s (that is, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n,\ \beta_1 + \cdots + \beta_n = s$) and for any $0 \leq s \leq 2m$, where we have set

$$D^\beta X_{(m)} := \sum_{|I|=q} \left(\frac{\partial^s X_{(m)_I}}{\partial x^\beta}\right) dx^I .$$

To this end, write $\varphi = \Box^{m+1}\Phi$ for some $\Phi \in C^{\infty,c}_q(\mathbb{R}^n)$, see Theorem 2.1; then

$$\left|\langle D^\beta X_{(m)}, \varphi\rangle\right| = \left|\langle D^\beta X_{(m)}, \Box^{m+1}\Phi\rangle\right|$$

Integrating the right-hand side of this identity by parts we find

$$\left|\langle D^\beta X_{(m)}, \varphi\rangle\right| = \left|\langle \Box^{m+1} X_{(m)}, D^\beta\Phi\rangle\right|$$

But $\Box^{m+1}X_{(m)} = d^*f$, see (13) and so

$$\left|\langle D^\beta X_{(m)}, \varphi\rangle\right| = \left|\langle d^*f, D^\beta\Phi\rangle\right| = \left|\langle f, dD^\beta\Phi\rangle\right|.$$

Applying Proposition 1.2 to $h := dD^\beta\Phi \in C^{\infty,c}_{q+1}(\mathbb{R}^n)$ we conclude

$$\left|\langle D^\beta X_{(m)}, \varphi\rangle\right| \leq C(n)\|f\|_{L^1_{q+1}}\|d^*dD^\beta\Phi\|_{L^n_q} \leq C(n)\|f\|_{L^1_{q+1}}\|\Phi\|_{W_q^{2(m+1),n}}$$

On the other hand, since we had chosen $\Phi = (\Box^{m+1})^{-1}\varphi$, Theorem 2.1 grants

$$\|\Phi\|_{W_q^{2(m+1),n}} \lesssim \|\varphi\|_{L^n_q}$$

which combines with the previous estimates to give the desired inequality.□

It should by now be clear that Propositions 1.1 and 1.2 are equivalent to one another: on the one hand, it is obvious that Proposition 1.2 $\Rightarrow$ Proposition 1.1 (because $\nabla h \in L^n_{q\pm 1} \Rightarrow dh \in L^n_{(q+1)\pm 1}$ and $d^*h \in L^n_{(q-1)\pm 1}$ and, moreover, $\|dh\|,\ \|d^*h\| \leq \|\nabla h\|$). On the other hand, it was proved in [13, page 61] that Proposition 1.1 $\Rightarrow$ Theorem 1.3 in the case $m = 0$ which in turn, as we have just seen, gives Theorem 1.3 for arbitrary m as well as Proposition 1.2.

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