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DIRICHLET-NEUMANN KERNEL FOR HYPERBOLIC-DISSIPATIVE SYSTEM IN HALF-SPACE

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Dedicated to Raghu Varadhan on the occasion of his 70th Birthday.

Abstract

The purpose of the present paper is to initiate a systematic study of the relation of the boundary values for hyperbolic-dissipative systems of partial differential equations. We introduce a general framework for explicitly deriving the boundary kernel for the Dirichlet-Neumann map. We first use the Laplace and Fourier transforms, and the stability consideration to derive the Master Relationship, the Dirichlet-Neumann relation in the transformed variables. New idea of Fourier-Laplace path and algebraic considerations are introduced for the explicit inversion of Fourier-Laplace transforms. We illustrate the basic ideas by carrying out the framework to models in the gas dynamics and the dissipative wave equations.

1. Introduction

The present paper is to introduce a general approach for the study of the relation of the boundary values for the solutions of hyperbolic-dissipative systems of partial differential equations. We aim at establishing a new algorithm for explicitly deriving the boundary kernel for the Dirichlet-Neumann map. In [5], explicit computations have been carried out for the Dirichlet-Neumann map for a system modeling the D'Alembert wave equation with

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viscosity. This helps to initiate the present effort of searching for a general framework for general systems of the hyperbolic-dissipative system:

$$\partial_t \boldsymbol{u} + \sum_{j=1}^{m} \boldsymbol{A}_j \partial_{x^j} \boldsymbol{u} = \sum_{j,i=1}^{m} \boldsymbol{B}_{ji} \partial^2_{x^j x^i} \boldsymbol{u}. \tag{1.1}$$

The dependence variables are $\boldsymbol{u} = \boldsymbol{u}(\vec{\boldsymbol{x}}, t) \in \mathbb{R}^n$. The space variables are $\vec{\boldsymbol{x}} \equiv (x^1, x^2, \dots, x^m) \equiv (x^1, \hat{\boldsymbol{x}}) \in \mathbb{R}^m_+$, so that the boundary of the spatial domain is

$$\partial \mathbb{R}^m_+ = \{\vec{\boldsymbol{x}}_\| : \ \vec{\boldsymbol{x}}_\| \equiv (0, \hat{\boldsymbol{x}}), \ \hat{\boldsymbol{x}} = (x^2, \dots, x^m) \in \mathbb{R}^{m-1}\}. \tag{1.2}$$

We are interested in the relation between the boundary value $\boldsymbol{u}(\vec{\boldsymbol{x}}_\|, t), \ t > 0$, and its normal derivatives $\partial_{x^1} \boldsymbol{u}(\vec{\boldsymbol{x}}_\|, t), \ t > 0$. The coefficients $\boldsymbol{A}_j, \ \boldsymbol{B}_{ij} \in M_n(\mathbb{R})$ are $n \times n$ matrices in $\mathbb{R}$ and satisfy, for any $(\omega^1, \dots, \omega^m)$ on the unit sphere $S^{m-1} \equiv \{\vec{\boldsymbol{x}} \in \mathbb{R}^m | \ |\vec{\boldsymbol{x}}| = 1\}$,

(Hyperbolicity) • All eigenvalues of $\sum_{j=1}^{m} \omega^j \boldsymbol{A}_j$ are real numbers,

(Dissipation) • The matrix $\sum_{i,j=1}^{m} \omega^i \omega^j \boldsymbol{B}_{ji}$ is a non-negative definite matrix.

For a number of the physically interesting models of the form (1.1), there is a satisfactory theory for the initial value problem, cf. [1], $\vec{\boldsymbol{x}} \in \mathbb{R}^m, \ t \geq 0$, and the Green's function can be constructed explicitly, c.f. [8, 6, 2, 3, 4]. As is well-known, the initial-boundary value problem can be solved explicitly only for some particularly designed boundary value condition, for which there is obvious symmetries to convert of the initial-boundary value problem to an initial value problem. In the general situation, such sharp harmonic analytic tools often fall short. Our focus is not on solving specific initial-boundary value problems. Instead, our goal here is to look for new techniques for explicitly constructing the kernel relating the Dirichlet and Neumann values on the boundary. The Dirichlet-Neumann kernel is of the same basic significance as the symbol for an differential operator, and the Green's function for an initial-boundary value problem can be readily constructed in terms of the kernel. The study of the boundary relations would be useful also when the boundary is virtual as in the numerical computations.

In the following, we outline the basic steps for the proposed framework:

I. Green's function, Fundamental Solution, and Dirichlet-Neumann map.

The Green's function for the initial value problem can be used to transfer the initial information to the boundary datum, so that the solution has certain homogeniety property in the direction normal to the boundary. The fundamental solution $\mathbb{G}(\vec{\boldsymbol{x}},t)$, the Green's function for the initial value problem for the system (1.1), is defined as

$$\begin{cases} \Big(\partial_t + \sum_{j=1}^{m} \boldsymbol{A}_j \partial_{x^j} - \sum_{i,j=1}^{m} \boldsymbol{B}_{ij} \partial^2_{x^i x^j}\Big)\mathbb{G}(\vec{\boldsymbol{x}},t) = 0, \\ \mathbb{G}(\vec{\boldsymbol{x}},0) = \delta(\vec{\boldsymbol{x}}) I_{n\times n}, \end{cases} \tag{1.3}$$

where $I_{n\times n}$ is the $n\times n$ identity matrix. Let $G(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)$ be the Green's function for a homogeneous initial-boundary value problem is defined as:

$$\begin{cases} \Big(\partial_t + \sum_{j=1}^{m} \boldsymbol{A}_j \partial_{x^j} - \sum_{i,j=1}^{m} \boldsymbol{B}_{ij} \partial^2_{x^i x^j}\Big) G = 0, \\ G(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)|_{\vec{\boldsymbol{x}}\in\partial\mathbb{R}^m_+} = 0, \\ G(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},0) = \delta(\vec{\boldsymbol{x}}) I_{n\times n}. \end{cases} \tag{1.4}$$

Their difference

$$H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t) \equiv G(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t) - \mathbb{G}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}},t)$$

satisfies

$$\begin{cases} \Big(\partial_t + \sum_{j=1}^{m} \boldsymbol{A}_j \partial_{x^j} - \sum_{i,j=1}^{m} \boldsymbol{B}_{ij} \partial^2_{x^i x^j}\Big) H = 0, \\ H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)|_{\vec{\boldsymbol{x}}\in\partial\mathbb{R}^m_+} = -\mathbb{G}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)|_{\vec{\boldsymbol{x}}\in\partial\mathbb{R}^m_+}, \\ H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},0) = 0. \end{cases} \tag{1.5}$$

By integrating (1.5) times $\mathbb{G}$ and applying (1.3), one can represent $H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)$ in terms of the fundamental function $\mathbb{G}$ and the boundary values:

$$H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t) = \int_0^t \int_{\partial\mathbb{R}^m_+} \mathbb{G}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{z}},t-\tau)\Big(\boldsymbol{A}_1 H(\vec{\boldsymbol{z}},\vec{\boldsymbol{y}},\tau) - \sum_{j=1}^{m} \boldsymbol{B}_{1j}\partial_{x^1} H(\vec{\boldsymbol{z}},\vec{\boldsymbol{y}},\tau)\Big) d\vec{\boldsymbol{z}}\, d\tau$$

$$+\int_0^t \int_{\partial\mathbb{R}_+^m} \boldsymbol{B}_{11}\partial_{x^1}\mathbb{G}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{z}},t-\tau)H(\vec{\boldsymbol{z}},\vec{\boldsymbol{y}},\tau)d\vec{\boldsymbol{z}}d\tau. \tag{1.6}$$

This gives the representation of $H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)$ in terms of the boundary Dirichlet values $H(\vec{\boldsymbol{z}},\vec{\boldsymbol{y}},\tau)$ and the Neumann values $\partial_{x^1}H(\vec{\boldsymbol{z}},\vec{\boldsymbol{y}},\tau)$, $\vec{\boldsymbol{z}}\in\mathbb{R}_+^m$. On the other hand, for a well-posed initial-boundary value problem, only some combination of the Dirichlet and the Neumann boundary values are given. A Dirichlet-Neumann relation would provide both the Dirichlet and the Neumann boundary values and thereby the explicit representation of $H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)$ through (1.6) and consequently yields the explicit construction of the Green's function

$$G(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)=H(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}},t)+\mathbb{G}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}},t).$$

In other words, the notion of the boundary relation is more basic than that of the Green's function for the initial-boundary value problem.

II. Laplace-Fourier transforms.

From the first step just outlined, for the study of the Dirichlet-Neumann map, we may consider, for simplicity, the problem with zero initial values:

$$\begin{cases}\partial_t\boldsymbol{u}+\displaystyle\sum_{j=1}^m \boldsymbol{A}_j\partial_{x^j}\boldsymbol{u}=\sum_{j,i=1}^m \boldsymbol{B}_{ji}\partial^2_{x^jx^i}\boldsymbol{u} \text{ for } x^1>0,\\ \boldsymbol{u}(\vec{\boldsymbol{x}},0)\equiv 0.\end{cases} \tag{1.7}$$

The Fourier and Laplace transforms are used to convert the initial-boundary value problem, with zero initial data, for a differential system into an algebraic, polynomial system. We note that this step and the next steps are also standard procedure for the study of initial-boundary value problem. As we are not confined to the study of the initial-boundary value problem, the boundary value at the boundary $x^1=0$ is not given. Nevertheless, it is known that when $\boldsymbol{B}_{11}$ is a full rank matrix, for well-posedness, the full boundary Dirichlet value can be given. In the general situation, there is no theory to provide the well-posed boundary Dirichlet data for a given $\boldsymbol{B}_{11}$. It is understood that the form of boundary data depends on the structure of the equations under consideration. The second example on the compressible Navier-Stokes equations in the present study amply illustrates this basic point. As in standard analysis for half space problem, cf. [9], [7], we take

Fourier transform $\mathscr{F}$ in the tangential directions $\hat{\boldsymbol{x}}$, with the transformed variable $\hat{\zeta} = (\zeta^2, \ldots, \zeta^m)$. We then take the Laplace transform $\mathbb{L}$ with respect to the time variable t, and finally another Laplace transform with respect to the spatial direction x^1 normal to the boundary:

$$\begin{cases} \boldsymbol{v}(x^1,\hat{\zeta},t) = \mathscr{F}[\boldsymbol{u}](x^1,\hat{\zeta},t) \equiv \int_{\mathbb{R}^{m-1}} \boldsymbol{u}(x^1,\hat{\boldsymbol{x}},t)e^{-i\hat{\boldsymbol{x}}\cdot\hat{\zeta}}d\hat{\boldsymbol{x}}, \\ \qquad\qquad\qquad\qquad \text{(Fourier transform in } \hat{\boldsymbol{x}}), \\ V(x^1,\hat{\zeta},s) = \mathbb{L}[\boldsymbol{v}](x^1,\hat{\zeta},s) \equiv \int_0^\infty e^{-st}\boldsymbol{v}(x^1,\hat{\zeta},t)dt, \\ \qquad\qquad\qquad\qquad \text{(Laplace transformation in } t), \\ \hat{V}(\xi,\hat{\zeta},s) = \mathbb{L}[V](\xi,\hat{\zeta},s) \equiv \int_0^\infty e^{-\xi x^1}V(x^1,\hat{\zeta},s)dx^1, \\ \qquad\qquad\qquad\qquad \text{(Laplace transform in } x^1). \end{cases} \tag{1.8}$$

The Dirichlet and Neumann values on the boundary are denoted by:

$$\begin{cases} \boldsymbol{u}^0(\hat{\boldsymbol{x}},t) \equiv \boldsymbol{u}(\vec{\boldsymbol{x}},t)|_{x^1=0} = \boldsymbol{u}(\vec{\boldsymbol{x}}_{\|},t);\ \boldsymbol{u}^0_{x^1}(\hat{\boldsymbol{x}},t) \equiv \dfrac{\partial \boldsymbol{u}}{\partial x^1}(\vec{\boldsymbol{x}},t)\Big|_{x^1=0}, \\ V^0(\hat{\zeta},s) \equiv V(0,\hat{\zeta},s);\ V^0_{x^1}(\hat{\zeta},s) \equiv \dfrac{\partial V}{\partial x^1}(x_1,\hat{\zeta},s)\Big|_{x^1=0}. \end{cases} \tag{1.9}$$

To be consistent with the zero initial condition, the boundary value is taken to be in the space$\mathscr{V}$:

$$\boldsymbol{u}^0(\vec{\boldsymbol{x}}_{\|},\cdot) \in \mathscr{V} \equiv \{g \in C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) | g^{[n]}(0) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}\}.$$

Take the Fourier transform of the system (1.7) with respect to the tangential variables $\hat{\boldsymbol{x}}$ to obtain a 1-D partial differential equation:

$$\begin{aligned} &(\partial_t + \boldsymbol{A}_1\partial_{x^1} - \boldsymbol{B}_{11}\partial^2_{x^1})\boldsymbol{v} \\ &\quad = \Big(i\sum_{j=2}^m \zeta^j(-\boldsymbol{A}_j + (\boldsymbol{B}_{1j} + \boldsymbol{B}_{j1})\partial_{x^1}) - \sum_{2\le k,j\le m} \zeta^k\zeta^l\Big)\boldsymbol{v}, \quad \text{for } x^1, t > 0. \end{aligned} \tag{1.10}$$

Next, we take the Laplace transform $\mathbb{L}$, first with respect to the time variable t and then with respect to the normal spatial direction x^1, so that the half space problem (1.7) turns to the following algebraic equations:

$$\Big(s + \xi\boldsymbol{A}_1 - \xi^2\boldsymbol{B}_{11} + i\sum_{j=2}^m \zeta^j(\boldsymbol{A}_j - (\boldsymbol{B}_{1j} + \boldsymbol{B}_{j1})\xi) + \sum_{2\le i,j\le m} \zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)\hat{V}(\xi,\hat{\zeta},s)$$

$$= -\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)+\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}-i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s). \qquad (1.11)$$

The algebraic system (2.6) is solved for the transformed variable in terms of the transformed boundary data:

$$\begin{aligned}
&\hat{V}(\xi,\hat{\zeta},s)\\
=\ &\Big(s+\xi\boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m\zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)^{-1}\\
&\cdot\Big(-\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)+\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}-i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s)\Big)\\
=\ &\frac{adj\Big(s+\xi\boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m\zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)}{p(\xi;\hat{\zeta},s)}\\
&\cdot\Big(-\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)+\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}-i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s)\Big)\\
\equiv\ &soln(\xi,\hat{\zeta},s;V^0,V^0_{x^1}). \qquad (1.12)
\end{aligned}$$

III. The Dirichlet-Neumann map in the transformed variable, the Master Relationship.

This step is to apply the stability, well-posedness analysis to identify the symbol of differential operator in the direction normal to the boundary in terms of that in the tangential direction.

One applies the Bromwich integral for the inversion of the Laplace transform:

$$\begin{cases} F(\beta)=\mathbb{L}[f](\beta)\equiv\displaystyle\int_0^\infty e^{-\beta\alpha}f(\alpha)d\alpha,\\ f(\alpha)=\mathbb{L}^{-1}[F](\beta)\equiv\displaystyle\frac{1}{2\pi i}\int_{Re(\beta)=0}e^{\alpha\beta}F(\beta)d\beta.\end{cases} \qquad (1.13)$$

The entries in (1.12) are rational functions with the characteristic polynomial $p(\xi;\hat{\zeta},s)$ as the denominator

$$p(\xi;\hat{\zeta},s)\equiv det\Big(s+\xi\boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^{m}\zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big). \tag{1.14}$$

For $\hat{\zeta}$ and s fixed, let $\xi=\lambda_j(\hat{\zeta},s)$ be the roots of the characteristic polynomial

$$p(\lambda_j(\hat{\zeta},s);\hat{\zeta},s)=0. \tag{1.15}$$

Suppose that each root is simple. Then these roots represent poles for the rational functions and the inverse Laplace transform in x^1 is given as the sum of residue at the poles by the Bromwich integral (1.13) for inverting the Laplace transform in x^1:

$$\hat{V}(x^1,\hat{\zeta},s)=\sum_{p(\lambda_j;\hat{\zeta},s)=0}e^{\lambda_j x^1}\operatorname*{Res}_{\xi=\lambda_j}(soln(\xi,\hat{\zeta},s;V^0,V^0_{x^1})). \tag{1.16}$$

The standard well-posedness condition that $\lim_{x^1\to\infty}V(\vec{\boldsymbol{x}},t)=0$ yields a system of algebraic equations on V^0 and $V^0_{x^1}$:

$$\operatorname*{Res}_{\substack{\xi=\lambda_j\\ p(\lambda_j;\hat{\zeta},s)=0\\ Re(\lambda_j)>0}} soln(\xi,\hat{\zeta},s;V^0,V^0_{x^1})=0. \tag{1.17}$$

The relation (1.17) is called the *Master Relationship.* The Master Relationship contains the vital information of the boundary behavior through the *symbol* $p(i\xi;\hat{\zeta},\sigma)$ of the partial differential equations (1.1). The Master Relationship is as basic as the symbol $p(i\xi;\hat{\zeta},s)$ of the system and is defined in terms of it. The boundary behavior is encoded in the *algebraic property* of the Master Relationship. The study of the algebraic property is one key ingredient of the present theory and needs to be explored for each particular system.

The Master Relationship (1.17) gives an implicit relation of the boundary Dirichlet and Neumann values in the transformed variables $(\hat{\zeta},s)\in\mathbb{R}^{m-1}\times\mathbb{R}_+$. Denote the roots of the characteristic polynomial with positive real part by $\lambda_{j_1},\ldots,\lambda_{j_l}$:

$$Re(\lambda_{j_1})>0,\ldots,Re(\lambda_{j_l})>0.$$

We rewrite the Master Relationship explicitly as Dirichlet-Neumann map in terms of the roots of the characteristic polynomial:

$$V^0_{x^1}(\hat{\zeta}, s) = (K_{ij}(\hat{\zeta}, s; \lambda_{j_1}, \ldots, \lambda_{j_l}))_{m\times m} V^0(\hat{\zeta}, s), \tag{1.18}$$

where the entries of matrix $(K_{ij}(\hat{\zeta}, s; \lambda_{j_1}, \ldots, \lambda_{j_l}))_{m\times m}$ are rational functions in $\lambda_{j_1}(\hat{\zeta}, s), cdots, \lambda_{j_l}(\hat{\zeta}, s)$. The main effort is to invert the transforms for the entries K_{ij}, first the Laplace transform in s and then the Fourier transform in $\hat{\zeta}$.

IV. Characteristic and non-characteristic interior wave operators.

If there is no branch point of the analytic function $\lambda_j(\hat{\zeta}, s)$ for any s with $Re(s) \geq 0$, then the inverse Laplace transform can be computed by the Bromwich integral (1.13), and the kernel function $\mathfrak{N}_{ij}(\hat{\boldsymbol{y}}, t)$ of Dirichlet-Neumann map in the physical variables is given as:

$$\begin{cases} \mathfrak{N}_{ij}(\hat{\boldsymbol{y}}, t) \equiv \mathscr{F}^{-1}\Big[\dfrac{1}{2\pi i}\displaystyle\int_{Re(s)=0} e^{st} K_{ij}(\hat{\zeta}, s; \lambda_{j_1}, \ldots, \lambda_{j_l}) ds\Big](\hat{\boldsymbol{y}}), \\ \boldsymbol{u}^0_{x^1}(\hat{\boldsymbol{y}}, t) = \int_0^t \int_{\mathbb{R}^{m-1}} \mathfrak{N}_{ij}(\hat{\boldsymbol{y}} - \vec{\boldsymbol{z}}, t-s) \boldsymbol{u}^0(\vec{\boldsymbol{z}}, s) d\vec{\boldsymbol{z}} ds. \end{cases} \tag{1.19}$$

In general, the operator $\mathfrak{N}_{ij}$ is a differential operator in the distributional sense and acts on functions defined in the tangential direction.

As $p(i\xi; \hat{\zeta}, s)$ is the symbol of the system, its root λ_j represents wave propagation for a *whole space* problem. We introduce the corresponding *Interior wave propagation operator on the boundary* $\partial\mathbb{R}^m_+$ as follow:

$$\mathfrak{L}_j(\hat{\boldsymbol{y}}, t) \equiv \mathscr{F}^{-1}\Big[\frac{1}{2\pi i}\int_{Re(s)=0} e^{st} \lambda_j(\hat{\zeta}, s) ds\Big](\hat{\boldsymbol{y}}),\ \hat{\boldsymbol{y}} \in \partial\mathbb{R}^m_+,\ j = j_1, \ldots, j_l. \tag{1.20}$$

Definition 1.1 (Characteristic and Non-Characteristic Interior Wave Operators). A root λ_j is non-characteristic if and only if there exist κ_1 and $\kappa_2 > 0$ such that the root $\lambda_j(\hat{\zeta}, s)$ is analytic in $Re(s) \in (-\kappa_1, 0)$ and any $|\hat{\zeta}| < \kappa_0$; otherwise, λ_j is characteristic. The corresponding operator $\mathfrak{L}_j$ is (non-)characteristic if and only if λ_j is (non-)characteristic.

For each non-characteristic λ_j, $\mathbb{L}^{-1}[\lambda_j](\hat{\zeta}, t,\ |\hat{\zeta}| \ll 1$ has a local in time, exponentially decaying structure. Note here that the distinction is for $|\hat{\zeta}|$

small, $|\hat{\zeta}| < \kappa_0$, that is, for the *long waves* Fourier component in the tangential directions. The short wave components automatically have localized property. In general, the roots λ_j may have quite varying behavior in this regard as $\hat{\zeta}$ varies. We will see that this notion of characteristic and non-characteristic interior wave operators draw a clear distinction between 1-D problems and multi-D problem. For example, for our second example of the 1-D compressible Navier-Stokes equation with a subsonic background velocity, the interiors wave operators are non-characteristic. On the other hand, all interiors wave operators for 2-D problem in our third example are characteristic. This has basic implication on the boundary wave propagation.

V. Laplace-Fourier paths for the characteristic interior wave operator.

A branch cut in the complex domain for the Laplace transformation of the time variable gives rise to the spectrum of the full Fourier transformation on the path along the cut. The notion of *Fourier-Lapalce paths* is a key element of our analysis to convert the inverse Laplace transform into an inverse Fourier transform, and so the symmetry is restored on this path and the harmonic analysis techniques can be applied to yield exponentially sharp wave structure for long wave component in a finite non-dissipative wave region. This conversion is to go from the Fourier-Laplace variable $(\hat{\zeta}, s)$, through the new Fourier variable ζ^1 for the Laplace variable s, to the combined Fourier variables $\vec{\zeta} = (\zeta^1, \hat{\zeta})$. The new Fourier variable is defined by $i\zeta^1 = \lambda_j(\hat{\zeta}, s)$, with the resulting implicit relation $s = \sigma(\zeta^1; \hat{\zeta}) = \sigma_j(\zeta^1; \hat{\zeta})$. The correspondence is illustrated for the 2-D case, $\vec{\boldsymbol{x}} = (x, y)$, $\vec{\zeta} = (\zeta^1, \zeta^2)$, in the following table.

The elementary waves are of the form $\alpha(s, \xi, \zeta^2)e^{\xi(s)x + i\zeta^2 y}$ because of the assumed the zero initial data $\boldsymbol{u}|_{t=0} = 0$.

The above two systems are related through the characteristic polynomial $p(\xi; \hat{\zeta}, s)$:

$$p(i\zeta^1; \hat{\zeta}, \sigma(\zeta^1; \hat{\zeta})) = 0 = p(\lambda_j(\hat{\zeta}, s); \hat{\zeta}, s). \tag{1.21}$$

We call

$$\Gamma_j \equiv \{s = \sum_j \sigma_j(\zeta^1; \hat{\zeta}) | \zeta^1 \in \mathbb{R}\}$$

	half space problem	whole space problem
D.E.	$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{A}_1 \partial_x \boldsymbol{u} \\ + \boldsymbol{A}_2 \partial_y \boldsymbol{u} - \Delta \boldsymbol{u} = 0 \\ U\|_{t=0} = 0 \end{cases}$	$\partial_t \boldsymbol{u} + \boldsymbol{A}_1 \partial_x \boldsymbol{u}$ $+ \boldsymbol{A}_2 \partial_y V - \Delta \boldsymbol{u} = 0$
Elementary form of solution in transform variables	$\alpha(s, \xi, \zeta^2) e^{\xi(s)x + i\zeta^2 y}$	$\beta(\zeta^1, \zeta^2) e^{\sigma(\eta)t} e^{i\zeta^1 x + i\zeta^2 y}$
characteristic equation	$\det((s - \xi^2 + \|\zeta^2\|^2) I$ $+ \xi \boldsymbol{A}_1 + i\zeta^2 \boldsymbol{A}_2) = 0$	$\det((\sigma(\vec{\zeta}) + \|\vec{\zeta}\|^2) I$ $+ i\zeta^1 \boldsymbol{A}_1 + i\zeta^2 \boldsymbol{A}_2) = 0,$ $\vec{\zeta} = (\zeta^1, \zeta^2)$
Independent transform variables in characteristic equation	(s, ζ^2)	(ζ^1, ζ^2)

the *Laplace-Fourier path.* The complex contour integral relates the inverse Laplace transformation to a pre-inverse Fourier transformation:

$$\begin{aligned}
\sum_j \frac{1}{2\pi i} \int_{Re(s)=0} e^{st} \partial_s \lambda_j(\hat{\zeta}, s) ds &= \frac{1}{2\pi i} \sum_j \int_{\Gamma_j} e^{st} \partial_s (i\zeta^1) \partial_{\zeta^1} s d\zeta^1 + O(1) e^{-\alpha t}, \\
\sum_j \frac{1}{2\pi i} \int_{\Gamma_j} e^{st} \partial_s (i\zeta^1) \partial_{\zeta^1} s d\zeta^1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sigma(\zeta^1; \hat{\zeta}) t} d\zeta^1 \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix^1 \zeta^1} e^{\sigma(\zeta^1; \hat{\zeta}) t} d\zeta^1 |_{x^1 = 0}.
\end{aligned}$$

One can treat the variable ζ^1 as an imaginary Fourier variable for the direction normal to the tangent direction $\partial \mathbb{R}^m_+$. That this relation works conveniently for $\partial_s \lambda_j$ and not for λ_j was implicitly hinted in our previous work [5] on detailed computations for a specific example.

The introduction of the Laplace-Fourier path for the inverse Laplace transformation is a major observation allowing one to take advantage of existing tools such as harmonic analysis or other developed tools for the whole space problem to obtain a sharp structure of $\mathfrak{L}_j(\hat{\boldsymbol{y}}, t)$.

VI. Introduction of the recombination operators.

We need to obtain the kernel functions $\mathfrak{N}_{ij}(\hat{\boldsymbol{y}},t)$ in the physical domain $(\hat{\boldsymbol{y}},t)$, (1.19), from the transformed kernel K_{ij}, (1.18), which is a rational function

$$K_{ij}(\hat{\zeta},s;\lambda_{j_1},\dots,\lambda_{j_l})=\frac{K_{ij}^N(\hat{\zeta},s;\lambda_{j_1},\dots,\lambda_{j_l})}{K_{ij}^D(\hat{\zeta},s;\lambda_{j_1},\dots,\lambda_{j_l})}. \tag{1.22}$$

The denominators K_{ij}^D are polynomials in $\hat{\zeta}$, s, λ_j. As the roots λ_j has branch cut, the denominators can obstruct the application of the complex contour integral in inverting the Laplace transform. As it turns out this is related to the rich wave phenomena around the boundary. For a non-characteristic root λ_j, the contour can be away from the imaginary axis and avoiding the branch cut. The characteristic roots are the main concern. Some algebraic manipulation is needed to rewrite the rational function as:

$$K_{ij}(\hat{\zeta};\lambda_{j_1},\dots,\lambda_{j_l})=\frac{K_{ij}^N(\hat{\zeta};\lambda_{j_1},\dots,\lambda_{j_l})\mathscr{D}_{ij}(\hat{\zeta};\lambda_{j_1},\dots,\lambda_{j_l})}{K_{ij}^D(\hat{\zeta};\lambda_{j_1},\dots,\lambda_{j_l})\mathscr{D}_{ij}(\hat{\zeta};\lambda_{j_1},\dots,\lambda_{j_l})}, \tag{1.23}$$

so that the new denominators are free of characteristic roots λ_j. This allows for the inversion of the transforms in the pointwise sense. The algebraic manipulation varies from model to model, as it ultimately reflects the wave propagation around the boundary.

VII. Long wave-short wave decomposition and energy estimates.

A weighted energy method in the direction normal to the boundary is designed to study the exponentially localized wave structures of the short wave component, for which the harmonic analysis sheds no light. The long-short wave decomposition design has been used for the study of Green's functions for the initial value problems of systems with physical viscosity, see [6, 4] and references therein. For the present study, the long-short wave decomposition technique is necessary for the case of multi spatial dimensions. For the short waves, the structure of $\lambda_j(\hat{\zeta},s)$ for $|\hat{\zeta}|\gg 1$, could be very different from that of the long waves, $|\hat{\zeta}|\ll 1$. Thus, for the general program applicable to various different equations and systems, we propose a long wave-short wave decomposition to the solution $V(\vec{\boldsymbol{x}}_{\|},t)$ of (1.7) as follows:

$$\begin{cases} \mathscr{I}^L[\mathsf{h}] \equiv \mathscr{F}^{-1,L}[\mathscr{F}[\mathsf{h}]],\ \mathscr{I}^S[\mathsf{h}] = \mathsf{h} - \mathscr{I}^L[\mathsf{h}], \\ \mathscr{F}^{-1,L}[\mathsf{g}](\vec{\boldsymbol{x}}_{\|}) \equiv \dfrac{1}{(2\pi)^{m-1}} \displaystyle\int_{|\hat{\zeta}|<\delta_1} e^{i\hat{\zeta}\cdot\vec{\boldsymbol{x}}_{\|}} \mathsf{g}(\hat{\zeta}) d\hat{\zeta}, \end{cases} \tag{1.24}$$

where δ_1 is a constant determined by the size of the branch cut for $\lambda_j(\hat{\zeta},s)$. We illustrate this in the third example of dissipative wave equations in two spatial dimensions. The inverse Laplace and Fourier transforms of the long wave part of λ_j, and $K_{ij}^N\mathscr{D}_{ij}$, $K_{ij}^D\mathscr{D}_{ij}$ are studied by converting to the combined inverse Fourier transform using their analytic properties and the idea of Laplace-Fourier path. We then use the Hadamard solution of the D'Alembert's wave equation in 2-D to construct the long wave component of $\mathscr{I}^L[\boldsymbol{u}(\cdot,t)](\vec{\boldsymbol{x}}_{\|})$ with $|\vec{\boldsymbol{x}}_{\|}| < K_0(1+t)$ for some $K_0 > 1$. The short wave component of the solution and its structure outside a finite Mach number region possess exponential decaying structures in both time and space variables. Though both the long and short waves structures are exponentially small for $|\vec{\boldsymbol{x}}_{\|}| > K_0(1+t)$, the transformation approach are too refined to apply to the short wave part. Instead, we apply the weighted energy method directly to the system (1.1) to assert the simple exponentially structure.

In Section 2, we will make precise the derivation of the Dirichlet-Neumann relation in the transformed variables for general systems. We then choose three examples to illustrate the basic ideas as stated above. The first example is the *Convected Heat Equation*:

$$\begin{cases} u_t + \Lambda u_x = u_{xx}, \\ u(0,t) = 0. \end{cases}$$

We carry out for this simple example the process of Laplace-Laplace transform, the stability criterion for obtaining the Master Relationship, and the inversion of these transforms, Section 3. The second example is the *Linearized compressible Navier-Stokes equations* in 1-D, with subsonic drifting speed $\Lambda \in (-1,1)$:

$$\begin{cases} \rho_t + \Lambda\rho_x + u_x = 0, \\ u_t + \Lambda u_x + \rho_x = u_{xx}. \end{cases}$$

The Dirichlet-Neumann relation for this example depends crucially on the sign of the speed Λ of the convection. In particular, the algebraic structure of the Master Relationsship for a system without a full rank dissipative matrix yields interesting boundary condition as function of Λ. This is done in Section 4. The third example is *System of Wave equations with dissipation in 2-D*, $\Lambda \in (-1,1)$.

$$\begin{cases} u_t + (\Lambda+1)u_x + v_y = u_{xx} + u_{yy}, \\ v_t + (\Lambda-1)v_x + u_y = v_{xx} + v_{yy}, \end{cases} \tag{1.25}$$
$$u(0,y,t) = v(0,y,t) = 0.$$

This example illustrates the importance of the Laplace-Fourier path and the short-long wave decomposition ideas, Section 5 and Section 6.

As an example, we now go to some details on the boundary relation for (1.25). As just mentioned, the Laplace-Fourier path relates the half space phenomena to the whole space phenomena. We consider first the Green's function, the fundamental solution $\mathbb{G}(x,y,t)$ of initial value problem for (1.25):

$$\begin{cases} \mathbb{G}_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \mathbb{G}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbb{G}_y = \Delta_2 \mathbb{G}, \ \vec{\boldsymbol{x}} \equiv (x,y), \\ \mathbb{G}(\vec{\boldsymbol{x}},0) = \delta(\vec{\boldsymbol{x}})\mathbb{I}, \end{cases}$$

where $\mathbb{I}$ is the 2×2 identity matrix. The Green's function can be computed easily as the dissipative version of the Kirchhoff and Hadamard method of descent for the wave equation. It is of the following shape:

$$\begin{aligned} &\mathbb{G} = (\mathbb{G}_{ij})_{2\times 2}, \\ &|\mathbb{G}_{ij}(\vec{\boldsymbol{x}},t)| \le O(1)\frac{1}{\sqrt{t}} W_2(\vec{\boldsymbol{x}} - (\Lambda t, 0), t; 2) \text{ for } i \ne j, \\ &|\mathbb{G}_{jj}(\vec{\boldsymbol{x}},t) \le O(1)\left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right) W_2(\vec{\boldsymbol{x}} - (\Lambda t, 0), t), t; 2) \text{ for } j = 1,2, \end{aligned} \tag{1.26}$$

where W_2 is defined by

$$W_2(\vec{\boldsymbol{x}},t;D_0)\equiv\begin{cases}\dfrac{1}{t-|\vec{\boldsymbol{x}}|} & \text{for } |\vec{\boldsymbol{x}}|\le t-\sqrt{D_0t},\\ \dfrac{1}{t^{3/4}D_0^{1/4}} & \text{for } ||\vec{\boldsymbol{x}}|-t|\le\sqrt{D_0t},\\ \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{4D_0t}}}{t^{3/4}D_0^{1/4}} & \text{for } |\vec{\boldsymbol{x}}|\ge t+\sqrt{D_0t}.\end{cases}\tag{1.27}$$

We now state the boundary relation for the system (1.25) for the case when the boundary Dirichlet value $\mathsf{u}^0(y,t)$ has bounded support. The Dirichlet-Neumann kernel is a generalized function and so the relation contains regular functions as well as generalized functions.

Theorem 1.2. *Suppose that the boundary Dirichlet value $\boldsymbol{u}^0(y,t)$ has unit support around $(y,t)=(0,0)$. Then the Dirichlet-Neumann relation $\boldsymbol{u}^0_x(y,t)=\mathfrak{N}\boldsymbol{u}^0(y,t)$ for the dissipative wave equations (1.25) is:*

$$\begin{aligned}\mathfrak{N}_{ij}\boldsymbol{u}^0(y,t) &= O(1)\left.\frac{\mathbb{W}(\vec{\boldsymbol{x}},t)}{t}\underset{\vec{\boldsymbol{x}}}{*}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C(t)}}}{t}\right|_{x=-\Lambda t}\|\boldsymbol{u}^0\|_{L^\infty_{(y,t)}}\\ &\quad+O(1)e^{-\frac{(y+t)}{C}}\sum_{|\alpha|=0}^{4}\|\partial^\alpha_{(y,t)}\boldsymbol{u}^0\|_{L^\infty_{(y,t)}},\ (i,j)\neq(2,1),\\ \partial_y\mathfrak{N}_{21}\boldsymbol{u}^0(y,t) &= O(1)\left.\frac{\mathbb{W}(\vec{\boldsymbol{x}},t)}{t}\underset{\vec{\boldsymbol{x}}}{*}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C(t)}}}{t}\right|_{x=-\Lambda t}\|\boldsymbol{u}^0\|_{L^\infty_{(y,t)}}\\ &\quad+O(1)e^{-\frac{(y+t)}{C}}\sum_{|\alpha|=0}^{4}\|\partial^\alpha_{(y,t)}\boldsymbol{u}^0\|_{L^\infty_{(y,t)}}.\end{aligned}\tag{1.28}$$

for some positive constant C and for $t>1$.

The ideas in the present paper should prove useful for the study of other hyperbolic-dissipative systems of physical interests. It would also be interesting to consider the Boltzmann equation in the kinetic theory for which the Green's function has been constructed explicitly for the initial value problem, [2], [3], [4]. These and possible applications are, however, left to the future.

2. Preliminaries

In this section, we will lay down the standard procedure for obtaining the Dirichlet-Neumann map in the transformed variables for the problem (1.7):

$$\begin{cases} \partial_t \boldsymbol{u} + \sum_{j=1}^{m} \boldsymbol{A}_j \partial_{x^j} \boldsymbol{u} = \sum_{j,i=1}^{m} \boldsymbol{B}_{ji} \partial^2_{x^j x^i} \boldsymbol{u} \text{ for } (x^1, \ldots, x^m) \in \mathbb{R}^m_+, \\ \boldsymbol{u}(\vec{\boldsymbol{x}}, 0) \equiv 0. \end{cases} \tag{2.1}$$

We also prepare some basic knowledge for converting the map from the transformed variables to the physical variables.

2.1. Transformations for half space problem

For a function $f(t)$ defined for $t \geq 0$, its Laplace transformation $F(s)$ and the inverse transformation, the Bromwich integral, are given as follows:

$$\begin{cases} F(s) = \mathbb{L}[f](s) \equiv \int_0^\infty e^{-st} f(t) dt \text{ for } s \in \{z \in \mathbb{C} | Re(s) \geq 0\}, \\ f(t) = \mathbb{L}^{-1}[F](t) \equiv \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-iT}^{iT} e^{st} F(s) ds \text{ for } t > 0. \end{cases} \tag{2.2}$$

Note that the Bromwich's integral makes sense when the function $f(s)$, defined for $s > 0$, can be analytically extended to $Re(s) > 0$ and that the Riemann integral converges over the imaginary axis.

One imposes a functional property for boundary values f for the consistency with zero initial data, $\boldsymbol{u}(\vec{\boldsymbol{x}}, 0) = 0$, and for an applicable condition for the Bromwich integral:

$$f \in \mathscr{V} \equiv \{g | \ \mathbb{L}[g](s) \text{ exists for } Re(s) > 0, \ g^{[n]}(0) = 0 \text{ for } n \in \mathbb{N} \cup \{0\}\}. \tag{2.3}$$

The independent variable s of $F(s)$ can be analytically extended to a simply connected subset of the complex plan.

With the functional property (2.3), one has the following properties of the Laplace transformation. These properties are used to extract the distributional values of the inverse Laplace transform in the case when $F(s)$ has nonzero values at $s=0$ or at $s=\infty$.

Lemma 2.1. *For f and $g\in\mathscr{V}$, their Laplace transformations $F=\mathbb{L}[f]$ and $G=\mathbb{L}[g]$ satisfy*

$$\begin{cases}\mathbb{L}[f^{[n]}]=s^nF(s),\\ (-t)^nf(t)=\mathbb{L}^{-1}\left[\frac{d^n}{ds^n}F\right](t),\\ [f*g](t)=\mathbb{L}^{-1}[F(s)G(s)],\\ \mathbb{L}^{-1}[F(s)G(s)](t)=\left[\mathbb{L}^{-1}\left[\frac{F(s)-F(0)}{s}\right]*g'\right](t)+F(0)g(t).\end{cases}\tag{2.4}$$

Proof. The first three properties are standard. The last one is a consequence of

$$\mathbb{L}^{-1}[F(s)G(s)]=\mathbb{L}^{-1}\left[\frac{F(s)-F(0)}{s}sG(s)+F(0)G(s)\right].\qquad\square$$

Lemma 2.2. *Suppose that $\mu(t)$ is generalized function defined for $t\geq 0$. Then its Laplace transformation $\hat{\mu}(s)=\mathbb{L}[\mu](s)$ satisfies*

$$\mu(t)+\frac{1}{t}\mathbb{L}^{-1}\left[\frac{d}{ds}\hat{\mu}(s)\right]=C\delta(t)\text{ for a constant }C\in\mathbb{C}.$$

Proof. This is a consequence of

$$\mathbb{L}[t(\mu+C\delta)]=\mathbb{L}[t\mu]=-\frac{d}{ds}\mathbb{L}[\mu]=-\frac{d}{ds}\hat{\mu}(s)\text{ for constant }C.\qquad\square$$

Definition 2.3. Let $g\in C[0,\infty]$.

$$Q[g](s)\equiv\frac{g(s)-g(0)}{s}\text{ for }s>0.$$

When the function $g(s)$ has sublinear growth in s as $s\to\infty$, the quotient $Q[g](s)$ decays in s and is convenient for the inversion of the Laplace transform by complex contour integration. More generally, if the function $g(s)$ has higher order of growth, one may need to consider the function $Q^n[g](s)$ for some $n>1$. We therefore will need to decompose a function into the *Taylor Series* in the following sense.

Proposition 2.4 (Taylor's Series). *For any* $g \in C^n[0,\infty)$,

$$g(s)=\sum_{l=0}^{n-1}\frac{g^{[l]}(0)}{l!}s^l+s^nQ^n[g](s)\ \textit{for}\ s>0.$$

Following (1.8), we first take the Fourier transformation $\mathscr{F}$ for the variables $\vec{\boldsymbol{x}}_\parallel$ parallel to the boundary $\partial\mathbb{R}^m_+$, then the Laplace transform $\mathbb{L}$ in time t, and finally the Laplace transform with the variable x^1 normal to the boundary to form the combined transform $\mathbb{J}$:

$$\begin{cases}\hat{\boldsymbol{x}}\equiv(x^2,\dots,x^m),\\ \vec{\boldsymbol{x}}_\parallel\equiv(0,\hat{\boldsymbol{x}}),,\\ \vec{\boldsymbol{\zeta}}\equiv(\zeta^1,\dots,\zeta^m),\\ \hat{\zeta}\equiv(\zeta^2,\dots,\zeta^m),\end{cases}$$

$$\begin{cases}\boldsymbol{v}(x^1,\hat{\zeta},t)\equiv\mathscr{F}[\boldsymbol{u}](x^1,\hat{\zeta},t)\equiv\displaystyle\int_{\mathbb{R}^{m-1}}\boldsymbol{u}(x^1,\hat{\boldsymbol{x}},t)e^{-i\hat{\boldsymbol{x}}\cdot\hat{\zeta}}d\hat{\boldsymbol{x}},\\ \boldsymbol{v}^0(\hat{\zeta},t)\equiv\mathscr{F}[\boldsymbol{u}^0](\hat{\zeta},t)\equiv\displaystyle\int_{\mathbb{R}^{m-1}}\boldsymbol{u}^0(\hat{\boldsymbol{x}},t)e^{-i\hat{\boldsymbol{x}}\cdot\hat{\zeta}}d\hat{\boldsymbol{x}},\\ V(x^1,\hat{\zeta},s)\equiv\mathbb{L}[\mathscr{F}[\boldsymbol{u}]](x^1,\hat{\zeta},s)=\mathbb{L}[\boldsymbol{v}](x^1,\hat{\zeta},s)\equiv\displaystyle\int_0^\infty e^{-st}\boldsymbol{v}(x^1,\hat{\zeta},t)dt,\\ V^0(\hat{\zeta},s)\equiv\mathbb{L}[\boldsymbol{v}^0](\hat{\zeta},s)\equiv\displaystyle\int_0^\infty e^{-st}\boldsymbol{v}^0(\hat{\zeta},t)dt,\\ \hat{V}(\xi,\hat{\zeta},s)\equiv\mathbb{J}[\mathscr{F}[\boldsymbol{u}]](\xi,\hat{\zeta},s)=\mathbb{L}[V](\xi,\hat{\zeta},s)\equiv\displaystyle\int_0^\infty e^{-\xi x^1}V(x^1,\hat{\zeta},s)dx^1.\end{cases}\tag{2.5}$$

The system (2.1) becomes a system of algebraic equations:

$$\begin{aligned}&\Big(s+\xi\boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m\zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)\hat{V}(\xi,\hat{\zeta},s)\\ &=\ -\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)-\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}+i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s).\end{aligned}\tag{2.6}$$

This is the system for the transformed variables in terms of the boundary Dirichlet and Neumann values, V^0 and $V^0_{x^1}$.

2.2. Characteristic polynomial, Dirichlet-Neumann map, and the Laplace-Fourier Path

From (2.6), the transformed function is then expressed as:

$$\begin{aligned}
&\hat{V}(\xi,\hat{\zeta},s) = soln(\xi,\hat{\zeta},s;V^0,V^0_{x^1}),\\
&soln(\xi;\hat{\zeta},s;V^0,V^0_{x^1})\\
&\equiv \frac{adj\Big(s+\xi \boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m \zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)}{\det\Big(s+\xi \boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m \zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)}\\
&\quad\Big(-\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)+\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}-i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s)\Big)\\
&= \frac{adj\Big(s+\xi \boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m \zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big)}{p(\xi;\hat{\zeta},s)}\\
&\quad\Big(-\boldsymbol{B}_{11}V^0_{x^1}(\hat{\zeta},s)+\Big(\boldsymbol{A}_1-\xi\boldsymbol{B}_{11}-i\sum_{j=2}^m\zeta^j(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\Big)V^0(\hat{\zeta},s)\Big)\\
&\equiv \frac{\mathbb{B}(\xi;\hat{\zeta},s;V^0,V^0_{x^1})}{p(\xi;\hat{\zeta},s)}.
\end{aligned}\tag{2.7}$$

Here the denominator is the degree n *characteristic polynomial* $p(\xi;\hat{\zeta},s)$, of degree n in the ξ variable:

$$p(\xi;\hat{\zeta},s)\equiv\det\Big(s+\xi \boldsymbol{A}_1-\xi^2\boldsymbol{B}_{11}+i\sum_{j=2}^m \zeta^j(\boldsymbol{A}_j-(\boldsymbol{B}_{1j}+\boldsymbol{B}_{j1})\xi)+\sum_{2\le i,j\le m}\zeta^i\zeta^j\boldsymbol{B}_{ij}\Big).\tag{2.8}$$

Its roots are denoted by:

$$\lambda_j=\lambda_j(\hat{\zeta},s),\ j=1,2,\dots,n;\ p(\lambda_j;\hat{\zeta},s)=0.\tag{2.9}$$

Note that, $p(i\xi;\hat{\zeta},s)$ is the *symbol* of the system (1.1) for the whole space $\vec{\boldsymbol{x}}\in\boldsymbol{R}^m$.

Assuming that the roots $\xi=\lambda_j$ of the characteristic polynomial $p(\xi;\hat{\zeta},s)$ are simple, then applying the complex contour integral (1.13) to the system (2.7), one yields its solution as the sum of the residues at the poles $\xi =$

$\lambda_j(\hat{\zeta},s),\ j=1,2,\ldots,n,$:

$$V(x^1,\hat{\zeta},s) = \sum_{j=1}^{n} e^{\lambda_j x^1} \operatorname*{Res}_{\xi=\lambda_j}(soln(\xi;\hat{\zeta},s;V^0,V^0_{x^1})) = \sum_{j=1}^{n} e^{\lambda_j x^1} \frac{\mathbb{B}(\lambda_j;\hat{\zeta},s;V^0,V^0_{x^1})}{p'(\lambda_j;\hat{\zeta},s)}. \tag{2.10}$$

The well-posedness of a differential equation requires the solution V to decay to zero as $x^1 \to \infty$. This implies that at $\xi=\lambda_j$ with $Re(\lambda_j)>0$ the residue of $soln(\xi;\hat{\zeta},s;V^0,V^0_{x^1}))$ vanishes. Denote the roots of the characteristic polynomial with positive real part by $\lambda_{j_1},\ldots,\lambda_{j_l}$:

$$Re(\lambda_{j_1})>0,\ldots,Re(\lambda_{j_l})>0.$$

The *Master Relationship* (1.17)

$$\operatorname*{Res}_{\substack{\xi=\lambda_j \\ p(\lambda_j;\hat{\zeta},s)=0,s>0 \\ Re(\lambda_j)>0}} (soln(\xi;\hat{\zeta},s;V^0,V^0_{x^1}))=0,$$

becomes

$$\mathbb{B}(\lambda_j;\hat{\zeta},s;V^0,V^0_{x^1})\Big|_{\substack{\lambda_j=\lambda_j(\hat{\zeta},s),\ j=j_1,\ldots,j_l, \\ p(\lambda_j;\hat{\zeta},s)=0,s>0, \\ Re(\lambda_j)>0.}} = 0, \tag{MR}$$

The Master Relationship (MR), the Dirichlet-Neumann relation in the transformed variables can be rewritten in the form:

$$\mathfrak{R}V^0_{x^1}(\hat{\zeta},s) = \mathfrak{R}K(\hat{\zeta},s;\lambda_1,\ldots,\lambda_l)\cdot V^0(\hat{\zeta},s). \tag{2.11}$$

The constant matrix $\mathfrak{R}$ may not be of full rank and depends on the structure of the dissipation matrix $\boldsymbol{B}_{ij}$. The main task is to convert this Dirichlet-Neumann relation in the transformed variables $(\hat{\zeta},s)$ to the physical variables $(\hat{\boldsymbol{y}},t)$ with exponentially sharp description.

The conversion of the transform variable using the Laplace-Fourier path idea is applied to the characteristic roots $\lambda_j(\hat{\zeta},s)$ according to Definition 1.1. A non-characteristic root $\lambda_j(\hat{\zeta},s)$ gives rise to the spectrum gap and one can replace the path integral along $Re(s)=0$ uniformly by $Re(s)=-\nu_0$

for some $\nu_0 > 0$ with respect to $\hat{\zeta}$ around 0. This results in an exponentially decaying structure in time, $e^{-\nu_0 t}$, for the long wave part, $|\hat{\zeta}| \ll 1$, in $\mathbb{L}^{-1}[\lambda_i(\hat{\zeta}, s)](t)$. For a characteristic root $\lambda_j(\hat{\zeta}, s)$, one substitutes the imaginary Fourier variable $i\zeta^1 = \xi$ into (1.15) so that s becomes a function of $s = \sigma(\vec{\zeta})$:

$$i\zeta^1 = \lambda_j(\hat{\zeta}, \sigma(\vec{\zeta})), \tag{2.12}$$

where $\vec{\zeta} = (\zeta^1, \zeta^2, \dots, \zeta^m)$, $\hat{\zeta} = (\zeta^2, \dots, \zeta^m)$. This function $\sigma(\vec{\zeta})$ satisfies

$$0 = \det\Big(-i\sum_{j=1}^{m} \boldsymbol{A}_j \zeta^j - \sum_{k,l=1}^{m} \boldsymbol{B}_{kl}\zeta^k\zeta^l - \sigma(\vec{\zeta}) I_{m\times m}\Big).$$

Thus, the function $\sigma(\vec{\zeta})$ is the spectrum of the full Fourier transformation in $\mathbb{R}^m$ of the operator $-\sum_{j=1}^{m} A_j\partial_{x^j} + \sum_{k,l=1}^{m} B_{kl}\partial^2_{x^k x^l}$. We will use the spectrum $\sigma(\vec{\zeta})$ to define the Laplace-Fourier path for the purpose of inverting the operator $\mathbb{L}$ in the time variable.

Definition 2.5 (Laplace-Fourier Path)**.** With a fixed $\hat{\zeta} = (\zeta^2, \zeta^3, \dots, \zeta^m) \in \mathbb{R}^{m-1}$, the Laplace-Fourier path is defined by

$$s = \sigma(\vec{\zeta}) \in \mathbb{C}, \vec{\zeta} \equiv (\zeta^1, \dots, \zeta^m),\ \zeta^1 \in \mathbb{R},$$

where $\sigma(\vec{\zeta})$ is the spectrum defined by the implicit function $0 = p(i\zeta^1; \hat{\zeta}, \sigma(\vec{\zeta}))$.

Remark 2.6. The name of this path is given to mean a path in the Laplace domain with the Fourier spectrum information.

The above procedures allow one to obtain the long wave component of the operator $\mathscr{F}^{-1,L}[\mathbb{L}^{-1}[\lambda_j(\hat{\zeta}, s)](\hat{\boldsymbol{y}}, t)$ in the wave region $|\hat{\boldsymbol{y}}| \le O(1)(1+t)$. Here $\mathscr{F}^{-1,L}$ is the long wave component of the inverse Fourier transformation defined in (1.24).

2.3. Basic operators for inverting the Dirichlet-Neumann kernel

Each entry $K_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)$ in the matrix K relating the Dirichlet-Neumann relation in the transformed variables, (2.11), is a rational function in $(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)$:

$$K_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l) = \frac{K_{ij}^N(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)}{K_{ij}^D(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)},$$

for some polynomials K_{ij}^D and K_{ij}^N. Recall that $\xi = \lambda_i$ are roots of $p(\xi; \hat{\zeta}, s) = 0$. In case the denominator K_{ij}^D depends on some characteristic root λ_j, we need to neutralize the singularity induced by λ_j in the denominator. This can be done algebraically by multiplying some polynomial $\mathscr{D}_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)$:

$$K_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l) = \frac{K_{ij}^N(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)\mathscr{D}_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)}{K_{ij}^D(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)\mathscr{D}_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)},$$

so that the new denominator $K_{ij}^D(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)\mathscr{D}_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)$ is independent of all the characteristic roots. Thus, the product has a non-characteristic divisor and one can have an exponential sharp decaying estimate in t for the long-wave operator $\mathscr{F}^{-1,L}[\mathbb{L}^{-1}[1/(\mathscr{D}_{ij}K_{ij}^D)]]$. The new numerator $K_{ij}^N(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)\mathscr{D}_{ij}(\hat{\zeta}, s; \lambda_1, \dots, \lambda_l)$ is a polynomial of the characteristic roots with analytic functions and non-characteristic roots as coefficients. Thus the focus is then on inverting the characteristic roots λ_j:

$$\mathfrak{L}_j(\hat{\boldsymbol{y}}, t) \equiv \mathscr{F}^{-1,L}[\mathbb{L}^{-1}[\lambda_j]](\hat{\boldsymbol{y}}, t), \ j = 1, 2, \dots, l.$$

In general one can obtain the structure of $\mathfrak{L}_j(\hat{\boldsymbol{y}}, t)$ only in some wave region $|\hat{\boldsymbol{y}}| \le O(1)(1+t)$. For some physical examples, a characteristic root $\lambda_j(\hat{\zeta}, s)$ can be analytically extended from $s > 0$ to $Re(s) \ge 0$ only when $|\hat{\zeta}|$ is small.

Since the product $K_{ij}^N \mathscr{D}_{ij}$ is a polynomial of λ_j, $j = 1, 2, \dots, l$, the operator $K_{ij}^N(\partial_{\hat{\boldsymbol{x}}}, \partial_t; \mathfrak{L}_1, \dots, \mathfrak{L}_l) * \mathscr{D}_{ij}(\partial_{\hat{\boldsymbol{x}}}, \partial_t; \mathfrak{L}_1, \dots, \mathfrak{L}_l)$ can be obtained from the operators $\mathfrak{L}_i$, and the long wave component $\mathscr{I}^I$, (1.24), of the Dirichlet-Neumann map $\mathfrak{N}$ is given by

$$\begin{aligned}\mathscr{I}^L[\mathfrak{R}\mathfrak{N}](\hat{\boldsymbol{y}}, t) \ \equiv\ & \mathscr{I}^L[K_{ij}^N(\partial_{\hat{\boldsymbol{x}}}, \partial_t; \mathfrak{L}_1, \dots, \mathfrak{L}_l)] * \mathscr{I}^L[\mathscr{D}_{ij}(\partial_{\hat{\boldsymbol{x}}}, \partial_t; \mathfrak{L}_1, \dots, \mathfrak{L}_l)] \\ & * \mathscr{I}^L \mathscr{F}^{-1,L}[\mathbb{L}^{-1}[1/(\mathscr{D}_{ij}K_{ij}^D)](\hat{\boldsymbol{y}}, t). \qquad (2.13)\end{aligned}$$

In the region $|\hat{\boldsymbol{y}}| \ge O(1)(1+t)$ away from the domain of influence of the associated inviscid system, there is only the diffusion effects of the system and the operator $\mathfrak{N}$ has only localized, exponentially decaying terms. This can be studied by the weighted energy method.

3. Convected Heat Equation

Consider the 1-D heat equation with a positive drifting speed $\Lambda > 0$:

$$\begin{cases} u_t + \Lambda u_x = u_{xx}, \ x, t \geq 0, \\ u(0,t) = u^0(t), \ u^0 \in \mathscr{V}, \\ u(x,0) = 0. \end{cases} \tag{3.1}$$

The fundamental solution $\mathbb{G}$ for the initial value problem is given in terms of the heat kernel $H(x,t)$:

$$\begin{cases} \mathbb{G}_t + \Lambda \mathbb{G}_x = \mathbb{G}_{xx}, \ \mathbb{G}_(x,0) = \delta(x), \\ \mathbb{G}(x,t) = H(x - \Lambda t, t), \ H(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \end{cases} \tag{3.2}$$

As the spatial dimension is one, there is no spatial direction parallel to the boundary and we don't need to take the Fourier transform. Let $U(x,s)$ be the Laplace transform of $u(x,t)$ in the time variable t, and $\hat{U}(\xi,s)$ its Laplace transform in the space variable x. The boundary values are denoted as $u^0 = u^0(t)$, $u_x^0 = u_x^0(t)$, $U^0 = U^0(s)$, $U_x^0 = U_x^0(s)$ as in (1.9). The algebraic equation for the Laplace-Laplace transform $\hat{U}(\xi,s)$ is

$$(s + \Lambda\xi - \xi^2)\hat{U} = (\Lambda - \xi)U^0(s) - U_x^0(s). \tag{3.3}$$

The characteristic polynomial $p(\xi,s)$

$$p(\xi,s) = s + \Lambda\xi - \xi^2,$$

has two roots

$$\lambda_1(s) = \frac{1}{2}[\Lambda - \sqrt{\Lambda^2 + 4s} < 0 < \lambda_2(s)] = \frac{1}{2}[\Lambda + \sqrt{\Lambda^2 + 4s}], \text{ for } s > 0.$$

We have

$$\begin{aligned}\hat{U}(\xi,s) &= soln(\xi,s;U^0,U^0_x) = \frac{(\Lambda-\xi)U^0(s)-U^0_x(s)}{p(\xi,s)}\\ &= \frac{U^0_x(s)-(\Lambda-\xi)U^0(s)}{(\xi-\lambda_1(s))(\xi-\lambda_1(s))}\\ &= \frac{U^0_x(s)-U^0(s)(\Lambda-\lambda_1(s))}{\sqrt{\Lambda^2+4s}(\xi-\lambda_1(s))}+\frac{U^0_x(s)-U^0(s)(\Lambda-\lambda_2(s))}{\sqrt{\Lambda^2+4s}(\xi-\lambda_2(s))}. \qquad (3.4)\end{aligned}$$

Invert the Laplace transform for x to obtain

$$U(x,s)=\frac{U^0_x(s)-U^0(s)(\Lambda-\lambda_1(s))}{\sqrt{\Lambda^2+4s}}e^{\lambda_1(s))x}+\frac{U^0_x(s)-U^0(s)(\Lambda-\lambda_1(s))}{\sqrt{\Lambda^2+4s}}e^{\lambda_2(s))x}.$$

The well-posedness condition requires that the coefficient of the exponential growing term $e^{\lambda_2(s)x}$ should be zero:

$$\frac{U^0_x(s)-U^0(s)(\Lambda-\lambda_1(s))}{\sqrt{\Lambda^2+4s}}=\frac{U^0_x(s)-\frac{1}{2}U^0(s)\left(\Lambda-\sqrt{\Lambda^2+4s}\right)}{\sqrt{\Lambda^2+4s}}=0. \qquad (3.5)$$

This gives the Master Relationship, (MR):

$$\begin{aligned} U^0_x(s)-\frac{1}{2}U^0(s)(\Lambda-\sqrt{\Lambda^2+4s}, \text{ or,}\\ U^0_x(s) &= \frac{1}{2}U^0(s)\left(\Lambda-\sqrt{\Lambda^2+4s}\right)=-\frac{s}{\Lambda/2+\sqrt{s+\Lambda^2/4)}}U^0(s). \qquad (3.6)\end{aligned}$$

Since $u^0\in\mathscr{V}$, from (2.4) the inverse Laplace transformation of the operator $-(s/(\Lambda+\sqrt{s+\Lambda^2/4}))$ on $\mathscr{V}$ is

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{st}\left(\frac{1}{\Lambda/2+\sqrt{s+\Lambda^2/4}}\right)ds*\partial_t\\ &= -\left(e^{-\frac{\Lambda^2 t}{4}}\left(\frac{1}{\sqrt{4\pi t}}-\Lambda e^{\Lambda^2 t}\mathrm{Erfc}\left(\Lambda\sqrt{t}\right)\right)\right)*\partial_t, \qquad (3.7)\end{aligned}$$

where

$$\mathrm{Erfc}(u)\equiv\int_u^\infty 2\frac{e^{-x^2}}{\sqrt{\pi}}dx.$$

This gives the following theorem:

Theorem 3.1. *The kernel function $\mathfrak{N}(t)$ of the Dirichlet-Neumann map for*

(3.1) *is*

$$\mathfrak{N}(t) = -\left(e^{-\frac{\Lambda^2 t}{4}}\left(\frac{1}{\sqrt{4\pi t}} - \Lambda e^{\Lambda^2 t} Erfc\left(\Lambda\sqrt{t}\right)\right)\right) * \partial_t. \tag{3.8}$$

Thus the Dirichlet-Neumann relation is

$$u_x^0(t) = -\int_0^t e^{-\frac{\Lambda^2(t-\tau)}{4}}\left(\frac{1}{\sqrt{4\pi(t-\tau)}} - \Lambda e^{\Lambda^2(t-\tau)} Erfc\left(\Lambda\sqrt{(t-\tau)}\right)\right)\partial_\tau u^0(\tau)d\tau. \tag{3.9}$$

4. Compressible Navier-Stokes Equations

Consider the isentropic compressible Navier-Stokes equations in one space dimension

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = (\mu v_x)_x. \end{cases} \tag{4.1}$$

Here ρ, $v, p(\rho)$ are the density, velocity, and pressure, respectively. The first equation, the continuity equation, has no dissipation. The second, the momentum equation, has the dissipation term $(\mu u_x)_x$, with the viscosity coefficient $\mu > 0$. Thus this system is not uniformly parabolic and is hyperbolic-parabolic. This is typical of many physical models of which (4.1) is the simplest. The inviscid model $\mu = 0$ is the isentropic Euler equations:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = 0. \end{cases} \tag{4.2}$$

The characteristic speeds for the Euler equations are $v - c$ and $v + c$, where c is the sound speed given by

$$c^2 = p'(\rho).$$

Linearize the Navier-Stokes equations around a constant state (ρ_0, v_0). We normalize the viscosity μ and the sound speed $c_0 = \sqrt{p'(\rho_0)}$ to be unity and take the base state to be $(\rho_0, v_0) = (1, \Lambda)$. For notational simplicity, the perturbed variables are still written as (ρ, v) so that the linearized system becomes

$$\begin{cases} \rho_t + \Lambda\rho_x + v_x = 0, \\ v_t + \rho_x + \Lambda v_x = v_{xx}, \end{cases} \tag{4.3}$$

which is also written in the matrix form, and as before, the initial values are taken to be zero,

$$\begin{cases} \boldsymbol{u}_t + \begin{pmatrix} \Lambda & 1 \\ 1 & \Lambda \end{pmatrix} \boldsymbol{u}_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{u}_{xx}, \\ \boldsymbol{u} \equiv \begin{pmatrix} \rho \\ v \end{pmatrix}, \\ \boldsymbol{u}(x,0) = 0. \end{cases} \tag{4.4}$$

The background velocity Λ is restricted to

$$\Lambda \in (-1, 1).$$

As the sound speed c has been normalized to be one, the above restriction means that the flow is *subsonic*. This is the interesting case, as the supersonic flows can be studied easily by weighted energy method, with up to exponentially in time accuracy. Also the well-posed boundary condition for the supersonic flows are straightforward and not considered here.

The Green's function for the initial value problem for (4.3) has been explicitly constructed in [8]. In fact, in one space dimension, the Green's function for general hyperbolic-parabolic in one space dimension has also been constructed in [6].

Following (1.8), we will denote the Laplace transform in t by U and the subsequent Laplace transform in x by $\hat{U}$:

$$U(x,s) \equiv \int_0^\infty e^{-st}\boldsymbol{u}(x,t)dt,\ \hat{U}(\xi,s) \equiv \int_0^\infty e^{-\xi x}\boldsymbol{u}(x,s)dx. \tag{4.5}$$

The transformed system of (4.4) is

$$s\hat{U} + \begin{pmatrix} \Lambda & 1 \\ 1 & \Lambda \end{pmatrix} (\xi\hat{U} + U^0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (\xi^2\hat{U} + U^0_x - \xi U^0), \tag{4.6}$$

or

$$\begin{pmatrix} \Lambda\xi + s & \xi \\ \xi & -\xi^2 + \Lambda\xi + s \end{pmatrix} \hat{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U_x^0 - \begin{pmatrix} \Lambda + \xi & 1 \\ 1 & \Lambda + \xi \end{pmatrix} U^0. \tag{4.7}$$

Thus characteristic polynomial $p(\xi, s)$ for this system is

$$p(\xi, s) = \det \begin{pmatrix} \Lambda\xi + s & \xi \\ \xi & -\xi^2 + \Lambda\xi + s \end{pmatrix} = s^2 + 2s\Lambda\xi + (\Lambda^2 - 1 - s)\xi^2 - \Lambda\xi^3. \tag{4.8}$$

For this polynomial, there is a root $\xi \sim O(1)s$ as $s \to 0$. Thus, one can ignore the terms $s\xi^2$ and $-\Lambda\xi^3$ in $p(\xi, s) = 0$ as $s \to 0$ for the purpose of approximating the root $\xi \sim O(1)s$ as $s \to 0$. Thus we consider the polynomial

$$p_0(\xi, s) \equiv s^2 + 2s\Lambda\xi - (1 - \Lambda^2)\xi^2,$$

with two roots $\xi = s/(1 - \Lambda)$ and $\xi = -s/(1 + \Lambda)$. These two roots give the asymptotic behavior of the roots of $p(\xi, s) = 0$ as $s \to 0$. Since the product of the three roots of $p(\xi, s) = 0$ in ξ is s^2/Λ, $\xi \to -(1 - \Lambda^2)/\Lambda$ is the asymptotic behavior of the third root of $p(\xi, s) = 0$ as $s \to 0$:

$$\begin{cases} \lambda_1(s) = -s/(1 + \Lambda) + O(1)s^2, \\ \lambda_2(s) = s/(1 - \Lambda) + O(1)s^2, \\ \lambda_3(s) = -\dfrac{1 - \Lambda^2}{\Lambda} + O(1)s. \end{cases}$$

Write

$$\lambda_3(s) \equiv -\frac{1 - \Lambda^2}{\Lambda} + \alpha,$$

and α satisfies

$$\begin{aligned} p_2(\alpha, s) \ \equiv \ & \alpha^3 - \frac{(2\Lambda^2 - s\Lambda^2 - 2\Lambda^4)}{\Lambda^3}\alpha^2 - \frac{(-\Lambda + 2s\Lambda + 2\Lambda^3 - \Lambda^5)}{\Lambda^3}\alpha \\ & - \frac{-s + s^2\Lambda^2 + s\Lambda^4}{\Lambda^3} = 0. \end{aligned}$$

The function $p_2(\alpha, s)$ satisfies

$$\begin{cases} p_2(0,0) = 0 \\ \dfrac{\partial}{\partial\alpha} p_2(0,0) = \dfrac{(1-\Lambda^2)^2}{\Lambda^2} \neq 0 \text{ for } |\Lambda| \neq 1. \end{cases}$$

Thus, the implicit function theorem applies to define an analytic root $\lambda_3(s) = -(1-\Lambda^2)/\Lambda + \alpha(s)$ around $s = 0$ with $\alpha(0) = 0$.

Next, one can use the Euclid's algorithm to find equation of the branch point of $p(\xi, s) = 0$. The necessary condition $p(\xi, s) = p_\xi(\xi, s) = 0$ for the branch point (ξ, s) yields

$$s(4s^3 + s^2\left(12 + \Lambda^2\right) + 4s\left(3 + 5\Lambda^2\right) + 4\left(-1 + \Lambda^2\right)^2) = 0.$$

Since $\lambda_3(s)$ is analytic around $s = 0$, this and the above yield that the possible branch point for $\lambda_3(s)$ are the roots of

$$4s^3 + s^2\left(12 + \Lambda^2\right) + 4s\left(3 + 5\Lambda^2\right) + 4\left(-1 + \Lambda^2\right)^2 = 0. \tag{4.9}$$

The following lemma is immediate from this expression.

Lemma 4.1. *For $\Lambda \in (-1, 1)$, there exists a positive constant $C_b > 0$ such that the roots s of the polynomial in* (4.9) *satisfies*

$$Re[s] < -(1 - |\Lambda|)/C_b < 0.$$

This lemma yields that $\lambda_3(s)$ is analytic in the region $Re(s) > -(1 - |\Lambda|)/C_b$ and, by iterating $p_2(\alpha, s)$, its asymptotic around $s = 0$ is

$$\lambda_3(s) = \frac{-1 + \Lambda^2}{\Lambda} + \frac{s\left(1 + \Lambda^2\right)}{\Lambda\left(-1 + \Lambda^2\right)} + \frac{s^2\left(-1 - 4\Lambda^2 - \Lambda^4\right)}{\Lambda\left(-1 + \Lambda^2\right)^3} + O(1)s^3. \tag{4.10}$$

With this asymptotic of $\lambda_3(s)$, one can factor out $(\xi - \lambda_3(s))$ from $p(\xi, s)$ to obtain:

$$\begin{aligned} p_3(\xi, s) \equiv \frac{p(\xi, s)}{-\Lambda(\xi - \lambda_3)} &= \xi^2 + \left(\lambda_3 + \frac{1}{\Lambda} + \frac{s}{\Lambda} - \Lambda\right)\xi + \lambda_3^2 - 2s + \frac{\lambda_3}{\Lambda} + \frac{\lambda_3 s}{\Lambda} - \lambda_3\Lambda \\ &= \frac{s^2\left(O(1)s + \left(-1 + \Lambda^2\right)^4\left(-1 - 2\Lambda^2 + \Lambda^4\right)\right)}{\Lambda^2\left(-1 + \Lambda^2\right)^6} \end{aligned}$$

$$-\frac{s\left(O(1)s-2\Lambda^2\left(-1+\Lambda^2\right)^2\right)\xi}{\Lambda\left(-1+\Lambda^2\right)^3}+\xi^2. \tag{4.11}$$

For the roots of $p_3(\xi,s)$ in ξ, one can rescale $\xi = s\beta$ so that the polynomial for β is given by

$$\begin{cases} P_3(\beta,s)=0, \\ P_3(\beta,s)\equiv\beta^2-\dfrac{\left(O(1)s-2\Lambda^2\left(-1+\Lambda^2\right)^2\right)}{\Lambda\left(-1+\Lambda^2\right)^3}\beta \\ \qquad\qquad +\dfrac{O(1)s+\left(-1+\Lambda^2\right)^4\left(-1-2\Lambda^2+\Lambda^4\right)}{\Lambda^2\left(-1+\Lambda^2\right)^6}. \end{cases}$$

This shows the roots $\beta = \beta(s)$ of $P_3(\beta, s) = 0$ are analytic around $s = 0$; and one has the asymptotic of $\beta(s)$ as follows:

$$\beta(s) = 1/(-\Lambda \pm 1) + O(1)s.$$

The above for λ_3 and β together yield the following lemma.

Lemma 4.2. *Let $|\Lambda| \in (0, 1)$, then there are three analytic roots $\xi = \lambda(s)$ of $p(\xi, s) = 0$ for $s \in \{Re(s) > -(1 - |\Lambda|)/C_b\}$, satisfying, for $|s| \ll 1$,*

$$\begin{cases} \lambda_1(s)=-\frac{s}{(1+\Lambda)}+O(1)s^2, \\ \lambda_2(s)=\frac{s}{(1-\Lambda)}+O(1)s^2, \\ \lambda_3(s)=-\dfrac{1-\Lambda^2}{\Lambda}+\dfrac{s\left(1+\Lambda^2\right)}{\Lambda\left(-1+\Lambda^2\right)}+O(1)s^2. \end{cases} \tag{4.12}$$

Here, the function $O(1)$ is an analytic function in the region $Re(s) > -(1 - |\Lambda|)/C_b$.

The behavior of the eigenvalues near $s = \infty$ will also be needed. First, it is easy to see that, for $s > 0$, the eigenvalues cannot have zero real part and so

$$Re(\lambda_1(s)) < 0,\ Re(\lambda_2(s)) > 0,\ Re(\lambda_3(s)) < 0,\ \text{for s} > 0. \tag{4.13}$$

Set

$$z = \frac{1}{\xi},\ \sigma = \frac{1}{s},$$

and write the characteristic polynomial (4.8) as

$$p = s^2\xi^3[z^3 + 2\Lambda\sigma z^2 + (\Lambda^2 - 1)\sigma^2 z - \sigma z - \Lambda\sigma^2] \equiv s^2\xi^3\tilde{p}(z,\sigma).$$

All the roots of the polynomial $\tilde{p}(z,\sigma)$ are zero at $\sigma = 0$. There is a root with linear leading term. Set

$$z \ = \ A\sigma + O(1)\sigma^2$$

and

$$\tilde{p}(z,\sigma) \ = \ \sigma^2(-A-\Lambda) + O(1)\sigma^3,$$

and so $A = -\Lambda$. This yields a root with negative real part. As we know from (4.13) that λ_1 is the only eigenvalue with negative real part, and so the above yields

$$\lambda_1(s) = -\frac{1}{\Lambda}s + O(1),\ \text{as s} \to \infty. \tag{4.14}$$

Straightforward calculations show that

$$\frac{\tilde{p}(z,\sigma)}{z - (A\sigma + O(1)\sigma^2)} = z^2 + (\Lambda\sigma + O(1)\sigma^2)z - \sigma + O(1)\sigma^2.$$

The roots of this polynomial is therefore of the form $O(1)\sigma + O(1)\sqrt{\sigma}$. This yields

$$\lambda_2(s),\ \lambda_3(s) = O(1)\sqrt{s},\ \text{as s} \to \infty. \tag{4.15}$$

We now construct the Dirichlet-Neumann relation according to the following three cases:

Case 1. $\Lambda \in (0,1)$.

The characteristic polynomial $p(\xi,s)$ has only one root $\xi = \lambda_2(s)$ with the property $Re(\lambda_2(s)) > 0$ for $s > 0$. For this root $\lambda = s\beta$, the function β

is a root of $P_3(\beta, s) = 0$. The Dirichlet-Neumann map is

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} U_x^0(s) = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} U^0(s), \tag{4.16}$$

where

$$\begin{pmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ \frac{s}{s+\lambda_2\Lambda} & \frac{s\Lambda-(\lambda_2)^2\Lambda+\lambda_2\left(-1-s+\Lambda^2\right)}{s+\lambda_2\Lambda} \end{pmatrix}.$$

In component form, this becomes

$$V_x^0 = \frac{s}{s+\lambda_2\Lambda} P^0 + \frac{s\Lambda-(\lambda_2)^2\Lambda+\lambda_2\left(-1-s+\Lambda^2\right)}{s+\lambda_2\Lambda} V^0, \tag{4.17}$$

where

$$P \equiv \mathbb{L}\rho, \ V \equiv \mathbb{L}v.$$

Proposition 4.3. *The functions*

$$\frac{s}{s+\Lambda\lambda_2} \ \textit{and} \ \frac{s\Lambda-(\lambda_2)^2\Lambda+\lambda_2\left(-1-s+\Lambda^2\right)}{s+\lambda_2\Lambda}$$

are analytic in the region $Re(s) > -(1-|\Lambda|)/C_b$.

Proof. By Lemma 4.2, the function $\lambda(s)$ is analytic in the region $Re(s) > -(1-|\Lambda|)/C_b$ and $\{s+\lambda\Lambda = 0\} \cap \{p(\lambda, s) = 0\}$ around $(\lambda, s) = (0, 0)$. The possible poles of the functions are at $s = 0$. Then, the asymptotic of λ_2 in Lemma 4.2 results in the analyticity of the functions $s/(s+\Lambda\lambda_2)$ and $(s\Lambda - \lambda^2\Lambda + \lambda_2\left(-1-s+\Lambda^2\right))/(s+\lambda_2\Lambda)$ around $s = 0$. Thus the pole is removable and both functions are analytic in the region $Re(s) > -(1-|\Lambda|)/C_b$. □

One expands $s/(s+\Lambda\lambda_2)$ at $s = 0$, $\lim_{s\to 0} s/(s+\Lambda\lambda_2) = 1/(1+\Lambda\lambda_2(0)) = 1-\Lambda$, as follows

$$\frac{s}{s+\Lambda\lambda_2} = 1 - \Lambda + s\, Q\left[\frac{s}{s+\Lambda\lambda_2}\right](s). \tag{4.18}$$

From (4.15), λ_2 grows at the rate of $\sqrt{s}$ as $s \to \infty$ and so

$$Q\left[\frac{s}{s+\Lambda\lambda_2}\right](s) = \frac{\frac{s}{s+\Lambda\lambda_2} - 1 + \Lambda}{s} = O(1)\frac{1}{s} \text{ as } s \to \infty. \tag{4.19}$$

Thus, by Lemmas 2.4 the inverse Laplace transformation $\mathbb{L}^{-1}[(s/(s+\lambda_2\Lambda))\mathbb{L}[u^0]](t)$ is

$$\mathbb{L}^{-1}\Big[\frac{s}{s+\lambda_2\Lambda}\mathbb{L}[u^0]\Big](t) = \int_0^t \mathbb{L}^{-1}\left[Q\left[s/(s+\lambda_2\Lambda)\right]\right](t-\tau)\partial_\tau u^0(\tau)d\tau + (1-\Lambda)u^0(t) \tag{4.20}$$

Since $Q[s/(s+\lambda_2\Lambda)]$ is analytic in the domain $Re(s) > -(1-|\Lambda|)/C_b$, the path integral of the Bromwich integral along $Re(s) = 0$ is the same as along $Re(s) = -(1-|\Lambda|)/C_b$. Thus from (4.19) there exist constants $C_0,\ C_1 > 0$ such that

$$\Big|\mathbb{L}^{-1}[\frac{s}{s+\lambda_2\Lambda}\mathbb{L}[u^0]](t) - (1-\Lambda)u^0(t)\Big| \le C_1 \int_0^t \frac{e^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}|\partial_\tau u^0(\tau)|d\tau. \tag{4.21}$$

Actually, from (4.19) one has the weaker singularity of $1/\log(t-\tau)$ rather than the singularity of $1/\sqrt{t-\tau}$ on the right hand side of (4.22). We only need the integrability of the singularity around $t = \tau$ so that the integral represents the localized effect of the boundary values. Thus we put in $1/\sqrt{t-\tau}$ for the notational uniformity. Similarly, we have

$$\begin{aligned}&\Big|\mathbb{L}^{-1}\Big[\frac{s\Lambda - (\lambda_2)^2\Lambda + \lambda_2\left(-1-s+\Lambda^2\right)}{s+\lambda_2\Lambda}\mathbb{L}[u^0]\Big](t) - (\Lambda-1)u^0(t)\Big| \\ &\quad \le\ C_1 \int_0^t \frac{e^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}|\partial_\tau u^0(\tau)|d\tau. \end{aligned} \tag{4.22}$$

We have from (4.17), (4.21), and (4.22) the following Dirichlet-Neumann relation for the compressible Navier-Stokes equations in the present case of $0 < \Lambda < 1$.

Theorem 4.4. *For the case of positive subsonic evaporation,* $0 < \Lambda < 1$, *there exists* $C_0 > 0$ *such that for* $(\rho^0, v^0) \in \mathscr{V}$

$$v_x^0(t) = (1-\Lambda)\rho^0(t) - (1-\Lambda)v^0(t) + O(1)\int_0^t \frac{e^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}(|\partial_\tau\rho^0(\tau)| + |\partial_\tau v^0(\tau)|)d\tau. \tag{4.23}$$

Remark 4.5. The relation (4.23) gives, within locally and exponentially decaying accuracy, the explicit expression of the Neumann value for the velocity in terms of the Dirichlet values of both density and velocity. From the continuity equation $\rho_t + \Lambda\rho_x + v_x = 0$ in (4.3) the Neumann value for the density is given in terms of the Dirichlet values also:

$$\begin{aligned}\rho_x^0(t) \;=\; & -\frac{1}{\Lambda}\partial_t\rho^0(t) - \frac{1-\Lambda}{\Lambda}\rho^0(t) + \frac{1-\Lambda}{\Lambda}v^0(t) \\ & + O(1)\int_0^t \frac{e^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}(|\partial_\tau\rho^0(\tau)| + |\partial_\tau v^0(\tau)|)d\tau. \end{aligned} \tag{4.24}$$

This way of using the continuity equation does not work for the case solid wall, $\Lambda = 0$, in Case 2. Although it also works for Case 3 of condensation, $-1 < \Lambda < 0$; the situation is quite different from the present case of evaporation.

Case 2. $\Lambda = 0$.

In this case, we write the system in the original (ρ, u)-coordinate:

$$\begin{cases}\rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}, \end{cases} \qquad \rho(x,0) = v(x,0) \equiv 0. \tag{4.25}$$

The first equation gives

$$sP(x,s) + V_x(x,s) = 0,$$

in particular, their boundary values satisfy

$$sP^0(s) + V_x^0(s) = 0. \tag{4.26}$$

For this case, the characteristic polynomial degenerates to

$$p(\xi, s) = s^2 - \xi^2 - s\xi^2,$$

and has only one root $\lambda = s/\sqrt{s+1}$ with $Re(\lambda) > 0$ for $s > 0$. Following the above procedure and use (4.26) to eliminate P from the Dirichlet-Neumann relation for the transformed variable, we obtain the relation for $V = \mathbb{L}[v]$

alone:

$$V_x^0(s) = -\frac{s}{\sqrt{1+s}}V(s). \tag{4.27}$$

This yields the Dirichlet-Neumann kernel

$$\mathfrak{N}_{22}(t) = -\partial_t\left(\frac{1}{2\pi i}\int_{Re(s)=0} e^{st}\frac{1}{\sqrt{1+s}}ds\right) = -\partial_t\left(\frac{e^{-t}}{\sqrt{\pi t}}\right). \tag{4.28}$$

Theorem 4.6. *When $\Lambda = 0$, The Dirichlet-Neumann relation for the system* (4.25) *is*

$$v_x^0(t) = -\int_0^t \partial_t\left(\frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}}\right)v^0(\tau)d\tau. \tag{4.29}$$

Remark 4.7. The boundary value for the density can be recovered from the system (4.25):

$$\rho^0(t) = -\int_0^t v_x^0(\tau)d\tau = \int_0^t\int_0^\tau \partial_\tau\left(\frac{e^{-(\tau-s)}}{\sqrt{\pi(\tau-s)}}\right)v^0(s)dsd\tau. \tag{4.30}$$

Thus the only one boundary value is needed for a well-posed initial-boundary value problem. This is different from the other two cases.

Case 3. $\Lambda \in (-1,0)$.

The analysis for this case is similar to Case 1, $\Lambda \in (0,1)$. For $s > 0$, two roots have positive real part, $Re(\lambda_j) > 0,\ j = 2,3$, and so the boundary relation (4.16) holds for $j = 2,3$:

$$\begin{cases}(s+\lambda_2\Lambda)V_x^0 = s\mathbb{L}P^0 + (s\Lambda - (\lambda_2)^2\Lambda + \lambda_2(-1-s+\Lambda^2))V^0,\\ (s+\lambda_3\Lambda)V_x^0 = s\mathbb{L}P^0 + (s\Lambda - (\lambda_3)^2\Lambda + \lambda_3(-1-s+\Lambda^2))V^0.\end{cases} \tag{4.31}$$

This leads to

$$\begin{aligned} V_x^0 &= \frac{-(\lambda_2+\lambda_3)+(-1-s+\Lambda^2)}{\Lambda}V^0 \\ &= -\left(\frac{\lambda_2+s}{s\Lambda}\right)sV^0 + \frac{(-1+\Lambda^2+\lambda_3)}{\Lambda}V^0 \\ &\equiv A(s)s\mathbb{L}[u_b] + \frac{(-1+\Lambda^2+\lambda_3)}{\Lambda}\mathbb{L}[u_b]. \end{aligned} \tag{4.32}$$

Apply similar computations as those for (4.23), one has the following theorem:

Theorem 4.8. *There exists $C_0 > 0$ such that*

$$\begin{aligned} v_x^0(t) &= \frac{(-1+\Lambda)(1+\Lambda)^2}{\Lambda^2} v^0(t) + \frac{2-\Lambda}{(-1+\Lambda)\Lambda} \partial_t v^0(t) \\ &\quad + O(1) \int_0^t \frac{e^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} (|\partial_\tau v^0(\tau)| + |\partial_\tau^2 v^0(\tau)|) d\tau. \end{aligned} \tag{4.33}$$

Remark 4.9. From (4.31), one can represent the boundary values for the density in terms of that of the velocity. Thus only one boundary value is needed for the well-posedness of the initial-boundary value problem. Note also from (4.33) that the Neumann value for the velocity depends not only on the first differential, but also on the second differential of the velocity.

5. A Viscous System in 2-D

We next study the multi-dimensional wave propagation, $\vec{\boldsymbol{x}} \in \mathbb{R}^m$, $m \geq 2$. We will consider in two space dimensions, $m = 2$, $\vec{\boldsymbol{x}} = (x, y)$, the viscous wave equations in half space, $x \geq 0, y \in \mathbb{R}$, $(x, y) \in \mathbb{R}_+^2$:

$$\boldsymbol{u}_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \boldsymbol{u}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{u}_y = \Delta_2 \boldsymbol{u},\ \boldsymbol{u} \in \mathbb{R}^2, x, t > 0, y \in \mathbb{R}. \tag{5.1}$$

As before, the boundary value $\boldsymbol{u}^0(y, \cdot) \equiv \boldsymbol{u}(0, y, \cdot)$ is $\mathscr{V}$-valued function, with

$$\mathscr{V} \equiv \{g \in C^\infty(\mathbb{R}_+) | \partial_t^n g(0) = 0 \text{ for } n \in \{0\} \cup \mathbb{N}\}.$$

This system is the D'Alembert's wave equation in 2-D with an artificial viscosity. The inviscid first order operator, after the Galilean transformation $x \to x - \Lambda t$, is the 2-D D'Alembert wave equation:

$$\left(\partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y \right) W(x, y, t) = 0, \tag{5.2}$$

or the standard D'Alembert wave equation in 2-D with the wave speed one:

$$(\partial_t^2 - \Delta_2) w = 0. \tag{5.3}$$

Here the speed Λ of the wave center is assumed to be subsonic:

$$-1 < \Lambda < 1. \tag{5.4}$$

Consider the Green's function, the fundamental solution $\mathbb{G}(x,y,t)$ of initial value problem for the wave equation:

$$\begin{cases} \mathbb{G}_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \mathbb{G}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbb{G}_y = \Delta_2 \mathbb{G}, \ \vec{\boldsymbol{x}} \equiv (x,y), \\ \mathbb{G}(\vec{\boldsymbol{x}},0) = \delta(\vec{\boldsymbol{x}})\mathbb{I}, \end{cases} \tag{5.5}$$

where $\mathbb{I}$ is the 2×2 identity matrix. Its Fourier transformations of $\hat{\mathbb{G}}(\vec{\boldsymbol{x}},t)$ can be computed easily and explicitly:

$$\begin{aligned} \hat{\mathbb{W}}(\vec{\zeta},t) &\equiv \int_{\mathbb{R}^2} e^{-i\vec{\zeta}\cdot\vec{\boldsymbol{x}}} \mathbb{W}(\vec{\boldsymbol{x}},t)d\vec{\boldsymbol{x}}, \ ;\vec{\zeta} = (\zeta^1,\zeta^2), \\ \hat{\mathbb{G}}(\vec{\zeta},t) &= e^{-|\vec{\zeta}|^2 t - i\Lambda\zeta^1 t} \begin{pmatrix} \cos(t|\vec{\zeta}|) - \frac{i\sin(t|\vec{\zeta}|)\zeta^1}{|\vec{\zeta}|} & -\frac{i\sin(t|\vec{\zeta}|)\zeta^2}{|\vec{\zeta}|} \\ -\frac{i\sin(t|\vec{\zeta}|)\zeta^2}{|\vec{\zeta}|} & \cos(t|\vec{\zeta}|) + \frac{i\sin(t|\vec{\zeta}|)\zeta^1}{|\vec{\zeta}|} \end{pmatrix}. \end{aligned} \tag{5.6}$$

From (5.6), the entries of the fundamental function consist of Sine transformation, Cosine transformation, Galilean translation, and heat kernel. Except for the heat kernel, the Fourier transform of the fundamental solution for the wave equation (5.3) also consists of these transforms. As the wave equation (5.3) has been solved explicitly by Hadamard's method of descent, we have the following structure of $\mathbb{G}(\vec{\boldsymbol{x}},t)$, Theorem 6.4:

$$\begin{aligned} &\mathbb{G} = (\mathbb{G}_{ij})_{2\times 2}, \\ &|\mathbb{G}_{ij}(\vec{\boldsymbol{x}},t)| \le O(1)\frac{1}{\sqrt{t}} W_2(\vec{\boldsymbol{x}} - (\Lambda t,0),t;2) \text{ for } i \ne j, \\ &|\mathbb{G}_{jj}(\vec{\boldsymbol{x}},t) \le O(1)\left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right) W_2(\vec{\boldsymbol{x}} - (\Lambda t,0),t),t;2) \text{ for } j = 1,2, \end{aligned} \tag{5.7}$$

where W_2 is defined by

$$W_2(\vec{\boldsymbol{x}},t;D_0)\equiv\begin{cases}\dfrac{1}{t-|\vec{\boldsymbol{x}}|} & \text{for } |\vec{\boldsymbol{x}}|\le t-\sqrt{D_0t},\\[2mm] \dfrac{1}{t^{3/4}D_0^{1/4}} & \text{for } ||\vec{\boldsymbol{x}}|-t|\le\sqrt{D_0t},\\[2mm] \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{4D_0t}}}{t^{3/4}D_0^{1/4}} & \text{for } |\vec{\boldsymbol{x}}|\ge t+\sqrt{D_0t}.\end{cases}\tag{5.8}$$

Unlike the first two examples with one spacial dimension, we now have two spacial diemsnsions and need to take the Fourier transform with respect to the tangential direction y first. We adopt the notations of (1.8), and, for simplicity, in the 2-D case here we also use $\vec{\boldsymbol{x}}=(x,y)$, $\eta\equiv\zeta^2$, $\vec{\zeta}=(\zeta^1,\zeta^2)$:

$$\begin{cases}\boldsymbol{v}(x,\zeta^2,t)=\boldsymbol{v}(x,\eta,t)=\mathscr{F}[\boldsymbol{u}](x,\eta,t)\equiv\int_{\mathbb{R}^{m-1}}\boldsymbol{u}(x,y,t)e^{-iy\eta}dy,\\ V(x,\zeta^2,s)=V(x,\eta,s)=\mathbb{L}[\boldsymbol{v}](x,\eta,s)\equiv\int_0^\infty e^{-st}\boldsymbol{v}(x,\eta,t)dt,\\ \qquad\qquad\text{Laplace transformation for } t,\\ \hat{V}(\xi,\zeta^2,s)=\hat{V}(\xi,\eta,s)=\mathbb{L}[V](\xi,\eta,s)\equiv\int_0^\infty e^{-\xi x}V(x,\eta,s)dx,\\ \qquad\qquad\text{Laplace transformation for } x,\\ \boldsymbol{u}^0(y,t)\equiv\boldsymbol{u}(0,y,t),\ \boldsymbol{v}^0(\zeta^2,t)\equiv\boldsymbol{v}(0,\zeta^2,t),\ V^0(\zeta^2,s)\equiv V(0,\zeta^2,s),\\ \qquad\qquad\text{boundary Dirichlet values},\\ \boldsymbol{u}^0_x(y,t)\equiv\boldsymbol{u}_x(0,y,t),\ \boldsymbol{v}^0_x(\zeta^2,t)\equiv\boldsymbol{v}_x(0,\zeta^2,t),\ V^0_x(\zeta^2,s)\equiv V_x(0,\zeta^2,s),\\ \qquad\qquad\text{boundary Neumann values.}\end{cases}\tag{5.9}$$

These variables satisfy:

$$\boldsymbol{v}_t+\begin{pmatrix}1+\Lambda & 0\\ 0 & -1+\Lambda\end{pmatrix}\boldsymbol{v}_x+\begin{pmatrix}0 & i\eta\\ i\eta & 0\end{pmatrix}\boldsymbol{v}=\boldsymbol{v}_{xx}-\eta^2\boldsymbol{v},\tag{5.10}$$

$$sV+\begin{pmatrix}1+\Lambda & 0\\ 0 & -1+\Lambda\end{pmatrix}V_x+\begin{pmatrix}0 & i\eta\\ i\eta & 0\end{pmatrix}V=V_{xx}-\eta^2V,\tag{5.11}$$

$$\begin{pmatrix} s+(1+\Lambda)\xi-\xi^2+|\eta|^2 & i\eta \\ i\eta & s+(-1+\Lambda)\xi-\xi^2+|\eta^2|^2 \end{pmatrix}\hat{V}(\xi,\eta,s)$$
$$=\begin{pmatrix} (1+\Lambda-\xi) & 0 \\ 0 & (-1+\Lambda-\xi) \end{pmatrix} V^0(\eta,s)-V^0_x(\eta,s). \quad (5.12)$$

Thus the characteristic polynomial $p(\xi;\eta,s)$ is

$$\begin{aligned} p(\xi;\eta,s) &= \det\begin{pmatrix} s+(1+\Lambda)\xi-\xi^2+|\eta|^2 & i\eta \\ i\eta & s+(-1+\Lambda)\xi-\xi^2+|\eta|^2 \end{pmatrix} \\ &= s^2+|\eta|^2+2s|\eta|^2+|\eta|^4+\left(2s\Lambda+2|\eta|^2\Lambda\right)\xi \\ &\quad +\left(-1-2s-2|\eta|^2+\Lambda^2\right)\xi^2-2\Lambda\xi^3+\xi^4. \end{aligned} \quad (5.13)$$

The characteristic polynomial $p(\xi;\eta,s)$ is a degree 4 polynomial in ξ. Its four roots $\lambda_j(\eta,s),\ j=1,\ldots,4$ can be computed explicitly when $\eta=0$ and $\Lambda\in(0,1)$:

$$\begin{cases} \lambda_1(0,s)=\dfrac{\Lambda+1+\sqrt{(\Lambda+1)^2+4s}}{2}, \\ \lambda_2(0,s)=\dfrac{\Lambda-1+\sqrt{(\Lambda-1)^2+4s}}{2}, \\ \lambda_3(0,s)=\dfrac{\Lambda+1-\sqrt{(\Lambda+1)^2+4s}}{2}, \\ \lambda_4(0,s)=\dfrac{\Lambda-1-\sqrt{(\Lambda-1)^2+4s}}{2}. \end{cases} \quad (5.14)$$

Thus for $s>0$, $\Lambda\in(0,1)$,

$$\lambda_1(0,s)>\lambda_2(0,s)>0>\lambda_3(0,s)>\lambda_4(0,s).$$

It is easy to check that any root $\lambda_j(\eta,s),\ 1\le j\le 4$, of the characteristic polynomial $p(\xi;\eta,s)$ has the property that, for any $s>0$ and real η, its real part does not vanish and therefore maintains their signs. Thus, for any $s>0$ and real η,

$$Re(\lambda_1(\eta,s)),\ Re(\lambda_2(\eta,s))>0>Re(\lambda_3(\eta,s)),\ Re(\lambda_4(\eta,s)).$$

Thus we may follow the same procedure as in Section 2 from (2.7) to (2.11) with $(j_1,j_2)=(1,2)$. After straightforward computations, the Master Relationship (MR), the Dirichlet-Neumann relations in the transformed variables, is given in terms of λ_1 and λ_2 as follows:

$$V^0_{x^1}(\eta, s) = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} V^0(\eta, s), \tag{5.15}$$

$$\begin{cases} K_{11} \equiv -\dfrac{1 - s - |\eta|^2 - \Lambda^2 + 2\lambda_1 + \lambda_1^2 + 2\lambda_2 + \lambda_1\lambda_2 + \lambda_2^2}{1 - \Lambda + \lambda_1 + \lambda_2}, \\ K_{12} \equiv -\dfrac{i\eta}{1 - \Lambda + \lambda_1 + \lambda_2}, \\ K_{21} \equiv -\dfrac{i\left(s + |\eta|^2 + (-1+\Lambda)\lambda_1 - \lambda_1^2\right)\left(s + |\eta|^2 + (-1+\Lambda)\lambda_2 - \lambda_2^2\right)}{\eta\,(1 - \Lambda + \lambda_1 + \lambda_2)}, \\ K_{22} \equiv -\dfrac{-1 + s + |\eta|^2 + 2\Lambda - \Lambda^2 + (-1+\Lambda)\lambda_2 + \lambda_1\,(-1 + \Lambda + \lambda_2)}{(1 - \Lambda + \lambda_1 + \lambda_2)}. \end{cases} \tag{5.16}$$

5.1. Laplace-Fourier paths and roots for the characteristic polynomial

The eigenvalues $\lambda_i = \lambda_i(\eta, s)$, $i = 1, 2, 3, 4$, are defined in terms of the Laplace variable $s > 0$ and Fourier variable $\eta \in \mathbb{R}$. For the inversion of the Laplace transform using the Bromwich integral (1.13), one needs to have an analytic extension of the Laplace variable $s \in \mathbb{R}_+$ to a simply connected subset containing $Re(s) = 0$ of $\mathbb{C}$. As we will see, the study of the analytic extension can be done only for $|\eta|$ small. At $\eta = 0$ we have explicit formula of $\lambda_i(0, s)$, (5.14). It is easy to see that both $\lambda_1(0, s)$ and $\lambda_4(0, s)$ are analytic in the domains $\boldsymbol{U}_1 \equiv \mathbb{C}\backslash(-\infty, -(1+\Lambda)^2/4)$ and the domain $\boldsymbol{U}_4 \equiv \mathbb{C}\backslash(-\infty, -(1-\Lambda)^2/4)$ respectively as illustrated in Figure a. Moreover, since both $\lambda_1(0, s)$ and $\lambda_4(0, s)$ are distinct from each other and other eigenvalues, both $\lambda_1(\eta, s)$ and $\lambda_4(\eta, s)$, $|\eta|$ small, are also analytic in the domains around $\boldsymbol{U}_1$ and $\boldsymbol{U}_4$ respectively. In particular, for $|\eta|$ small, the eigenvalues $\lambda_1(\eta, s)$ and $\lambda_4(\eta, s)$ have spectral gap and are non-characteristic according to Definition 1.1.

The situation is different for the eigenvalues $\lambda_2(\eta, s)$ and $\lambda_3(\eta, s)$, as their values coincide and are zero for $(\eta, s) = (0, 0)$, (5.14). For $\eta = 0$, $(\bar{\xi}, \bar{s}) = (0, 0)$ is a removable branch point for both λ_2 and λ_3 and $\lambda_1(0, s)$ and $\lambda_4(0, s)$ can also be analytically extended to the domain $\boldsymbol{U}_4$. However,

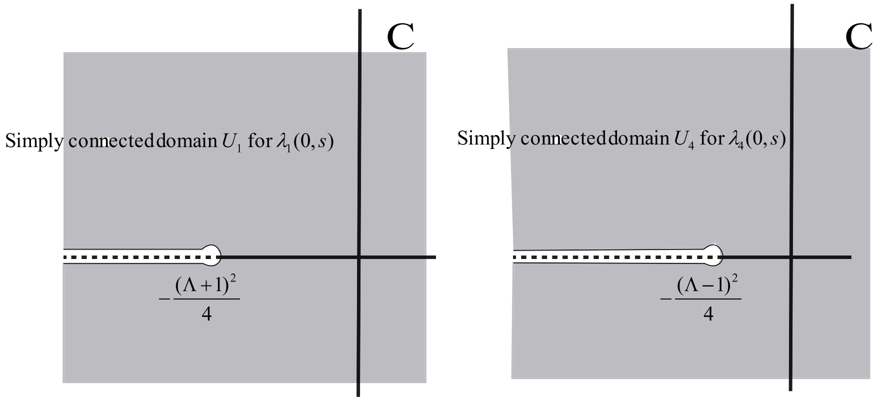

Figure a

for $\eta \neq 0$, the qualitative structure of the non-removable branch points changes and $(\bar{\xi}, \bar{0}) = (0,0)$ branches into two points. To study this, we first consider the general study of the branch points (η, s) at which $\lambda_i(\eta, s)$ is not differentiable in s. As λ_i is defined implicitly by $p(\lambda_i; \eta, s) = 0$, we have $\partial_s \lambda_i = -\partial_s p(\lambda_i; \eta, s)/\partial_\xi p(\lambda_i; \eta, s)$. The necessary condition for the branch point at $\eta = 0$ is

$$\begin{cases} p(\bar{\xi}, 0, \bar{s}) = 0, \\ p_\xi(\bar{\xi}, 0, \bar{s}) = 0. \end{cases} \tag{5.17}$$

The solutions are

$$(\bar{\xi}, \bar{s}) \in \left\{ \left(\frac{\Lambda+1}{2}, -\frac{(\Lambda+1)^2}{4} \right),\ (0,0),\ \left(\frac{\Lambda-1}{2}, -\frac{(\Lambda-1)^2}{4} \right) \right\}. \tag{5.18}$$

Thus, $(\xi, s) = (0,0)$ is the only possible characteristic branching point for $|\eta|$ small. For $\eta \neq 0$, there is no simple explicit expression for the eigenvalues. To analyze the two branch points bifurcating from $(\bar{\xi}, \bar{s}) = (0,0)$, we consider an implicit expression for the roots. For the case $\Lambda = 0$, studied in [5], the roots of the characteristic polynomial have explicit expression. Motivated by this, we replace $\Lambda\xi$ in the right hand side of (5.13) by $\Lambda\lambda$:

$$\begin{aligned} p(\xi; \eta, s) \ &= \ s^2 + |\eta|^2 + 2s|\eta|^2 + |\eta|^4 + 2s\Lambda\lambda + 2|\eta|^2\Lambda\lambda + (-1 - 2s - 2|\eta|^2)\xi^2 \\ &\quad + \Lambda^2\lambda^2 - 2\Lambda\lambda\xi^2 + \xi^4 = 0, \end{aligned}$$

for roots $\xi = \lambda$ of the characteristic polynomial. From this we obtain the implicit expression of the root $\xi = \lambda$:

$$\lambda(\eta, s) = \pm\sqrt{(\sqrt{(s + \Lambda\lambda(\eta, s) + \frac{1}{4})} - \frac{1}{2})^2 + |\eta|^2}. \tag{5.19}$$

With this expression, one considers branch cut for the outer square root along the negative real axis. In other words, the right hand side of (5.19) is a pure imaginary number, denoted by the new variable $i\zeta^1$, $\zeta^1 \in \mathbb{R}$:

$$\begin{aligned} i\zeta^1 &= \pm\sqrt{(\sqrt{(s + \Lambda i\zeta^1 + \frac{1}{4})} - \frac{1}{2})^2 + |\eta|^2} \\ &= \sqrt{(\sqrt{(s + \Lambda i\zeta^1 + \frac{1}{4})} - \frac{1}{2})^2 + |\zeta^2|^2}, \end{aligned} \tag{5.20}$$

recalling that we have written, for simplicity, $\eta = \zeta^2$. The equation (5.20) can be solved explicitly for s:

$$\begin{aligned} s &= s_\pm(\zeta^1; \zeta^2) \equiv -(|\zeta^1|^2 + |\zeta^2|^2) \pm i\sqrt{|\zeta^1|^2 + |\zeta^2|^2} - i\Lambda\zeta^1 \\ &= -|\vec{\zeta}|^2 \pm i|\vec{\zeta}| - i\Lambda\zeta^1, \end{aligned} \tag{5.21}$$

and we define the path $\Gamma_\pm^{\zeta^2}$:

$$\Gamma_+^{\zeta^2} \equiv \{s_+(\zeta^1; \zeta^2) | \zeta^1 \in \mathbb{R}\}, \quad \Gamma_-^{\zeta^2} \equiv \{s_-(\zeta^1; \zeta^2) | \zeta^1 \in \mathbb{R}\}.$$

Here, the paths $\Gamma_\pm^{\zeta^2}$ are named as the *Laplace-Fourier path*, due to the fact that the function $s_\pm(\zeta^1; \zeta^2)$ also satisfies

$$p(i\zeta^1; \zeta^2, s_\pm) = 0.$$

It turns out that $s_\pm(\zeta^1; \zeta^2)$ is the spectrum of the Fourier transformation of (5.1) with respect to the wave number $(i\zeta^1, i\zeta^2)$:

$$s_\pm(\zeta^1; \zeta^2) = \sigma(\vec{\zeta}).$$

The known structure for $\lambda_i(0, s)$ allows us to carry out the perturbation analysis of possible branch points for $|\zeta^2| \ll 1$. The condition $|\zeta^2| \ll 1$ is necessary for study of the spectral information in the Laplace-Frourier analysis. This is because there does exist a branch point in $Re(s) > 0$ when

ζ^2 is not small. Computational result based on $p = p_\xi = 0$, c.f. (5.17), shows that branch point does appear in the region $Re(s) > 0$ for the case of $(\Lambda, \zeta^2) = (1/2, 1/2)$, Figure b.

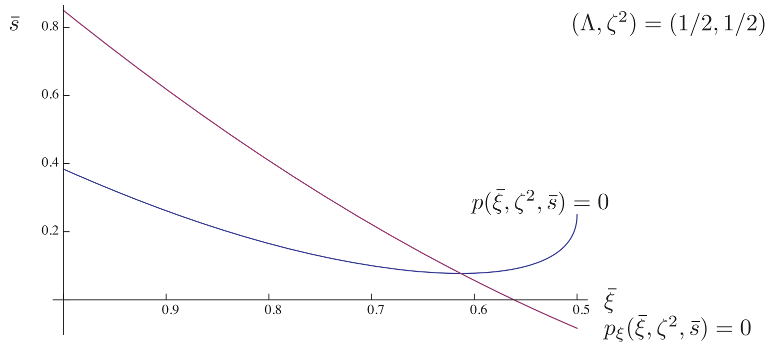

Figure b

Thus the size $|\zeta^2|$ of the Fourier variable needs to be restricted to be sufficiently small so that all branch points reside in the domain $Re(s) < 0$. In other words, we are forced to have the *long-short waves decomposition* in our analysis. The analysis for the short waves, $|\eta|$ not small, will be done later using the energy method, and not the Laplace-Fourier path method presented here.

Consider a small $\delta-$neighborhood of the path zero, Γ_0^δ:

$$\Gamma_0^\delta \equiv \{s_0(x)|x \in \mathbb{R}\},$$
$$s_0(x) \equiv \begin{cases} x + 2\delta - \dfrac{(1-|\Lambda|^2)}{4} + i\delta \text{ for } x < -\delta, \\ -\dfrac{(1-|\Lambda|^2)}{4} + \delta - ix \text{ for } x \in (-\delta, \delta), \\ -x + 2\delta - \dfrac{(1-|\Lambda|^2)}{4} - i\delta \text{ for } x > \delta. \end{cases}$$

The region is designed to bound the branch points, with small perturbation of $|\zeta|^2$, around $(\bar{\xi}, \bar{s}) = ((\Lambda+1)/2, -(\Lambda+1)^2/4)$ and $(\bar{\xi}, \bar{s}) = ((\Lambda-1)/2, -(\Lambda-1)^2/4)$ for $\zeta^2 = 0$, (5.18). The following lemmas in this subsection follow from the above discussions, we omit their proofs.

Lemma 5.1. *There exist $\delta_0 > 0$ and $\delta_1 > 0$ such that for all $|\zeta^2| < \delta_1$ the branch points of $(\bar{\xi}, \bar{s})$ of $p(\xi, \zeta^2, s) = 0$ can not occur on $\Gamma_0^{\delta_0}$.*

The paths $\Gamma_{\pm}^{\zeta^2}$ and $\Gamma_0^{\delta_0}$ give complete branch cuts for $\lambda_2(\zeta^2,s)$ and $\lambda_3(\zeta^3,s)$ in s for each given small ζ^2, $|\zeta^2|\le\delta_0$; and the path $\Gamma_0^{\delta_0}$ gives the branch cut for $\lambda_1(\zeta^2,s)$ and $\lambda_4(\zeta^2,s)$ as illustrated in Figure c.

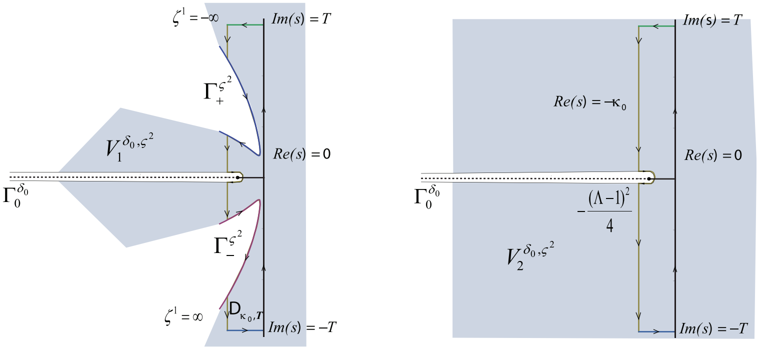

Figure c

Definition 5.2. With δ_0 and δ_1 given by Lemma 5.1, for each given ζ^2 satisfying $|\zeta^2|<\delta_1$ one denotes:

$\boldsymbol{V}_1^{\delta_0,\zeta^2}$: the connected component of $\mathbb{C}\backslash(\Gamma_+\cup\Gamma_0^{\delta_0}\cup\Gamma_-)$ connecting $\mathbb{R}_+$.
$\boldsymbol{V}_2^{\delta_0,\zeta^2}$: the connected component of $\mathbb{C}\backslash\Gamma_0^{\delta_0}$ connecting $\mathbb{R}_+$.

Lemma 5.3. *Suppose that $\Lambda\neq 0$. Then, there exist δ_0 and $\delta_1>0$ such that for any $|\zeta^2|<\delta_1$ the roots of the characteristic polynomial $p(\xi;\zeta^2,s)$ branching from (5.14) have the property that $\lambda_2(\zeta^2,s)$ and $\lambda_3(\zeta^2,s)$ are analytic for $s\in\boldsymbol{V}_1^{\delta_0,\zeta^2}$; and $\lambda_1(\zeta^2,s)$ and $\lambda_4(\zeta^2,s)$ are analytic for $s\in\boldsymbol{V}_2^{\delta_0,\zeta^2}$. On the paths $\Gamma_{\pm}^{\zeta^2}$, the root $\lambda_3(\zeta^2,s)$ satisfies*

$$\begin{cases}\lambda_3(\zeta^2,s_+(\zeta^1;\zeta^2))=\lambda_3(\zeta^2,s_-(\zeta^1;\zeta^2))=i\zeta^1,\\ s_\pm(\zeta^1;\zeta^2)\equiv i(-\Lambda\zeta^1\pm\sqrt{|\zeta^1|^2+|\zeta^2|^2})-(|\zeta^1|^2+|\zeta^2|^2).\end{cases}\tag{5.22}$$

Furthermore, with ζ^2 fixed, the asymptotics of the roots λ_i, $i=2,3$, satisfy

$$\lim_{\substack{s\to\pm\infty\\ Re(s)=0}}\frac{\lambda_j(\zeta^2,s)}{\sqrt{s}}=\begin{cases}1 \text{ for } j=1,2,\\ -1 \text{ for } j=3,4.\end{cases}\tag{5.23}$$

Remark 5.4. Since $\lambda_3(\zeta^2,s)$ is an exponent of a solution of the inverse Laplace transformation, $e^{\lambda_3 x}$ with respect to x-variable, on $\Gamma_{\pm}^{\zeta^2}$ the exponent satisfies $\lambda_3(\zeta^2,\sigma_\pm(\zeta^1;\zeta^2))=i\zeta^1$ and the solution becomes $e^{\lambda_3 x}=$

$e^{i\zeta^1 x}$. Thus, one can consider the parameter ζ^1 of the parametric curve $s = s_\pm(\zeta^1; \zeta^2)$ as an *imaginary Fourier variable* for x-direction, the direction of the normal to the boundary.

Since there is no branch point in $\boldsymbol{V}_2^{\delta_0,\zeta^2}$ for $|\zeta^2| < \delta_1$, it follows by the definition that

$$\begin{cases} \lambda_2, \lambda_3: \text{ characteristic,} \\ \lambda_1, \lambda_4: \text{ non-characteristic when } |\Lambda| < 1. \end{cases} \tag{5.24}$$

Definition 5.5. $\mathfrak{L}_j(y,t)$: the interior wave operators is defined as

$$\mathfrak{L}_j(y,t) \equiv \mathscr{F}^{-1}[\mathbb{L}^{-1}[\lambda_j]](y,t),\ j = 1,2,3,4. \tag{5.25}$$

$\mathfrak{L}_j$ is defined as a characteristic operator if and only if λ_j is characteristic.

As before, the operators $\mathfrak{L}_j(y,t)$ are generalized functions due to the non-decaying property of the roots $\lambda_j(\zeta^2, s)$ as $|s| \to \infty$. For small t, we form the decaying quotient $Q[\lambda_j(\zeta^2,s)] = [\lambda_j(\zeta^2,s) - \lambda_j(\zeta^2,0)]/s$, Definition 2.3, to decompose $\mathfrak{L}_j(y,t)$ into compositions of differential operators and integral operators, (2.4),

$$\lambda_j(\zeta^2,s) = s\, Q[\lambda_j(\zeta^2,\cdot)] + \lambda_j(\zeta^2,0).$$

Lemma 5.6. *The interior wave operators $\mathfrak{L}_j(y,t)$ on $\mathscr{V}$ satisfies*

$$\mathfrak{L}_j = \mathscr{F}^{-1}\left[\mathbb{L}^{-1}\left[Q[\lambda_j(\zeta^2,s)]\right]\right] \underset{t}{*} \partial_t + \mathscr{F}^{-1}[\lambda_j(\zeta^2,0)]\delta(t) \text{ for } j = 1,\cdots,4. \tag{5.26}$$

The operators $\underset{t}{*}$ is the convolution operators respect t variable.

The above lemma is similar to Lemma 2.2. For large t, we form instead the function $\partial_s\lambda_j$. As we will see the inverse transform of this decaying function can be computed by the Laplace-Fourier path and gives explicit description of $\mathfrak{L}_j(y,t)$ for large t.

Lemma 5.7. *There exists a function $C(\zeta^2)$ such that the interior wave operator $\mathfrak{L}_j(y,t)$ satisfies*

$$\mathfrak{L}_j = -\frac{1}{t}\mathscr{F}^{-1}[\mathbb{L}^{-1}[\partial_s\lambda_j]] + \mathscr{F}^{-1}[C(\zeta^2)]\delta(t). \tag{5.27}$$

5.2. Kernel functions for interior wave operators

With Lemmas 5.6 and 5.7, we now estimate the pointwise structure of the following functions:

$$\mathbb{L}^{-1}\left[Q[\lambda_j(\zeta^2,s)]\right],\ \mathbb{L}^{-1}[\partial_s\lambda_j].$$

First we study the non-characteristic interior wave operators $\mathfrak{L}_j$, $j=1,4$.

Lemma 5.8 (Non-characteristic root). *For $\Lambda\in(-1,1)$ there exist $\beta_0>0$ and $\delta_1>0$ such that for (ζ^2,s) in $\{s|-\beta_0<Re(s)<0\}\cap\{\zeta^2||\zeta^2|<\delta_1\}$, the roots $\lambda_j(\zeta^2,s)$, $j=1,4$, are analytic in s and satisfy*

$$\begin{aligned}|Q[\lambda_j(\zeta^2,s)](s)|=\left|\frac{\lambda_j(\zeta^2,s)-\lambda_j(\zeta^2,0)}{s}\right|\le\frac{O(1)}{\sqrt{1+|s|}},\\ |\partial_s\lambda_j(\zeta^2,s)|\le\frac{O(1)}{\sqrt{1+|s|}}.\end{aligned}\tag{5.28}$$

Proof. This follows from the perturbation, for $|\zeta^1|\ll 1$, of (5.14). □

Theorem 5.9 (Global estimates for non-characteristic operators). *For $\Lambda\in(-1,1)$ there exist positive constants C and δ_1 such that the non-characteristic roots λ_1 and λ_4 satisfy for $|\zeta^2|<\delta_1$*

$$\left|\mathbb{L}^{-1}\left[Q[\lambda_j(\zeta^2,s)]\right](t)\right| = O(1)\frac{e^{-t/C}}{\sqrt{t}}\tag{5.29}$$

$$\left|\mathbb{L}^{-1}\left[\partial_s\lambda_j(\zeta^2,s)\right](t)\right| = O(1)\frac{e^{-t/C}}{\sqrt{t}}.\tag{5.30}$$

Proof. For the function $\mathscr{F}^{-1}[\mathbb{L}^{-1}[Q[\lambda_j(\zeta^2,s)]]](y,t)$, we have, by the fact that λ_1 and λ_4 are analytic in $s\in\boldsymbol{V}_2^{\delta_0,\zeta^2}$ and that (5.28) holds,

$$\begin{aligned}\left|\int_{Re(s)=0}e^{st}\frac{\lambda_j(\zeta^2,s)-\lambda_j(\zeta^2,0)}{s}ds\right| &= \left|\int_{Re(s)=-\beta_0}e^{st}\frac{\lambda_j(\zeta^2,s)-\lambda_j(\zeta^2,0)}{s}ds\right|\\ &\le O(1)\frac{e^{-t/C}}{\sqrt{t}}.\end{aligned}\tag{5.31}$$

Similarly, one has (5.30). □

We now come to one of the main parts of the present analysis, the study of *using the Laplace-Fourier path to relate the characteristic interior wave operators $\mathfrak{L}_2$ and $\mathfrak{L}_3$ to the solution of D'Alembert wave equation.*

Lemma 5.10. *For each given Λ with $|\Lambda|<1,\ \Lambda\neq 0,$ there exist positive constants C and δ_1 such that for $|\zeta^2|<\delta_1$ the characteristic roots $\lambda_j(\zeta^2,s)$, $j=2,3,$ satisfy*

$$\begin{aligned}&\left|\mathbb{L}^{-1}[\partial_s\lambda_3](\zeta^2,t)+sign(\Lambda)\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-i\Lambda\zeta^1t-|\vec{\zeta}|^2t}\cos(|\vec{\zeta}|\ t)d\zeta^1\right|\\&\quad=O(1)\frac{e^{-t/C}}{\sqrt{t}},\end{aligned}\tag{5.32}$$

$$\begin{aligned}&\left|\mathbb{L}^{-1}[\partial_s\lambda_2](\zeta^2,t)-sign(\Lambda)\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-i\Lambda\zeta^1t-|\vec{\zeta}|^2t}\cos(|\vec{\zeta}|\ t)d\zeta^1\right|\\&\quad=O(1)\frac{e^{-t/C}}{\sqrt{t}}.\end{aligned}\tag{5.33}$$

Proof. We only consider the case $\Lambda>0$; other cases are similar. By the Cauchy's integral formula for the analytic function $e^{st}\lambda_3(\zeta^2,s)$ in $s\in\boldsymbol{V}_1^{\delta_0,\zeta^2}$, one has

$$\mathbb{L}^{-1}[\partial_s\lambda_3(\zeta^2,s)](t)=-\int_{\Gamma_-^{\zeta^2}+\Gamma_0^{\delta_0}+\Gamma_+^{\zeta^2}}e^{st}\partial_s\lambda_3(\zeta^2,s)ds,\tag{5.34}$$

by taking the limit of

$$\lim_{\kappa_0,T\to\infty}\oint_{\partial D_{\kappa_0,T}}e^{st}\partial_s\lambda_3 ds=0,$$

where $D_{\kappa_0,T}$ is the complex domain illustrated in Figure c. The sum of two path integrals on the Laplace-Fourier paths, $\Gamma_\pm^{\zeta^2}\equiv\{s=s_\pm(\zeta^1;\zeta^2)|\zeta^1\in\mathbb{R}\}$,

$$s_\pm(\zeta^1;\zeta^2)\equiv i(-\Lambda\zeta^1\pm|\vec{\zeta}|)-|\vec{\zeta}|^2,$$

can be parametrized by ζ^1. From the property $\lambda_3=i\zeta^1$ on the Laplace-Fourier path, we have the *key observation*

$$\partial_s\lambda_3\frac{ds}{d\zeta^1}=\frac{d(i\zeta^1)}{ds}\frac{ds}{d\zeta^1}=i.\tag{5.35}$$

This results in

$$\begin{aligned}
&\int_{\Gamma_-^{\zeta^2}+\Gamma_+^{\zeta^2}} e^{st}\partial_s\lambda_3(s,\zeta^2)ds \\
&= \int_{-\infty}^{\infty} e^{s_-(\zeta^1;\zeta^2)t}\partial_s\lambda_3(s,\zeta^2)\frac{ds}{d\zeta^1}d\zeta^1 + \int_{-\infty}^{\infty} e^{s_+(\zeta^1;\zeta^2)t}\partial_s\lambda_3(s,\zeta^2)\frac{ds}{d\zeta^1}d\zeta^1 \\
&= i\int_{-\infty}^{\infty}(e^{s_-(\zeta^1;\zeta^2)t}+e^{s_+(\zeta^1;\zeta^2)t})id\zeta^1 \\
&= 2i\int_{-\infty}^{\infty} e^{-i\Lambda\zeta^1 t-|\vec{\zeta}|^2 t}\cos(|\vec{\zeta}|t)d\zeta^1. \qquad (5.36)
\end{aligned}$$

We have $|\partial_s\lambda_j(\zeta^2,s)| \le O(1)/\sqrt{1+|s|}$ on $\Gamma_0^{\delta_0,\zeta^2}$ for $|\zeta^2| \ll 1$ and $\sup Re(\Gamma_0^{\delta_0,\zeta^2}) < -(1-\Lambda)^2/4+\delta_0$, and so

$$\left|\int_{\Gamma_0^{\delta_0,\zeta^2}} e^{st}\partial_s\lambda_3(\zeta^2,s)ds\right| \le O(1)\frac{e^{-t/C}}{\sqrt{t}}. \qquad (5.37)$$

This proves (5.32). From $\lambda_2+\lambda_3 = 2\Lambda-\lambda_1-\lambda_4$ and that both λ_1 and λ_4 are non-characteristic, we conclude (5.33) from (5.30) and (5.32). □

The following lemma will be used for local in time estimates later.

Lemma 5.11. *For $\Lambda \in (-1,1)\backslash\{0\}$ and $|\zeta^2| \ll 1$, $i = 2,3$,*

$$\left|\mathbb{L}^{-1}\left[Q[\lambda_i(\zeta^2,s)]\right](\zeta^2,t)\right| = O(1)\left(|\log(t)|+|\log(|\zeta^2|)|\right)e^{-|\zeta^2|^2 t}+O(1)\frac{e^{-t/C}}{\sqrt{t}}. \qquad (5.38)$$

Proof. Again we only consider the case $\Lambda > 0$. We only need to show the case for $i = 3$; and use the property $\lambda_2+\lambda_3 = 2\Lambda-\lambda_1-\lambda_4$ again to conclude the case for $i = 2$.
Similar to (5.34) and (5.37),

$$\begin{aligned}
&\mathbb{L}^{-1}\left[\frac{\lambda_3(\zeta^2,s)-\lambda_3(\zeta^2,0)}{s}\right](\zeta^2,t) \\
&= \frac{1}{2\pi i}\left(\int_{\Gamma_0^{\delta_0,\zeta^2}}+\int_{\Gamma_+^{\zeta^2}+\Gamma_-^{\zeta^2}}\right)e^{st}\frac{\lambda_3(\zeta^2,s)-\lambda_3(\zeta^2,0)}{s}ds. \qquad (5.39)
\end{aligned}$$

One defines

$$G_\pm(\zeta^2,t) \equiv \int_{\Gamma_+^{\zeta^2}+\Gamma_-^{\zeta^2}}\frac{e^{st}}{s}\lambda_3(\zeta^2,s)ds.$$

From the definition of $s_\pm(\zeta^1;\zeta^2)$ and $\Gamma_\pm^{\zeta^2}$, one has

$$G_\pm(\zeta^2,t)=\int_{\mathbb{R}} L_\pm(\zeta^2,\zeta^1,t)d\zeta^1, \tag{5.40}$$

where

$$\begin{cases} L_+(\zeta^1,\zeta^2,t)\equiv \\ \dfrac{e^{-t|\vec{\zeta}|^2}(\zeta^1+2i|\vec{\zeta}|\zeta^1-|\vec{\zeta}|\Lambda)(\cos(t(|\vec{\zeta}|-\zeta^1\Lambda))+i\sin(t(|\vec{\zeta}|-\zeta^1\Lambda)))}{|\vec{\zeta}|(|\vec{\zeta}|+i|\vec{\zeta}|^2-\zeta^1\Lambda)}, \\ L_-(\zeta^1,\zeta^2,t)\equiv \\ \dfrac{e^{-t|\vec{\zeta}|^2}(\zeta^1-2i|\vec{\zeta}|\zeta^1+|\vec{\zeta}|\Lambda)(\cos(t(|\vec{\zeta}|-\zeta^1\Lambda))-i\sin(t(|\vec{\zeta}|-\zeta^1\Lambda)))}{|\vec{\zeta}|(|\vec{\zeta}|-i|\vec{\zeta}|^2+\zeta^1\Lambda)}. \end{cases} \tag{5.41}$$

By straightforwardly evaluating the integral

$$\int_{\mathbb{R}} L_\pm(\zeta^1,\zeta^2,t)d\zeta^1=\left(\int_{|\zeta^1|<2|\zeta^2|}+\int_{|\zeta^1|>2|\zeta^2|}\right)L_\pm(\zeta^1,\zeta^2,t)d\zeta^1,$$

one has that

$$|G_\pm(\zeta^2,t)|\le O(1)\left(|\log(t)|+|\log(|\zeta^2|)|\right)e^{-|\zeta^2|^2t}.$$

This, (5.39), and (5.37) yield (5.38). □

5.3. The recombination operator for $\mathbb{L}^{-1}[1/(1-\Lambda+\lambda_1+\lambda_2)]$.

We now invert the transforms for the expression of the Dirichlet-Neumann relation (5.16). In that expression, the matrix K_{ij} have the common divisor of $1/(1-\Lambda+\lambda_1+\lambda_2)$. As the roots λ_j, $j=1,\ldots,4$, are not analytic, the divisor has the undesirable singularity. There is the standard algebraic method of regularizing this through multiplying the symmetric factors of $1/(1-\Lambda+\lambda_1+\lambda_2)$ to form the symmetric expression

$$\begin{aligned}\frac{1}{1-\Lambda+\lambda_1+\lambda_2} &= \frac{\Pi_{(j,k)\neq(1,2)}(1-\Lambda+\lambda_j+\lambda_k)}{g(\lambda_1,\lambda_2,\lambda_3,\lambda_4)},\ g(\lambda_1,\lambda_2,\lambda_3,\lambda_4) \\ &\equiv \Pi_{j,k=1}^4(1-\Lambda+\lambda_j+\lambda_k),\end{aligned} \tag{5.42}$$

so that new denominator, the polynomial g is invariant under the permutation of the roots. A basic theorem in algebra says that the polynomial g is a polynomial of the coefficients of the characteristic polynomial p and thereby analytic. However, this overall symmetrization may be redundant, as it creates too big a polynomial and the denominator g may becomes zero in some domain under consideration of the contour integrations. Instead, we notice that the roots $\lambda_1,\ \lambda_4$ are non-characteristic, (5.24), and so give rise to local operator. Thus the symmetrization should be minimal and needs only to aim at converting the divisor into a polynomial in λ_2 with coefficients rational functions of λ_1 and λ_4. This way the divisor becomes a polynomial in λ_2 over a ring spanned by analytic functions in $s \in \boldsymbol{V}_2^{\delta_0,\zeta^2}$. We start with the first two symmetric expressions:

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2\Lambda, \\ \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_3\lambda_4 \\ = (\lambda_1+\lambda_4)(\lambda_2+\lambda_3)+\lambda_1\lambda_4+\lambda_2\lambda_3 = 1+2s+2\eta^2-\Lambda^2. \end{cases}$$

Thus we have

$$\begin{cases} \lambda_2+\lambda_3 = 2\Lambda-(\lambda_1+\lambda_4), \\ \lambda_2\lambda_3 = -1-2s-2\eta^2+\Lambda^2-2\Lambda\lambda_1+\lambda_1^2-2\Lambda\lambda_4+\lambda_1\lambda_4+\lambda_4^2. \end{cases}$$

With this, one can regularize the rational function $1/(1-\Lambda+\lambda_1+\lambda_2)$ as following

$$\begin{aligned} \frac{1}{(1-\Lambda+\lambda_1+\lambda_2)} &= \frac{(1-\Lambda+\lambda_1+\lambda_3)}{(1-\Lambda+\lambda_1+\lambda_2)(1-\Lambda+\lambda_1+\lambda_3)} \\ &= \frac{(1+\Lambda-\lambda_4)}{\lambda_1^2+\lambda_4^2-(-1+\Lambda)\lambda_1-(1+\Lambda)\lambda_4-2\,(s+\eta^2)} \\ &\quad -\frac{1}{\lambda_1^2+\lambda_4^2-(-1+\Lambda)\lambda_1-(1+\Lambda)\lambda_4-2\,(s+\eta^2)}\lambda_2 \\ &\equiv \mathscr{A}(\zeta^2,s)+\mathscr{B}(\zeta^2,s)\lambda_2. \end{aligned} \tag{5.43}$$

Lemma 5.12. *For $\Lambda \in (-1,1)$, $s \in \boldsymbol{V}_2^{\delta_0,\zeta^2}$ and $|\zeta^2| \ll 1$ the roots $\lambda_1,\ \lambda_2$, and λ_3 satisfy $(1-\Lambda+\lambda_1+\lambda_2)(1-\Lambda+\lambda_1+\lambda_3) \neq 0$.*

Proof. Under the condition $|\zeta^2| \ll 1$, the roots $\lambda_j(\zeta^2,s)$ are approximations to $\lambda_j(0,s)$. One has that

$$\begin{aligned} &Re\left((1-\Lambda+\lambda_1+\lambda_2)(1-\Lambda+\lambda_1+\lambda_3)|_{\zeta^2=0}\right)\\ &= 2+Re\left(\sqrt{(\Lambda+1)^2+4s}+\sqrt{(\Lambda-1)^2+4s}\right)\\ &\geq 2 \text{ for } s\in \boldsymbol{V}_2^{\delta_1,\zeta^2}. \end{aligned}$$

With this gap and $|\lambda_i(\zeta^2,s)-\lambda_i(0,s)| = O(1)|\zeta^2|/\sqrt{1+|s|}$, this proves this lemma. ☐

Lemma 5.13. *The coefficients $\mathscr{A}$ and $\mathscr{B}$ given in* (5.43) *are analytic functions in $\boldsymbol{V}_2^{\delta_1,\zeta^2}$. Moreover, K_{ij} in* (5.16) *can be uniquely separated into:*

$$\begin{cases} K_{ij} = & \mathscr{A}_{ij}+\mathscr{B}_{ij}\lambda_2 \text{ for } (i,j)\neq(2,1),\\ K_{21} = & \dfrac{\mathscr{A}_{21}+\mathscr{B}_{21}\lambda_2}{\zeta^2}, \end{cases} \tag{5.44}$$

with the coefficients $\mathscr{A}_{ij}$ and $\mathscr{B}_{ij}$ analytic in $s\in\boldsymbol{V}_2^{\delta_0,\zeta^2}$.

Proof. Lemma 5.12 concludes that the meromorphic functions $\mathscr{A}$ and $\mathscr{B}$ defined on $\boldsymbol{V}^{\delta_1,\zeta^2}$ contain no pole. This proves the first statement of the lemma. The second statement follows immediately from the first and the form of K_{ij} in (5.16). ☐

From (5.14), it is easy to show that, for $|\eta|\ll 1$,

$$\begin{aligned} &\mathscr{A}(\eta,s) \underset{s\to\pm\infty}{\longmapsto} \begin{pmatrix} -\sqrt{s}/2 & -i\eta/4\\ is-\frac{1}{2}i\left(\eta^2-\Lambda\right)\sqrt{s} & -\sqrt{s}/2 \end{pmatrix},\\ &\mathscr{B}(\eta,s) \underset{s\to\pm\infty}{\longmapsto} \begin{pmatrix} -1/2 & 0\\ -i\sqrt{s} & -1/2 \end{pmatrix}. \end{aligned} \tag{5.45}$$

With the asymptotic of $\mathscr{A}_{ij}(\eta,s)$ and $\mathscr{B}_{ij}(\eta,s)$ as $s\to\pm i\infty$ and the analytic property in $s\in\boldsymbol{V}_2^{\delta_0,\zeta^2}$ (the spectrum gap property), one has the following decomposition for $\mathscr{A}$ and $\mathscr{B}$.

$$\begin{cases} \mathfrak{b}_{12}(\eta,s)\equiv\mathscr{B}_{12}(\eta,s),\\ \mathfrak{a}_{12}(\eta,s)\equiv\mathscr{A}_{21}(\eta,s)+\frac{i\eta}{4},\\ \mathfrak{b}_{11}(\eta,s)\equiv\mathscr{B}_{11}(\eta,s)+\frac{1}{2},\\ \mathfrak{b}_{22}(\eta,s)\equiv\mathscr{B}_{22}(\eta,s)+\frac{1}{2}, \end{cases} \quad \begin{cases} \mathfrak{a}_{11}(\eta,s)\equiv Q[\mathscr{A}_{11}(\eta,s)],\\ \mathfrak{a}_{22}(\eta,s)\equiv Q[\mathscr{A}_{22}(\eta,s)],\\ \mathfrak{b}_{21}(\eta,s)\equiv Q[\mathscr{B}_{21}(\eta,s)],\\ \mathfrak{a}_{21}(\eta,s)\equiv Q[\mathscr{A}_{21}(\eta,s)-i\,s]. \end{cases} \tag{5.46}$$

Or, we have

$$\begin{pmatrix} \mathscr{A}_{11}=\mathscr{A}_{11}(0)+\mathfrak{a}_{11}s & \mathscr{A}_{12}=\mathfrak{a}_{12}-\frac{i\eta}{4} \\ \mathscr{A}_{21}=\mathscr{A}_{21}(0)+\mathfrak{a}_{21}s+is & \mathscr{A}_{22}=\mathscr{A}_{22}(0)+\mathfrak{a}_{22}s \end{pmatrix}, \tag{5.47}$$

and

$$\begin{pmatrix} \mathscr{B}_{11}=\mathfrak{b}_{11}-\frac{1}{2} & \mathscr{B}_{12}=\mathfrak{b}_{12} \\ \mathscr{B}_{21}=\mathscr{B}_{21}(0)+\mathfrak{b}_{21}s & \mathscr{B}_{22}=\mathfrak{b}_{22}-\frac{1}{2} \end{pmatrix}, \tag{5.48}$$

for functions $\mathfrak{a}_{ij}$ and $\mathfrak{b}_{ij}$ satisfying, for $|\eta|\ll 1$,

$$\begin{cases} \lim\limits_{s\to\pm i\infty}\mathfrak{a}_{ij}(\eta,s)=0, \\ \lim\limits_{s\to\pm i\infty}\mathfrak{b}_{ij}(\eta,s)=0. \end{cases} \tag{5.49}$$

The last decaying property makes it possible to evaluate the inverse Laplace transform of the non-characteristic $\mathfrak{a}_{ij}$, $\mathfrak{b}_{ij}$ by the contour integration. Thus the above decomposes the operators into into compositions of differential operators and integral operators according to Section 2, (2.4). The integral operators have the kernels as regular functions. We formulate this as a theorem.

Theorem 5.14. *There exist δ_0, $\delta_1>0$, and $C>0$ such that for all $|\zeta^2|<\delta_1$ the operators $\mathscr{A}_{ij}$ and $\mathscr{B}_{ij}$ satisfy*

$$\begin{cases} \mathbb{L}^{-1}[\mathscr{A}_{ij}(\zeta^2,s)]=\mathscr{A}_{ij}(\zeta^2,0)\delta(t)+\mathbb{L}^{-1}\left[\mathfrak{a}_{ij}\right]\underset{t}{*}\partial_t \text{ for } (i,j)=(1,1),\ (2,2) \\ \mathbb{L}^{-1}[\mathscr{A}_{12}(\zeta^2,s)]=-\dfrac{i\eta}{4}\delta(t)+\mathbb{L}^{-1}\left[\mathfrak{a}_{12}\right] \\ \mathbb{L}^{-1}[\mathscr{A}_{21}(\zeta^2,s)]=i\delta'(t)+\mathscr{A}_{21}(\zeta^2,0)\delta(t)+\mathbb{L}^{-1}\left[\mathfrak{a}_{21}\right]\underset{t}{*}\partial_t, \\ \mathbb{L}^{-1}[\mathscr{B}_{12}(\zeta^2,s)]=\mathbb{L}^{-1}\left[\mathfrak{b}_{12}\right], \\ \mathbb{L}^{-1}[\mathscr{B}_{ij}(\zeta^2,s)]=-\dfrac{1}{2}\delta(t)+\mathbb{L}^{-1}\left[\mathfrak{b}_{ij}\right] \text{ for } (i,j)=(1,1),\ (2,2), \\ \mathbb{L}^{-1}\left[\mathscr{B}_{21}(\zeta^2,s)\right]=\mathscr{B}_{2,1}(\zeta^2,0)\delta(t)+\mathbb{L}^{-1}[\mathfrak{b}_{21}]\underset{t}{*}\partial_t, \end{cases} \tag{5.50}$$

and for $i=0,1$

$$|\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{a}_{11}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{a}_{22}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{a}_{21}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{b}_{21}]|\le O(1)\frac{e^{-t/C}}{\sqrt{t}}, \tag{5.51}$$

$$|\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{a}_{12}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{b}_{12}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{b}_{11}]|,\ |\partial^i_{\zeta^2}\mathbb{L}^{-1}[\mathfrak{b}_{22}]|\le O(1)e^{-t/C}. \tag{5.52}$$

5.4. Composition with characteristic operator and Long Wave-Short Wave decomposition

From (5.44) and Theorem 5.14, it remains to study the following kernel functions of the integral operators in t-variable with small wave number ζ^2 for the purpose of obtaining $\mathscr{F}^{-1}[\mathbb{L}^{-1}[K_{ij}]](y,t)$:

$$\begin{aligned} \mathfrak{l}_2(\zeta^2,t) &\equiv \mathbb{L}^{-1}\left[Q[\lambda_2(\zeta^2,s)]\right], \\ \mathfrak{l}_2^s(\zeta^2,t) &\equiv \mathbb{L}^{-1}\left[\partial_s\lambda_2(\zeta^2,s)\right]. \end{aligned} \tag{5.53}$$

The operators $\mathfrak{l}_2$ and $\mathfrak{l}_2^s$ are related to characteristic roots. The other operators $\mathfrak{a}_{ij}$ and $\mathfrak{b}_{ij}$ are *regular* in the sense that the operator is an inverse Laplace transformation of some analytic function defined in the domain $s\in \boldsymbol{V}_2^{\delta_0,\zeta^2}$. Thus, one has the following proposition from the expression of $\mathfrak{a}_{ij},\ \mathfrak{b}_{ij}$. In the following analysis, for the local in time estimates, we will use (5.38).

Proposition 5.15. *There exist $\delta_1>0$ and $C>0$ such that*

$$\begin{cases} \displaystyle\int_{|\zeta^2|<\delta_1}(|\mathbb{L}^{-1}[\mathfrak{a}_{ij}](\zeta^2,t)|+|\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t)|)d\zeta^2 \le \delta_1 C\frac{e^{-t/C}}{\sqrt{t}}, \\ \displaystyle\int_{|\zeta^2|<\delta_1}|\mathfrak{l}_2(\zeta^2,t)|d\zeta^2 \le \delta_1 C\frac{1}{\sqrt{t}} \ \textit{for}\ t\in(0,1). \end{cases} \tag{5.54}$$

For any boundary value $\boldsymbol{v}^0(\zeta^2,t)$ with $\boldsymbol{v}^0(\zeta^2,\cdot)\in\mathscr{V}$ and transformed value $V^0(\zeta^2,s)=\mathbb{L}[\boldsymbol{v}^0](\zeta^2,s)$, (5.9), the identities (5.50) can be rewritten as follows:

Case 1. For $\mathscr{A}_{11}$, $\mathscr{A}_{22}$, and $\mathscr{B}_{21}$:

$$\begin{cases} \mathbb{L}^{-1}[\mathscr{A}_{ij}V^0](t)=\mathscr{A}_{ij}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,t)+\displaystyle\int_0^t \mathfrak{a}_{ij}(\zeta^2,\tau)\partial_\tau\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau, \\ \mathbb{L}^{-1}[\mathscr{B}_{ij}V^0](t)=\mathscr{B}_{ij}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,t)+\displaystyle\int_0^t \mathfrak{b}_{ij}(\zeta^2,\tau)\partial_\tau\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau. \end{cases} \tag{5.55}$$

Case 2. For $\mathscr{A}_{21}$.

$$\begin{aligned} \mathbb{L}^{-1}[\mathscr{A}_{21}V^0](t) &= \mathscr{A}_{21}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,t)-i\partial_t\boldsymbol{v}^0(\zeta^2,t) \\ &\quad+\int_0^t \mathfrak{a}_{21}(\zeta^2,\tau)\partial_\tau\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau. \end{aligned} \tag{5.56}$$

Case 3. For $\mathscr{A}_{12}$, $\mathscr{B}_{11}$, $\mathscr{B}_{22}$.

$$\begin{cases} \mathbb{L}^{-1}[\mathscr{A}_{12}V^0](t) = -i\dfrac{\zeta^2}{4}\boldsymbol{v}^0(\zeta^2,t) + \displaystyle\int_0^t \mathfrak{a}_{12}(\zeta^2,\tau)\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau, \\ \mathbb{L}^{-1}[\mathscr{B}_{ij}V^0](t) = -\dfrac{1}{2}\boldsymbol{v}^0(\zeta^2,t) + \displaystyle\int_0^t \mathfrak{b}_{ij}(\zeta^2,\tau)\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau, \end{cases} \tag{5.57}$$

Case 4. For $\mathscr{B}_{12}$.

$$\mathbb{L}^{-1}[\mathscr{B}_{ij}V^0](t) = \int_0^t \mathfrak{b}_{ij}(\zeta^2,\tau)\boldsymbol{v}^0(\zeta^2,t-\tau)d\tau. \tag{5.58}$$

When an operator $U \in \mathscr{V}$ convolves with the characteristic root $\lambda_2(\zeta^2,s)$, we need to decompose $\mathbb{L}^{-1}[\lambda_2]$ into generalized and regular functions as in Lemma 5.6 and Lemma 5.7:

$$\begin{aligned} &\mathbb{L}^{-1}[\lambda_2] * U(t) \\ &= \left(\int_0^{\max(t-1,t/2)} + \int_{\max(t-1,t/2)}^t\right)\mathbb{L}^{-1}[\lambda_2](t-\tau)U(\tau)d\tau \\ &= \int_0^{\max(t-1,t/2)}\left(-\frac{\mathbb{L}^{-1}[\partial_s\lambda_2](t-\tau)}{t-\tau} + C(\zeta^2)\delta(t-\tau)\right)U(\tau)d\tau \\ &\quad + \int_{\max(t-1,t/2)}^t \mathbb{L}^{-1}\left[\frac{\lambda_2(s,\zeta^2)-\lambda_2(0,\zeta^2)}{s}\right](t-\tau)U'(\tau)d\tau + \lambda_2(0,\zeta^2)U(t) \\ &= \int_0^{\max(t-1,t/2)} -\frac{\mathbb{L}^{-1}[\partial_s\lambda_2](t-\tau)}{t-\tau}U(\tau)d\tau \\ &\quad + \int_{\max(t-1,t/2)}^t \mathbb{L}^{-1}\left[\frac{\lambda_2(s,\zeta^2)-\lambda_2(0,\zeta^2)}{s}\right](t-\tau)U'(\tau)d\tau + \lambda_2(0,\zeta^2)U(t). \end{aligned} \tag{5.59}$$

From this we have:

Case 1. For $(i,j)=(2,1)$.

$$\begin{aligned} &\mathbb{L}^{-1}[\lambda_2(\zeta^2,s)\mathscr{B}_{ij}V^0](t) \\ &= -\int_0^{\max(t-1,t/2)} \frac{\mathfrak{l}_2^s(\zeta^2,t-\tau)}{t-\tau}\Big(\mathscr{B}_{ij}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,\tau) \\ &\qquad + \int_0^\tau \mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\partial_\sigma\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\Big)d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_{\max(t-1,t/2)}^{t} \mathfrak{l}_2(\zeta^2,t-\tau)\Big(\mathscr{B}_{ij}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,\tau)\\
&+\int_0^{\tau}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\partial_\sigma\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\Big)d\tau\\
&+\lambda_2(\zeta^2,0)\left(\mathscr{B}_{ij}(\zeta^2,0)\boldsymbol{v}^0(\zeta^2,t)+\int_0^t\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\partial_\sigma\boldsymbol{v}^0(\zeta^2,t-\sigma)d\sigma\right).
\end{aligned}
\tag{5.60}
$$

Case 2. $(i,j)=(1,1),\ (2,2)$

$$
\begin{aligned}
&\mathbb{L}^{-1}[\lambda_2(\zeta^2,s)\mathscr{B}_{ij}V^0](t)\\
&=-\int_0^{\max(t-1,t/2)}\frac{\mathfrak{l}_2^s(\zeta^2,t-\tau)}{t-\tau}\Big(-\boldsymbol{v}^0(\zeta^2,\tau)/2\\
&\quad+\int_0^{\tau}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\Big)d\tau\\
&\quad+\int_{\max(t-1,t/2)}^{t}\mathfrak{l}_2(\zeta^2,t-\tau)\Big(-\boldsymbol{v}^0(\tau,\zeta^2)/2\\
&\quad+\int_0^{\tau}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\Big)d\tau\\
&\quad+\lambda_2(\zeta^2,0)\left(-\boldsymbol{v}^0(\zeta^2,t)/2+\int_0^t\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,t-\sigma)d\sigma\right).
\end{aligned}
\tag{5.61}
$$

Case 3. $(i,j)=(1,2)$

$$
\begin{aligned}
&\mathbb{L}^{-1}[\lambda_2(\zeta^2,s)\mathscr{B}_{ij}V^0](t)\\
&=-\int_0^{\max(t-1,t/2)}\frac{\mathfrak{l}_2^s(\zeta^2,t-\tau)}{t-\tau}\left(\int_0^{\tau}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\right)d\tau\\
&\quad+\int_{\max(t-1,t/2)}^{t}\mathfrak{l}_2(\zeta^2,t-\tau)\left(\int_0^{\tau}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,\tau-\sigma)d\sigma\right)d\tau\\
&\quad+\lambda_2(\zeta^2,0)\left(\int_0^t\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\sigma)\boldsymbol{v}^0(\zeta^2,t-\sigma)d\sigma\right).
\end{aligned}
\tag{5.62}
$$

Let χ the characteristic function:

$$
\chi(\eta)=\begin{cases}1 \text{ if } |\eta|<\delta_1,\\ 0 \text{ else.}\end{cases}
$$

The *long wave-short wave decomposition* is defined through the Fourier transformation as follows:

$$\begin{cases} \mathsf{g} = \mathscr{I}^L[\mathsf{g}] + \mathscr{I}^S[\mathsf{g}], \\ \mathscr{F}^{-1} \equiv \mathscr{F}^{-1,L} + \mathscr{F}^{-1,S}, \\ \mathscr{F}^{-1,L}[\mathsf{h}](y) \equiv \mathscr{F}^{-1,S}[\mathsf{h}\chi](y) = \dfrac{1}{2\pi}\displaystyle\int_{|\eta|<\delta_1} e^{i\eta y}\mathsf{h}(\eta)d\eta, \\ \mathscr{F}^{-1,S}[\mathsf{h}](y) \equiv \mathscr{F}^{-1,S}[\mathsf{h}(1-\chi)](y) = \dfrac{1}{2\pi}\displaystyle\int_{|\eta|>\delta_1} e^{i\eta y}\mathsf{h}(\eta)d\eta, \\ \mathscr{I}^L[\mathsf{g}](y) \equiv \mathscr{F}^{-1,L}[\mathscr{F}[\mathsf{g}]](y), \text{ (Long wave component)} \\ \mathscr{I}^S[\mathsf{g}](y) \equiv \mathscr{F}^{-1,S}[\mathscr{F}[\mathsf{g}]](y), \text{ (Short wave component)} \end{cases}$$

where the positive constant δ_1 is chosen with the property that, for any $|\zeta^2| < \delta_1$ there is no branch point for the functions inside $\boldsymbol{V}_2^{\delta_0,\zeta^2}$.

Lemma 5.16. *There exist $\delta_1 > 0$ and $C > 0$ such that for any given $\mathscr{V}$-valued function $\boldsymbol{v}^0(y,\cdot)$,*

Case 1. $(i,j) \neq (2,1)$.

$$\begin{aligned} &\Big\| \mathscr{I}^L[\mathfrak{N}_{ij}\boldsymbol{v}^0](y,t) + \int_0^{\max(t-1,t/2)} \mathscr{F}^{-1,L}\Big[\frac{\mathfrak{l}_2^s(\zeta^2,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t-\tau)}{t-\tau}\Big] \\ &\qquad (y) \underset{y}{*} \boldsymbol{v}^0(y,\tau)d\tau \Big\|_{L^\infty_y} \\ &\le O(1) \max_{\substack{\tau\in(\max(t-1,t/2),t) \\ |\zeta^2|<\delta_1}} |\mathscr{F}[\boldsymbol{v}^0](\zeta^2,\tau)| \\ &\qquad + \int_0^t \frac{e^{-(t-\tau)/C}}{\sqrt{t-\tau}} \|\mathscr{I}^L[\partial_\tau \boldsymbol{v}^0](y,\tau)\|_{L^\infty_y} d\tau. \end{aligned} \tag{5.63}$$

Case 2. $(i,j) = (2,1)$.

$$\begin{aligned} &\Big\| \mathscr{I}^L[\mathfrak{N}_{ij}\boldsymbol{v}^0](y,t) + \int_0^{\max(t-1,t/2)} \mathscr{F}^{-1,L}\Big[\frac{\mathfrak{l}_2^s(\zeta^2,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{21}](\zeta^2,t-\tau)}{\zeta^2(t-\tau)}\Big] \\ &\qquad \cdot(y) \underset{y}{*} \boldsymbol{v}^0(y,\tau)d\tau \Big\|_{L^\infty_y} \\ &\le O(1) \max_{\substack{\tau\in(\max(t-1,t/2),t) \\ |\zeta^2|<\delta_1}} |\mathscr{F}[\boldsymbol{v}^0](\zeta^2,\tau)| \\ &\qquad + \int_0^t \frac{e^{-(t-\tau)/C}}{\sqrt{t-\tau}} \|\mathscr{I}^L[\partial_\tau \boldsymbol{v}^0](y,\tau)\|_{L^\infty_y} d\tau. \end{aligned} \tag{5.64}$$

Proof. Since the operator $\mathscr{I}^L$ is a projection operator,

$$\begin{aligned}&\mathscr{F}^{-1,L}[\mathfrak{l}_2^s(\zeta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t-\tau)\mathscr{F}[\boldsymbol{v}^0](\zeta^2,\tau)]\\ &\quad= \mathscr{F}^{-1}\left[\chi\left(\zeta^2\right)\mathfrak{l}_2^s(\zeta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t-\tau)\mathscr{F}[\boldsymbol{v}^0](\zeta^2,\tau)\right]\\ &\quad= \mathscr{F}^{-1}\left[\chi\left(\zeta^2\right)\mathfrak{l}_2^s(\zeta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t-\tau)\right](y)\underset{y}{*}\boldsymbol{v}^0(y,\tau)\\ &\quad= \mathscr{F}^{-1,L}[\mathfrak{l}_2^s(\zeta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,t-\tau)](y)\underset{y}{*}\boldsymbol{v}^0(y,\tau). \qquad (5.65)\end{aligned}$$

Then, by (5.54), (5.60) and (5.61), the lemma follows. □

Re-organize the operator $\chi(\eta)\mathfrak{l}_2^s(\eta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}(\eta,\tau)]$, using (5.33), (5.51), (5.52), and one has

$$\begin{aligned}&\chi(\eta)\mathfrak{l}_2^s(\eta,t-\tau)\mathbb{L}^{-1}[\mathfrak{b}_{ij}(\eta,\tau)]\\ &\quad=\frac{1}{\pi}\int_{\mathbb{R}}e^{-i\Lambda\kappa(t-\tau)}\cos(\sqrt{\kappa^2+\eta^2}(t-\tau))\left(e^{-(t-\tau)(\kappa^2+\eta^2)}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau)\right)\chi(\eta)d\kappa\\ &\qquad+O(1)\chi(\eta)e^{-t/C}. \qquad (5.66)\end{aligned}$$

One also has that

$$\begin{aligned}&\int_{\mathbb{R}}e^{iy\eta}\int_{\mathbb{R}}e^{-i\Lambda\kappa(t-\tau)}\cos(\sqrt{\kappa^2+\eta^2}(t-\tau))\left(e^{-(t-\tau)(\kappa^2+\eta^2)}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau)\right)\chi(\eta)d\kappa d\eta\\ &\quad=4\pi^2\mathscr{F}_2^{-1}\left[e^{-i\Lambda\kappa(t-\tau)}\cos(\sqrt{\kappa^2+\eta^2}(t-\tau))\right]\\ &\qquad\underset{(x,y)}{*}\mathscr{F}_2^{-1}\left[\left(e^{-(t-\tau)(\kappa^2+\eta^2)}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau)\right)\chi(\eta)\right]\Big|_{x=0}, \qquad (5.67)\end{aligned}$$

where $\mathscr{F}_2$ is the 2-dimensional Fourier transformation with the Fourier variables (κ,η). By (5.51), (5.52) and the analyticity of $\mathbb{L}^{-1}[\mathfrak{b}_{ij}]$ in η, one can show that

$$\begin{aligned}&\mathscr{F}_2^{-1}\left[\left(e^{-(t-\tau)(\kappa^2+\eta^2)}\mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau)\right)\chi(\eta)\right]\\ &\qquad\le O(1)\begin{cases}\dfrac{e^{-\frac{x^2+y^2}{C(t-\tau)}-\frac{\tau}{C}}}{1+t-\tau} & \text{for } |x|+|y|<8(t-\tau)+1\ ,\\[2ex] \dfrac{e^{-t/C}}{1+t-\tau} & \text{else.}\end{cases} \qquad (5.68)\end{aligned}$$

Lemma 5.17. *For $|y| \le 4(1+t)$ and $t \ge 1$, there exists $C > 0$ such that*

$$\left| \int_0^{\max(t-1,t/2)} \frac{1}{t-\tau} \mathscr{F}^{-1,L} \left[\mathfrak{l}_2^s(\zeta^2, t-\tau) \mathbb{L}^{-1}[\mathfrak{b}_{ij}](\zeta^2,\tau) \right](y) \right|$$

$$= O(1)\ \mathbb{W}(\vec{\boldsymbol{x}}, t) \underset{\vec{\boldsymbol{x}}}{*} \left. \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{Ct}}}{t} \right|_{\vec{\boldsymbol{x}}=(-\Lambda t, y)}, \tag{5.69}$$

$$\left| \mathscr{F}^{-1,L} \left[\mathfrak{l}_2^s(\zeta^2, t) \mathscr{B}_{ij}(\zeta^2, 0) \right](y) \right| = O(1)\ \mathbb{W}(\vec{\boldsymbol{x}}, t) \underset{\vec{\boldsymbol{x}}}{*} \left. \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{Ct}}}{t} \right|_{\vec{\boldsymbol{x}}=(-\Lambda t, y)}. \tag{5.70}$$

where $\mathbb{W}(\vec{\boldsymbol{x}}, t)$ is the solution of initial value problem of the D'Alembert wave equation in 2-D:

$$\begin{cases} \mathbb{W}_{tt} - \mathbb{W}_{xx} - \mathbb{W}_{yy} = 0, \\ \mathbb{W}(x,y,0) = \delta(x,y), \\ \mathbb{W}_t(x,y,0) = 0. \end{cases}$$

Proof. We show (5.69) only, since the proofs for (5.69) and (5.70) are similar.

The function $U(x,y,t) \equiv \cos\left(\sqrt{\kappa^2+\eta^2}\, t\right) e^{-i\Lambda\kappa t}$ is the Fourier transformation of the D'Alembert's wave equation in 2-D:

$$\begin{cases} ((\partial_t - \Lambda\partial_x)^2 - \Delta) U(x,y,t) = 0, \\ U(x,y,0) = \delta(x,y), \\ U_t(x,y,0) = -\Lambda\partial_x \delta(x,y) \end{cases}.$$

Thus, the function $U(x,y,t)$ can be identified with $U(x,y,t) = \mathbb{W}(x-\Lambda t, y, t)$ and so

$$\int_{\mathbb{R}} e^{iy\eta} \int_{\mathbb{R}} e^{-i\Lambda\kappa(t-\tau)} \cos(\sqrt{\kappa^2+\eta^2}(t-\tau)) \Big(e^{-(t-\tau)(\kappa^2+\eta^2)} \mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau) \Big) \chi(\eta) d\kappa d\eta$$
$$= 4\pi^2 \mathbb{W}(x - \Lambda(t-\tau), y, t-\tau)$$
$$\underset{(x,y)}{*} \mathscr{F}_2^{-1} \left[\left(e^{-(t-\tau)(\kappa^2+\eta^2)} \mathbb{L}^{-1}[\mathfrak{b}_{ij}](\eta,\tau) \right) \chi(\eta) \right] \Big|_{x=0}. \tag{5.71}$$

By (5.68) and (5.71), the lemma follows. $\square$

Theorem 5.18 (Wave in finite Mach number region)**.** *For $|y|<3t$ there exist $C>0$ such that, for any given $\mathscr{V}$-valued function $\boldsymbol{v}^0(y,\cdot)$,*

$$\begin{aligned}
&|\partial_y \mathscr{I}^L[\mathfrak{N}_{21}\boldsymbol{v}^0](y,t)| + \sum_{(i,j)\neq(2,1)} |\mathscr{I}^L[\mathfrak{N}_{ij}\boldsymbol{v}^0](y,t)| \\
&= O(1)\int_0^{\max(t/2,t-1)} \left(\frac{|\mathbb{W}|(\vec{\boldsymbol{x}},t-\tau)}{t-\tau} \underset{\vec{\boldsymbol{x}}}{*} \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C(t-\tau)}}}{t-\tau}\Bigg|_{\vec{\boldsymbol{x}}=(-\Lambda(t-\tau),y)} \right) \underset{y}{*} |\boldsymbol{v}^0|(y,\tau)d\tau \\
&\quad + O(1) \max_{\substack{\tau\in(\max(t-1,t/2),t)\\ |\zeta^2|<\delta_1}} |\mathscr{F}[\boldsymbol{v}^0](\zeta^2,\tau)| \\
&\quad + \int_0^t e^{-(t-\tau)/C}\left(\|\mathscr{I}^L[\partial_\tau \boldsymbol{v}^0](y,\tau)\|_{L^\infty_y} + \|\mathscr{I}^L[\boldsymbol{v}^0](y,\tau)\|_{L^\infty_y}\right) d\tau. \qquad (5.72)
\end{aligned}$$

Proof. This theorem is a consequence of Lemmas 5.16 and 5.17. Note that there is a factor of η in the denominator of K_{21}, (5.16), and this accounts for the partial differentiation with respect to y in the first term of (5.72). □

Remark 5.19. For simplicity, we may consider the input function $\boldsymbol{v}^0(y,t)$ to have unit compact support in (y,t) around $(0,0)$. In this case, (5.72) can be simplified to, for $t>1$, and some $C>0$,

$$\begin{aligned}
|\mathscr{I}^L[\mathfrak{N}_{ij}\boldsymbol{v}^0](y,t)| &= O(1)\, \frac{\mathbb{W}|(\vec{\boldsymbol{x}},t)}{t} \underset{\vec{\boldsymbol{x}}}{*} \frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C(t)}}}{t}\Bigg|_{x=-\Lambda t} \|\boldsymbol{v}^0\|_{L^\infty_{(y,t)}} \\
&\quad + O(1)e^{-\frac{(y+t)}{C}}\left(\|\partial_t \boldsymbol{v}^0\|_{L^\infty_{(y,t)}} + \|\boldsymbol{v}^0\|_{L^\infty_{(y,t)}}\right),\ (i,j)\neq(2,1). \qquad (5.73)
\end{aligned}$$

6. The Global Structure

In Theorem 5.18 we have established the long wave structure of $\mathfrak{N}_{ij}\boldsymbol{v}^0(y,t)$ in a finite Mach number region $|y|<3t$. In this section we will establish the global exponential time decaying structure of the short wave component and the global exponential decaying structure in the space variable outside the finite Mach number region using directly the structure of differential equation through simple, standard energy estimates. These structures are obtained as part of the solution, and not given in terms of the inverse Laplace-Fourier process for the components $\mathfrak{N}_{ij}\boldsymbol{v}^0$, (1.19). Only by considering the whole system (5.1) we can obtain the exact correlation between components and the natural cancellations between them to yield the sharp estimates. The component-wise approach could work if the Laplace transformation approach works. However, the diagram in Figure c indicates the existence of a branch point in $Re(s) > 0$. This prevents the Laplace transformation approach through the complex analysis to work when wave number is large, i.e. $|\zeta^2| \gg 1$.

6.1. Global exponential decay of the short wave component

Lemma 6.1. *The Neumann value* $\boldsymbol{v}_x^0(\zeta^2,t)$ *of* (5.1) *satisfies*

$$\begin{aligned}|\boldsymbol{v}_x^0(\zeta^2,t)|^2 \;=\; & O(1)\int_0^t \left(|\zeta^2|^2+\frac{1}{|\zeta^2|^2}\right)\Big((1+|\zeta^2|^2)|\boldsymbol{v}^0(\zeta^2,\tau)|^2 \\ & \qquad +|\partial_\tau \boldsymbol{v}^0(\zeta^2,\tau)|^2\Big)e^{-|\zeta^2|^2(t-\tau)}d\tau \\ & +O(1)\int_0^t \frac{1+|\zeta^2|^2}{|\zeta^2|^2}\Big((1+|\zeta^2|^2)|\partial_\tau \boldsymbol{v}^0(\zeta^2,\tau)|^2 \\ & \qquad +|\partial_\tau^2 \boldsymbol{v}^0(\zeta^2,\tau)|^2\Big)e^{-|\zeta^2|^2(t-\tau)}d\tau \\ & +O(1)\int_0^t \frac{1+|\zeta^2|^2}{|\zeta^2|^2}\Big((1+|\zeta^2|^2)|\partial_\tau^2 \boldsymbol{v}^0(\zeta^2,\tau)|^2 \\ & \qquad +|\partial_\tau^3 \boldsymbol{v}^0(\zeta^2,\tau)|^2\Big)e^{-|\zeta^2|^2(t-\tau)}d\tau. \qquad (6.1)\end{aligned}$$

Proof. We break this proof into two parts. The first part is the interior energy estimates and the second one is the boundary gradient estimates.

Interior Estimates

We start with the system (5.10):

$$\left(\partial_t+\begin{pmatrix}1+\Lambda & 0\\ 0 & -1+\Lambda\end{pmatrix}\partial_x+\begin{pmatrix}|\zeta^2|^2 & i\zeta^2\\ i\zeta^2 & |\zeta^2|^2\end{pmatrix}-\partial_x^2\right)\boldsymbol{v}=0. \qquad (6.2)$$

Now, one extracts the boundary value by subtracting $\boldsymbol{v}^0(\zeta^2,t)e^{-x}$ from $\boldsymbol{v}$ to result in:

$$\mathsf{U}(x,\zeta^2,t)\equiv \boldsymbol{v}(x,\zeta^2,t)-\boldsymbol{v}^0(\zeta^2,t)e^{-x}, \qquad (6.3)$$

$$\begin{cases} \left(\partial_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \partial_x + \begin{pmatrix} |\zeta^2|^2 & i\zeta^2 \\ i\zeta^2 & |\zeta^2|^2 \end{pmatrix} - \partial_x^2 \right) \mathsf{U} = \boldsymbol{S}(x,\zeta^2,t), \\ \mathsf{U}(0,\zeta^2,t) = 0, \\ \mathsf{U}(x,\zeta^2,0) = 0, \end{cases} \tag{6.4}$$

where the inhomogeneous term $\boldsymbol{S}(x,\zeta^2,t)$ is given by

$$\boldsymbol{S}(x,\zeta^2,t) \equiv -e^{-x}\left(\partial_t + \begin{pmatrix} -2-\Lambda+|\zeta^2|^2 & i\zeta^2 \\ i\zeta^2 & -\Lambda+|\zeta^2|^2 \end{pmatrix}\right) \boldsymbol{v}^0(\zeta^2,t). \tag{6.5}$$

We now perform the energy estimate, starting with

$$\begin{aligned} &\int_0^\infty \bar{\mathsf{U}}^{\boldsymbol{t}} \left(\partial_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \partial_x + \begin{pmatrix} |\zeta^2|^2 & i\zeta^2 \\ i\zeta^2 & |\zeta^2|^2 \end{pmatrix} - \partial_x^2 \right) \mathsf{U} dx \\ &\qquad + \int_0^\infty \mathsf{U}^{\boldsymbol{t}} \left(\partial_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \partial_x + \begin{pmatrix} |\zeta^2|^2 & -i\zeta^2 \\ -i\zeta^2 & |\zeta^2|^2 \end{pmatrix} - \partial_x^2 \right) \bar{\mathsf{U}} dx \\ &= 2\int_0^\infty Re(\bar{\mathsf{U}}^{\boldsymbol{t}} \boldsymbol{S}) dx, \end{aligned} \tag{6.6}$$

to yield

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty |\mathsf{U}|^2 dx + \int_0^\infty 2|\zeta^2|^2|\mathsf{U}|^2 + 2|\partial_x \mathsf{U}|^2 dx \\ &\quad = 2\int_0^\infty Re(\mathsf{U}^{\boldsymbol{t}} \boldsymbol{S}) dx \le \int_0^\infty \left(|\zeta^2|^2|\mathsf{U}|^2 + \frac{|\boldsymbol{S}|^2}{|\zeta^2|^2}\right) dx. \end{aligned} \tag{6.7}$$

This leads to

$$\frac{d}{dt}\int_0^\infty |\mathsf{U}|^2 dx + \int_0^\infty |\zeta^2|^2|\mathsf{U}|^2 + 2|\partial_x \mathsf{U}|^2 dx \le \int_0^\infty \frac{|\boldsymbol{S}|^2}{|\zeta^2|^2} dx, \tag{6.8}$$

and

$$\begin{aligned} &\int_0^\infty |\mathsf{U}(x,\tau)|^2 dx \Big|_{\tau=t} + \int_0^t \int_0^\infty e^{-|\zeta^2|^2(t-\tau)} |\partial_x \mathsf{U}(x,\tau)|^2 dx d\tau \\ &\quad \le \int_0^t \int_0^\infty e^{-|\zeta^2|^2(t-\tau)} \frac{|\boldsymbol{S}|^2}{|\zeta^2|^2} dx d\tau. \end{aligned} \tag{6.9}$$

Similarly, one has that

$$\int_0^\infty |\partial_t^l \mathsf{U}(x,\tau)|^2 dx\Big|_{\tau=t} + \int_0^t \int_0^\infty e^{-|\zeta^2|^2(t-\tau)} |\partial_x \partial_t^l \mathsf{U}(x,\tau)|^2 dx d\tau \\ \leq \int_0^t \int_0^\infty e^{-|\zeta^2|^2(t-\tau)} \frac{|\partial_t^l \boldsymbol{S}|^2}{|\zeta^2|^2} dx d\tau, \tag{6.10}$$

for $l \geq 0$.

Boundary Gradient Estimate

By the zero boundary value (6.3), $\mathsf{U}(0,\zeta^2,t) = 0$,

$$\int_0^\infty \bar{\mathsf{U}}_x^{\boldsymbol{t}} \left(\partial_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \partial_x + \begin{pmatrix} |\zeta^2|^2 & i\zeta^2 \\ i\zeta^2 & |\zeta^2|^2 \end{pmatrix} - \partial_x^2 \right) \mathsf{U} dx \\ + \int_0^\infty \mathsf{U}_x^{\boldsymbol{t}} \left(\partial_t + \begin{pmatrix} 1+\Lambda & 0 \\ 0 & -1+\Lambda \end{pmatrix} \partial_x + \begin{pmatrix} |\zeta^2|^2 & -i\zeta^2 \\ -i\zeta^2 & |\zeta^2|^2 \end{pmatrix} - \partial_x^2 \right) \bar{\mathsf{U}} dx \\ = 2\int_0^\infty Re(\bar{\mathsf{U}}_x^{\boldsymbol{t}} \boldsymbol{S}) dx. \tag{6.11}$$

This yields

$$\frac{1}{2}|\mathsf{U}_x(0,\zeta^2,t)|^2 + \int_0^\infty (Re\left(\bar{\mathsf{U}}_x^{\boldsymbol{t}} \mathsf{U}_t\right) + |\zeta^2|^2 |\mathsf{U}|^2) dx \leq \int_0^\infty |\mathsf{U}_x|^2 + |\boldsymbol{S}|^2 dx. \tag{6.12}$$

This, (6.9), and (6.10) with $l = 1$ result in

$$\frac{1}{2}\int_0^t |\mathsf{U}_x(0,\zeta^2,\tau)|^2 e^{-|\zeta^2|^2(t-\tau)} d\tau \\ \leq \int_0^t \int_{\mathbb{R}} \left(|\mathsf{U}_t|^2 + |\zeta^2|^2|\mathsf{U}|^2 + |\mathsf{U}_x|^2 + |\boldsymbol{S}|^2 \right) e^{-|\zeta^2|^2(t-\tau)} dx d\tau \\ \leq 8\int_0^t \int_{\mathbb{R}} \left(\left(1 + \frac{1}{|\zeta^2|^2}\right) |\boldsymbol{S}|^2 + \frac{1}{|\zeta^2|^2} |\boldsymbol{S}_t|^2 \right) e^{-|\zeta^2|^2(t-\tau)} dx d\tau. \tag{6.13}$$

Similarly, one has

$$\frac{1}{2}\int_0^t |\partial_t \mathsf{U}_x(0,\zeta^2,\tau)|^2 e^{-|\zeta^2|^2(t-\tau)} d\tau \\ \leq 8\int_0^t \int_{\mathbb{R}} \left(\left(1 + \frac{1}{|\zeta^2|^2}\right) |\boldsymbol{S}_t|^2 + \frac{1}{|\zeta^2|^2} |\boldsymbol{S}_{tt}|^2 \right) e^{-|\zeta^2|^2(t-\tau)} dx d\tau. \tag{6.14}$$

From (6.13) and (6.14),

$$|\mathsf{U}_x(0,\zeta^2,t)|^2 \le O(1)\int_0^t\int_{\mathbb{R}}\Big(\Big(|\zeta^2|^2+\frac{1}{|\zeta^2|^2}\Big)|\boldsymbol{S}|^2+\frac{1+|\zeta^2|^2}{|\zeta^2|^2}|\boldsymbol{S}_t|^2 + \frac{1+|\zeta^2|^2}{|\zeta^2|^2}|\boldsymbol{S}_{tt}|^2\Big)e^{-|\zeta^2|^2(t-\tau)}dxd\tau. \tag{6.15}$$

By substituting (6.5) into (6.15), the lemma follows. □

6.2. Global time pointwise structure outside a finite Mach number region

For $|y|>3(1+t)$, one uses the direct structure of the differential equation. Similar to the estimate for the short wave component, one subtracts $\boldsymbol{u}^0(y,t)e^{-x}$ from $\boldsymbol{u}(x,y,t)$:

$$\tilde{\mathsf{U}}(x,y,t)\equiv \boldsymbol{u}(x,y,t)-\boldsymbol{u}^0(y,t)e^{-x},$$

then (5.1) becomes

$$\begin{cases}\partial_t\tilde{\mathsf{U}}+(\boldsymbol{A}_1\partial_x+\boldsymbol{A}_2\partial_y)\,\tilde{\mathsf{U}}-\Delta_2\tilde{\mathsf{U}}=\boldsymbol{S}(x,y,t)\text{ for } x>0,\\ \tilde{\mathsf{U}}(y,t)=0,\\ \tilde{\mathsf{U}}(\vec{\boldsymbol{x}},0)\equiv 0,\end{cases} \tag{6.16}$$

where

$$\boldsymbol{A}_1 \equiv \begin{pmatrix}1+\Lambda & 0\\ 0 & -1+\Lambda\end{pmatrix},\quad \boldsymbol{A}_2\equiv\begin{pmatrix}0&1\\1&0\end{pmatrix},$$
$$\boldsymbol{S}(x,y,t) \equiv \left(-\partial_t+\boldsymbol{A}_1+I-\boldsymbol{A}_2\partial_y+\partial_y^2\right)\boldsymbol{u}^0(y,t)e^{-x}.$$

One considers the following weighted energy estimates:

$$\int_0^\infty\int_{\mathbb{R}}\tilde{\mathsf{U}}\cdot\Big(\partial_t\tilde{\mathsf{U}}+(\boldsymbol{A}_1\partial_x+\boldsymbol{A}_2\partial_y)\,\tilde{\mathsf{U}}-\Delta_2\tilde{\mathsf{U}}\Big)\,e^{\alpha(y-\beta t)}dydx = \int_0^\infty\int_{\mathbb{R}}\tilde{\mathsf{U}}\cdot\boldsymbol{S}e^{\alpha(y-\beta t)}dydx. \tag{6.17}$$

This leads to

$$\frac{d}{dt}\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}\tilde{\mathsf{U}}\cdot\tilde{\mathsf{U}}e^{\alpha(y-\beta t)}dydx$$
$$+\int_0^\infty\int_{\mathbb{R}}\left(\frac{\alpha}{2}\tilde{\mathsf{U}}\cdot\begin{pmatrix}\beta-\alpha & -1\\ -1 & \beta-\alpha\end{pmatrix}\tilde{\mathsf{U}}+|\tilde{\mathsf{U}}_x|^2+|\tilde{\mathsf{U}}_y|^2\right)e^{\alpha(y-\beta t)}dydx$$
$$\leq \int_0^\infty\int_{\mathbb{R}}|\boldsymbol{S}(x,y,t)||\tilde{\mathsf{U}}(x,y,t)|e^{\alpha(y-\beta t)}dydx. \tag{6.18}$$

Assume that

$$\beta>\alpha+1,\ \alpha>0,$$

then (6.18) yields

$$\frac{d}{dt}\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}\tilde{\mathsf{U}}\cdot\tilde{\mathsf{U}}e^{\alpha(y-\beta t)}dydx$$
$$+\int_0^\infty\int_{\mathbb{R}}\left(\frac{\alpha(\beta-\alpha-1)}{4}|\tilde{\mathsf{U}}|^2+|\tilde{\mathsf{U}}_x|^2+|\tilde{\mathsf{U}}_y|^2\right)e^{\alpha(y-\beta t)}dydx$$
$$\leq \int_0^\infty\int_{\mathbb{R}}\frac{8}{\alpha(\beta-\alpha-1)}|\boldsymbol{S}(x,y,t)|^2e^{\alpha(y-\beta t)}dydx, \tag{6.19}$$

and so

$$\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}|\tilde{\mathsf{U}}(x,y,\tau)|^2e^{\alpha(y-\beta\tau)}dydx\Big|_{\tau=t}$$
$$+\int_0^t\int_0^\infty\int_{\mathbb{R}}(|\tilde{\mathsf{U}}_x|^2+|\tilde{\mathsf{U}}_y|^2)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau$$
$$\leq\int_0^t\int_0^\infty\int_{\mathbb{R}}\frac{8}{\alpha(\beta-\alpha-1)}|\boldsymbol{S}(x,y,\tau)|^2e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.20}$$

Similarly, one has

$$\int_0^\infty\int_{\mathbb{R}}\frac{1}{2}|\bar{\partial}^\gamma\tilde{\mathsf{U}}(x,y,\tau)|^2e^{\alpha(y-\beta\tau)}dydx\Big|_{\tau=t}$$
$$+\int_0^t\int_0^\infty\int_{\mathbb{R}}(|\bar{\partial}^\gamma\tilde{\mathsf{U}}_x|^2+|\bar{\partial}^\gamma\tilde{\mathsf{U}}_y|^2)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau$$

$$\le \int_0^t\int_0^\infty\int_{\mathbb{R}}\frac{8}{\alpha(\beta-\alpha-1)}|\bar{\partial}^\gamma \boldsymbol{S}(x,y,\tau)|^2 e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.21}$$

where the operator $\bar{\partial}^\gamma$ is a differential operator in the space $(y,t)\in\mathbb{R}\times\mathbb{R}_+$:

$$\bar{\partial}^\gamma = \partial_t^{\gamma_1}\partial_y^{\gamma_2},\ |\gamma| = \gamma_1+\gamma_2. \tag{6.22}$$

Next we consider the following:

$$\int_0^t\int_0^\infty\int_{\mathbb{R}}\tilde{\mathsf{U}}_x\cdot\Big(\partial_\tau\tilde{\mathsf{U}}+(\boldsymbol{A}_1\partial_x+\boldsymbol{A}_2\partial_y)\tilde{\mathsf{U}}-\Delta_2\tilde{\mathsf{U}}\Big)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau$$
$$= \int_0^t\int_0^\infty\int_{\mathbb{R}}\tilde{\mathsf{U}}_x\cdot\boldsymbol{S}e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.23}$$

This yields that

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}}|\tilde{\mathsf{U}}_x(0,y,t)|^2 e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dyd\tau$$
$$+\int_0^t\int_0^\infty\int_{\mathbb{R}}\left(\tilde{\mathsf{U}}_t\cdot\tilde{\mathsf{U}}_x+\tilde{\mathsf{U}}_x\cdot\begin{pmatrix}\alpha & 1\\ 1 & \alpha\end{pmatrix}\tilde{\mathsf{U}}_y\right)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dxdyd\tau$$
$$\le \int_0^t\int_0^\infty\int_{\mathbb{R}}\left(|\tilde{\mathsf{U}}_x|^2+|\boldsymbol{S}|^2\right)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.24}$$

This, (6.20), and (6.21) with $\bar{\partial}^\gamma = \partial_t$ together result in

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}}|\tilde{\mathsf{U}}_x(0,y,t)|^2 e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dyd\tau$$
$$\le O(1)\int_0^t\int_0^\infty\int_{\mathbb{R}}\left(|\boldsymbol{S}_\tau|^2+|\boldsymbol{S}|^2\right)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.25}$$

The same procedure will give the higher order energy estimate

$$\frac{1}{2}\int_0^t\int_{\mathbb{R}}|\bar{\partial}^\gamma\tilde{\mathsf{U}}_x(0,y,t)|^2 e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dyd\tau$$
$$\le O(1)\int_0^t\int_0^\infty\int_{\mathbb{R}}\Big(|\bar{\partial}^\gamma\boldsymbol{S}_\tau|^2+|\bar{\partial}^\gamma\boldsymbol{S}|^2\Big)e^{\alpha(y-\beta\tau)-\frac{\alpha(\beta-\alpha-1)}{4}(t-\tau)}dydxd\tau. \tag{6.26}$$

From (6.25) and (6.26) with $|\gamma|\ge 2$, one has the pointwise estimates by the

Sobolev's inequality, for $(y,t)\in\mathbb{R}\times\mathbb{R}_+$:

$$\sup_{(y,t)\in\mathbb{R}\times\mathbb{R}_+}|\tilde{\mathsf{U}}_x(0,y,t)|e^{\frac{\alpha(y-\beta t)}{2}+\frac{\alpha(\beta-\alpha-1)t}{8}}$$

$$\leq\ O(1)\Big(\int_0^t\int_0^\infty\int_{\mathbb{R}}\Big(\sum_{|\gamma|\leq 2}|\bar{\partial}^\gamma\boldsymbol{S}_\tau|^2+|\bar{\partial}^\gamma\boldsymbol{S}|^2\Big)e^{\alpha(y-\beta\tau)+\frac{\alpha(\beta-\alpha-1)}{4}\tau}dydxd\tau\Big)^{1/2}. \quad (6.27)$$

Thus, for $y>2\beta t$,

$$|\tilde{\mathsf{U}}_x(0,y,t)|\leq O(1)e^{-\frac{\alpha|y|}{4}-\frac{\alpha(\beta-\alpha-1)t}{8}}$$
$$\cdot\Big(\int_0^t\int_0^\infty\int_{\mathbb{R}}\Big(\sum_{|\gamma|\leq 2}|\bar{\partial}^\gamma\boldsymbol{S}_\tau|^2+|\bar{\partial}^\gamma\boldsymbol{S}|^2\Big)e^{\alpha(y-\beta\tau)+\frac{\alpha(\beta-\alpha-1)}{4}\tau}dydxd\tau\Big)^{1/2}. \quad (6.28)$$

Similarly, one has, for $y<-2\beta t$

$$|\tilde{\mathsf{U}}_x(0,y,t)|\leq O(1)e^{-\frac{\alpha|y|}{4}-\frac{\alpha(\beta-\alpha-1)t}{8}}$$
$$\cdot\Big(\int_0^t\int_0^\infty\int_{\mathbb{R}}\Big(\sum_{|\gamma|\leq 2}|\bar{\partial}^\gamma\boldsymbol{S}_\tau|^2+|\bar{\partial}^\gamma\boldsymbol{S}|^2\Big)e^{-\alpha(y+\beta\tau)+\frac{\alpha(\beta-\alpha-1)}{4}\tau}dydxd\tau\Big)^{1/2}. \quad (6.29)$$

From (6.28) and (6.29), one has the following lemma for wave structure in $|y|\geq 2\beta$ with $\beta>3/2$:

Lemma 6.2. *For a given $\beta>3/2$ there exists $\alpha>0$ with $\beta-\alpha-1>0$ so that for any $|y|>2\beta$ the Neumann value $\boldsymbol{u}_x^0(y,t)\in\mathbb{R}^2$ satisfies*

$$|\boldsymbol{u}_x^0(y,t)|\ \leq\ O(1)e^{-\frac{\alpha|y|}{4}-\frac{\alpha(\beta-\alpha-1)t}{8}}\cdot\Big(\int_0^t\int_{\mathbb{R}}\Big(\sum_{|\gamma|\leq 4}|\bar{\partial}^\gamma\boldsymbol{u}^0(y,\tau)|^2\Big)$$
$$\times e^{-\alpha(y+\beta\tau)+\frac{\alpha(\beta-\alpha-1)}{4}\tau}dyd\tau\Big)^{1/2}, \quad (6.30)$$

where $\bar{\partial}^\gamma$ is the differential operator on $(y,t)\in\mathbb{R}\times\mathbb{R}_+$ defined in (6.22).

Remark 6.3. We have thus obtained the global wave structure for the boundary relation. Note that the smoothness requirement in the last lemma is higher. We state this for the case when the input function has compact support as in Theorem 1.2:

$$|\mathscr{I}^L[\mathfrak{N}_{ij}\boldsymbol{u}^0](y,t)|\ =\ O(1)\ \frac{\mathbb{W}|(\vec{\boldsymbol{x}},t)}{t}\underset{\vec{\boldsymbol{x}}}{*}\frac{e^{-\frac{|\vec{\boldsymbol{x}}|^2}{C(t)}}}{t}\Bigg|_{x=-\Lambda t}\|\boldsymbol{u}^0\|_{L^\infty_{(y,t)}}$$

$$+O(1)e^{-\frac{(y+t)}{C}}\sum_{|\alpha|=0}^{4}\|\bar{\partial}^{\alpha}\boldsymbol{u}^0\|_{L^{\infty}_{(y,t)}}. \tag{6.31}$$

6.3. Sine-Cosine transformation and Wave equations

Consider the wave equation in $m+1$ dimensional space with initial datum U_0 and U_1:

$$\begin{cases}\partial_t^2 U=\Delta_m U,\\ U(\vec{\boldsymbol{x}},0)=U_0(\vec{\boldsymbol{x}}),\\ \partial_t U(\vec{\boldsymbol{x}},0)=U_1(\vec{\boldsymbol{x}}).\end{cases} \tag{6.32}$$

The solution of $U(\vec{\boldsymbol{x}},t)$ in Fourier variable $\vec{\eta}\in\mathbb{R}^m$ is

$$\hat{U}(\hat{\zeta},t)=\cos(|\vec{\eta}|t)\hat{U}_0(\vec{\eta})+\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\hat{U}_1(\vec{\eta}).$$

This identifies t $\sin(|\vec{\eta}|t)/|\vec{\eta}|$ as the Fourier transformation of the solution of $U(\vec{\boldsymbol{x}},t)$ for (6.32) with $U_0(\vec{\boldsymbol{x}})\equiv 0$ and $U_1(\vec{\boldsymbol{x}})=\delta(\vec{\boldsymbol{x}})$.

Theorem 6.4. *Let $m\in\mathbb{N}$ and D_0 be a given positive number. Then, for any $\vec{\boldsymbol{x}}\in\mathbb{R}^m$, the inverse Fourier transformation of $e^{-D_0|\vec{\eta}|^{\boldsymbol{t}}}\sin(|\vec{\eta}|t)/|\vec{\eta}|$ satisfies:*

Case 1. $m=3$

$$\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2t}\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right](\vec{\boldsymbol{x}}) = \frac{1}{4\pi t}\int_{\substack{\hat{\boldsymbol{y}}\in\mathbb{R}^3\\ |\hat{\boldsymbol{y}}-\vec{\boldsymbol{x}}|=t}}\frac{e^{-\frac{|\hat{\boldsymbol{y}}|^2}{4D_0t}}}{(4D_0\pi t)^{3/2}}dS \tag{6.33}$$

and

$$\left|\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2t}\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right](\vec{\boldsymbol{x}})\right|\le O(1)W_3(\vec{\boldsymbol{x}},t;D_0), \tag{6.34}$$

$$\left|\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2t}\frac{i|\zeta^2|^j\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right](\vec{\boldsymbol{x}})\right|\le O(1)\frac{1}{\sqrt{D_0t}}W_3(\vec{\boldsymbol{x}},t;2D_0)\quad \textit{for } j=1,2,3, \tag{6.35}$$

$$\left|\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2t}\cos(|\vec{\eta}|t)\right](\vec{\boldsymbol{x}})\right|\le O(1)\left(\frac{1}{D_0}+1\right)\frac{1}{t}W_3(\vec{\boldsymbol{x}},t;2D_0) \tag{6.36}$$

where

$$W_3(\vec{\boldsymbol{x}},t;D_0) \equiv \begin{cases} \dfrac{1}{t^{3/2}\sqrt{D_0}} \ \textit{for}\ ||\vec{\boldsymbol{x}}|-t| \le \sqrt{D_0 t}, \\ \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{4tD_0}}}{t^{3/2}\sqrt{D_0}} \ \textit{for}\ ||\vec{\boldsymbol{x}}|-t| \ge \sqrt{D_0 t}. \end{cases} \tag{6.37}$$

Case 2. $m=2$.

$$\left|\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2 t}\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right](\vec{\boldsymbol{x}})\right| \le O(1)W_2(\vec{\boldsymbol{x}},t;D_0 0), \tag{6.38}$$

$$\left|\mathscr{F}^{-1}\left[i|\zeta^2|^j e^{-D_0|\vec{\eta}|^2 t}\frac{\sin(|\vec{\eta}|t)}{|\vec{\eta}|}\right](\vec{\boldsymbol{x}})\right| \le O(1)\frac{1}{\sqrt{D_0 t}}W_2(\vec{\boldsymbol{x}},t;D_0 0), \tag{6.39}$$

$$\left|\mathscr{F}^{-1}\left[e^{-D_0|\vec{\eta}|^2 t}\cos(|\vec{\eta}|t)\right](\vec{\boldsymbol{x}})\right| \le O(1)\left(\frac{1}{D_0}+1\right)\frac{1}{t}W_2(\vec{\boldsymbol{x}},t;D_0 0), \tag{6.40}$$

where

$$W_2(\vec{\boldsymbol{x}},t;D_0) \equiv \begin{cases} \dfrac{1}{t-|\vec{\boldsymbol{x}}|} & \textit{for}\ |\vec{\boldsymbol{x}}| \le t-\sqrt{D_0 t}, \\ \dfrac{1}{t^{3/4}D_0^{1/4}} & \textit{for}\ ||\vec{\boldsymbol{x}}|-t| \le \sqrt{D_0 t}, \\ \dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{4D_0 t}}}{t^{3/4}D_0^{1/4}} & \textit{for}\ |\vec{\boldsymbol{x}}| \ge t+\sqrt{D_0 t}. \end{cases} \tag{6.41}$$

Proof. The inverse transformation in (6.33) is a convolution of the inverse of $e^{-D_0|\vec{\eta}|^2}$ and the solution of the wave equation. The solution of the wave equation in 3-D can be expressed by the Kirchhoff's formula. Thus the identity (6.33) follows as the convolution of the heat kernel in R^3, $e^{-\frac{|\vec{\boldsymbol{x}}|^2}{4D_0 t}}/\sqrt{(4D_0\pi t)^3} = \mathscr{F}^{-1}[e^{-D_0|\vec{\eta}|^2 t}](\vec{\boldsymbol{x}})$ and the Kirchhoff solution. With (6.33), it is straightforward to obtain (6.34). (6.35) follows from the estimate, $|\mathscr{F}^{-1}(|\zeta^2|^j e^{-D_0|\vec{\eta}|^2})| \le O(1)e^{-\frac{|\vec{\eta}|^2}{8D_0 t}}/(D_0 t)^2$. For (6.36), one uses the property $\cos(|\vec{\eta}|t) = \frac{d}{dt}\sin(|\vec{\eta}|t)/|\vec{\eta}|$.

For the 2-D case, it is a consequence of the identity (6.33) and the Hadamard's method of descend. □

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