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APPROXIMATE QUATERNARY JORDAN DERIVATIONS ON BANACH QUATERNARY ALGEBRAS

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Abstract

We show that a quaternary Jordan derivation on a quaternary Banach algebra associated with the equation

$$f(\frac{x+y+z}{4})+f(\frac{3x-y-4z}{4})+f(\frac{4x+3z}{4})=2f(x) \ .$$

is satisfied in generalized Hyers–Ulam stability.

1. Introduction

A quaternary algebra is a real or complex linear space, endowed with a linear mapping the so-called a quaternary product $(x,y,z,t)\to[xyzt]_A$ of $A\times A\times A\times A$ into A such that $[[xyzt]_A\ wvu]_A=[x[yztw]_A\ vu]_A=[xy[ztwv]_A\ u]_A=[xyz[twvu]_A]_A$ for all $x,y,z,t,w,v,u\in A$. If $(A,.)$ is a usual binary algebra, then an induced quaternary multiplication can be, of course, defined by $[xyzt]_A=((x.y).z).t=(x.(y.z)).t=x.((y.z).t)=x.(y.(z.t))$. Hence the quaternary algebra is a natural generalization of the

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binary case. If a quaternary algebra $(A, [\ ]_A)$ has a unit, i.e., an element $e \in A$ such that $x = [xeee]_A = [eeex]_A$ for all $x \in A$, then A with the binary product $x.y = [xeey]_A$, is a usual algebra.

A normed quaternary algebra is a quaternary algebra with a norm $\|.\|$ such that $\|[xyzt]_A\| \leq \|x\|\|y\|\|z\|\|t\|$ for $x, y, z, t \in A$. A Banach quaternary algebra is a normed quaternary algebra such that the normed linear space with norm $\|.\|$ is complete. Assume that A and B are real or complex quaternary algebras. A linear map $h : A \to B$ is said to be a quaternary homomorphism if $h[xyzt]_A = [h(x)h(y)h(z)h(t)]_B$ holds for all $x, y, z, t \in A$.

Let A be a Banach quaternary algebra and X be a Banach space. Then X is called a quaternary Banach A-module, if module operations $A \times A \times A \times X \to X$, $A \times A \times X \times A \to X$, $A \times X \times A \times A \to X$,and $X \times A \times A \times A \to X$, which are $\mathbb{C}$-linear in every variable. Moreover satisfy

$$[[xabc]_X\ def]_X = [x[abcd]_A\ ef]_X = [xa[bcde]_A\ f]_X = [xab[cdef]_A]_X,$$

$$[[axbc]_X\ def]_X = [a[xbcd]_X\ ef]_X = [ax[bcde]_A\ f]_X = [axb[cdef]_A]_X,$$

$$[[abxc]_X\ def]_X = [a[bxcd]_X\ ef]_X = [ab[xcde]_X\ f]_X = [abx[cdef]_A]_X,$$

$$[[abcx]_X\ def]_X = [a[bcxd]_X\ ef]_X = [ab[cxde]_X\ f]_X = [abc[xdef]_X]_X,$$

$$[[abcd]_A\ xef]_X = [a[bcdx]_X\ ef]_X = [ab[cdxe]_X\ f]_X = [abc[dxef]_X]_X,$$

$$[[abcd]_A\ exf]_X = [a[bcde]_A\ xf]_X = [ab[cdex]_X\ f]_X = [abc[dexf]_X]_X,$$

$$[[abcd]_A\ efx]_X = [a[bcde]_A\ fx]_X = [ab[cdef]_A\ x]_X = [abc[defx]_X]_X,$$

for all $x \in X$ and all $a, b, c, d, e, f \in A$,

$$\max\{\|[xabc]_X\|, \|[axbc]_X\|, \|[abxc]_X\|, \|[abcx]_X\|\} \leq \|a\|\|b\|\|c\|\|x\|$$

for all $x \in X$ and all $a, b, c \in A$.

Let $(A, [\ ]_A)$ be a Banach quaternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and $(X, [\ ]_X)$ be a quaternary Banach A-module. A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a quaternary derivation, if

$$D([xyzt]_A) = [D(x)yzt]_X + [xD(y)zt]_X + [xyD(z)t]_X + [xyzD(t)]_X$$

for all $x, y, z, t \in A$.

A linear mapping $D : (A, [\,]_A) \to (X, [\,]_X)$ is called a quaternary Jordan derivation, if

$$D([xxxx]_A) = [D(x)xxx]_X + [xD(x)xx]_X + [xxD(x)x]_X + [xxxD(x)]_X$$

for all $x \in A$.

The stability of functional equations was first introduced by S. M. Ulam [1] in 1940. In 1941, D. H. Hyers [2] gave a partial solution of $Ulam's$ problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [3] is called the Hyers–Ulam–Rassias stability. According to Th. M. Rassias Theorem:

Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{2}$$

for all $x \in E$. If $p < 0$ then inequality (1) *holds for all $x, y \neq 0$, and* (2) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.*

On the other hand J. M. Rassias [4, 5], generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms. According to J. M. Rassias Theorem:

Theorem 1.2. *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E^{'}$ is a map from a norm space E into a Banach space $E^{'}$ such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in E$, *then there exists a unique additive mapping* $T : E \to E'$ *such that*

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p,$$

for all $x \in E$. *If in addition for every* $x \in E$, $f(tx)$ *is continuous in real* t *for each fixed* x, *then* T *is linear* (*see* [7]-[13]).

Stability problems of functional equations have been investigated extensively during the last decade. A large list of references concerning the stability of functional equations can be found in [14], [15], [16, 17, 18], [19] and [20]-[24].

Recently, R. Badora [25] and T. Miura et al. [26] proved the Ulam–Hyers stability, the Isac and Rassias-type stability [27], the Hyers–Ulam–Rassias stability and the Bourgin–type superstability of ring derivations on Banach algebras. On the other hand, C. Park [28], C. Park and M. E. Gordji [29] and Bavand et al. [30] have contributed works to the stability problem of ternary homomorphisms and ternary derivations. For more details about the results concerning stability of functional equations the reader is referred to [31]–[70].

The main purpose of the present paper is to offer the Ulam–Hyers stability of quaternary Jordan derivations on Banach quaternary algebras associated with the following functional equation

$$f(\frac{x+y+z}{4}) + f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) = 2f(x)\ . \tag{1.1}$$

2. Quaternary Jordan Derivations on Banach Quaternary Algebras

In this section, we investigate quartenary Jordan derivations on Banach quaternary algebras.

Throughout this section, assume that $(A, [\ ]_A)$ is a Banach quaternary algebra and $(X, [\ ]_X)$ is a quaternary Banach A-module.

Lemma 2.1 ([31]). *Let* V *and* W *be linear spaces and let* $f : V \to W$ *be an additive mapping such that* $f(\mu x) = \mu f(x)$ *for all* $x \in V$ *and all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}\ ; |\lambda| = 1\}$. *Then the mapping* f *is* $\mathbb{C}$*-linear.*

Lemma 2.2 ([32]). *Let $f : A \to X$ be a mapping such that*

$$f(\frac{x+\mu y+z}{4}) + \mu f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) = 2f(x),$$

for all $x, y, z \in A$. Then f is $\mathbb{C}$-linear.

Theorem 2.3. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f : A \to X$ be a mapping such that*

$$f(\frac{x+\mu y+z}{4}) + \mu f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) = 2f(x), \tag{2.1}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([yyyy]_A) - [f(y)yyy]_X - [yf(y)yy]_X - [yyf(y)y]_X - [yyyf(y)]_X\| \leq \theta \|y\|^{4p} \tag{2.2}$$

for all $y \in A$. Then the mapping $f : A \to X$ is a quaternary Jordan derivation.

Proof. Assume $p < 1$. By Lemma 2.2, the mapping $f : A \to X$ is $\mathbb{C}$-linear. It follows from (2.2) that

$$\begin{aligned} &\|f([yyyy]_A) - [f(y)yyy]_X - [yf(y)yy]_X - [yyf(y)y]_X - [yyyyf(y)]_X\| \\ &= \frac{1}{n^4}\|f([(ny)(ny)(ny)(ny)]_A) - [f(ny)(ny)(ny)(ny)]_X - [(ny)f(ny)(ny)(ny)]_X \\ &\quad -[(ny)(ny)f(ny)(ny)]_X - [(ny)(ny)(ny)f(ny)]_X\| \\ &\leq \frac{\theta}{n^4} n^{4p} \|y\|^{4p} \end{aligned}$$

for all $y \in A$. Thus, since $p < 1$, by letting n tend to ∞ in last inequality, we obtain

$$f([yyyy]_A) = [f(y)yyy]_X + [yf(y)yy]_X + [yyf(y)y]_X + [yyyf(y)]_X$$

for all $y \in A$. Hence the mapping $f : A \to X$ is a quaternary Jordan derivation. Similarly, one obtains the result for the case $p > 1$. □

We prove the following Ulam stability problem for functional equation (1.1) controlled by the mixed type product-sum function

$$(x, y) \to \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p) \qquad (p = p_1 + p_2 + p_3)$$

introduced by J. M. Rassias (see [23]).

Theorem 2.4. *Let p, p_1, p_2, p_3 be real numbers such that $p < 1$, $p_1+p_2+p_3 < 1$, and $\theta > 0$. Suppose $f : A \to X$ satisfies*

$$\begin{aligned} &\|f(\frac{x+\mu y+z}{4}) + \mu f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) - 2f(x)\| \\ &\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p), \end{aligned} \tag{2.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xxxx]_A) - [f(x)xxx]_X - [xf(x)xx]_X - [xxf(x)x]_X - [xxxf(x)]_X\| \leq \theta\|x\|^{4p} \tag{2.4}$$

for all $x \in A$. Then there exists a unique quaternary Jordan derivation $D : A \to X$ satisfying

$$\|f(x) - D(x)\| \leq 2\theta\frac{2^p}{2-2^p}\|x\|^p \tag{2.5}$$

for all $x \in A$.

Proof. Setting $\mu = 1$ and $x = y = z = 0$ in (2.3), yields $f(0) = 0$. Let us take $\mu = 1$, $z = 0$ and $y = x$ in (2.3). Then we obtain

$$\|2f(\frac{x}{2}) - f(x)\| \leq 2\theta\|x\|^p , \tag{2.6}$$

for all $x \in A$. In (2.6), replacing $\frac{x}{2}$ by x and then dividing by 2, we get

$$\|f(x) - \frac{1}{2}f(2x)\| \leq 2^p\theta\|x\|^p , \tag{2.7}$$

for all $x \in A$. We easily prove that by induction that

$$\|f(x) - \frac{1}{2^n}f(2^n x)\| \leq 2\theta\|x\|^p\sum_{i=1}^{n}2^{i(p-1)}. \tag{2.8}$$

In order to show that the functions $D_n(x) = \frac{1}{2^n}f(2^n x)$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace x by $2^m x$ and divide by 2^m in (2.8), where m is an arbitrary positive integer. We find that

$$\|\frac{1}{2^m}f(2^m x)-\frac{1}{2^{m+n}}f(2^{m+n}x)\|\le 2\theta\|x\|^p\sum_{i=m+1}^{m+n}2^{i(p-1)}$$

for all positive integers. Hence by the Cauchy criterion the limit $D(x)=\lim_{n\to\infty}D_n(x)$ exists for each $x\in A$. By taking the limit as $n\to\infty$ in (2.8) we see that

$$\|f(x)-D(x)\|\le 2\theta\|x\|^p\sum_{i=1}^{\infty}2^{i(p-1)}$$

and (2.5) holds for all $x\in A$. Now, we have

$$\begin{aligned}
&\|D(\frac{x+\mu y+z}{4})+\mu D(\frac{3x-y-4z}{4})+D(\frac{4x+3z}{4})-2D(x)\|\\
&= \lim_{n\to\infty}\frac{1}{2^n}\|f(\frac{2^n x+\mu 2^n y+2^n z}{4})+\mu f(\frac{3.2^n x-2^n y-4.2^n z}{4})\\
&\quad +f(\frac{4.2^n x+3.2^n z}{4})-2f(2^n x)\|_A\le \lim_{n\to\infty}\frac{1}{2^n}\theta(\|2^n x\|^{p_1}\|2^n y\|^{p_2}\|2^n z\|^{p_3}\\
&\quad +\|2^n x\|^p+\|2^n y\|^p+\|2^n z\|^p)\\
&= \lim_{n\to\infty}2^{n(p_1+p_2+p_3-1)}\theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{P_3})\\
&\quad +\lim_{n\to\infty}2^{n(p-1)}\theta(\|x\|^p+\|y\|^p+\|z\|^P)=0
\end{aligned}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$. Hence

$$D(\frac{x+\mu y+z}{4})+\mu D(\frac{3x-y-4z}{4})+D(\frac{4x+3z}{4})=2D(x)$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in A$. So by Lemma (2.2), D is $\mathbb{C}$-linear. On the other hand

$$\begin{aligned}
&\|D([xxxx]_A)-[D(x)xxx]_X-[xD(x)xx]_X-[xxD(x)x]_X-[xxxD(x)]_X\|\\
&= \lim_{n\to\infty}\frac{1}{16^n}\|f([(2^n x)(2^n x)(2^n x)(2^n x)]_A)-[f(2^n x)(2^n x)(2^n x)(2^n x)]_X\\
&\quad -[(2^n x)f(2^n x)(2^n x)(2^n x)]_X-[(2^n x)(2^n x)f(2^n x)(2^n x)]_X\\
&\quad -[(2^n x)(2^n x)(2^n x)f(2^n x)]_X\|\\
&\le \lim_{n\to\infty}\frac{\theta}{16^n}\|2^n x\|^{4p}\\
&= \lim_{n\to\infty}\theta 16^{n(p-1)}\|x\|^{4p}=0
\end{aligned}$$

for all $x \in A$, which means that

$$D([xxxx]_A) = [D(x)xxx]_X + [xD(x)xx]_X + [xxD(x)x]_X + [xxxD(x)]_X.$$

Therefore, we conclude that D is a quaternary Jordan derivation. Suppose that there exists another quaternary Jordan derivation $D^{'} : A \to X$ satisfying (2.5). Since $D^{'}(x) = \frac{1}{2^n} D^{'}(2^n x)$, we see that

$$\begin{aligned} \|D(x) - D^{'}(x)\| &= \frac{1}{2^n}\|D(2^n x) - D^{'}(2^n x)\| \\ &\leq \frac{1}{2^n}(\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - D^{'}(2^n x)\|) \\ &\leq 4\theta \frac{2^p}{2 - 2^p} 2^{n(p-1)} \|x\|^p \ , \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. Therefore $D^{'} = D$ as claimed and the proof of the theorem is complete. □

Theorem 2.5. *Let p, p_1, p_2, p_3 be real numbers such that $p > 1$, $p_1+p_2+p_3 > 1$, and $\theta > 0$. Suppose $f : A \to X$ satisfies*

$$\|f(\frac{x + \mu y + z}{4}) + \mu f(\frac{3x - y - 4z}{4}) + f(\frac{4x + 3z}{4}) - 2f(x)\| \tag{2.9}$$

$$\leq \theta(\|x\|^{p_1}\|y\|^{p_2}\|z\|^{p_3} + \|x\|^p + \|y\|^p + \|z\|^p), \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$,

$$\|f([xxxx]_A) - [f(x)xxx]_X - [xf(x)xx]_X - [xxf(x)x]_X - [xxxf(x)]_X\| \leq \theta\|x\|^{4p} \tag{2.11}$$

for all $x \in A$. Then there exists a unique quaternary Jordan derivation $D : A \to X$ satisfying

$$\|D(x) - f(x)\| \leq 2\theta \frac{2^p}{2^p - 2}\|x\|^p \tag{2.12}$$

for all $x \in A$.

Proof. Setting $\mu = 1$ and $x = y = z = 0$ in (2.9), yields $f(0) = 0$. Let us take $\mu = 1$, $z = 0$ and $y = x$ in (2.9). Then we obtain

$$\|2f(\frac{x}{2}) - f(x)\| \leq 2\theta\|x\|^p \ , \tag{2.13}$$

for all $x \in A$. By induction, we get

$$\|2^n f(\frac{x}{2^n}) - f(x)\| \le 2\theta \|x\|^p \sum_{i=0}^{n-1} 2^{i(1-p)}. \tag{2.14}$$

In order to show that the functions $D_n(x) = 2^n f(\frac{x}{2^n})$ form a convergent sequence, we use the Cauchy convergence criterion. Indeed, replace x by $\frac{x}{2^m}$ and multiply by 2^m in (2.14), where m is an arbitrary positive integer. We find that

$$\|2^{m+n} f(\frac{x}{2^{m+n}}) - 2^m f(\frac{x}{2^m})\| \le 2\theta \|x\|^p \sum_{i=m}^{m+n-1} 2^{i(1-p)}$$

for all positive integers. Hence by the Cauchy criterion the limit $D(x) = \lim_{n\to\infty} D_n(x)$ exists for each $x \in A$. By taking the limit as $n \to \infty$ in (2.14) we see that

$$\|D(x) - f(x)\| \le 2\theta \|x\|^p \sum_{i=0}^{\infty} 2^{i(1-p)}$$

and (2.11) holds for all $x \in A$. Thus, we have

$$\begin{aligned}
&\|D(\frac{x+\mu y+z}{4}) + \mu D(\frac{3x-y-4z}{4}) + D(\frac{4x+3z}{4}) - 2D(x)\| \\
&= \lim_{n\to\infty} 2^n \|f(\frac{2^{-n}x+\mu 2^{-n}y+2^{-n}z}{4}) + \mu f(\frac{3.2^{-n}x-2^{-n}y-4.2^{-n}z}{4}) \\
&\quad + f(\frac{4.2^{-n}x+3.2^{-n}z}{4}) - 2f(2^{-n}x)\| \\
&\le \lim_{n\to\infty} 2^n \theta (\|2^{-n}x\|^{p_1} \|2^{-n}y\|^{p_2} \|2^{-n}\|^{p_3} \\
&\quad + \|2^{-n}x\|^p + \|2^{-n}y\|^p + \|2^{-n}z\|^p) = \lim_{n\to\infty} 2^{n(1-p_1+p_2+p_3)} \theta (\|x\|^{p_1} \|y\|^{p_2} \|z\|^{P_3}) \\
&\quad + \lim_{n\to\infty} 2^{n(1-p)} \theta (\|x\|^p + \|y\|^p + \|z\|^P) = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Hence

$$D(\frac{x+\mu y+z}{4}) + \mu D(\frac{3x-y-4z}{4}) + D(\frac{4x+3z}{4}) = 2D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. So by Lemma 2.2, D is $\mathbb{C}$-linear. Thus, we have

$$\begin{aligned}
&\|D([xxxx]_A) - [D(x)xxx]_X - [xD(x)xx]_X - [xxD(x)x]_X - [xxxD(x)]_X\| \\
&= \lim_{n\to\infty} 16^n \|f([2^{-n}x(2^{-n}x)(2^{-n}x)(2^{-n}x)]_A) - [f(2^{-n}x)(2^{-n}x)(2^{-n}x)(2^{-n}x)]_X \\
&\quad -[(2^{-n}x)f(2^{-n}x)(2^{-n}x)(2^{-n}x)]_X - [(2^{-n}x)(2^{-n}x)f(2^{-n}x)(2^{-n}x)]_X \\
&\quad -[(2^{-n}x)(2^{-n}x)(2^{-n}x)f(2^{-n}x)]_X\| \\
&\le \lim_{n\to\infty} 16^n\theta\|\frac{x}{2^n}\|^{4p} \\
&= \lim_{n\to\infty} \theta 16^{n(1-p)}\|x\|^{4p} = 0
\end{aligned}$$

for all $x \in A$, which means that

$$D([xxxx]_A) = [D(x)xxx]_X + [xD(x)xx]_X + [xxD(x)x]_X + [xxxD(x)]_X.$$

Therefore, we conclude that D is a quaternary Jordan derivation. Suppose that there exists another quaternary Jordan derivation $D^{'} : A \to X$ satisfying (2.11). Since $D^{'}(x) = 2^n D^{'}(\frac{x}{2^n})$, we see that

$$\begin{aligned}
\|D(x) - D^{'}(x)\| &= 2^n\|D(\frac{x}{2^n}) - D^{'}(\frac{x}{2^n})\| \\
&\le 2^n(\|f(\frac{x}{2^n}) - D(\frac{x}{2^n})\| + \|f(\frac{x}{2^n}) - D^{'}(\frac{x}{2^n})\|) \\
&\le 4\theta\frac{2^p}{2^p-2}2^{n(1-p)}\|x\|^p\ ,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. Hence, $D^{'} = D$ as claimed and proof of theorem is complete. □

We are going to investigate the Hyers–Ulam–Rassias stability problem for functional equation (1.1).

Corollary 2.6. *Let* $P \in (-\infty, 1) \cup (1, \infty)$, $\theta > 0$. *Suppose* $f : A \to X$ *satisfies*

$$\Big\|f(\frac{x+\mu y+z}{4})+\mu f(\frac{3x-y-4z}{4})+f(\frac{4x+3z}{4})-2f(x)\Big\| \le \theta(\|x\|^p+\|y\|^p+\|z\|^p),$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$,

$$\|f([xxxx]_A)-[f(x)xxx]_X-[xf(x)xx]_X-[xxf(x)x]_X-[xxxf(x)]_X\| \le \theta\|x\|^{4p}$$

for all $x \in A$. *Then there exists a unique quaternary Jordan derivation* $D : A \to X$ *satisfying*

$$\|f(x) - D(x)\| \leq 2\theta \frac{2^p}{|2-2^p|}\|x\|^p$$

for all $x \in A$.

By Theorems 2.4 and 2.5 we solve the following Hyers-Ulam stability problem for functional equation (1.1).

Corollary 2.7. *Let* θ *be a positive real number. Suppose* $f : A \to X$ *satisfies*

$$\Big\|f(\frac{x+\mu y+z}{4}) + \mu f(\frac{3x-y-4z}{4}) + f(\frac{4x+3z}{4}) - 2f(x)\Big\| \leq \theta$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$,

$$\|f([xxxx]_A) - [f(x)xxx]_X - [xf(x)xx]_X - [xxf(x)x]_X - [xxxf(x)]_X\| \leq \theta\|x\|$$

for all $x \in A$. *Then there exists a unique quaternary Jordan derivation* $D : A \to X$ *satisfying*

$$\|f(x) - D(x)\| \leq \theta$$

for all $x \in A$.

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