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A REMARK ON BEALE-NISHIDA'S PAPER

YASUSHI HATAYA

Department of Mathematics, Yamaguchi University, Yamaguchi 753-8512, Japan.
E-mail: hataya@yamaguchi-u.ac.jp

Abstract

We discuss decay properties of solutions to viscous surface waves with capillarity given in Beale-Nishida's article [2]. We study their problem more precisely and make some remarks on their results.

1. Introduction

The aim of the present paper is to discuss Beale-Nishida's results [2] more precisely and to give a complete proof to their decay estimates. J. T. Beale and T. Nishida studied the decay properties of solutions to viscous surface waves with capillarity more than twenty years ago in [2], based upon the result of existence of smooth solution to a nonlinear problem [1]. They gave delicate analysis on linearized operators by showing that a branch of continuous spectra of negative real numbers accumulate at the origin, to conclude decay of the solutions in algebraic orders. They applied the theory of analytic perturbation to a family of two-point boundary value problems of ODE's, but they omitted writing details in [2]. The author considers that their results are still significant and play an important role in the analysis of nonlinear boundary value problems close to a constant state. To the author's knowledge, no one has given complete proofs to their results, and we give supplementary remarks on their approach.

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Following [2], we state the problem as follows. We denote a fluid domain of constant depth $b\,(>0)$ by Ω, and

$$\Omega = \{(\boldsymbol{x}, y) \in \mathbb{R}^3 \colon \boldsymbol{x} \in \mathbb{R}^2,\ -b < y < 0\}.$$

We consider the fluid domain bounded by a free surface S_F from above, and by a rigid flat bottom S_B from below. We denote the fluid velocity by $\boldsymbol{u} = (u_1, u_2, u_3)(t, \boldsymbol{x}, y)$ and the pressure by $q(t, \boldsymbol{x}, y)$, and we suppose the elevation of free surface to be given by a graph $y = \eta(t, \boldsymbol{x})$. Then our aimed system linearized around the equilibrium is written as follows:

$$\frac{\partial \eta}{\partial t} - u_3 = 0 \quad \text{on } S_F, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + \nabla q = 0 \quad \text{in } \Omega, \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{3}$$

$$\frac{\partial u_i}{\partial y} + \frac{\partial u_3}{\partial x_i} = 0 \quad i = 1, 2 \quad \text{on } S_F, \tag{4}$$

$$q - 2\nu \frac{\partial u_3}{\partial y} - (g - \beta\Delta)\eta = 0 \quad \text{on } S_F, \tag{5}$$

$$\boldsymbol{u} = 0 \quad \text{on } S_B. \tag{6}$$

Here ν, g and β are given positive constants. The inhomogeneous functions in (2) and (5) are neglected here for simplicity. This system is accompanied by an initial data

$$(\eta, \boldsymbol{u}) = (h, \boldsymbol{f}) \quad \text{at } t = 0. \tag{7}$$

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2. Resolvents of the Linearized Operator

We formulate (1)-(6) in an operator form according to [2]. Let an operator P be the Helmholtz projection defined as

$$L^2(\Omega) = PL^2(\Omega) \oplus \{\nabla\phi \colon \phi \in H^1(\Omega),\ \phi = 0 \text{ on } S_F\},$$

and decompose the pressure term ∇q as $P\nabla q = \nabla\pi^{(1)} + \nabla\pi^{(2)}$, where

$$\Delta\pi^{(i)} = 0 \quad \text{in } \Omega, \quad \frac{\partial\pi^{(i)}}{\partial y} = 0 \quad \text{on } S_B \ (i = 1, 2),$$
$$\pi^{(1)} = 2\nu\frac{\partial u_3}{\partial y}, \quad \pi^{(2)} = (g - \beta\Delta)\eta \quad \text{on } S_F.$$

The system (1)-(6) is reduced to the following evolution equations in $H^1(\mathbb{R}^2) \times PL^2(\Omega)$

$$\frac{\partial\eta}{\partial t} - R\boldsymbol{u} = 0 \quad \text{on } S_F,$$
$$\frac{\partial\boldsymbol{u}}{\partial t} + A\boldsymbol{u} + R^*(g - \beta\Delta)\eta = 0 \quad \text{in } \Omega.$$

Here we define $R\boldsymbol{u} := u_3|_{S_F}$, $A\boldsymbol{u} := -\nu P\Delta\boldsymbol{u} + \nabla\pi^{(1)}$ and $R^*(g - \beta\Delta)\eta := \nabla\pi^{(2)}$.

We introduce a formal operator G by

$$G\begin{pmatrix}\eta\\ \boldsymbol{u}\end{pmatrix} = \begin{pmatrix}0 & R\\ -R^*(g-\beta\Delta) & -A\end{pmatrix}\begin{pmatrix}\eta\\ \boldsymbol{u}\end{pmatrix} \qquad \text{in } H^1(\mathbb{R}^2)\times PL^2(\Omega),$$

$\mathcal{D}(G) \supset W = \{{}^t(\eta, \boldsymbol{u}) \in H^{5/2}(\mathbb{R}^2) \times PH^2(\Omega)\colon \boldsymbol{u}$ satisfies (3), (4) and (6)$\}$, and we consider

$$(\lambda - G)\begin{pmatrix}\eta\\ \boldsymbol{u}\end{pmatrix} = \begin{pmatrix}h\\ \boldsymbol{f}\end{pmatrix} \tag{8}$$

for $(h, \boldsymbol{f}) \in H^1(\mathbb{R}^2) \times PL^2(\Omega)$. Since G is a dissipative operator, we have (i) the right half plane belongs to the resolvent set, (ii) G has a closed extension, which is also denoted by G, and it generates a contraction semigroup e^{tG}.

Now we turn to discuss the solvability of equation (8) under the restriction $(h, \boldsymbol{f}) \in H^{5/2}(\mathbb{R}^2) \times PL^2(\Omega)$.

Lemma 1. ([2, Lemma 3.3]) *For any $\varepsilon_0, \varepsilon_1 > 0$, there exists $c_0 = c_0(\varepsilon_0, \varepsilon_1) > 0$ such that the operator $(\lambda - G)$ has a bounded inverse satisfying*

$$\|\boldsymbol{u}\|_{H^2(\Omega)} + |\lambda|\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\lambda^{-1}R\boldsymbol{u}\|_{H^{5/2}(S_F)} + \|\eta\|_{H^{5/2}(\mathbb{R}^2)} + |\lambda|\|\eta\|_{H^{3/2}(\mathbb{R}^2)}$$
$$\le c_0\left(\|h\|_{H^{5/2}(\mathbb{R}^2)} + \|\boldsymbol{f}\|_{L^2(\Omega)}\right)$$

for $\lambda \in \{\lambda \in \mathbb{C} \colon |\lambda| > \varepsilon_0, |\arg\lambda| < \pi - \varepsilon_1\}$.

We should refer to the resolvent near $\lambda = 0$. We denote by $\hat{f}$ the partial Fourier transform of f with respect to $\boldsymbol{x}$.

Lemma 2. ([2, Lemma 3.4]) *Let* $\operatorname{supp}\hat{h}$ *and* $\operatorname{supp}\hat{\boldsymbol{f}}(\cdot, y)$ *belong to* $\{\xi \in \mathbb{R}^2 \colon |\xi| > \xi_0\}$ *for* $\xi_0 > 0$. *Then there exist constants* $r_0 > 0$ *and* $c_1 = c_1(\xi_0, r_0) > 0$ *such that for* $|\lambda| < r_0$, *equation* (8) *has a solution* ${}^t(\eta, \boldsymbol{u})$ *satisfying*

$$\|\boldsymbol{u}\|_{H^2(\Omega)} + \|\eta\|_{H^{5/2}(\mathbb{R}^2)} \le c_1 \left(\|h\|_{H^{5/2}(\mathbb{R}^2)} + \|\boldsymbol{f}\|_{L^2(\Omega)} \right).$$

We remark that the function $(\eta, \boldsymbol{u})$ given in Lemma 1 and Lemma 2 can be considered holomorphic with respect to λ as in the case of usual resolvent problems.

In order to analyze the spectrum near the origin, we apply the partial Fourier transform to (8) with respect to $\boldsymbol{x}$ to obtain a family of ODE's parametrized by $\xi = (\xi_1, \xi_2)$:

$$\lambda\hat{\eta} - \hat{u}_3 = \hat{h} \quad \text{on } y = 0, \tag{9}$$

$$\lambda\hat{\boldsymbol{u}} - \nu(D^2 - |\xi|^2)\hat{\boldsymbol{u}} + (i\xi, D)\hat{q} = \hat{\boldsymbol{f}} \quad \text{in } I, \tag{10}$$

$$(i\xi, D) \cdot \hat{\boldsymbol{u}} = 0 \quad \text{in } I, \tag{11}$$

$$D\hat{u}_j + i\xi_j\hat{u}_3 = 0 \quad j = 1, 2 \quad \text{on } y = 0, \tag{12}$$

$$-2\nu D\hat{u}_3 + \hat{q} - (g + \beta|\xi|^2)\hat{\eta} = 0 \quad \text{on } y = 0, \tag{13}$$

$$\hat{\boldsymbol{u}} = 0 \quad \text{on } y = -b, \tag{14}$$

where $I = (-b, 0)$ and $D = \partial/\partial y$. We rewrite the system of these equations in an operator form such as

$$(\lambda - \hat{G}(\xi)) \begin{pmatrix} \hat{\eta} \\ \hat{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} \hat{h}(\xi) \\ \hat{\boldsymbol{f}}(\xi, y) \end{pmatrix} \quad \text{in } P_\xi X,$$

where

$$X = \{{}^t(\hat{\eta}(\xi), \hat{\boldsymbol{u}}(\xi, \cdot)) \in \mathbb{C} \times L^2(I)\},$$
$$P_\xi X = \{{}^t(\hat{\eta}(\xi), \hat{\boldsymbol{u}}(\xi, \cdot)) \in X \colon i\xi_1\hat{u}_1 + i\xi_2\hat{u}_2 + D\hat{u}_3 = 0 \text{ in } I\},$$

$$\mathcal{D}(\hat{G}(\xi)) \supset \{{}^t(\hat{\eta}, \hat{\boldsymbol{u}}) \in \mathbb{C} \times H^2(I) \colon \hat{\boldsymbol{u}} \text{ satisfies } (11), (12) \text{ and } (14)\}.$$

For each $\xi \in \mathbb{R}^2$, the operator $\hat{G}(\xi)$ in $P_\xi X$ is dissipative, and it has a closed extension which we denote by $\hat{G}(\xi)$ again. The spectra of $\hat{G}(\xi)$ is determined in the next proposition which is a slight modification of [2, Lemma 3.5].

Proposition 3. *There exist $\xi_0 > 0$ and $0 < r_1 < \nu(\pi/2b)^2$ such that if $|\xi| < \xi_0$, then the spectrum of $\hat{G}(\xi)$ contained in $\{\lambda \in \mathbb{C} \colon |\lambda| < r_1\}$ consists of a simple eigenvalue. Furthermore, the eigenvalue and the eigenvector are analytic in ξ and have the following expansions.*

$$\begin{cases} \lambda = -\dfrac{gb^3}{3\nu}|\xi|^2 + O(|\xi|^3), \\ \hat{\eta}^e = 1 + O(|\xi|), \\ \hat{u}_j^e = i\dfrac{g}{2\nu}(y^2 - b^2)\xi_j + O(|\xi|^2), \quad j = 1, 2, \\ \hat{u}_3^e = \dfrac{g}{2\nu}\left(\dfrac{y^3}{3} - b^2 y - \dfrac{2b^3}{3}\right)|\xi|^2 + O(|\xi|^3). \end{cases} \tag{15}$$

This proposition is a key to conclude decay properties of the solutions, but Beale-Nishida omitted its proof in [2]. The author will complete its proof in the present paper. We will prove this proposition in several steps. Firstly we have the following preparatory lemma.

Lemma 4. *The spectrum of $\hat{G}(0)$ in $\{\lambda \in \mathbb{C} \colon |\lambda| < \nu(\pi/2b)^2\}$ consists of a simple eigenvalue $\lambda = 0$ associated with eigenvector $(\hat{\eta}(0), \hat{\boldsymbol{u}}(0, y)) = (1, 0)$.*

Proof. If we put $\xi = 0$ in (9)–(14), we have

$$\hat{u}_3(0, y) = 0, \quad \lambda\hat{\eta}(0) = \hat{h}(0), \quad \hat{q}(0, y) = \int_0^y \hat{f}(0, z)dz + g\hat{\eta}(0),$$

and

$$(\lambda - \nu D^2)\hat{u}_j(0, y) = \hat{f}_j(0, y) \qquad j = 1, 2 \quad \text{in } I, \tag{16}$$

$$\hat{u}_j(0, -b) = 0 \qquad j = 1, 2, \tag{17}$$

$$D\hat{u}_j(0, 0) = 0 \qquad j = 1, 2. \tag{18}$$

For (16)–(18), the largest spectrum of the operator D^2 with the boundary conditions (17) and (18) is an eigenvalue $-(\pi/2b)^2$. Hence, the set $\{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\nu(\pi/2b)^2\} \setminus \{0\}$ is in resolvent set of (9)–(14).

On the other hand, we see that $\lambda = 0$ is an eigenvalue associated with eigen vector $(\hat{\eta}(0), \hat{\boldsymbol{u}}(0,y)) = (1, \mathbf{0})$. □

Before proceeding the next step, we recall the definition of a holomorphic family of unbounded operators ([5, VII §1]).

Definition 5. Let X, Y be Hilbert spaces.

(1) A family of bounded operators $\{S(\xi)\} \subset \mathscr{B}(X,Y)$ are said to be bounded-holomorphic, if each ξ has a neighborhood in which $S(\xi)$ is bounded and a complex valued function $(S(\xi)U, V)_Y$ is holomorphic in ξ for every $U \in X$ and $V \in Y$.

(2) A family of closed operators $\{\hat{G}(\xi)\} \subset \mathscr{C}(X)$ are said to be holomorphic, if there are a Hilbert space Y and two families of bounded-holomorphic operators $\{S(\xi)\} \subset \mathscr{B}(Y,X)$, $\{T(\xi)\} \subset \mathscr{B}(Y,X)$ such that $S(\xi)$ maps Y to $\mathcal{D}(\hat{G}(\xi))$ bijectively and $\hat{G}(\xi)S(\xi) = T(\xi)$.

Lemma 6. $\{\hat{G}(\xi)\}$ *are holomorphic in* ξ *near the origin* $\xi = 0$.

Proof. In order to avoid the boundary conditions of the domain $\mathcal{D}(\hat{G}(\xi))$ depending on ξ, we adopt the associated sesqui-linear form

$$\mathfrak{g}(\xi)[U,V] = (g + \beta|\xi|^2)\{(\hat{u}_3, \hat{\theta})_{\mathbb{C}} - (\hat{\eta}, \hat{v}_3)_{\mathbb{C}}\} + \frac{\nu}{2}\int_I \widehat{\mathbf{S}(\boldsymbol{u})} : \overline{\widehat{\mathbf{S}(\boldsymbol{v})}}\, dy,$$

where $U = {}^t(\hat{\eta}, \hat{\boldsymbol{u}}), V = {}^t(\hat{\theta}, \hat{\boldsymbol{v}}) \in \mathcal{D}(\mathfrak{g}(\xi))$, $\mathbf{S}(\boldsymbol{u})_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$ and $(a_{ij}) : (b_{ij}) = \sum_{ij} a_{ij}b_{ij}$. It is easy to see that the sesqui-linear form $\mathfrak{g}$ has the following property.

Lemma 7. $\mathfrak{g}(\xi)$ *is a densely defined, closed, and m-sectorial sesqui-linear form in* $P_\xi X$*, whose domain is*

$$\mathcal{D}(\mathfrak{g}(\xi)) = \{{}^t(\hat{\eta}, \hat{\boldsymbol{u}}) \in P_\xi X\colon \hat{\boldsymbol{u}}(\xi, \cdot) \in {}_0H^1(I)\},$$

where ${}_0H^1(I) = \{\hat{\boldsymbol{u}}(\xi, \cdot) \in H^1(I)\colon \hat{\boldsymbol{u}}(\xi, -b) = 0\}$.

We are ready to construct a bounded-holomorphic operator which maps $\mathcal{D}(\mathfrak{g}(\xi))$ onto X. To this end, define a sesqui-linear form $\mathfrak{h}(\xi) := 1 + \operatorname{Re}\mathfrak{g}(\xi)$ with its associated operator $S(\xi) := S_{\mathfrak{h}(\xi)}$, and we have

$$S(\xi)U = \begin{pmatrix} \hat{\eta} \\ \hat{\boldsymbol{u}} \end{pmatrix} + \begin{pmatrix} 0 \\ -\nu(D^2 - |\xi|^2)\hat{\boldsymbol{u}} + (i\xi, D)\pi^{(1)} \end{pmatrix}.$$

Then it is easy to check the following three facts: (i) $S(\xi)$ is essentially self-adjoint, and bounded from below. (ii) $\{S^{-1}(\xi)\}$ are bounded-holomorphic on X. (iii) $\{S^{-1/2}(\xi)\}$ are bounded-holomorphic, and map X to $\mathcal{D}(\mathfrak{g}(\xi))$ bijectively.

Since

$$S^{-1/2}(\xi)U \in \mathcal{D}(S^{1/2}(\xi)) = \mathcal{D}(\mathfrak{h}(\xi)) = \mathcal{D}(\operatorname{Re}\mathfrak{g}(\xi)) \quad \text{for all } U \in X,$$

we define a form $\mathfrak{g}_0(\xi)$ on X by

$$\mathfrak{g}_0(\xi)[U, V] = \mathfrak{g}(\xi)[S^{-1/2}(\xi)U, S^{-1/2}(\xi)V].$$

The sesqui-linear form $\mathfrak{g}_0(\xi)$ is closed, sectorial, and defined everywhere in X, and its family are bounded-holomorphic on X. Thus we have a family of bounded-holomorphic operators $\{\hat{G}_0(\xi)\}$ by

$$\mathfrak{g}_0(\xi)[U, V] = (\hat{G}_0(\xi)U, V)_X,$$

and

$$\hat{G}(\xi)S^{-1}(\xi) = S^{1/2}(\xi)\hat{G}_0(\xi)S^{-1/2}(\xi). \tag{19}$$

Since the right side of (19) is holomorphic and the left side is bounded, $\{\hat{G}(\xi)\}$ are holomorphic. This completes the proof of Lemma 6. □

We now continue the proof of Proposition 3. From Lemma 4 and Lemma 6, the spectrum of $\hat{G}(\xi)$ near the origin consists of a simple eigenvalue $\lambda(\xi)$ associated with the eigenvector $\varphi(\xi)$, both of which are holomorphic in ξ by the following Lemma. This is the special case of general analytic perturbation theory. For the proof, the reader may refer to [5, II §1–1 and VII §1–3], and also to [4, II §5–7] for holomorphy in several variables.

Lemma 8. *Let $\{T(\xi)\}$ be a family of holomorphic operators in X. Assume that the spectra $\Sigma(0)$ of $T(0)$ is separated into $\{\lambda(0)\}\cup\Sigma''(0)$ by a rectifiable, simple closed curve Γ_c, and assume that $\lambda(0)$ is a simple eigenvalue. Then the spectra $\Sigma(\xi)$ of $T(\xi)$ are also separated by Γ_c into $\{\lambda(\xi)\}\cup\Sigma''(\xi)$ for $|\xi| < r_1$ with the associated decomposition $X = \mathrm{span}\langle\varphi(\xi)\rangle \oplus P''(\xi)X$. In particular $\lambda(\xi)$ is a simple eigenvalue associated with the eigenvector $\varphi(\xi)$, both of which are holomorphic in ξ.*

Since we know $\lambda(0)$ is simple, we can choose an eigenvector and its eigenvalue which are holomorphic in ξ:

$$\begin{cases} \lambda(\xi) = \sum_{|j|\geq 1} \lambda_j \cdot \xi^j, & \hat{\boldsymbol{u}}(\xi, y) = \sum_{|j|\geq 1} \hat{\boldsymbol{u}}_j(y) \cdot \xi^j, \\ \hat{\eta}(\xi) = 1 + \sum_{|j|\geq 1} \hat{\eta}_j \cdot \xi^j, & \hat{q}(\xi, y) = \hat{q}(0, y) + \sum_{|j|\geq 1} \hat{q}_j(y) \cdot \xi^j. \end{cases} \tag{20}$$

Here we adopt the vector notation $j = (j_1, j_2) \in \mathbb{N}_0^2$, $\lambda_j \cdot \xi^j = \lambda_{j_1}\xi_1^{j_1} + \lambda_{j_2}\xi_2^{j_2}$, and so on. If we put (20) in (9)–(14) and set $(\hat{h}, \hat{\mathbf{f}}) = (0, \mathbf{0})$, we have $\hat{q}(0, y) = g$. We next calculate the coefficients of order ξ of (9)–(14) and obtain for $|j| = 1$:

$$\lambda_j = 0, \quad \sum_{|j|=1} \hat{\boldsymbol{u}}_j(y) \cdot \xi^j = (i\frac{g}{2\nu}(y^2 - b^2)\xi_1, i\frac{g}{2\nu}(y^2 - b^2)\xi_2, 0),$$

and of order $|\xi|^2$ for $|j| = 2$:

$$\lambda_j = -\frac{gb^3}{3\nu}|\xi|^2, \quad \sum_{|j|=2} \hat{\boldsymbol{u}}_3(y) \cdot \xi^j = \frac{g}{2\nu}(\frac{y^3}{3} - b^2 y - \frac{2}{3}b^3)|\xi|^2.$$

Hence we have determined some of the coefficients of the power series in ξ. We complete the proof of Proposition 3. □

Remark 9. The eigenvalues $\{\lambda(\xi)\}$ of the family of ordinary differential operators $\{\hat{G}(\xi)\}$ given in Proposition 3 correspond to continuous spectra of the partial differential operator G. Indeed, from the proof of Proposition 3, (i) if $\lambda(\xi_0)$ is an eigenvalue of G, then the eigenvector $U(\boldsymbol{x}, y)$ does not vanish. However $\hat{U}$ is supported only on $\xi = \xi_0$, and $\hat{U} = 0$ holds for almost every ξ. Hence $\lambda(\xi)$ is not an eigenvalue of G. (ii) We see that the range of $(\lambda(\xi_0) - G)$ is dense in $H^{5/2}(\mathbb{R}^2) \times PL^2(\Omega)$, and it implies that $\lambda(\xi_0)$ is not a residual spectrum of G. (iii) Meanwhile, we can choose $\{F_n\} \subset H^{5/2}(\mathbb{R}^2) \times PL^2(\Omega)$ which satisfy $\|F_n\|_{H^{5/2}(\mathbb{R}^2)\times PL^2(\Omega)} = 1$ and $\|(\lambda(\xi_0) - G)^{-1} F_n\|_{L^2(\mathbb{R}^2)\times PL^2(\Omega)} \to \infty$ as $n \to \infty$. Therefore, $\{\lambda(\xi)\}$ are a family of continuous spectra of the partial differential operator G.

By Lemma 1 and Lemma 2, the inverse $(\lambda - \hat{G}(\xi))^{-1}$ is holomorphic in $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda > -r_0\}$ for $|\xi| \geq \xi_0$, and we take the path of Dunford integral in the left half plane to obtain

$$\begin{aligned}\begin{pmatrix}\hat{\eta}(t,\xi)\\ \hat{\boldsymbol{u}}(t,\xi,y)\end{pmatrix} &= e^{t\hat{G}(\xi)}\begin{pmatrix}\hat{\eta}_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}\\ &= \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{-r_0-i\tau}^{-r_0+i\tau} e^{\lambda t}(\lambda - \hat{G}(\xi))^{-1}\begin{pmatrix}\hat{\eta}_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix} d\lambda. \qquad (21)\end{aligned}$$

On the other hand, by Lemma 1 and Proposition 3, the inverse $(\lambda - \hat{G}(\xi))^{-1}$ is holomorphic in $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda > -r_1\}$ except the pole $\lambda = \lambda(\xi)$ for $|\xi| < \xi_0$, and the integral path should be modified ([3, VII.4 Theorem 22]) as

$$\begin{aligned}\begin{pmatrix}\hat{\eta}(t,\xi)\\ \hat{\boldsymbol{u}}(t,\xi,y)\end{pmatrix} &= \frac{1}{2\pi i}\left\{\oint_{C_\xi} + \lim_{\tau\to\infty}\int_{-r_1-i\tau}^{-r_1+i\tau}\right\} e^{\lambda t}(\lambda - \hat{G}(\xi))^{-1}\begin{pmatrix}\hat{\eta}_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix} d\lambda,\\ &= e^{\lambda(\xi)t}P'(\xi)\begin{pmatrix}\hat{\eta}_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}\\ &\quad + \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{-r_1-i\tau}^{-r_1+i\tau} e^{\lambda t}(\lambda - \hat{G}(\xi))^{-1}\begin{pmatrix}\hat{\eta}_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix} d\lambda. \qquad (22)\end{aligned}$$

Here we note that C_ξ is a positively-oriented small circle enclosing $\lambda = \lambda(\xi)$ but excluding the line $\{\lambda \in \mathbb{C}\colon \mathrm{Re}\lambda = -r_1\}$, and we denote by $P'(\xi)$ the eigenprojection associated with the eigenvalue $\lambda = \lambda(\xi)$, which is holomorphic in ξ:

$$P'(\xi)\begin{pmatrix}\hat{\eta}_0\\ \hat{\boldsymbol{u}}_0\end{pmatrix} = \frac{\langle(\hat{\eta}_0, \hat{\boldsymbol{u}}_0), (\hat{\eta}^e, \hat{\boldsymbol{u}}^e)\rangle_{\mathbb{C}\times L^2(I)}}{|\hat{\eta}^e(\xi)|^2 + \|\hat{\boldsymbol{u}}^e(\xi,\cdot)\|^2_{L^2(I)}}\begin{pmatrix}\hat{\eta}^e\\ \hat{\boldsymbol{u}}^e\end{pmatrix}.$$

Here by virtue of (15),

$$\left|\frac{\langle(\hat{\eta}_0, \hat{\boldsymbol{u}}_0), (\hat{\eta}^e, \hat{\boldsymbol{u}}^e)\rangle_{\mathbb{C}\times L^2(I)}}{|\hat{\eta}^e(\xi)|^2 + \|\hat{\boldsymbol{u}}^e(\xi,\cdot)\|^2_{L^2(I)}}\right| \leq c_1(|\hat{\eta}_0(\xi)| + |\xi| \cdot \|\hat{\boldsymbol{u}}_0(\xi,\cdot)\|_{L^2(I)}). \qquad (23)$$

We denote ${Q_\eta}^t(\eta,\boldsymbol{u})=\eta$ and ${Q_u}^t(\eta,\boldsymbol{u})=\boldsymbol{u}$. If we take (21), (22) and (23) into account, we have for $\alpha\in[0,5/2]$ and $r_2=\min(r_0,r_1)$,

$$\begin{aligned}
&\|\eta(t)\|^2_{\dot H^\alpha(\mathbb{R}^2_x)} = \||\xi|^\alpha\hat\eta(t,\xi)\|^2_{L^2(\mathbb{R}^2_\xi)}\\
\le\ & 2c_1^2\int_{|\xi|<\xi_0}|\xi|^{2\alpha}e^{2\lambda(\xi)t}(|\hat\eta_0(\xi)|^2+|\xi|^2\|\hat{\boldsymbol{u}}_0(\xi,\cdot)\|^2_{L^2(I)})|\hat\eta^e(\xi)|^2d\xi\\
&+\int_{|\xi|<\xi_0}\frac{|\xi|^{2\alpha}}{4\pi^2}\lim_{\tau\to\infty}\left|\int_{-r_1-i\tau}^{-r_1+i\tau}e^{\lambda t}Q_\eta(\lambda-\hat G(\xi))^{-1}\begin{pmatrix}\hat\eta_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}d\lambda\right|^2d\xi\\
&+\int_{|\xi|\ge\xi_0}\frac{|\xi|^{2\alpha}}{4\pi^2}\lim_{\tau\to\infty}\left|\int_{-r_0-i\tau}^{-r_0+i\tau}e^{\lambda t}Q_\eta(\lambda-\hat G(\xi))^{-1}\begin{pmatrix}\hat\eta_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}d\lambda\right|^2d\xi\\
\le\ & c_2\Big\{\|\hat\eta_0\|^2_{L^\infty(\mathbb{R}^2_\xi)}\int_{|\xi|<\xi_0}|\xi|^{2\alpha}e^{2\lambda(\xi)t}d\xi\\
&+t^{-\alpha-1}\int_{|\xi|<\xi_0}(t|\xi|^2)^{\alpha+1}e^{2\lambda(\xi)t}\|\hat{\boldsymbol{u}}_0(\xi,\cdot)\|^2_{L^2(I)}d\xi\Big\}\\
&+c_3e^{-2r_2t}(\|\eta_0\|^2_{H^{5/2}(\mathbb{R}^2_x)}+\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}),\\
\le\ & c_4t^{-\alpha-1}(\|\eta_0\|^2_{H^{5/2}(\mathbb{R}^2_x)}+\|\eta_0\|^2_{L^1(\mathbb{R}^2_x)}+\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}).
\end{aligned}$$

On the other hand, since $\hat{\boldsymbol{u}}^e(\xi,\cdot)$ is $O(|\xi|)$, we have for $\beta\in[0,2]$,

$$\begin{aligned}
&\|\partial_x^\beta\boldsymbol{u}(t)\|^2_{L^2(\Omega)} = \||\xi|^\beta\hat{\boldsymbol{u}}(t,\xi,y)\|^2_{L^2(\mathbb{R}^2_\xi\times I)}\\
\le\ & 2c_1^2\int_{|\xi|<\xi_0}|\xi|^{2\beta}e^{2\lambda(\xi)t}(|\hat\eta_0(\xi)|^2+|\xi|^2\|\hat{\boldsymbol{u}}_0(\xi,\cdot)\|^2_{L^2(I)})\|\hat{\boldsymbol{u}}^e(\xi,\cdot)\|_{L^2(I)}d\xi\\
&+\int_{\{|\xi|<\xi_0\}\times I}\frac{|\xi|^{2\beta}}{4\pi^2}\lim_{\tau\to\infty}\left|\int_{-r_1-i\tau}^{-r_1+i\tau}e^{\lambda t}Q_u(\lambda-\hat G(\xi))^{-1}\begin{pmatrix}\hat\eta_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}d\lambda\right|^2d\xi dy\\
&+\int_{\{|\xi|\ge\xi_0\}\times I}\frac{|\xi|^{2\beta}}{4\pi^2}\lim_{\tau\to\infty}\left|\int_{-r_0-i\tau}^{-r_0+i\tau}e^{\lambda t}Q_u(\lambda-\hat G(\xi))^{-1}\begin{pmatrix}\hat\eta_0(\xi)\\ \hat{\boldsymbol{u}}_0(\xi,y)\end{pmatrix}d\lambda\right|^2d\xi dy\\
\le\ & c_5\Big\{\|\hat\eta_0\|^2_{L^\infty(\mathbb{R}^2_\xi)}\int_{|\xi|<\xi_0}|\xi|^{2(\beta+1)}e^{2\lambda(\xi)t}d\xi\\
&+t^{-\beta-2}\int_{|\xi|<\xi_0}(t|\xi|^2)^{\beta+2}e^{2\lambda(\xi)t}\|\hat{\boldsymbol{u}}_0(\xi,\cdot)\|^2_{L^2(I)}d\xi\Big\}\\
&+c_3e^{-2r_2t}(\|\eta_0\|^2_{H^{5/2}(\mathbb{R}^2_x)}+\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}),\\
\le\ & c_6t^{-\beta-2}(\|\eta_0\|^2_{H^{5/2}(\mathbb{R}^2_x)}+\|\eta_0\|^2_{L^1(\mathbb{R}^2_x)}+\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}).
\end{aligned}$$

As for $\|\partial_y^\beta\boldsymbol{u}(t)\|_{L^2(\Omega)}$, if we apply y-derivatives to $\hat{\boldsymbol{u}}^e$, we have no gain of the order of ξ. Hence $\|\partial_y^\beta \boldsymbol{u}(t)\|_{L^2(\Omega)}$ is $O(t^{-1})$ for $\beta \in [0,2]$. We have thus obtained the following theorem, which is one of the main results of Beale-Nishida's paper [2].

Theorem 10. ([2, Theorem 3.1]) *Let* $E_2 = \|\eta_0\|_{L^1(\mathbb{R}^2)} + \|\eta_0\|_{H^{5/2}(\mathbb{R}^2)} + \|\boldsymbol{u}_0\|_{L^2(\Omega)}$*. Then the solution to* (1)-(7) *has the decay rate:*

$$\begin{aligned} \|\partial_x^\alpha \eta(t)\|_{L^2(\mathbb{R}^2)} &\leq c_0 E_2 t^{-(1+\alpha)/2}, \quad 0 \leq \alpha \leq 5/2, \\ \|\boldsymbol{u}(t)\|_{H^2(\Omega)} &\leq c_0 E_2 t^{-1}. \end{aligned}$$

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