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VISCOUS SHOCK PROPAGATION WITH BOUNDARY EFFECT

LI-CHENG TSAI

Institute of Mathematics, Academia Sinica, Taipei.

E-mail: lctsai@math.sinica.edu.tw

1. Introduction

In this paper, the boundary effect on the propagation of viscous shock waves is investigated. For convex viscous scalar conservation law:

$$u_t + f(u)_x = u_{xx}, f'' > 0$$

the shock with velocity α is a travelling wave solution $u(x,t) = \varphi(x - \alpha t; \alpha)$:

$$-\alpha\varphi'(x - \alpha t; \alpha) + f(\varphi(x - \alpha t; \alpha))_x = \varphi''(x - \alpha t; \alpha).$$

In free space the shock velocity α is determined by the end states $u_\pm := \varphi(\pm\infty)$ through the Rankine-Hugoniot condition:

$$\alpha = \frac{f(u_-) - f(u_+)}{u_- - u_+}. \tag{1}$$

However, in space with boundary, the boundary affects the propagation of the shock. The case with $\alpha > 0$ has been studied by Lan, et al. [1] through Green's function approach. Liu and Nishihara [2] have studied the cases with $\alpha > 0, \alpha < 0$ through the energy method. In this paper, the critical case in which the shock is stationary with respect to the boundary, i.e. $\alpha = 0$, is considered. This case is critical for the reason that the boundary effect becomes the most dramatic: while the Rankine–Hugoniot condition predicts a zero velocity and a constant location of the shock, the shock actually

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propagates away from the boundary as time goes on, with a vanishing speed and sub-linear position, under the continuing boundary effect.

For simplicity the Burgers equation is studied, with the stationary Burgers shock ϕ with end states $\phi(\pm\infty) = \mp 1$. This Burgers shock is:

$$\phi(x) = -\tanh\left(\tfrac{x}{2}\right). \tag{2}$$

Thus, the initial boundary value (IBV) problem for our study is:

$$u_t + uu_x = u_{xx};\ \ x, t > 0 \tag{3a}$$

$$u(0, t) = u_- = 1 \tag{3b}$$

$$u(\infty, t) = \phi(\infty) = u_+ = -1 \tag{3c}$$

This problem has been investigated by Liu and Yu [3]. They have proved that the shock propagates away from the boundary with position of order $\log t$, time asymptotically. [3] regards the solution u as a perturbation of the translated stationary shock, $\phi(x - Y(t))$, where the shock location $Y(t)$ is defined by conservation law:

$$\int_0^\infty u(x,t) - \phi\left(x - Y(t)\right) dx = 0. \tag{4}$$

Since $\phi(x - Y(t))$ does not satisfy boundary condition, i.e. $\phi(-Y(t)) < 1$ regardless of the value of $Y(t)$, the boundary value discrepancy of $\phi(x-Y(t))$ prohibits the shock from staying stationary.

The main purpose of this paper is to gain a better qualitative understanding of the drifting of the shock. The solution u is not treaded as a perturbation of the translated stationary shock profile $\phi(x - Y(t))$, but as a perturbation of an evolving shock profile $\psi(x - X(t); \alpha)$. $X(t)$ is similarly defined by conservation law:

$$\int_0^\infty u(x,t) - \psi\left(x - X(t); \alpha\right) dx = 0. \tag{5}$$

As before, $\psi(\infty) = -1$. However, this time, in order for ψ to satisfy the boundary condition:

$$\phi(-X(t); \alpha) = 1, \tag{6}$$

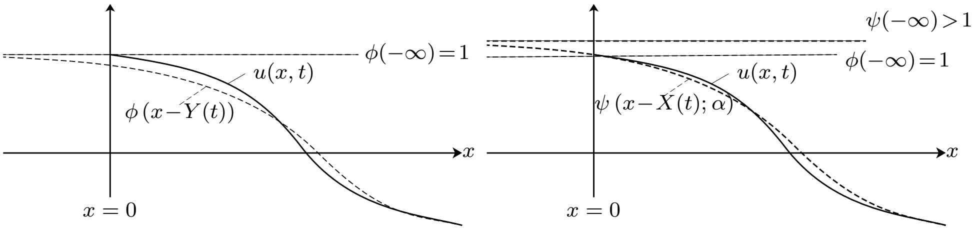

Figure 1. Approximations as a stationary and a travelling shock.

$\psi(-\infty)$ is set to be greater then 1. See Figure 1. Consequently, it results in a positive drifting velocity α of ψ, which is required by the Rankine–Hugoniot condition for Burgers shock waves: $\alpha = (\psi(-\infty) + \psi(\infty))/2$. The Burgers shock with velocity α and $\psi(\infty) = 1$ is:

$$\psi(\xi;\alpha) = \alpha - (\alpha+1)\tanh\left(\tfrac{1+\alpha}{2}\xi\right).$$

α can be expressed in terms of $X(t)$ by solving from the boundary condition (6) we imposed.

$$\alpha = \alpha(X(t)) = e^{-X(t)} + O(X(t)e^{-2X(t)}), \tag{7}$$

cf. (10). It is then natural to equate α with the true velocity $X'(t)$ of the shock. Consequently, we obtain a time evolution equation for $X(t)$:

$$X'(t) = e^{-X(t)} + \text{ higher order term}, \tag{8}$$

which properly describes the time asymptotic behavior of $X(t)$: $X(t) \approx X(0) + \log(1 + te^{-X(0)})$.

Unlike ϕ, ψ satisfies the boundary condition $\psi(x = 0) = 1$. Therefore, the drifting of the shock is now understood as the global effect of the conservation law, namely the Rankine–Hugoniot condition, coupling to the boundary. ψ approximates the solution u more naturally and closer than ϕ, and therefore is more suitable for our study of the propagation of the shock.

As we regard u as a perturbation of ψ, (4) becomes the natural definition of shock location $X(t)$. In order to compare our result with the previous one [3], we estimate $X(t) - Y(t)$ in Proposition 12, where a $\log t/t$ decay rate of $X(t) - Y(t)$ is obtained, close to the optimal rate $1/t$, cf. Remark 3.

Our goal is time asymptotic description of the solution, so the condition $L_0 := X(0) \gg 1$ is assumed throughout the analysis. The following are our main results: the first and the second main theorem describe time asymptotic behavior of u and $X(t)$ respectively.

Theorem 1 (First Main Theorem)**.** *Define* $\psi(x,t) = \psi(x-\alpha(X(t)); \alpha(X(t)))$. *Put* $X(0) = L_0$. *Suppose that for some* $1/2 < \beta < 3/2$, *and* $C > 0$,

$$|u(x,0) - \psi(x,0)| \leq \frac{Ce^{-\beta L_0}}{\cosh\left(\frac{x-L_0}{2}\right)}.$$

Then, for all L_0 *large enough, solution of* (3) *exists and satisfies*

$$\begin{aligned} u(x,t) - \psi(x,t) &= \frac{O(1)(C+1)e^{-\beta X(t)}}{\cosh\left(\frac{x-X(t)}{2}\right)} \\ u_x(x,t) - \psi_x(x,t) &= \frac{O(1)(C+1)e^{-\beta X(t)}}{\cosh\left(\frac{x-X(t)}{2}\right)}\left(1+\frac{1}{\sqrt{t}}\right). \end{aligned}$$

Theorem 2 (Second Main Theorem)**.**

$$\begin{aligned} X(t) &= X(0) + \log\left(1+e^{-X(0)}t\right) + \frac{O(1)e^{-X(0)}\left(t+e^{X(0)}\right)^{\left(\frac{3}{2}-\beta\right)}}{1+e^{-X(0)}t} \\ &= X(0) + \log\left(1+e^{-X(0)}t\right) + \frac{O(1)}{t^{\beta-\frac{1}{2}}}. \end{aligned} \tag{9}$$

Remark 3. ψ is not only suitable for qualitative explanation, but also for quantitative analysis. By Theorem 1, near the boundary ($x \ll 1$), the difference between u and ψ is of order $e^{-(\beta+1/2)X(t)}$, smaller order than the difference $1-\phi(-X(t)) \sim e^{-X(t)}$ between ϕ and u near the boundary. This confirms with our previous statement that ψ approximates ϕ closer. This fact also allows us to obtain the convergence rate of $X(t)$ to $X(0)+\log(1+te^{-X(0)})$.

The convergence rate of $X(t)$ to $\log(1+te^{-X(0)})$ cannot surpass $1/t$. To see this, consider any $0 < a < b$. By some direction computations,

$$a + \log(1+te^{-a}) - \left(b + \log(1+te^{-b})\right) \sim \frac{1-e^{a-b}}{1+t}.$$

$X(t)$ is not always equal to $\log(1+te^{-X(0)})$, and the induced error cannot decay at a faster rate than $1/t$. Consequently, our estimate of convergence rate is close to the optimal one.

2. Basic Framework

For a given shock location L, as stated before, the condition $\psi(-L;\alpha)=1$ is required. Namely:

$$\alpha(L)=\frac{1+\alpha(L)}{2}\tanh\left(\tfrac{1+\alpha(L)}{2}L\right). \tag{10}$$

Solving $\alpha(L)$ from (10),

$$\alpha(L)=e^{-L}(1+O(1)Le^{-L}), \tag{11a}$$
$$\alpha'(L)=O(1)e^{-2L}. \tag{11b}$$

Write u as a perturbation of ψ: $u(x,t)=\psi(x,t)+v(x,t)$. The equation of v follows from the equation (3) of u:

$$v_t+\left(\psi(x,t)v+\frac{v^2}{2}\right)_x=v_{xx} \tag{12a}$$
$$v(0,t)=0 \tag{12b}$$
$$v(x,0)\text{ given, and }\int_0^\infty v(x,0)dx=0. \tag{12c}$$

To estimate pointwisely, we adopt the Green's function approach. However, since $X(t)$ varies with time, the Green's function valid for all time is difficult to obtain. Therefore, the problem is temporally localized: we will choose some $T>0$, specified in the proof of Theorem 1, and slice the temporal coordinate into intervals of length T. Furthermore, we make the intervals overlap with their consecutive neighbors in a length of $T/2$. More explicitly,

$$I_n:=[T_n,T_{n+2}]:=\left[\frac{n}{2}T,\frac{n+2}{2}T\right].$$

Within each time interval I_n, u is not approximated exactly as $\psi(x,t)$, but as a travelling shock $\psi^{[n]}$ with *constant* velocity α_n. To be precise, let

$L_n = X(T_n)$, $\alpha(t) = \alpha(X(t))$, $\alpha_n = \alpha(T_n)$.

$$\psi^{[n]}(x - L_n - \alpha_n(t - T_n)) := \psi(x - L_n - \alpha_n(t - T_n); L_n)$$
$$u(x,t) = \psi^{[n]}(x - L_n - \alpha_n(t - T_n)) + v^{[n]}(x, t - T_n).$$

Consequently, $\psi^{[n]}$ satisfies boundary condition $\psi^{[n]}(x = 0) = 1$ only at $t = T_n$. There is an artificial boundary value discrepancy for $t > T_n$. This strategy allows us to obtain the Green's function, at the cost of causing artificial boundary value discrepancy. However, the discrepancy $\psi^{[n]}(x = 0) - 1$ is small (bounded by $O(1)e^{-2L_n}T$, much smaller than $1 - \phi(-L_n) \sim e^{-L_n}$), and turns out to be negligible.

Within I_n, the equation for $v^{[n]}$ is:

$$(v^{[n]})_t + \left(\psi^{[n]}(x - L_n - \alpha_n(t - T_n))\, v^{[n]} + \frac{(v^{[n]})^2}{2}\right)_x = (v^{[n]})_{xx} \tag{13a}$$

$$v^{[n]}(0, t - T_n) = a^{[n]}(t - T_n), \text{ where } a^{[n]}(t) := \psi^{[n]}(x - L_n - \alpha_n(t - T_n)) - 1 \tag{13b}$$

$$v^{[n]}(x, 0) \text{ given, and } \int_0^\infty v^{[n]}(x,t)dx = 0. \tag{13c}$$

It is convenient to consider the anti-derivative $w^{[n]}(x, t - T_n) = -\int_x^\infty v(y, t - T_n)dy$. After integrating (13), we have the following equation for $w^{[n]}$.

$$(w^{[n]})_t + \psi(x - L_n - \alpha_n(t - T_n))\, (w^{[n]})_x + \frac{{(w^{[n]})_x}^2}{2} = (w^{[n]})_{xx} \tag{14a}$$
$$(w^{[n]})_x(0, t - T_n) = a^{[n]}(t - T_n) \tag{14b}$$
$$(w^{[n]})(0,0) = 0. \tag{14c}$$

The IBV problem (13) will be solved within each I_n, and the estimate of both $v^{[n]}$ and $(v^{[n]})_x$ will be obtained. We estimate $(v^{[n]})_x$ because, as it turns out, $v_x(0,t)$ is crucial in locating the shock. Subsequently, the smallness of $v^{[n]}$ is passed to the next time interval for us to solve (13) for the next index $n + 1$.

At the beginning of each time interval, smoothness of the initial data $v^{[n]}(x, 0)$ is not presumed, and the derivative $(v^{[n]})_x$ is obtained by differentiating the Green's function. The resulting derivative $(v^{[n]})_x(x, t - T_n)$ blows up at a rate of $1/\sqrt{t - T_n}$ as $t \to T_n^+$, which is similar the heat equation.

Therefore, the intervals overlap is made: the $T/2$ amount of time is in place for $(v^{[n]})_x$ to settle down. Note that this blow-up is artificial. From the Green's function, we know v is smooth for any $t > 0$. However, differentiating the Green's function for $v^{[n]}$ is easier than finding the Green's function for $(v^{[n]})_x$, so the prescribed scheme is adopted at the cost of an artificial initial singularity.

In Section 3, we obtain an approximate form of Green's function within each time interval. Followed by, in Section 4, with this approximate form of Green's function, the IBV problem (13) within each time interval is solved. Lastly, in Section 5, we patch up results obtained in Section 4 to solve (12) completely and subsequently obtain the time asymptotic expression of $X(t)$. To simplify notation, the index n is omitted in Section 3 and Section 4. $\psi(s)$, $v(x,t)$, L, α, and t stand for $\psi^{[n]}(s)$, $v^{[n]}(x,t)$, L_n, α_n, and $t - T_n$, respectively.

3. An Approximate Form of the Green's Function

To simplify notation, the index n is omitted in this section: $\psi(\xi)$, $v(x,t)$, L, α, t, and $a(s)$ stand for $\psi^{[n]}(\xi)$, $v^{[n]}(x, t-T_n)$, L_n, α_n, $t-T_n$, and $a^{[n]}(s)$ respectively.

First, an approximate form of the Green's function—the Green's function of linearized equation of (14) — is derived:

$$w_t + \psi\left(x - L - \alpha t\right) w_x - w_{xx} = 0 \tag{15a}$$

$$w_x(0,t) = a(t) \tag{15b}$$

$$w(0,0) = 0. \tag{15c}$$

Although (15) has an inhomogeneous boundary condition, by Duhamel's principle, Theorem 4, it suffices to consider the Green's function for the homogeneous boundary condition:

$$G_t + \psi(x - L - \alpha t)G_x = G_{xx},\ 0 < s < t \tag{16a}$$

$$G_x(0,t;y,s) = 0 \tag{16b}$$

$$G(x,s;y,s) = \delta(x-y). \tag{16c}$$

The proof the following standard PDE theorem is done by simple integration by parts.

Theorem 4 (Duhamel's principle). *Let $b(x,t), S(x,t), d(t), D(t)$ be continuous functions. Suppose $\mathcal{G}(x,t;y,r)$ is the Green's function for the following IBV problem:*

$$f_t + b(x,t)f_x - f_{xx} = 0,\ \ x,t > 0 \tag{17a}$$
$$f(x,0)\ \text{given} \tag{17b}$$
$$f_x(0,t) + d(t)f(0,t) = 0\ \text{for}\ t > 0. \tag{17c}$$

If h is the solution of the following inhomogeneous form of (17):

$$\begin{aligned} &h_t + b(x,t)h_x - h_{xx} = S(x,t),\ \ x,t > 0 \\ &h(x,0)\ \text{given} \\ &h_x(0,t) + d(t)h(0,t) = D(t)\ \text{for}\ t > 0, \end{aligned}$$

then h can be solved by $\mathcal{G}$ as following:

$$\begin{aligned} h(x,t) \ &= \ \int_0^\infty \mathcal{G}(x,t;y,0)h(y,0)dy - \int_0^t D(r)\mathcal{G}(x,t;0,r)dr \\ &+ \int_0^t \int_0^\infty \mathcal{G}(x,t;y,r)S(y,r)dydr. \end{aligned}$$

To simplify notation, let us set

$$F(x,t;y,s) = e^{\frac{\alpha(x-y)}{2} - \frac{((1+\alpha)^2+\alpha^2)(t-s)}{4}} \frac{\cosh\left(\frac{(1+\alpha)}{2}(y - L - \alpha s)\right)}{\cosh\left(\frac{(1+\alpha)}{2}(x - L - \alpha t)\right)}.$$

Theorem 5. *For any given $T > 0$ and any $0 \le s < t \le T$, provided that L is large enough, the Green's function $G(x,t;y,s)$ exists, and satisfies*

$$\begin{aligned} G(x,t;y,s) = F(x,t;y,s) \times \Bigg(& k(x-y,t-s) + k(x+y,t-s) \\ & + \int_y^\infty e^{\frac{1+a(s)}{2}(z-y)} k(x+z,t-s)dz + \mathcal{E}(x,t;y,s)e^{-\frac{y}{2}} \Bigg), \end{aligned}$$

where $k(x,t) = e^{-\frac{x^2}{4t}}/\sqrt{4\pi t}$ *is the heat kernel, and the residual term* $\mathcal{E}$ *is bounded by* $O(1)(T+1)^2e^Te^{-2L}$.

Proof. Consider the following IBV problem:

$$\overline{w}_t + \psi(x-L-\alpha t)\overline{w}_x = \overline{w}_{xx},\ t>s \tag{19a}$$

$$\overline{w}(x,s) \text{ given} \tag{19b}$$

$$\overline{w}_x(0,t) = 0,\ t>s. \tag{19c}$$

We are going to transform (19) into the heat equation. Observe that $B(x,t) = \cosh\left(\frac{(1+\alpha)}{2}(x-L-\alpha t)\right)e^{\frac{(1+\alpha)^2}{4}t}$ satisfies $B_t+\alpha B_x = B_{xx}$. By multiplying (19) with $B(x,t)$,

$$(\overline{w}B)_t + \alpha(\overline{w}B)_x = (\overline{w}B)_{xx}.$$

Put $W = e^{-\frac{\alpha x}{2}+\frac{\alpha^2 t}{4}}\overline{w}B$. W then satisfies

$$W_t = W_{xx} \tag{20a}$$

$$W(x,s) \text{ given} \tag{20b}$$

$$W_x(0,t) + \frac{1+a(s)}{2}W(0,t) = -\frac{a(t)-a(s)}{2}W(0,t) \text{ for } t>s. \tag{20c}$$

We have transformed equation (19) of $\overline{w}$ into the heat equation (20) of W. This is a generalization of linearized Hopf-Cole transform, cf. [4]. Note that $G = FH$, where $H(x,t;y,s)$ is the Green's function for (20). Hence our goal is to find H. The boundary condition of (20) involves a linear combination of W and W_x, with a variable coefficient, $-(a(t)-a(s))/2$. In general, this IBV problem has no explicit expression for solution. However, with $a(t)-a(s) = O(1)e^{-2L}(t-s)$ being small, H can be solved by iteration. To be more precise, let $K(x,t;y,r;s)$ be the Green's function for the following IBV problem:

$$\tilde{W}_t = \tilde{W}_{xx},\ s\le r<t \tag{21a}$$

$$\tilde{W}(x,r) \text{ given} \tag{21b}$$

$$\tilde{W}_x(0,t) + \frac{1+a(s)}{2}\tilde{W}(0,t) = 0. \tag{21c}$$

K can be solve explicitly as following, [5]:

$$\begin{aligned} K(x,t;y,r;s) &= k(x-y,t-r)+k(x+y,t-r) \\ &\quad + (1+a(s))\int_y^\infty e^{\frac{1+a(s)}{2}(z-y)}k(x+z,t-r)dz \qquad (22a) \\ &= k(x-y,t-r)-k(x+y,t-r) \\ &\quad -2\int_y^\infty e^{\frac{1+a(s)}{2}(z-y)}k_z(x+z,t-r)dz, \qquad (22b) \end{aligned}$$

where $s \le r < t$. By Theorem 4, (20) is transformed to the following integral equation:

$$W(x,t) = W^{(0)}(x,t) + \int_s^t \frac{a(r)-a(s)}{2} W(0,r)K(x,t;0,r;s)dr. \qquad (23)$$

The zeroth order part $W^{(0)}$ of W can be expressed as

$$W^{(0)}(x,t) = \int_0^\infty K(x,t;y,s;s)W(y,s)dy := \int_0^\infty H^{(0)}(x,t;y,s)W(y,s)dy. \qquad (24)$$

Subsequently, we solve (23) by iterations.

$$\begin{aligned} W^{(0)}(x,y) &= \int_0^\infty H^{(0)}(x,t;y,s)W(y,s)dy, \\ W^{(m)}(x,t) &= \int_s^t \frac{a(r)-a(s)}{2} W^{(m-1)}(0,r)K(x,t;0,r;s)dr, \text{ for } m>0. \qquad (25) \end{aligned}$$

The series $\sum W^{(m)}$ solves (23) provided that it converges uniformly.

The corresponding m-th order Green's function $H^{(m)}$ is defined by

$$\begin{aligned} W^{(m)}(x,t) &= \int_0^\infty H^{(m)}(x,t;y,s)W(y,s)dy, \\ H^{(m)}(x,t;y,s) &= \int_s^t \frac{a(r)-a(s)}{2} H^{(m-1)}(0,r;y,s)K(x,t;0,r;s)dr. \qquad (26) \end{aligned}$$

The uniform convergence of $\sum W^{(m)}$ follows from the following claim, which will be proved by induction.

Claim: For all $m > 0$

$$H^{(m)}(x,t;y,s) = e^T\left(O(1)(T+1)^2 e^{-2L}\right)^m e^{-\frac{y}{2}}. \qquad (27)$$

By (22b) and (24),

$$
\begin{aligned}
\left|H^{(0)}(0,r;y,s)\right| \le & 2\int_y^{\infty} e^{\frac{1+a(s)}{2}(z-y)}\left|k_z(z,r-s)\right|dz \\
=&O(1)e^{\frac{-(1+a(s))}{2}y+\frac{(1+a(s))^2}{4}(r-s)} \\
&\times\int_{-\infty}^{\infty}\left(\frac{|z-(1+a(s))(r-s)|}{r-s}+1+a(s)\right)\frac{e^{-\frac{(z-(1+a(s))(r-s))^2}{4(r-s)}}}{\sqrt{r-s}}dz \\
=&O(1)e^{\frac{T}{2}}e^{\frac{-y}{2}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{r-s}}+1\right)\frac{e^{-\frac{(z-(1+a(s))(r-s))^2}{8(r-s)}}}{\sqrt{r-s}}dz \\
=&O(1)e^{\frac{T}{2}}\left(\frac{1}{\sqrt{r-s}}+1\right)e^{-\frac{y}{2}}. \qquad (28)
\end{aligned}
$$

Note that $a(s)=O(1)Te^{-2L}<1$, provided that L is large enough. Similarly, by (22a),

$$
K(x,t;0,r;s)=O(1)\left(\frac{1}{\sqrt{t-r}}+e^{\frac{T}{2}}e^{-\frac{x}{2}}\right)=O(1)e^{\frac{T}{2}}\left(\frac{1}{\sqrt{t-r}}+1\right). \qquad (29)
$$

Plugging (28) and (29) into (26),

$$
H^{(1)}(x,t;y,s)=O(1)e^{T}(T+1)^2e^{-2L}e^{-\frac{y}{2}}.
$$

This proves (27) for $m=1$. Since the right hand side of (27) does not involve t, it is clear by induction that (27) holds for all m. This proves the claim. Consequently, $\sum W^{(m)}$ solves (23), and therefore for all L large enough $H=\sum H^{(m)}$. By (27), the residual term $\sum_1^{\infty} H^{(m)}$ is bounded by $O(1)e^{T}(T+1)^2e^{-2L}e^{-y/2}$. □

G is the Green's function for w. To estimate v and v_x, G_x and G_{xx} are needed. We first estimate $\mathcal{E}_x$ and $\mathcal{E}_{xx}$:

Lemma 6. *Under the same requirement of Theorem* 5, $\mathcal{E}_x,\mathcal{E}_{xx}=O(1)e^{\frac{3T}{2}}(T+1)^2e^{-2L}$.

Proof. Differentiate the recursion relation (26) to obtain

$$
(H^{(m)})_x(x,t;y,s)=\int_s^t\frac{a(r)-a(s)}{2}H^{(m-1)}(0,r;y,s)K_x(x,t;0,r;s)dr
$$

$$
\begin{aligned}
&= -\int_s^t \frac{a(r)-a(s)}{2} H^{(m-1)}(0,r;y,s)\Big((1+a(s))k(x,t-r) \\
&\quad + \frac{(1+a(s))^2}{2}\int_y^\infty e^{\frac{1+a(s)}{2}(z-y)}k(x+z,t-r)dz\Big)dr \\
&\quad - \frac{1}{\sqrt{4\pi}}\int_{\frac{x}{\sqrt{t-s}}}^\infty \left(a\left(t-\tfrac{x^2}{\sigma^2}\right)-a(s)\right) H^{(m-1)}\left(0,t-\tfrac{x^2}{\sigma^2};y,s\right)e^{-\frac{\sigma^2}{4}}d\sigma \qquad (30)\\
&= \int_s^t (a(r)-a(s))\, H^{(m-1)}(0,r;y,s)\left(O(1)\frac{1}{\sqrt{r-s}}+O(1)e^{\frac{T}{2}}e^{-\frac{y}{2}}\right)dr \\
&\quad - \frac{1}{\sqrt{4\pi}}\int_{\frac{x}{\sqrt{t-s}}}^\infty \left(a\left(t-\tfrac{x^2}{\sigma^2}\right)-a(s)\right) H^{(m-1)}\left(0,t-\tfrac{x^2}{\sigma^2};y,s\right)e^{-\frac{\sigma^2}{4}}d\sigma. \qquad (31)
\end{aligned}
$$

Since $a(r)-a(s)=O(1)e^{-2L}(r-s)$, by (27) and (28),

$$
(a(r)-a(s))\, H^{(m-1)}(0,r;y,s) = e^T(T+1)e^{-2L}\left(O(1)(T+1)^2e^{-2L}\right)^{m-1}e^{-\frac{y}{2}}. \qquad (32)
$$

Plugging (32) into (31),

$$
\begin{aligned}
\mathcal{E}_x e^{-\frac{y}{s}} &= \left(\sum_{m=1}^\infty H^{(m)}(x,t;y,s)\right)_x = \sum_{m=1}^\infty (H^{(m)})_x(x,t;y,s) \\
&= O(1)e^{\frac{3T}{2}}(T+1)^2e^{-2L}e^{-\frac{y}{2}},
\end{aligned}
$$

provided that L is large enough.

Next, with the aid of(27) and (28), after differentiating (30),

$$
\begin{aligned}
&(H^{(m)})_{xx}(x,t;y,s) \\
&= (1+T)e^{\frac{3T}{2}}\left(O(1)(1+T)^2e^{-2L}\right)^m e^{-\frac{y}{2}} \\
&\quad +O(1)\int_{\frac{x}{\sqrt{t-s}}}^\infty \left(a\left(t-\tfrac{x^2}{\sigma^2}\right)-a(s)\right)\frac{x}{\sigma^2}(H^{(m-1)})_{xx}\left(0,t-\tfrac{x^2}{\sigma^2};y,s\right)e^{-\frac{\sigma^2}{4}}d\sigma. \qquad (33)
\end{aligned}
$$

From (22b) and (24),

$$
\begin{aligned}
(H^{(0)})_{xx}(0,r;y,s) &= \int_y^\infty e^{\frac{1+a(s)}{2}(z-y)}k_{zr}(z,r-s)dz \\
&= O(1)e^{\frac{T}{2}}\left(\frac{1}{(r-s)^{\frac{3}{2}}}+1\right)e^{-\frac{y}{2}}. \qquad (34)
\end{aligned}
$$

Plugging (34) into (33),

$$
\begin{aligned}
&H_{xx}^{(1)}(x,t;y,s) = e^{\frac{3T}{2}}\left(O(1)(1+T)^2e^{-2L}\right)e^{-\frac{y}{2}}\\
&+O(1)e^{\frac{T}{2}}e^{-2L}e^{-\frac{y}{2}}\int_{\frac{x}{\sqrt{t-s}}}^{\infty}\left(t-s-\frac{x^2}{\sigma}\right)\frac{x}{\sigma^2}\left(\frac{1}{\left(t-s-\frac{x^2}{\sigma}\right)^{\frac{3}{2}}}+1\right)e^{-\frac{\sigma^2}{4}}d\sigma\\
=&e^{\frac{3T}{2}}\left(O(1)(1+T)^2e^{-2L}\right)e^{-\frac{y}{2}}\\
&+O(1)e^{\frac{T}{2}}e^{-2L}e^{-\frac{y}{2}}\int_0^{t-s}\left(\frac{1}{\sqrt{t-s-\eta}}+(t-s-\eta)\right)e^{-\frac{x^2}{4\eta}}d\eta\\
=&e^{\frac{3T}{2}}\left(O(1)(1+T)^2e^{-2L}\right)e^{-\frac{y}{2}}.
\end{aligned}
$$

Now, it is clear by induction and by (33) that:

$$H_{xx}^{(m)}(x,t;y,s) = (1+T)e^{\frac{3T}{2}}\left(O(1)(1+T)^2e^{-2L}\right)^m e^{-\frac{y}{2}}\ (m>0),$$

which implies $\mathcal{E}_{xx} = O(1)e^{\frac{3T}{2}}(T+1)^2e^{-2L}$. □

With Lemma 6, the following approximate form of G_x and G_{xx} is obtained:

Corollary 7. *Under the same requirement of Theorem* 5,

$$
\begin{aligned}
G_x(x,t;y,s) = &\frac{e^{\frac{1}{2}(y-L-\alpha s)}+e^{-\frac{1+2\alpha}{2}(y-L-\alpha s)}}{e^{\frac{1}{2}(x-L-\alpha t)}+e^{-\frac{1+2\alpha}{2}(x-L-\alpha t)}}\left[O(1)\left(e^{-\frac{(1+2\alpha)(t-s)}{4}}+e^{-\frac{L}{2}}e^{-\frac{y}{2}}\right)\right.\\
&\left.+e^{-\frac{(1+2\alpha)(t-s)}{4}}\Big(k_x(x-y,t-s)+k_x(x+y,t-s)\Big)\right], \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
G_{xx}(x,t;y,s) = &\frac{e^{\frac{1}{2}(y-L-\alpha s)}+e^{-\frac{1+2\alpha}{2}(y-L-\alpha s)}}{e^{\frac{1}{2}(x-L-\alpha t)}+e^{-\frac{1+2\alpha}{2}(x-L-\alpha t)}}\\
&\times\left[O(1)\Big(e^{-\frac{(1+2\alpha)(t-s)}{4}}(k(x-y,t-s)\right.\\
&+|k_x(x-y,t-s)|+|k_x(x+y,t-s)|)+e^{-\frac{L}{2}}e^{-\frac{y}{2}}\Big)\\
&\left.+e^{-\frac{(1+2\alpha)(t-s)}{4}}\left(k_{xx}(x-y,t-s)+k_{xx}(x+y,t-s)\right)\right]. \qquad (36)
\end{aligned}
$$

Proof. By direct differentiation and Lemma 6,

$$\begin{aligned} G_x &= (FH^{(0)} + F\mathcal{E})_x = (FH^{(0)})_x + O(1)F \times (\mathcal{E} + \mathcal{E}_x) \\ &= (FH^{(0)})_x + O(1)F \times (e^{-\frac{L}{2}} e^{-\frac{y}{2}}) \\ G_{xx} &= (FH^{(0)})_{xx} + O(1)F \times (\mathcal{E} + \mathcal{E}_x + \mathcal{E}_{xx}) \\ &= (FH^{(0)})_{xx} + O(1)F \times (e^{-\frac{L}{2}} e^{-\frac{y}{2}}), \end{aligned}$$

provided that L is large enough. Hence it remains only to estimate $(FH^{(0)})_x$ and $(FH^{(0)})_{xx}$. We first establish two inequalities. Assume that L is so large that $e^{\alpha T}, e^{a(t)T}, e^{a(t)^2 T} < 1/2$:

$$1 + \tanh\left(\tfrac{1+\alpha}{2}(x - L - \alpha t)\right) = O(1)\frac{1}{1 + e^{(1+\alpha)(L-x)}} = O(1)e^{-\frac{L}{2}} e^{\frac{x}{2}}, \quad (37)$$

$$\begin{aligned} &\int_y^\infty e^{\frac{1+a(s)}{2}(z-y)} k(x+z, t-s) dz \\ &= e^{\frac{(1+a(s))^2}{4}(t-s)} e^{-\frac{(1+a(s))}{2}(x+y)} \int_y^\infty k(x + z - (1+a(s))(t-s), t-s) dz \\ &= O(1)e^{\frac{t-s}{4}} e^{-\frac{x+y}{2}}. \end{aligned} \quad (38)$$

Differentiate $(FH^{(0)})_x$ to obtain:

$$\begin{aligned} \frac{\left(FH^{(0)}\right)_x}{F} &= \left[\frac{\alpha}{2} - \frac{1+\alpha}{2} \tanh\left(\tfrac{1+\alpha}{2}(x - L - \alpha t)\right)\right] \times H^{(0)}(x, t; y, s) \\ &\quad + \Bigg[k_x(x-y, t-s) + k_x(x+y, t-s) - (1+a(s))k(x+y, t-s) \\ &\quad - \frac{(1+a(s))^2}{2} \int_y^\infty e^{\frac{1+a(s)}{2}(z-y)} k(x+z, t-s) dz \Bigg]. \end{aligned}$$

For convenience, those functions whose values and first order derivatives in x are bounded are denoted with $\mathcal{O}(x,t)$. From the above we have

$$\begin{aligned} &\frac{\left(FH^{(0)}\right)_x}{F} \\ &= \mathcal{O}(x,t)k(x-y, t-s) + \mathcal{O}(x,t)k(x+y, t-s) \\ &\quad + k_x(x-y, t-s) + k_x(x+y, t-s) \end{aligned}$$

$$+\Big[\mathcal{O}(x,t)\alpha-\Big(a(s)+\frac{a(s)^2}{2}\Big)-\frac{1}{2}\left(1+\tanh\left(\tfrac{1+\alpha}{2}(x-L-\alpha t)\right)\right)\Big]$$
$$\times\int_y^\infty e^{\frac{1+a(s)}{2}(z-y)}k(x+z,t-s)dz. \tag{39}$$

With the aid of (37) and (38), (39) yields

$$\begin{aligned}\frac{\left(FH^{(0)}\right)_x}{F} &= O(1)k(x-y,t-s)+O(1)k(x+y,t-s)+k_x(x-y,t-s)\\ &\quad +k_x(x+y,t-s)+O(1)e^{\frac{t-s}{4}}e^{-\frac{L}{2}}e^{-\frac{y}{2}},\end{aligned}$$

which proves (35). Note that $0<k(x+y,t-s)\le k(x-y,t-s)$ and that

$$F(x,t;y,s)=\frac{e^{\frac{1}{2}(y-L-\alpha s)}+e^{-\frac{1+2\alpha}{2}(y-L-\alpha s)}}{e^{\frac{1}{2}(x-L-\alpha t)}+e^{-\frac{1+2\alpha}{2}(x-L-\alpha t)}}\times e^{-\frac{1+2\alpha}{4}(t-s)}.$$

After differentiating (39), $(FH^{(0)})_{xx}$ can be treated similarly. The details are omitted here for simplicity. □

4. Estimates of The Solution and Its Spacial Derivative

In this section we solve (13). To simplify notation, the index n is omitted in this section: $\psi(\xi)$, $v(x,t)$, L, α, t, and $a(s)$ stand for $\psi^{[n]}(\xi)$, $v^{[n]}(x,t-T_n)$, L_n, α_n, $t-T_n$, and $a^{[n]}(s)$ respectively.

By Theorem 4, (14) can be transformed into the following integral equation:

$$\begin{aligned}w(x,t) &= \int_0^\infty G(x,t;y,0)v(y,0)dyds-\int_0^t a(s)G(x,t;0,s)ds\\ &\quad -\int_0^t\int_0^\infty G(x,t;y,s)\frac{1}{2}w_y(y,s)^2dyds.\end{aligned} \tag{40}$$

Differentiate (40) to yield

$$\begin{aligned}v(x,t) &= \int_0^\infty G_x(x,t;y,0)w(y,0)dy-\int_0^t a(s)G_x(x,t;0,s)ds\\ &\quad -\int_0^t\int_0^\infty \frac{v(y,s)^2}{2}G_x(x,t;y,s)dyds.\end{aligned} \tag{41}$$

Theorem 8. *Suppose that for some* $1/2 < \beta < 3/2$*, and* $C > 0$*, the initial value satisfies:*

$$|v(x,0)| \leq \frac{Ce^{-\beta L}}{\cosh\left(\frac{x-L}{2}\right)}.$$

Then, fix any $T > 0$*, for all* L *large enough, the solution of* (13) *exists in* $[0,T]$ *and satisfies*

$$v(x,t) = O(1)(T+1)(1+C)e^{-\beta L}\frac{e^{-\frac{t}{4}} + e^{-\left(\frac{3}{2}-\beta\right)L} + e^{-\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}. \tag{42}$$

Proof. We will solve (13) by Picard iteration through (41). The terms containing initial and boundary value are treated as the first order part:

$$v^{(1)}(x,t) = \int_0^\infty G_x(x,t;y,0)w(y,0)dy - \int_0^t a(s)G_x(x,t;0,s)ds := I_1 + I_2$$

Note that, since $\int_0^\infty v(x,0) = 0$,

$$w(x,0) = \frac{O(1)Ce^{-\beta L}}{\cosh\left(\frac{x-L}{2}\right)}.$$

With the help of (35),

$$\begin{aligned} I_1 &= \frac{O(1)Ce^{-\beta L}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-\frac{t}{4}} + e^{-\frac{L}{2}}\right) - e^{-\frac{1+2\alpha}{4}t}\int_0^\infty (k(x-y,t) \\ &\quad -k(x+y,t))\left(\frac{e^{\frac{1}{2}(y-L)} + e^{-\frac{1+2\alpha}{2}(y-L)}}{e^{\frac{1}{2}(x-L-\alpha t)} + e^{-\frac{1+2\alpha}{2}(x-L-\alpha t)}}w(y,0)\right)_y dy \\ &= \frac{O(1)Ce^{-\beta L}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-\frac{t}{4}} + e^{-\frac{L}{2}}\right), \end{aligned} \tag{43}$$

$$\begin{aligned} I_2 &= O(1)\int_0^t \frac{a(s)e^{\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-\frac{t-s}{4}}\frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{t-s}} + e^{-\frac{L}{2}} + e^{-\frac{t-s}{4}}\frac{x}{(t-s)^{\frac{3}{2}}}e^{-\frac{x^2}{4(t-s)}}\right)ds \\ &= \frac{O(1)e^{-\frac{3L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}(1+e^{-\frac{L}{2}}t) = \frac{O(1)e^{-\frac{3L}{2}}}{\cosh\left(\frac{x-L}{2}\right)} = \frac{O(1)e^{-\beta L}}{\cosh\left(\frac{x-L}{2}\right)}e^{-(\frac{3}{2}-\beta)L}. \end{aligned} \tag{44}$$

Hence

$$v^{(1)}(x,t)=O(1)(1+C)e^{-\beta L}N(x,t), \text{ where } N(x,t):=\frac{e^{-\frac{t}{4}}+e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}.$$

Next define higher order parts $v^{(m)}$ recursively by

$$v^{(m+1)}(x,t)=-\int_0^t\int_0^\infty \frac{1}{2}v^{(m)}(y,s)^2G_x(x,t;y,s)dyds. \tag{45}$$

For any $0\le t\le T$, the following estimate (46) with be proved by induction.

$$v^{(m)}(x,t)=\Big(O(1)(1+C)(T+1)e^{-\beta L}\Big)^m(1+C)e^{-\beta L}N(x,t). \tag{46}$$

First

$$\begin{aligned}
&\int_0^t\int_0^\infty G_x(x,t;y,s)N(y,s)dyds\\
=&\frac{O(1)}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\Bigg(\int_0^t\int_0^\infty e^{-\frac{t}{4}}(k(x-y,t-s)+k(x+y,t-s))dyds\\
&+\left(e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)\int_0^t\int_0^\infty k(x-y,t-s)+k(x+y,t-s)dyds\\
&+e^{-\frac{L}{2}}\int_0^t\int_0^\infty e^{-\frac{y}{2}}\left(e^{-\frac{s}{4}}+e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)dyds\\
&+\int_0^t\int_0^\infty e^{-\frac{t}{4}}\left(|k_x(x-y,t-s)|+|k_x(x+y,t-s)|\right)dyds\\
&+\left(e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)\int_0^t\int_0^\infty |k_x(x-y,t-s)|+|k_x(x+y,t-s)|dyds\Bigg)\\
=&\frac{O(1)(T+1)}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-\frac{t}{4}}+e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)=O(1)(T+1)N(x,t).
\end{aligned}\tag{47}$$

Consequently,

$$\begin{aligned}
v^{(2)}(x,t)&=(O(1)(1+C)e^{-\beta L})^2O(1)(T+1)N(x,t)\\
&=(O(1)(1+C)e^{-\beta L})^2N(x,t).
\end{aligned}\tag{48}$$

If (46) holds for $m = k > 1$:

$$\begin{aligned} v^{(k)}(y,s)^2 &= (O(1)(T+1)(1+C)e^{-\beta L})^{2k} O(1) N(y,s) \\ &= (O(1)(T+1)(1+C)e^{-\beta L})^{k+1} O(1)(T+1)(1+C)e^{-\beta L} N(y,s), \end{aligned}$$

then, by similar arguments,

$$\begin{aligned} v^{(k+1)}(x,t) &= (O(1)(T+1)(1+C)e^{-\beta L})^{k+1} O(1)(T+1)^2(1+C)e^{-\beta L} N(x,t) \\ &= (O(1)(T+1)(1+C)e^{-\beta L})^{k+1} e^{-\beta L} N(x,t). \end{aligned}$$

This completes the proof of (46).

Hence, by (46), provided that L is large enough, the solution v exists in $0 \le t \le T$ and satisfies

$$v(x,t) = \sum_{m=0}^{\infty} v^{(m)}(x,t) = O(1)(T+1)(1+C)e^{-\beta L} \frac{e^{-\frac{t}{4}} + e^{-\left(\frac{3}{2}-\beta\right)L} + e^{-\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}. \tag{49}$$

□

Next we estimate v_x.

Theorem 9. *Under the requirement of Theorem 8, for all $0 < t \le T$,*

$$v_x(x,t) = \left(\frac{1}{\sqrt{t}} + 1\right) \frac{O(1)(1+C)(T+1)e^{-\beta L}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}. \tag{50}$$

In particular,

$$v_x(0,t) = \left(\frac{1}{\sqrt{t}} + 1\right) O(1)(1+C)(T+1)e^{-\left(\beta+\frac{1}{2}\right)L}. \tag{51}$$

Proof. We prove the following estimates:

$$(v^{(1)})_x(x,t) = \left(\frac{1}{\sqrt{t}} + 1\right) \frac{O(1)(1+C)(T+1)e^{-\beta L}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)} \tag{52a}$$

$$(v^{(m)})_x(x,t) = \frac{\left(O(1)(1+C)(T+1)e^{-\beta L}\right)^m}{\cosh\left(\frac{x-L-\alpha t}{2}\right)} \text{ for } m > 1, \tag{52b}$$

which imply (50), provided that L is large enough.

First, differentiate (45) to obtain:

$$(v^{(m+1)})_x(0,t) = -\frac{1}{2}\int_0^t\int_0^\infty G_{xx}(0,t;y,s)v^{(m)}(y,s)^2dyds. \tag{53}$$

With the help of (36) and (46), the right hand side of (53) can be transformed into

$$\begin{aligned}
&\frac{\big(O(1)(T+1)(C+1)e^{-\beta L}\big)^{2m}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\\
&\quad\times\int_0^t\int_0^\infty\Bigg(e^{-\frac{t-s}{4}}\frac{1}{\sqrt{t-s}}e^{-\frac{(x-y)^2}{4(t-s)}}+e^{-\frac{L}{2}}e^{-\frac{y}{2}}+\sum_{\pm}e^{-\frac{t-s}{4}}\frac{|x\pm y|}{(t-s)^{\frac{3}{2}}}e^{-\frac{(x\pm y)^2}{4(t-s)}}\Bigg)\\
&\quad\times\left(e^{-\frac{s}{4}}+e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)dyds\\
&\quad+\frac{O(1)}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\frac{\partial}{\partial x}\int_0^\infty\int_0^t(k_x(x-y,t-s)+k_x(x+y,t-s))\\
&\quad\times\left(e^{\frac{1}{2}(y-L-\alpha s)}+e^{-\frac{1+2\alpha}{2}(y-L-\alpha s)}\right)v^{(m)}(y,s)^2dyds\\
=&\ \frac{\big(O(1)(T+1)(C+1)e^{-\beta L}\big)^{m+1}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}(1+t)\left(e^{-\frac{t}{4}}+e^{-(\frac{3}{2}-\beta)L}+e^{-\frac{L}{2}}\right)\\
&\quad+\frac{\big(O(1)(T+1)(C+1)e^{-\beta L}\big)^{m}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\\
&\quad\times\int_0^\infty\int_0^t\Big(|k_x(x-y,t-s)|+|k_x(x+y,t-s)|\Big)\left|(v^{(m)})_y(y,s)\right|dyds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(v^{(m+1)})_x(x,t) =&\ \frac{\big(O(1)(T+1)(C+1)e^{-\beta L}\big)^{m}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\Bigg((T+1)(C+1)e^{-\beta L}\\
&+\int_0^\infty\int_0^t\Big(|k_x(x-y,t-s)|+|k_x(x+y,t-s)|\Big)\left|(v^{(m)})_y(y,s)\right|dyds\Bigg).
\end{aligned}\tag{54}$$

Put

$$(v^{(1)})_x(0,t) = \int_0^\infty G_{xx}(0,t;y,0)w(y,0)dy - \int_0^t G_{xx}(x,t;0,s)a(s)ds := J_1+J_2.$$

With the aid of (36), estimate of J_1 follows from routine computations:

$$J_1 = \left(1+\frac{1}{\sqrt{t}}\right)\frac{O(1)Ce^{-\beta L}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-\frac{t}{4}} + e^{-\left(\frac{3}{2}-\beta\right)L} + e^{-\frac{L}{2}}\right).$$

For J_2,

$$\begin{aligned}
J_2 &= O(1)\frac{e^{\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\Bigg[\int_0^t a(s)\Bigg(e^{-\frac{t-s}{4}}\frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{t-s}} + e^{-\frac{L}{2}} \\
&\quad + e^{-\frac{t-s}{4}}\frac{x}{(t-s)^{\frac{3}{2}}}e^{-\frac{x^2}{4(t-s)}}\Bigg)ds + \frac{\partial}{\partial x}\left(\int_0^t a(s)k_x(x,t-s)ds\right)\Bigg] \\
&= O(1)\frac{e^{\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left[e^{-2L}(1+e^{-\frac{L}{2}}t) + \left(\int_{\frac{x}{\sqrt{t}}}^{\infty}\frac{x}{\sigma^2}e^{-\frac{\sigma^2}{4}}a'\left(t-\frac{x^2}{\sigma^2}\right)d\sigma\right)\right] \\
&= O(1)\frac{e^{\frac{L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}\left(e^{-2L}(1+e^{-\frac{L}{2}}t) + e^{-2L}\right) = O(1)\frac{e^{\frac{L}{2}}e^{-\frac{3L}{2}}}{\cosh\left(\frac{x-L-\alpha t}{2}\right)}.
\end{aligned}$$

Hence (52a) follows.

For $m > 1$, it is clear by induction and by (54) that (52b) follows. □

5. Global Solution and Propagation of the Shock

In this section we patch up results from previous sections to solve the equation (12) completely. Suppose that (13) has been solved by Theorem 8 within the time interval $I_n = [T_n, T_{n+2}]$. In order to apply Theorem 8 again, we have to estimate how much the center of $\psi^{[n]}$ has deviated from the true shock location $X(t)$.

Lemma 10. *Assume that* (51) *holds. Namely assume that*

$$(v^{[n]})_x(0, t-T_n) = \left(\frac{1}{\sqrt{t-T_n}}+1\right)O(1)(1+C)(T+1)e^{-\left(\beta+\frac{1}{2}\right)L_n}. \quad (55)$$

With large enough L_n,

$$X(t) - L_n - \alpha_n(t-T_n) = (T+1)^2(1+C)e^{-\left(\frac{1}{2}+\beta\right)L_n}, \text{ for all } T_n \le t \le T_{n+2}. \quad (56)$$

Proof. From the definition (5) of shock location,

$$\begin{aligned} 0 &= \int_0^\infty u(x,t)-\psi(x,t)dx \\ &= \int_0^\infty \Big(\psi^{[n]}(x,t-T_n)-\psi(x,t)\Big)dx+\int_0^\infty v^{[n]}(x,t-T_n)dx \end{aligned} \tag{57}$$

First, by (55),

$$\begin{aligned} &\int_0^\infty (v^{[n]})_t(x,t-T_n)dx = \left[-\left(\psi^{[n]}v^{[n]}+\frac{(v^{[n]})^2}{2}\right)+(v^{[n]})_x\right]\Bigg|_{(0,t-T_n)} \\ &= O(1)\left[e^{-2L_n}(t-T_n)+(T+1)(1+C)\left(1+\frac{1}{\sqrt{t-T_n}}\right)e^{-\left(\frac{1}{2}+\beta\right)L_n}\right] \\ &= O(1)(T+1)(1+C)\left(1+\frac{1}{\sqrt{t-T_n}}\right)e^{-\left(\frac{1}{2}+\beta\right)L_n}. \end{aligned} \tag{58}$$

Consequently

$$\begin{aligned} \int_0^\infty v^{[n]}(x,t-T_n)dx &= \int_{T_n}^t O(1)(T+1)(1+C)\left(1+\frac{1}{\sqrt{s-T_n}}\right)e^{-\left(\frac{1}{2}+\beta\right)L_n}ds \\ &= O(1)(T+1)^2(1+C)e^{-\left(\frac{1}{2}+\beta\right)L_n}. \end{aligned} \tag{59}$$

Next, by some direct computations,

$$\begin{aligned} &\int_0^\infty \Bigg[\alpha_n-(\alpha_n+1)\tanh\left(\tfrac{\alpha_n+1}{2}(x-L_n-\alpha_n(t-T_n))\right) \\ &\qquad -\Big(\alpha_n-(\alpha_n+1)\tanh\left(\tfrac{\alpha_n+1}{2}(x-X(t))\right)\Bigg]dx \\ &\sim (L_n+\alpha_n(t-T_n))-X(t), \end{aligned} \tag{60}$$

and

$$\begin{aligned} &\int_0^\infty \Bigg[\alpha_n-(\alpha_n+1)\tanh\left(\tfrac{\alpha_n+1}{2}(x-X(t))\right) \\ &\qquad -\Big(\alpha(t)-(\alpha(t)+1)\tanh\Big(\tfrac{\alpha(t)+1}{2}(x-X(t))\Big)\Big)\Bigg]dx \\ &= O(1)\left(\alpha_n-\alpha(t)\right)(1+X(t)). \end{aligned} \tag{61}$$

Plugging (59), (60), and (61) into (57),

$$\begin{aligned}
&(L_n + \alpha_n(t - T_n)) - X(t)\\
&\;=\; O(1)\,(\alpha_n - \alpha(t))\,(1 + X(t)) + O(1)(T+1)^2(1+C)e^{-\left(\frac{1}{2}+\beta\right)L_n}\\
&:=\; h_n(t) + O(1)(T+1)^2(1+C)e^{-\left(\frac{1}{2}+\beta\right)L_n}. \qquad (62)
\end{aligned}$$

Note that $L_n = X(T_n)$ and that $h_n(T_n) = 0$. Moreover, since $\alpha'(L) = O(1)e^{-L}$, there exists ε_n such that

$$\begin{aligned}
&h(t) < \frac{1}{2}\left|(L_n + \alpha_n(t - T_n)) - X(t)\right|, \text{ whenever}\\
&|(L_n + \alpha_n(t - T_n)) - X(t)| < \varepsilon_n.
\end{aligned}$$

Consequently, provided that L_n is large enough, by the continuity of $h_n(t)$, (62) implies (56). □

With the aid of Lemma 10, we estimate the difference between $\psi(x,t)$ and $\psi^{[n]}(x - L_n - \alpha_n(t - T_n))$:

Proposition 11. *Under the same requirement of Lemma 10, for all $T_n \leq t \leq T_{n+2}$,*

$$\psi(x,t) - \psi^{[n]}(x - L_n - \alpha_n(t - T_n)) = \frac{O(1)(T+1)^2(1+C)e^{-\left(\beta+\frac{1}{2}\right)L_n}}{\cosh\left(\frac{x - L_n - \alpha_n(t-T_n)}{2}\right)} \qquad (63a)$$

$$\frac{\partial}{\partial x}\left(\psi(x,t) - \psi^{[n]}(x - L_n - \alpha_n(t - T_n))\right) = \frac{O(1)(T+1)^2(1+C)e^{-\left(\beta+\frac{1}{2}\right)L_n}}{\cosh\left(\frac{x - L_n - \alpha_n(t-T_n)}{2}\right)}. \qquad (63b)$$

Proof.

$$\begin{aligned}
&\psi(x,t) - \psi^{[n]}(x - L_n - \alpha_n(t - T_n))\\
&= (\alpha(t) - \alpha_n)\left(1 - \tanh\left(\tfrac{1+\alpha(t)}{2}(x - X(t))\right)\right)\\
&\quad + (1+\alpha_n)\left(\tanh\left(\tfrac{1+\alpha(t)}{2}(x - X(t))\right) - \tanh\left(\tfrac{1+\alpha_n}{2}(x - L_n - \alpha_n(t - T_n))\right)\right)\\
&= O(1)e^{-2L_n}\left|L_n + \alpha_n(t - T_n) - X(t)\right|\left(1 - \tanh\left(\tfrac{1+\alpha(t)}{2}(x - X(t))\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+O(1)\text{sech}^2\left(\frac{x-L_n-\alpha_n(t-T_n)}{2}\right)\Big(|\alpha(t)-\alpha_n|+|X(t)-L_n-\alpha_n(t-T_n)|\Big)\\
=&\ \frac{O(1)e^{-\frac{3}{2}L_n}}{\cosh\left(\frac{x-L_n-\alpha_n(t-T_n)}{2}\right)}+\frac{O(1)(T+1)^2(1+C)e^{-\left(\beta+\frac{1}{2}\right)L_n}}{\cosh\left(\frac{x-L_n-\alpha_n(t-T_n)}{2}\right)}.
\end{aligned}
$$

This proves (63a). (63b) can be proved similarly. The details are omitted here for simplicity. □

Now, Theorem 8 and Theorem 9 together with Proposition 11 yields the first main theorem:

Proof of the First Main Theorem. Assume the same requirement of Theorem 8. By Theorem 8, Lemma 10 and Proposition 11,

$$
\begin{aligned}
v^{[n+1]}(x,0) &= u(x,T_{n+1})-\psi(x,T_{n+1})\\
&= v^{[n]}\left(x,\tfrac{T}{2}\right)+\psi^{[n]}\left(x-L_n-\alpha_n\tfrac{T}{2}\right)-\psi(x,T_{n+1})\\
&= \frac{O(1)(C+1)(1+T)^2}{\cosh\left(\frac{x-L_n-\alpha_n(T/2)}{2}\right)}e^{-\beta L_n}\left(e^{-\frac{T/2}{4}}+e^{-\frac{L_n}{2}}+e^{-\left(\beta-\frac{3}{2}\right)L_n}\right)\\
&\quad+\frac{O(1)(C+1)(1+T)^2}{\cosh\left(\frac{x-L_n-\alpha_n(T/2)}{2}\right)}e^{-\beta L_n}e^{-\frac{L}{2}}\\
&= \frac{e^{-\beta L_{n+1}}}{\cosh\left(\frac{x-L_{n+1}}{2}\right)}D(1+C)(1+T)^2\Big(e^{-\frac{T}{8}}+e^{-\frac{L_n}{2}}+e^{-\left(\beta-\frac{3}{2}\right)L_n}\Big), \qquad (64)
\end{aligned}
$$

for some constant D. Now we choose a fixed T so that $D(C+1)(T+1)^2e^{-T/8}<C/2$. Therefore, provided that L_n is large enough,

$$
\begin{aligned}
\left|v^{[n+1]}(x,0)\right| &\le \frac{e^{-\beta L_{n+1}}}{\cosh\left(\frac{x-L_{n+1}}{2}\right)}\left(\frac{C}{2}+D(1+C)(1+T)^2\left(e^{-\frac{L_n}{2}}+e^{-\left(\beta-\frac{3}{2}\right)L_n}\right)\right)\\
&\le \frac{Ce^{-\beta L_{n+1}}}{\cosh\left(\frac{x-L_{n+1}}{2}\right)}. \qquad (65)
\end{aligned}
$$

Note that by Lemma 10, L_n increases with n. Consequently, each L_n will be large enough so long as L_0 is. We can therefore inductively apply Theorem 8 to solve (12). The estimate of $(u-\psi)_x$ follows directly from Theorem 8, Proposition 11 and Lemma 10. □

Proof of the Second Main Theorem. With the aid of (58), differentiate (57) to obtain

$$\begin{aligned}&\int_0^\infty (\psi^{[n]})_t(x-L_n-\alpha_n(t-T_n))-\psi_t(x,t)dx\\ =& -\int_0^\infty (v^{[n]})_t(x,t-T_n)dx = O(1)\left(1+\frac{1}{\sqrt{t-T_n}}\right)e^{-\left(\beta+\frac{3}{2}\right)L_n}\\ =& \ O(1)e^{-\left(\beta+\frac{3}{2}\right)L_n}, \qquad (66)\end{aligned}$$

for all $T_{n+1}\le t\le T_{n+2}$. By some direct computations and by Lemma 10, (66) yields

$$X'(t)=\alpha_n+\left(1+\frac{1}{\sqrt{T_n-t}}\right)e^{-\left(\beta+\frac{1}{2}\right)L_n},\ \text{for } T_n\le t\le T_{n+2},\ \text{for all } n.$$

$$\Longrightarrow X'(t)=e^{-X(t)}+O(1)\left(1+\frac{1}{\sqrt{t}}\right)e^{-\left(\beta+\frac{1}{2}\right)X(t)},\ \text{for all } t. \qquad (67)$$

Next, integrate (67) to obtain

$$e^{X(t)}=e^{X(0)}+t+O(1)\int_0^t\left(1+\frac{1}{\sqrt{s}}\right)\frac{ds}{e^{\left(\beta-\frac{1}{2}\right)X(s)}}.$$

Put $E(t)=e^{X(t)}-e^{X(0)}-t$. Then

$$E(t)=O(1)\int_0^t\left(1+\frac{1}{\sqrt{s}}\right)\frac{ds}{\left(E(s)+e^{X(0)}+s\right)^{\left(\beta-\frac{1}{2}\right)}}. \qquad (68)$$

Besides, by (67), we already have

$$|E(t)|\le O(1)(t+1)e^{-\left(\beta-\frac{1}{2}\right)X(0)},$$

which together with (68) implies $E(t)=O(1)(t+e^{X(0)})^{\frac{3}{2}-\beta}$. Hence

$$X(t)=X(0)+\log\left[1+e^{-X(0)}\left(t+O(1)\left(t+e^{-X(0)}\right)^{\frac{3}{2}-\beta}\right)\right]. \qquad \square$$

Lastly, we estimate the difference between $X(t)$ and $Y(t)$.

Proposition 12.

$$|X(t)-Y(t)|=O(1)X(t)e^{-X(t)}.$$

Proof. The equation that relates $X(t)$ with $Y(t)$ is

$$\int_0^\infty -\tanh\left(\tfrac{x-Y(t)}{2}\right) - \left(\alpha(t) - (1+\alpha(t))\tanh\left(\tfrac{1+\alpha(t)}{2}(x-X(t))\right)\right) dx.$$

Since

$$\begin{aligned}&\int_0^\infty \tanh\left(\tfrac{x-X(t)}{2}\right) - \tanh\left(\tfrac{x-Y(t)}{2}\right) dy \sim (X(t)-Y(t)),\\ &\int_0^\infty \alpha(t) - (1+\alpha(t))\tanh\left(\tfrac{1+\alpha(t)}{2}(x-X(t))\right) + \tanh\left(\tfrac{x-X(t)}{2}\right) dx\\ &\quad = O(1)X(t)e^{-X(t)},\end{aligned}$$

we have

$$|X(t)-Y(t)| = O(1)X(t)e^{-X(t)}. \qquad \square$$

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