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DISJUNCTIVE ELEMENTS OF INVERSE MONOIDS

BY

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Abstract

An element of a semigroup is called *disjunctive* if it does not form a class modulo some non-identity congruence. We first study identity elements of inverse monoids that are disjunctive. It turns out that a disjunctive identity element can be determined by a basis. We continue to analyze other disjunctive elements, and find out that disjunctive elements in some D-classes can be determined by a special element.

1. Introduction

Throughout this paper we shall use the terminology and notation of [1] and refer to [8] for basic properties of groups. Disjunctive elements play a role in automata theory, and in the congruence theory of (inverse) semigroups. The term *disjunctive* was introduced by Schein [9]. An element of a semigroup is called *disjunctive* if it does not form a class modulo some non-identity congruence. For a homomorphism ϕ of a group G onto $\phi(G)$, the inverse images of elements of $\phi(G)$ partition G into classes of a congruence ρ. As a consequence, ρ is uniquely determined not only by the ρ-class containing the identity of G, which is a normal subgroup of G, but also by each of its congruence classes that are the cosets of this normal subgroup. It is obvious that the order of each non-identity congruence class is more than one. And it turns out that every element of the group G is disjunctive. However, the situation is more complicated and interesting for inverse semigroups, the

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semigroups that are "closet" to groups. For example, the identity element of an infinite lower semillatice with identity is not disjunctive. The purpose of this paper is to investigate disjunctiveness in inverse monoids.

There are already some discussions on disjunctive elements of semigroups in [9]. Now we focus on disjunctive elements of inverse monoids. We recall the following definitions. A semigroup S is called an *inverse semigroup* if every element a of S has a unique inverse a^{-1} such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. An inverse monoid is an inverse semigroup with an identity element. Two elements a and b of an inverse semigroup S are called R-equivalent if there exists $x \in S$ such that $aa^{-1} = xx^{-1}$, and L-equivalent if $b^{-1}b = x^{-1}x$. Also, a and b are called D-equivalent if there exists $x \in S$ such that a and x are R-equivalent and x and b are L-equivalent.

The *syntactic* congruence τ^K of a subset K of a semigroup S is defined by

$$(a, b) \in \tau^K \text{ if and only if } (xay \in K \Leftrightarrow xby \in K)$$

for all $x, y \in S^1$.

If K is the union of some ρ-classes, then $\rho \subseteq \tau^K$ (see [6]). For a subset K of an inverse semigroup S, it is easy to show that the definition of τ^K is equivalent to

$$(a, b) \in \tau^K \text{ if and only if } (xay \in K \Leftrightarrow xby \in K)$$

for all $x, y \in S$.

When $K = \{a\}$ for $a \in S$, it is clear that a is disjunctive if and only if $\tau^{\{a\}} = 1_S$. There is an in-depth study in [10] of the relationship between disjunctiveness and syntactic congruences in inverse monoids.

The main aim of the second section of this paper is to investigate disjunctive identity elements of inverse monoids. We recall the following definitions. Let I be a proper ideal of a semigroup S. Then ρ_I is called a *Rees* congruence on S if

$$\rho_I = (I \times I) \cup 1_S.$$

We note that the set of ρ_I-classes consists entirely of the ideal I and singletons which are not contained in I. A semigroup S is called *simple* if it has no proper ideals. Then S is simple if and only if for every $a, b \in S$

there exist $x, y \in S$ such that $xay = b$. Also, it can be seen in ([2]) that the *principal filter* $\uparrow a$ of an ordered set S is

$$\uparrow a = \{s \in S : s \geq a\}.$$

The syntactic congruence $\tau^{\uparrow a}$ of the subset $\uparrow a$ of a simple inverse semigroup S is the identity congruence if and only if D_a is a basis (Theorem 2.1). This result leads immediately to a necessary and sufficient condition for the identity element of an inverse monoid (Corollary 2.2).

The third and final section concerns the other disjunctive elements. A maximal idempotent f of an E-unitary inverse semigroup is disjunctive if and only if every element in D_f is disjunctive (Theorem 3.1). To obtain more results, we study further completely 0-simple semigroups. A *completely 0-simple semigroup* S is a 0-simple semigroup such that every idempotent z of S has the property that $zf = fz = f \neq 0$ implies $z = f$. As explained in [3] and [5], every completely 0-simple inverse semigroup is isomorphic to a Brandt semigroup. A *Brandt* semigroup $S = \mathcal{M}^0[G; I, I; \triangle]$ is a semigroup $S = (I \times G \times I) \cup \{0\}$ whose mutiplication is given by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, a\delta_{\lambda j}b, \mu) & \text{if } \delta_{\lambda j} = e \\ 0 & \text{if } \delta_{\lambda j} = 0 \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0,$$

where G is a group with an identity element e, I is a non-empty set, and $\triangle = (\delta_{ij})$ is the $I \times I$ matrix given by,

$$\delta_{ij} = \begin{cases} e & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Finally, we investigate disjunctive elements of finite inverse semigroups. Every non-zero element of a finite inverse semigroup is disjunctive if and only if it is isomorphic to a Brandt semigroup (Theorem 3.2). This leads to an identification of finite inverse monoids with disjunctive identity elements.

2. Disjunctive Identity Elements of Inverse Monoids

For an element a of a semigroup S, clearly, a is disjunctive if and only if $\tau^{\{a\}}$ is the identity congruence on S. We begin to investigate the syntactic congruence $\tau^{\uparrow a}$ of the principal filter $\uparrow a$ of a simple inverse semigroup S, especially when $\tau^{\uparrow a}$ is the identity congruence on S. By virtue of the definition of $\tau^{\uparrow a}$, we now produce two non-empty identical subsets of a simple inverse semigroup S.

$$\begin{aligned} X_{b,a} &= \{x^{-1}ay^{-1} : xby \geq a \; for \, x, y \in S\}. \\ A_{b,a} &= \{t \in D_a : t \leq b\}. \end{aligned}$$

Before we state our main theorem, it is important to prove the following lemma.

Lemma 2.1. *Let S be a simple inverse semigroup, and let $a, b \in S$. Then $X_{b,a}$ and $A_{b,a}$ are two non-empty identical subsets of S.*

Proof. Since S is simple, it is clear that $X_{b,a}$ and $A_{b,a}$ are non-empty. We now show that $X_{b,a} \subseteq A_{b,a}$. Suppose that $x^{-1}ay^{-1} \in X_{b,a}$. Then $xby \geq a$ for $x, y \in S$. It follows that $b \geq x^{-1}xbyy^{-1} \geq x^{-1}ay^{-1}$. Next, we verify that $x^{-1}ay^{-1} \in D_a$. First, it is necessary to show that $xx^{-1} \geq aa^{-1}$ and $y^{-1}y \geq a^{-1}a$. Indeed,

$$xx^{-1} \geq xx^{-1}(xbyy^{-1}b^{-1}x^{-1}) = xbyy^{-1}b^{-1}x^{-1} = (xby)(xby)^{-1} \geq aa^{-1}. \quad (2.1)$$

Similary, we obtain

$$y^{-1}y \geq a^{-1}a. \quad (2.2)$$

According to (2.1) and (2.2), $(x^{-1}ay^{-1})^{-1}(x^{-1}ay^{-1}) = ya^{-1}xx^{-1}ay^{-1} = ya^{-1}(aa^{-1})(xx^{-1})ay^{-1} = (ay^{-1})^{-1}(ay^{-1})$, and $aa^{-1} = a(y^{-1}y)(a^{-1}a)a^{-1} = ay^{-1}ya^{-1} = (ay^{-1})(ay^{-1})^{-1}$. Hence $X_{b,a} \subseteq A_{b,a}$. Conversely, suppose that $t \in D_a$, and $t \leq b$. Then $t^{-1}t = u^{-1}u$, and $aa^{-1} = uu^{-1}$ for some $u \in S$. Now, $t = bt^{-1}t = bu^{-1}uu^{-1}u = (ub^{-1})^{-1}a(u^{-1}a)^{-1}$. Also, $(ub^{-1})b(u^{-1}a) = u(b^{-1}bu^{-1}u)u^{-1}a = u(t^{-1}t)u^{-1}a = u(u^{-1}u)u^{-1}a \geq a$. We are done. □

Before we show Theorem 2.1, we recall the following definitions. A non-empty subset Y of an ordered set S is called a *basis* if every element of S is the least upper bound of some subset of Y. For a non-empty subset K of a semigroup S, the relation δ_K on S is defined by

$$(a, b) \in \delta_K \text{ if and only if } (xay \in K \Rightarrow xby \in K)$$

for all $x, y \in S^1$.

Theorem 2.1. *Let S be a simple inverse semigroup with $a \in S$. Then the following statements are equivalent:*

(1) $\tau^{\uparrow a} = 1_S$.

(2) D_a *is a basis.*

(3) $b = \vee X_{b,a}$ *for any* $b \in S$.

(4) $\delta_{\uparrow a}$ *is the natural relation* $\leq$ *on* S.

Proof. (1) $\Rightarrow$ (2). Suppose that $\tau^{\uparrow a} = 1_S$. Let $b \in S$. First, we claim that $b = \vee X_{b,a}$. Indeed, if b is not the least upper bound of $X_{b,a}$, then there exists $c \in S$ such that $c < b$ and $x^{-1}ay^{-1} \leq c$. It follows that $(c, b) \in \tau^{\uparrow a}$. Indeed, if $xcy \geq a$, then $b > c$ implies that $xby \geq xcy$ and hence $xby \geq a$. Conversely, if $xby \geq a$, then $x^{-1}ay^{-1} \in X_{b,a}$. Since $x^{-1}ay^{-1} \leq c$, we have $xcy \geq xx^{-1}ay^{-1}y$. By (2.1) and (2.2) just proved above we have

$$xx^{-1}ay^{-1}y \geq (xx^{-1}aa^{-1})a(a^{-1}ay^{-1}y) = a.$$

Hence $xcy \geq a$. But $(c, b) \in \tau^{\uparrow a}$ contradicts the assumption. So $b = \vee X_{b,a}$. By Lemma 2.1, $b = \vee A_{b,a}$. Therefore D_a is a basis.

(2) $\Rightarrow$ (3). If D_a is a basis and $b \in S$, then b is the least upper bound of some subset K_b of D_a. We claim that b is the least upper bound of the subset $A_{b,a}$ of D_a. First, b is an upper bound of $A_{b,a}$. Next, let h be an upper bound of $A_{b,a}$. Since $K_b \subseteq A_{b,a}$, it is clear that h is also an upper bound of K_b. Thus by the hypothesis we obtain $b \leq h$. It follows that $b = \vee A_{b,a} = \vee X_{b,a}$.

(3) $\Rightarrow$ (4). Suppose that $(c, d) \in \delta_{\uparrow a}$ with $c, d \in S$. Then $X_{c,a} \subseteq X_{d,a}$. It follows easily that d is an upper bound of $X_{c,a}$. By the hypothesis we obtain $c = \vee X_{c,a} \leq d$.

(4) $\Rightarrow$ (1). Suppose that $\delta_{\uparrow a}$ is the natural relation $\leq$ on S. Let $(b,c) \in \tau^{\uparrow a}$. Then

$$xby \geq a \ if \ and \ only \ if \ xcy \geq a \tag{2.3}$$

for all $x, y \in S$.

Now we claim that $(b,c) \in \delta_{\uparrow a}$ and $(c,b) \in \delta_{\uparrow a}$. First, we verify that $xby \geq a$ implies $xcy \geq a$ for all $x, y \in S^1$. Let $by \geq a$. Then $(bb^{-1})by \geq a$. According to (2.3), $bb^{-1}cy \geq a$. So $cc^{-1}bb^{-1}cy = bb^{-1}cc^{-1}cy = bb^{-1}cy \geq a$. It follows again from (2.3) that $cc^{-1}bb^{-1}by \geq a$. Clearly, $cc^{-1}by \geq a$ implies that $cc^{-1}cy \geq a$. So $cy \geq a$. Similarly, if $xb \geq a$, then $xc \geq a$. Also, if $b \geq a$, then $b(b^{-1}b) \geq a$ and hence $c \geq cb^{-1}b \geq a$. Thus, $xby \geq a$ implies that $xcy \geq a$ for all $x, y \in S^1$. Similarly,we can verify that $xcy \geq a$ implies $xby \geq a$ for all $x, y \in S^1$. Now, $(b,c) \in \delta_{\uparrow a}$ and $(c,b) \in \delta_{\uparrow a}$. By the hypothesis, $b \leq c$ and $c \leq b$. So $b = c$and therefore $\tau^{\uparrow a} = 1_S$. □

Suppose that e is an identity element of an inverse monoid S. Then it is obvious that $e = \uparrow e$. If $\tau^{\uparrow e} = 1_S$, then an inverse monoid S is simple. Indeed, if I is a proper ideal of S, then $e \notin I$. So there exists a non-identity Rees congruence ρ_I such that $e\rho_I = \{e\}$. It follows that the identity element e is not disjunctive, and $\tau^{\uparrow e} \neq 1_S$. A useful further specialization of Theorem 2.1 is provided by the following corollary. We omit the proof.

Corollary 2.2. *Let S be an inverse monoid with an identity element e. Then the following statements are equivalent:*

(1) *e is disjunctive.*

(2) $\tau^{\{e\}} = 1_S$.

(3) *D_e is a basis.*

(4) $b = \vee\{x^{-1}y^{-1} : xby = e \ for \, x, y \in S\}$ *for any* $b \in S$.

(5) *$\delta_{\uparrow e}$ is the natural relation $\leq$ on S.*

Remark. We make a further observation about inverse monoids whose semilattices of idempotents form a chain. Let $\mathcal{C}_\omega = \{e_0, e_1, e_2, \ldots\}$, with $e_0 \geq e_1 \geq \cdots$. Bisimple semigroups whose semilattices of idempotents are isomorphic to $\mathcal{C}_\omega$ are so-called bisimple *ω-semigroups* (see [7]). If we apply Corollary 2.2 to bisimple inverse ω-monoids, we find out the following interesting result. *An inverse ω-monoid S is bisimple if and only if its identity element is disjunctive.*

There are several different forms of the identity $xby = e$ in Corollary 2.2 (4). We discuss them in the following propositions.

Proposition 2.1. *Let S be an inverse monoid with an identity element e, and let $x, y \in S$. Then the following statements are equivalent:*

(1) $xby = e$.

(2) $x \in R_e$, $x^{-1}x \leq bb^{-1}$, *and* $x^{-1} \leq by$.

(3) $xbb^{-1}x^{-1} = e$, *and* $x^{-1} \leq by$.

(4) $y \in L_e$, $yy^{-1} \leq b^{-1}b$, *and* $y^{-1} \leq xb$.

(5) $y^{-1}b^{-1}by = e$, *and* $y^{-1} \leq xb$.

Proof. First we show that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). If $xby = e$, then $x^{-1} = x^{-1}e = x^{-1}(xby) = (x^{-1}x)by \leq by$. Next, $xx^{-1} = (xx^{-1})e = (xx^{-1})xby = xby = e$. Before we show that $x^{-1}x \leq bb^{-1}$, we first verify that $y \in L_e$. Indeed, $y^{-1}y = e(y^{-1}y) = xbyy^{-1}y = xby = e$. Now, $x^{-1}y^{-1} = x^{-1}(xby)y^{-1} = (x^{-1}x)b(yy^{-1}) \leq b$. It follows that $x^{-1}x = x^{-1}(y^{-1}y)x = (x^{-1}y^{-1})(x^{-1}y^{-1})^{-1} \leq bb^{-1}$.

(2) $\Rightarrow$ (3). It suffices to show that $xbb^{-1}x^{-1} = x(x^{-1}xbb^{-1}x^{-1}x)x^{-1} = x(x^{-1}xx^{-1}x)x^{-1} = xx^{-1} = e$.

(3) $\Rightarrow$ (1). Since $x^{-1} \leq by$, we have $x^{-1} = zby$ for some idempotent z. Now, $xx^{-1} = (y^{-1}b^{-1}z)zby = (y^{-1}b^{-1}z)by = xby$. Also, $e = xbb^{-1}x^{-1} = x(x^{-1}xbb^{-1})x^{-1} = xx^{-1}(xbb^{-1}x^{-1}) = xx^{-1}$. Thus, $xby = xx^{-1} = e$.

Similarly, (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). We are done. □

Let ρ be a congruence on an inverse semigroup S with semilattice E_S of idempotents. We note that $tr\rho$ denotes the restriction of ρ to E_S. The following proposition is a special case of Corollary 2.2.

Proposition 2.2. *Let S be an inverse monoid with an identity element e. Then $tr\tau^{\{e\}} = 1_{E_S}$ if and only if $f = \vee\{x^{-1}x : xfy = e\ for\ x, y \in S\}$ for any $f \in E_S$.*

Proof. Let $tr\tau^{\{e\}} = 1_{E_S}$, and let $f \in E_S$. If $f \neq \vee\{x^{-1}x : xfy = e\ for\ x, y \in S\}$, then there exists $w \in S$ such that $w < f$, and $x^{-1}x \leq w$

with $xfy = e$ for $x, y \in S$. We claim that $(f, w) \in \tau^{\{e\}}$. Indeed, if $xfy = e$ for some $x, y \in S$, then $e = xfy = xf(yy^{-1})y = xy(y^{-1}fy) \leq xy$. This implies that $xy = e$. By the assumption, $xwy = x(x^{-1}x)wy = xy = e$. Next, suppose that $xwy = e$. Since $w < f$, we have $e = xwy \leq xfy$. So $xfy = e$, and hence $(f, w) \in \tau^{\{e\}}$. This is a contradiction of $tr\tau^{\{e\}} = 1_{E_S}$. Conversely, let $(f, w) \in \tau^{\{e\}}$ for some f, w in E_S. Then $\{x^{-1}x : xfy = e \ for \ x, y \in S\}$ $= \{x^{-1}x : xwy = e \ for \ x, y \in S\}$. So $f = \vee\{x^{-1}x : xfy = e \ for \ x, y \in S\} = \vee\{x^{-1}x : xwy = e \ for \ x, y \in S\} = w$. □

Since every element of an inverse semigroup has a unique inverse, $xfy = e$ implies that $y = x^{-1}$ in Proposition 2.2. An inverse semigroup S is called *E-unitary* if, for any $x \in S$ and $f \in E_S$, $x \leq f$ implies that $x \in E_S$ (see [4]). The next proposition is of some interest.

Proposition 2.3. *Let S be a simple E-unitary inverse monoid with an identity element e. Then $\tau^{\{e\}}$ is an idempotent-pure congruence.*

Proof. Let $(b, f) \in \tau^{\{e\}}$ for some $b \in S$ and $f \in E_S$. Since S is simple, there exist $x, y \in S$ such that $xby = e$. We note that $(b, bb^{-1}) \in \tau^e$. It follows that $xbb^{-1}y = e$. According to Proposition 2.1, $e = xbb^{-1}y = xx^{-1}xbb^{-1}y = xbb^{-1}x^{-1}xy = xy$. Since x has a unique inverse element of S, we obtain $y = x^{-1}$. Now $x^{-1}x = x^{-1}ex = x^{-1}(xbx^{-1})x \leq b$. Since S is E-unitary, we have $b \in E_S$. So $\tau^{\{e\}}$ is an idempotent-pure congruence. □

Example 2.1. We recall an inverse ω-semigroup $B_2 = \{(m, n) \in \mathbb{N}^0 \times \mathbb{N}^0 : m \equiv n (\bmod 2)\}$. It is shown in [3] that B_2 is a simple inverse monoid with an identity element (0,0), and the idempotents of B_2 are of the form (m, m). Clearly, B_2 is E-unitary. Indeed, if $(m, m) \leq (p, q)$ for some $p, q \in \mathbb{N}^0$, then $(m, m) = (m, m)(p, q)^{-1}(m, m) = (q - p + max(p - q + max(m, q)), max(p - q + max(m, q)))$. It follows that $p = q$. We note that $\tau^{\{(0,0)\}} \neq 1_{B_2}$, and $\tau^{\{(0,0)\}}$ is an idempotent-pure congruence on B_2.

3. Further Properties of Disjunctive Elements of Inverse Semigroups

In this section we first examine certain elements which can affect other disjunctive elements in the same D-class. We will study maximal idempotents of E-unitary inverse semigroups.

Theorem 3.1. *Let S be an E-unitary inverse semigroup with semilattice E_S of idempotents, and let f be a maximal idempotent. Then f is disjunctive if and only if every element in D_f is disjunctive.*

Proof. Case 1. Suppose that S is a monoid with an identity element e. We claim that e is disjunctive if and only if every element in D_e is disjunctive. Indeed, let $a \in D_e$. Then $a^{-1}a = u^{-1}u$ and $e = uu^{-1}$ for some $u \in S$. We show that $\tau^{\{a\}} \subseteq \tau^{\{e\}}$. If $\tau^{\{a\}} = 1_S$, we are done. Now, let us assume that $(b, e) \in \tau^{\{a\}}$ with $b \in S$. Then $au^{-1}eu = a$ implies that $au^{-1}bu = a$. It follows that $u^{-1}bu = u^{-1}u$. Therefore, $b = ebe = uu^{-1}buu^{-1} = u(u^{-1}bu)u^{-1} = e$. We note that $\tau^{\{e\}}$ is the greatest congruence on S having a congruence class $\{e\}$. Thus, $\tau^{\{a\}} \subseteq \tau^{\{e\}}$. If e is disjunctive, then $\tau^{\{a\}} = 1_S$, and hence a is disjunctive. This completes the direct part of the proof.

Case 2. Suppose that S is an E-unitary inverse semigroup without an identity element. Let $a \in D_f$. Then $a^{-1}a = u^{-1}u$ and $f = uu^{-1}$ for some $u \in S$. We show that $\tau^{\{a\}} \subseteq \tau^{\{f\}}$. If $\tau^{\{a\}} = 1_S$, we are done. Now, let us assume that $(b, f) \in \tau^{\{a\}}$ with $b \in S$. Then $au^{-1}fu = a$ implies that $au^{-1}bu = a$. It follows that $f = ua^{-1}au^{-1} = ua^{-1}(au^{-1}bua^{-1}a)u^{-1} = fbf \leq b$. Since S is E-unitary, we obtain $b \in E_S$. Also, we note that f is a maximal idempotent. Thus $f = b$. Since $\tau^{\{f\}}$ is the greatest congruence on S having a congruence class $\{f\}$, it follows that $\tau^{\{a\}} \subseteq \tau^{\{f\}}$. If f is disjunctive, then $\tau^{\{f\}} = 1_S$, and hence a is disjunctive. □

Next, we recall 0-tight semigroups in Schein [9]. A semigroup S with zero is 0-*tight* if each of the congruences is uniquely determined by each of its congruence classes which do not contain zero. By the definition of 0-tight semigroups we obtain the following result of disjunctive elements of finite inverse semigroups with zero.

Theorem 3.2. *Let S be a finite inverse semigroup with zero. Then the following statements are equivalent:*

(1) *S is 0-tight.*

(2) *Every non-zero element of S is disjunctive.*

(3) *Every non-zero idempotent of S is disjunctive.*

(4) *S is isomorphic to a Brandt semigroup.*

Proof. We shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that S is a 0-tight inverse semigroup. Then 1_S is uniquely determined by each of the congruence classes which do not contain zero. So every non-zero element of S is disjunctive.

(2) $\Rightarrow$ (3). By the hypothesis it is clear that every non-zero idempotent of S is disjunctive.

(3) $\Rightarrow$ (4). Suppose that every non-zero idempotent of S is disjunctive. First, we claim that S is 0-simple. Indeed, if S is not 0-simple, then there exists a proper ideal I of S. Obviously, there exists an idempotent $aa^{-1} \notin I$ for some $a \notin I$. Otherwise, all idempotents belong to I, and this leads to $I = S$, in contradiction to the already noted fact that I is a proper ideal. It follows that there exists a non-identity Rees congruence ρ_I such that $aa^{-1}\rho_I = \{aa^{-1}\}$. This contradicts the hypothesis. So S is 0-simple. Now, S is a finite 0-simple inverse semigroup. As proved in ([3]), S is a completely 0-simple inverse semigroup, and hence S is isomorphic to a Brandt semigroup.

(4) $\Rightarrow$ (1). Suppose that S is isomorphic to a Brandt semigroup $T = \mathcal{M}^0[G; I, I; \triangle]$. We claim that T has exactly two D-classes, namely $\{0\}$ and $D = T\backslash\{0\}$. First, we note that every non-zero idempotent of T is of the form (i, e, i), where e is the identity element of G. For every $(i, e, i), (j, e, j) \in E_T$,

$$(i, e, i) = (i, e\delta_{jj}e, i) = (i, e, j)(j, e, i) = (i, e, j)(i, e, j)^{-1},$$

$$(j, e, j) = (j, e\delta_{ii}e, j) = (j, e, i)(i, e, j) = (i, e, j)^{-1}(i, e, j).$$

It follows that (i, e, i) and (j, e, j) are in the same D-class for every pair $(i, e, i), (j, e, j) \in E_T$, and hence (since every non-zero element of T is D-equivalent—indeed R- or L-equivalent—to an idempotent) S is 0-bisimple. As proved in Schein [9], every 0-bisimple inverse semigroup is 0-tight. We are done. □

A semigroup S is *tight* if each of the congruences is uniquely determined by each of its congruence classes (see [9]). The following corollary can be proved in some way analogous to Theorem 3.2. We omit the proof.

Corollary 3.3. *Let S be a finite inverse semigroup. The following statements are equivalent:*

(1) *S is tight.*

(2) *Every element of S is disjunctive.*

(3) *Every idempotent of S is disjunctive.*

(4) *S is a group.*

Remark. If in Theorem 3.2 and Corollary 3.3, S is a finite inverse monoid with an identity element e, then we obtain the following equivalent statements.

(1) S is [0]-tight.

(2) e is disjunctive.

(3) S is a [0]-group.

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