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SOME THEOREMS ON WEIGHTED MEAN SUMMABILITY

BY

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Abstract

In this paper we have proved some theorems on weighted mean summability method by using analytical and summability techniques, which also extends the well known result of Hardy on the Cesàro summability.

1. Introduction

Let $\sum x_v$ be a given infinite series of complex number with $s = (s_n)$ as the sequence its n-th partial sum and let (p_n) be a sequence of positive numbers where for $n = 0, 1, 2, \ldots$.

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \quad \text{as} \quad n \to \infty, \qquad P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \qquad n = 0, 1, 2, \ldots \tag{1}$$

defines the sequence (T_n) of $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) [2]. The series $\sum x_k$ is said to be summable $(\overline{N}, p_n)$ to a number λ if

$$T_n \to \lambda \quad \text{as} \quad n \to \infty.$$

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It is well known that $(\overline{N}, p_n)$ summability method is regular if and only if $P_n \to \infty$ as $n \to \infty$. In the special case where $p_v = 1$ for $v = 0, 1, 2, \ldots$, it reduces to the Cesàro summability $(C, 1)$.

In [1] Hardy proved the following theorem on Cesàro summability.

Theorem 1.1. *The series* $\sum x_v$ *is summable* $(C, 1)$ *to a finite number* λ *if and only if* $\sum b_n$ *converges to* λ, *where*

$$b_n = \sum_{v=n}^{\infty} \frac{x_v}{v+1}, \quad n = 0, 1, \ldots .$$

We need the following lemma to prove the main theorems.

Lemma 1.2. *An infinite matrix* $B = (b_{nv})$ *regular if and only if* (*see* [2])

$$\text{(a)}\ \sup_n \sum_{v=0}^{\infty} |b_{nv}| < \infty, \quad \text{(b)}\ \lim_n b_{nv} = 0\ (v = 0, 1, \ldots), \quad \text{(c)}\ \lim_n \sum_{v=0}^{\infty} b_{nv} = 1.$$

2. The Main Results

In this section we have proved the following theorems on weighted mean summability method using analytical and summability techniques, which also extend the well known result of Hardy on the Cesàro summability.

Theorem 2.1. *Let* (p_n) *and* (q_n) *be two sequences of positive numbers satisfying the following conditions:*

$$P_n \to \infty, \quad Q_n \to \infty \ \text{ as } \ n \to \infty. \tag{2}$$

Then the series $\sum x_v$ *is summable* $(\overline{N}, p_n)$ *to* λ *whenever* $\sum b_n$ *converges to a finite number* λ *if and only if*

$$\sum_{v=1}^{n} \left| \frac{p_v Q_{v+1}}{q_{v+1}} - \frac{p_{v-1} Q_{v-1}}{q_v} \right| = O(P_n) \tag{3}$$

where

$$b_n = q_n \sum_{v=n}^{\infty} \frac{x_v}{Q_v}, \quad n = 0, 1, \ldots. \tag{4}$$

Theorem 2.2. *Let* (p_n) *and* (q_n) *be two sequences of positive numbers satisfying the following conditions:*

$$\text{(a) } P_n = (p_n Q_n) \text{ and (b) } P_n \to \infty, \quad Q_n \to \infty \text{ as } n \to \infty. \tag{5}$$

Then the series $\sum b_n$ *converges to* λ *whenever the series* $\sum x_v$ *is summable* $(\overline{N}, p_n)$ *to a finite number* λ *if and only if*

$$\sum_{v=n}^{\infty} \frac{P_v}{Q_{v+1}} \left| \frac{q_{v+1}}{p_v Q_v} - \frac{q_{v+2}}{p_{v+1} Q_{v+2}} \right| = O(1/Q_n), \tag{6}$$

where b_n *is defined by* (4).

It is noticed that, if take $p_n = q_n = 1$ for $v = 0, 1, \ldots$ in Theorem 2.1 and Theorem 2.2, then we obtain Theorem 1.1.

Proof of Theorem 2.1. Let $B_n = \sum_{v=0}^{n} b_v \to \lambda$. It follows from (4) that

$$x_n = Q_n \Big(\frac{b_n}{q_n} - \frac{b_{n+1}}{q_{n+1}} \Big), \ n = 0, 1, \ldots \quad \text{and} \quad \frac{b_n}{q_n} = \sum_{v=n}^{\infty} \frac{x_v}{Q_v}$$

and so

$$s_m = \sum_{v=0}^{m} Q_v \Big(\frac{b_v}{q_v} - \frac{b_{v+1}}{q_{v+1}} \Big) = B_m - Q_m \frac{b_{m+1}}{q_{m+1}}.$$

Since the series $\sum_{v=n}^{\infty} \frac{x_v}{Q_v}$ is convergent, we have $\frac{b_n}{q_n} = \sum_{v=n}^{\infty} \frac{x_v}{Q_v} \to 0$ as $n \to \infty$, which implies

$$\frac{s_m}{Q_m} = \frac{B_m}{Q_m} - \frac{b_{m+1}}{q_{m+1}} \to 0 \quad \text{as} \quad n \to \infty,$$

by virtue of (2). Hence, for $n \geq 0$,

$$b_n \ = \ q_n \sum_{v=n}^{\infty} \frac{x_v}{Q_v} = \lim_m q_n \sum_{v=n}^{m} \frac{s_v - s_{n-1}}{Q_v}, \quad (s_{-1} = 0)$$

$$
\begin{aligned}
&= \lim_m q_n \left\{ \frac{s_m}{Q_m} + \sum_{v=n}^{m-1} \left(\frac{1}{Q_v} - \frac{1}{Q_{v+1}} \right) s_v - \frac{s_{n-1}}{Q_n} \right\} \\
&= -\frac{q_n s_{n-1}}{Q_n} + q_n \sum_{v=n}^{\infty} c_v s_v, \quad \text{where} \quad c_v = \frac{1}{Q_v} - \frac{1}{Q_{v+1}}.
\end{aligned}
$$

On the other hand, using Abel's summation by parts gives us

$$
\begin{aligned}
B_n &= \sum_{v=0}^{n} q_v \frac{b_v}{q_v} = \sum_{v=0}^{n-1} Q_v \frac{x_v}{Q_v} + Q_n \frac{b_n}{q_n} = s_{n-1} + Q_n \Big(-\frac{s_{n-1}}{Q_n} + \sum_{v=n}^{\infty} c_v s_v \Big) \\
&= Q_n \sum_{v=n}^{\infty} c_v s_v \Rightarrow \frac{B_n}{Q_n} = \sum_{v=n}^{\infty} c_v s_v \Rightarrow c_n s_n = \frac{B_n}{Q_n} - \frac{B_{n+1}}{Q_{n+1}}, \ (n = 0, 1, \ldots).
\end{aligned}
$$

By the last equality, we write

$$
\begin{aligned}
T_n &= \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v = \frac{1}{P_n} \sum_{v=0}^{n} p_v \frac{1}{c_v} \Big(\frac{B_v}{Q_v} - \frac{B_{v+1}}{Q_{v+1}} \Big) \\
&= \frac{1}{P_n} \left\{ \frac{p_0}{c_0 q_0} B_0 + \sum_{v=1}^{n-1} \frac{1}{Q_v} \left(\frac{p_v}{c_v} - \frac{p_{v-1}}{c_{v-1}} \right) B_v - \frac{p_n}{c_n Q_{n+1}} B_{n+1} \right\} \\
&= \sum_{v=0}^{\infty} c_{nv} B_v,
\end{aligned}
$$

where the matrix $C = (c_{nv})$ is defined by

$$
c_{nv} = \begin{cases}
\dfrac{1}{P_n} \cdot \dfrac{p_0}{c_0 Q_0}, & v = 0 \\
\dfrac{1}{P_n} \Big(\dfrac{p_v}{c_v} - \dfrac{p_{v-1}}{c_{v-1}} \Big) \dfrac{1}{Q_v}, & 1 \le v \le n \\
-\dfrac{1}{P_n} \dfrac{p_n}{c_n Q_{n+1}}, & v = n+1 \\
0, & v \ge n.
\end{cases}
$$

Now, it is clear that $c_{nv} \to 0$ as $n \to \infty$ and

$$
\sum_{v=0}^{\infty} c_{nv} = \frac{1}{P_n} \left\{ \frac{p_0}{c_0 Q_0} + \sum_{v=1}^{n} \frac{1}{Q_v} \left(\frac{p_v}{c_v} - \frac{p_{v-1}}{c_{v-1}} \right) - \frac{p_n}{c_n Q_{n+1}} \right\} = 1.
$$

By Lemma 1.2, $T_n \to \lambda$ as $n \to \infty$ if and only if the matrix C is regular, or equivalently

$$\sum_{v=0}^{\infty}|c_{nv}| = \frac{p_0}{P_n c_0 Q_0}+\frac{1}{P_n}\sum_{v=1}^{n}\frac{1}{Q_v}\left|\frac{p_v}{c_v}-\frac{p_{v-1}}{c_{v-1}}\right|+\frac{p_n}{P_n c_n Q_{n+1}} = O(1) \text{ as } n\to\infty. \tag{7}$$

Because of that the boundedness of the middle term implies the other term, (7) is equivalent to (3), whence the result.

Proof of Theorem 2.2. Suppose that if $T_n \to \lambda$ as $n \to \infty$, then $\sum b_n$ converges to λ. Then, by (1), since

$$s_0 = T_0, \quad s_n = \frac{1}{p_n}(P_n T_n - P_{n-1}T_{n-1}), \quad n = 1, 2, \ldots,$$

we have

$$\frac{s_m}{Q_m} \to 0 \text{ as } m \to \infty,$$

by virtue of (5a) and (5b). Hence, for $n \geq 0$,

$$\begin{aligned} b_n &= q_n\sum_{v=n}^{\infty}\frac{x_v}{Q_v} = \lim_m q_n\sum_{v=n}^{m}\frac{s_v - s_{n-1}}{Q_v}, \quad (s_{-1}=0) \\ &= \lim_m q_n\left\{\frac{s_m}{Q_m}+\sum_{v=n}^{m-1}\left(\frac{1}{Q_v}-\frac{1}{Q_{v+1}}\right)s_v - \frac{s_{n-1}}{Q_n}\right\} \\ &= -\frac{q_n s_{n-1}}{Q_n} + q_n\sum_{v=n}^{\infty}c_v s_v, \quad \text{where } c_v = \frac{1}{Q_v}-\frac{1}{Q_{v+1}}. \end{aligned}$$

On the other hand, it follows from Abel's summation by parts that

$$\begin{aligned} B_n &= \sum_{v=0}^{n} q_v\frac{b_v}{q_v} = \sum_{v=0}^{n-1}Q_v\frac{x_v}{Q_v} + Q_n\frac{b_n}{q_n} = s_{n-1} + Q_n\left(-\frac{s_{n-1}}{Q_n}+\sum_{v=n}^{\infty}c_v s_v\right) \\ &= Q_n\sum_{v=n}^{\infty}c_v s_v = Q_n\lim_m\sum_{v=n}^{m}c_v s_v = Q_n\lim_m\sum_{v=n}^{m}c_v\frac{1}{p_v}(P_v T_v - P_{v-1}T_{v-1}) \\ &= Q_n\lim_m\left\{c_m\frac{P_m}{p_m}T_m - c_n\frac{P_{n-1}}{p_n}T_{n-1}+\sum_{v=n}^{m-1}P_v\left(\frac{c_v}{p_v}-\frac{c_{v+1}}{p_{v+1}}\right)T_v\right\}. \end{aligned} \tag{8}$$

Now define

$$\overline{N} \ = \ \Big\{(x_k) : \sum_{k=0}^{\infty} x_k \ \text{ is summable } \ (\overline{N}, p_n)\Big\}$$

and

$$B \ = \ \Big\{(x_k) : \Big(\sum_{k=0}^{n} b_k\Big) \ \text{ is convergent}\Big\}.$$

These are BK-spaces (i.e., Banach spaces with continuous coordinates) with respect to the norms

$$\|x\|_{\overline{N}} \ = \ \sup_n |T_n| = \sup_n \Big|\frac{1}{P_n}\sum_{k=0}^{n} p_k s_k\Big| \tag{9}$$

and

$$\|x\|_B \ = \ \sup_n \Big|\sum_{k=0}^{n} b_k\Big| = \sup_n \Big|\sum_{k=0}^{n} q_k \sum_{v=k}^{\infty} \frac{x_v}{Q_v}\Big|, \tag{10}$$

respectively. By the Banach-Steinhaus theorem, there exists a constant $M > 0$ such that

$$\|x\|_B \leq M\|x\|_{\overline{N}} \tag{11}$$

for all $x \in \overline{N}$. Applying (9) and (10) to the special sequence

$$x_n = \begin{cases} 1, & n = k \\ -1, & n = k+1, \qquad k = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\|x\|_{\overline{N}} = \frac{p_k}{P_k} \ \text{ and } \ \|x\|_B = Q_k c_k.$$

It follows from (11) that for $k = 0, 1, 2, \ldots$

$$Q_k c_k \leq M\frac{p_k}{P_k} \Leftrightarrow c_k \cdot \frac{P_k}{p_k} = O\Big(\frac{1}{Q_k}\Big), \tag{12}$$

which implies

$$c_m \frac{P_m}{p_m} \to 0 \ \text{ as } \ m \to \infty, \tag{13}$$

by virtue of (5b). Thus, considering (8) we write

$$B_n = -\frac{Q_n c_n P_{n-1}}{p_n} T_{n-1} + Q_n \sum_{v=n}^{\infty} P_v \Big(\frac{c_v}{p_v} - \frac{c_{v+1}}{p_{v+1}}\Big) T_v = \sum_{v=0}^{\infty} b_{nv} T_v, \quad (14)$$

where the matrix $B = (b_{nv})$ is defined by

$$b_{nv} = \begin{cases} 0, & 0 \le v < n-1, \\ -\dfrac{Q_n c_n P_{n-1}}{p_n}, & v = n-1, \\ Q_n P_v \Big(\dfrac{c_v}{p_v} - \dfrac{c_{v+1}}{p_{v+1}}\Big), & v \ge n. \end{cases} \quad (15)$$

By hypothesis, the matrix B is regular. Now it is clear that $\lim_n b_{nv} = 0$ and $\sum_{v=0}^{\infty} b_{nv} = 1$. Therefore it follows from Lemma 1.2 that B is regular if and only if

$$\sum_{v=0}^{\infty} |b_{nv}| = \frac{Q_n c_n P_{n-1}}{p_n} + Q_n \sum_{v=n}^{\infty} P_v \Big|\frac{c_v}{p_v} - \frac{c_{v+1}}{p_{v+1}}\Big| = O(1) \quad \text{as} \quad n \to \infty. \quad (16)$$

By considering (13) we obtain

$$Q_n P_{n-1} \frac{c_n}{p_n} \le Q_n \sum_{v=n}^{\infty} P_v \Big|\frac{c_v}{p_v} - \frac{c_{v+1}}{p_{v+1}}\Big|$$

and so (16) is equivalent to

$$Q_n \sum_{v=n}^{\infty} P_v \Big|\frac{c_v}{p_v} - \frac{c_{v+1}}{p_{v+1}}| = O(1) \quad \text{as} \quad n \to \infty.$$

Thus, the condition (6) is necessary.

Conversely, if the condition (6) is satisfied, then the series

$$\sum_{v=0}^{\infty} P_v \left(\frac{c_v}{p_v} - \frac{c_{v+1}}{p_{v+1}}\right)$$

converges. Hence it follows that the sequence $\Big(P_n \dfrac{c_{n+1}}{p_{n+1}}\Big)$ converges, which implies that

$$\frac{c_n}{p_n} \to 0 \text{ as } n \to \infty$$

by virtue of (5b). By considering (6) again, we have $\dfrac{P_n c_n}{p_n} \to 0$ as $n \to \infty$ and so (14) is valid. Therefore the result is seen by the regularity of the matrix B, and completes the proof. □

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