Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **5** (2010), No. 1, pp. 41-53

APPROXIMATION OF NONLINEAR STABILITY AND DYNAMICS FOR SOLIDIFICATION OF A DILUTE BINARY ALLOY (KURAMOTO-SIVASHINSKY EQUATION) USING HPM

BY

M. KAZEMINIA, S. A. ZAHEDI AND N. TOLOU

Abstract

The objective of this paper is to present an investigation on the nonlinear stability and dynamics for solidification of a dilute binary alloy that has been represented via well known Kuramoto-Sivashinsky equation. The analysis has been carried out using a semi-analytical method, called homotopy perturbation method (HPM), which did not need small parameters. The perturbation method depends on assumption of small parameter and the obtained results, in most cases, end up with a non-physical result, furthermore, the numerical method may leads to inaccurate results. Homotopy Perturbation Method (HPM) clearly overcame the above shortcomings and furthermore it was very convenient and effective method.

1. Introduction

In many driven nonequilibrium systems, primary instabilities generate periodic patterns that become unstable to secondary instabilities which generate chaotic or disordered structures [1]. Spatiotemporal chaos is a complex phenomenon that arises in many driven nonequilibrium systems such as directional solidification, parametrically driven surface waves, electro convec-

Received September 22, 2008 and in revised form November 4, 2008.

Key words and phrases: solidification, dilute binary alloy, Homotopy perturbation method (HPM), kuramoto-sivashinsky equation.

tion and directional viscous fingering. These examples illustrate the ubiquitous and diverse nature of spatiotemporal chaos [1].

A classic example of this problem is in directional solidification in which a liquid-solid system is driven through a temperature gradient at constant velocity such that the liquid is continuously converted to a solid. If solidification is accompanied by impurity rejection, the buildup of impurities at the interface can lead to a primary instability known as the Mullins-Sekerka instability. At small pulling velocities which tend to select a periodic cellular interface with characteristic wavelength [1]. This situation enables one to derive an asymptotic nonlinear partial differential equation (PDE) of the fourth-order which directly describes the dynamics of the onset and stabilization of cellular structure as follows:

$$u_t + u_{xxxx} + \alpha u + ((2-u)u_x)_x = 0, \quad t \in (0,T) \tag{1}$$

where $\alpha > 0$ and $T > 0$.

This is called the Kuramoto-Sivashinsky equation, see [2, 3]. The Kuramoto-Sivashinsky equation plays an important role as a low-dimensional prototype for complicated fluid dynamics systems which have been studied due to its chaotic pattern forming behavior and is one of the simplest one-dimensional PDE's which exhibits complex dynamical behavior [4]. As an evolution equation, it arises in a number of applications including concentration waves and plasma physics, flame propagation and reaction diffusion combustion dynamics, free surface film-flows and two-phase flows in cylindrical or plane geometries [5].

This equation was introduced by Kuramoto (1976) in one-spatial dimension, for the study of phase turbulence in the Belousov- Zhabotinsky reaction. Sivashinsky derived it independently in the context of small thermal dilutive instabilities for laminar flame fronts. It and related equations have also been used to model directional solidification and, in multiple spatial dimensions, weak fluid turbulence [6].

The K-S equation is non-integrable, therefore the exact solution of this equation is not obtainable and only three numerical schemes have been proposed for the solutions of the Kuramoto-Sivashinsky equation, see [7, 8, 22]. The authors make their investigations on a finite interval $X = [0, 1]$ and they add some initial and boundary conditions in order to obtain the approximate numerical solutions.

Partial differential equations which arise in real-world physical problems are often too complicated to be solved exactly and even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or difficult to interpret the outcome. Very recently, some practical approximate analytical solutions are proposed, such as Exp-function method [9, 10], Adomian decomposition method [11, 12], variational iteration method (VIM) [13, 14] and homotopy-perturbation method (HPM) [15, 16]. Other methods are reviewed in Refs. [17, 18].

HPM is the most effective and convenient one for both linear and nonlinear equations and extremely accessible to non-mathematicians and engineers. This method does not depend on a small parameter or linearization, the solution procedure is very simple, and only few iterations lead to high accurate solutions which are valid for the whole solution domain [19], and can freely choose initial solutions [20]. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". This method was successfully applied to various engineering and physics problems [21, 22, 23]. This paper is motivated to solve problem (1) by means of homotopy perturbation method.

This paper is organized as follows: Fundamentals of the proposed method are presented in Section 2. Following that, in Section 3, some illustrating examples are given in order to assess the benefits of this method and the results of HPM are portrayed graphically. The conclusions are then made in the final Section.

2. Basic Idea of the HPM

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2}$$

with the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{3}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Eq. (3) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega. \tag{4}$$

Homotopy perturbation structure is shown as follows:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{5}$$

where,

$$v(r, p) : \Omega \times [0, 1] \to R. \tag{6}$$

In Eq. (6), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (6) can be written as a power series in p, as following:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{7}$$

and the best approximation for solution is:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \tag{8}$$

3. The Illustrative Examples

Example 3.1. Consider the Sivashinsky equation

$$u_t + u_{xxxx} + \alpha u + [(2-u)u_x]_x = 0, \quad t \in (0, T). \tag{9}$$

Subjected to the initial condition:

$$u(x, 0) = \operatorname{sech}^2\left(\frac{1}{4}x\right), \tag{10}$$

with $\alpha = 0.5$.

Substituting Eq. (7) into Eq. (5) and rearranging based on powers of P-terms, we have the coefficient of P^0:

$$P^0 : \frac{\partial u_0}{\partial t} = 0, \tag{11}$$

with implementation of boundary condition and solution for u_0 we have:

$$u_0(x,t) = \text{sech}^2\left(\frac{1}{4}x\right). \tag{12}$$

The coefficient of P^1:

$$\begin{aligned} P^1 \ : \ & \left\{\frac{11}{8}\text{sech}\left(\frac{1}{4}x\right)^2 \tanh\left(\frac{1}{4}x\right)^2 \left[\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{1}{4}x\right)^2\right]^2 \right. \\ & +\text{sech}\left(\frac{1}{4}x\right)^2 \left[\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{1}{4}x\right)^2\right]^2 + 0.5\text{sech}\left(\frac{1}{4}x\right)^2 \\ & -0.25\text{sech}\left(\frac{1}{4}x\right)^2 \tanh\left(\frac{1}{4}x\right)^2 + 0.5\text{sech}\left(\frac{1}{4}x\right)^2 \tanh\left(\frac{1}{4}x\right)^2 \\ & -\text{sech}\left(\frac{1}{4}x\right)^2 \left[\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{1}{4}x\right)^2\right] - 1.\text{sech}\left(\frac{1}{4}x\right)^2 \left\{\frac{1}{4}\text{sech}\left(\frac{1}{4}x\right)^2 \right. \\ & \left.\left. \times\tanh\left(\frac{1}{4}x\right)^2 - \frac{1}{2}\text{sech}\left(\frac{1}{4}x\right)^2 \left[\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{1}{4}x\right)^2\right]\right\}\right. \\ = \ & 0, \end{aligned} \tag{13}$$

and solution for u_1:

$$u_1(x,t) = -\frac{t\left[-42\cosh\left(\frac{1}{2}x\right) + 81 + 17\cosh(x)\right]}{4\left[\cosh\left(\frac{3}{2}x\right) + 6\cosh(x) + 15\cosh\left(\frac{1}{2}x\right) + 10\right]}, \tag{14}$$

by considering coefficient of P^2 and solving for $u_2(x,t)$, we have:

$$u_2(x,t) = \frac{1}{1024\cosh\left(\frac{1}{4}x\right)^{10}}\left\{\left[-4287\cosh\left(\frac{1}{4}x\right)^6 + 14325\cosh\left(\frac{1}{4}x\right)^2\right.\right.$$

$$+578\cosh\Big(\frac{1}{4}x\Big)^8+9135\Big]t^2\Big\}. \tag{15}$$

The final solution is:

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots \\ u(x,t) &= \operatorname{sech}\Big(\frac{1}{4}x\Big)^2-\frac{t\Big[-42\cosh\Big(\frac{1}{2}x\Big)+81+17\cosh(x)\Big]}{4\Big[\cosh\Big(\frac{3}{2}x\Big)+6\cosh(x)+15\cosh\Big(\frac{1}{2}x\Big)+10\Big]} \\ &\quad +\frac{1}{1024\cosh\Big(\frac{1}{4}x\Big)^{10}}\Big\{\Big[-4287\cosh\Big(\frac{1}{4}x\Big)^6+14325\cosh\Big(\frac{1}{4}x\Big)^2 \\ &\quad +578\cosh\Big(\frac{1}{4}x\Big)^8+9135\Big]t^2\Big\}+\cdots . \end{aligned} \tag{16}$$

Our approximate solution is given by:

$$u_{app}(x,t)=\sum_{i=0}^{2}u_i(x,t). \tag{17}$$

The behavior of $u_{app}(x,t)$ has been illustrated in Figure 1 and Figure 2 and which are obviously accurate to those of [24].

It's illustrated from Figure 1 and Figure 2 that the wave spreads symmetrically from the center of solidification. In fact, it's a main property in the nature of solidification and both Kuramoto-Sivashinsky equation that was possible.

Example 3.2. Let us consider again the Sivashinsky equation

$$u_t+u_{xxxx}+\alpha u+[(2-7)u_x]_x=0, \quad t\in(0,T). \tag{18}$$

Solution 1. We next consider the initial condition as follows:

$$u_0(x,t)=\cos\Big(\frac{1}{2}x\Big) \tag{19}$$

and $\alpha=0.5$.

Substituting Eq. (7) into Eq. (5) and rearranging based on powers of p-terms, we have the coefficient of p^0:

$$p^0 : \frac{\partial u_0}{\partial t} = 0. \tag{20}$$

Similar to previous example with implementation of boundary condition and solution for u_0 we have:

$$u_0(x,t) = \cos\left(\frac{1}{2}x\right). \tag{21}$$

The coefficient of p^1:

$$p^1 : \left(\frac{\partial}{\partial t}u_1(x,t)\right) - 0.25\cos\left(\frac{1}{2}x\right)^2 + 0.5625\cos\left(\frac{1}{2}x\right) + 0.25\sin\left(\frac{1}{2}x\right)^2 = 0. \tag{22}$$

And solution for u_1

$$u_1(x,t) = \frac{1}{4}t\cos(x) - \frac{9}{16}t\cos\left(\frac{1}{2}x\right). \tag{23}$$

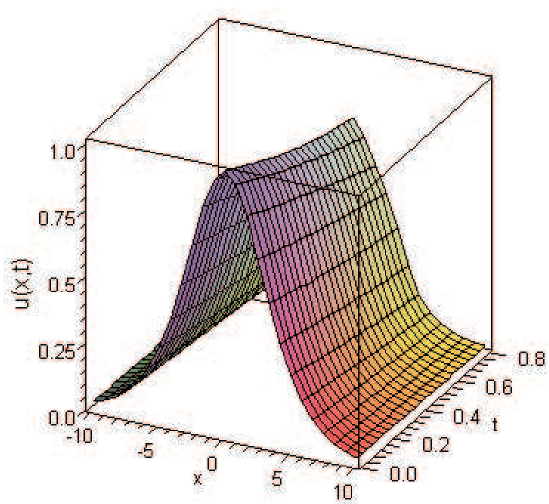

Figure 1. The graph of the approximate solution for Example 1.

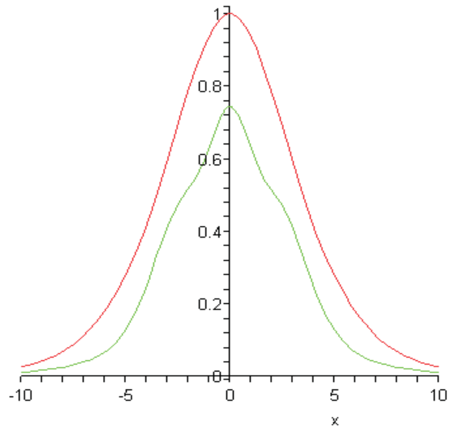

Figure 2. The graph of the approximate solution for Example 1 at $t = 0$ (red line) and $t = 0.5$ (green line).

The coefficient of p^2:

$$\begin{aligned} p^2 \ : \ & \left(\frac{\partial}{\partial t}u_2(x,t)\right) + \cos\left(\frac{1}{2}x\right)\left(-\frac{1}{4}t\cos(x) + \frac{9}{64}t\cos\left(\frac{1}{2}x\right)\right) \\ & -\sin\left(\frac{1}{2}x\right)\left(-\frac{1}{4}t\sin(x) + \frac{9}{32}t\sin\left(\frac{1}{2}x\right)\right) + 0.375t\cos(x) \\ & -0.3164t\cos\left(\frac{1}{2}x\right) - 0.25\left(\frac{1}{4}t\cos(x) - \frac{9}{16}t\cos\left(\frac{1}{2}x\right)\right)\cos\left(\frac{1}{2}x\right) = 0 \quad (24) \end{aligned}$$

and solution for u_2:

$$u_2(x,t) = \frac{89}{512}t^2\cos\left(\frac{1}{2}x\right) + \frac{9}{64}t^2\cos\left(\frac{3}{2}x\right) - \frac{21}{64}t^2\cos(x). \qquad (25)$$

The final solution is:

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ u(x,t) &= \cos\left(\frac{1}{2}x\right) + \frac{1}{4}t\cos(x) - \frac{9}{16}t\cos\left(\frac{1}{2}x\right) + \frac{89}{512}t^2\cos\left(\frac{1}{2}x\right) \\ & \quad + \frac{9}{64}t^2\cos\left(\frac{3}{2}x\right) - \frac{21}{64}t^2\cos(x) + \cdots . \qquad (26) \end{aligned}$$

The numerical results for the approximate solution of Example 2 by using HPM according to Eq. (17) are portrayed in Figure 3 and Figure 4. These figures well agree to those of [24].

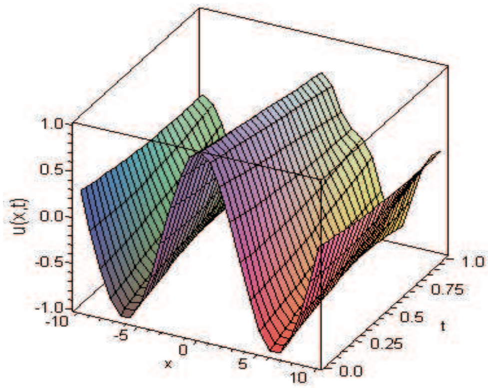

Figure 3. The graph of the approximate solution for Example 2.

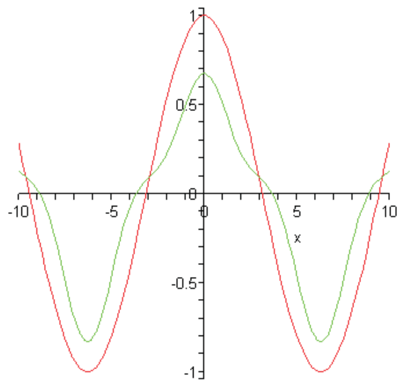

Figure 4. The graph of the approximate Solution 1 for Example 2 at $t = 0$ (red line) and $t = 0.5$ (green line).

As illustrated in Figure 3 and Figure 4 the qualitative behavior of the KS equation is quite simple. Cellular structures are generated due to the linear instability. These cells then interact chaotically with each other via the nonlinear spatial coupling to form the spatiotemporal chaos (STC) steady state [25].

Solution 2. As mentioned, the advantage of the homotopy perturbation method is that it can freely choose initial solutions, therefore this selection is efficacious on the length of calculation.

Here, the initial condition is assumed with unknown parameter:

$$u_0(x,t) = \cos(0.5x + bt). \tag{27}$$

Where b is the unknown parameter and the cosine form is due to the symmetric shape of physical properties in solidification.

According to the initial guess and Eq. (18), a homotopy should be constructed:

$$\begin{aligned} &u_t(x,t) + 0.5u(x,t) - u_{0_t}(x,t) - 0.5u_0(x,t) - p\lfloor u_{0_t}(x,t) + 0.5u_0(x,t)\rfloor \\ &\quad +p\{u_{xxxx}(x,t) + [(2 - u(x,t))_x]_x\} = 0. \end{aligned} \tag{28}$$

Where $p \in [0,1]$ is embedding parameter and it is obvious that when $p = 0$, Eq. (28) becomes a linear equation and; when $p = 1$ it becomes the original nonlinear one.

Using p as an expanding parameter as that one in classic perturbation method, we have:

$$u_{0_t}(x,t) + u_0(x,t) + b\sin(0.5x + bt) - \cos(0.5x + bt) = 0 \tag{29}$$

$$\begin{aligned} &u_{1_t}(x,t) + u_1(x,t) - b\sin(0.5x + bt) + \cos(0.5x + bt) + u_{0_{xxxx}}(x,t) \\ &\quad +u_{0_x}(x,t)^2 + 2u_{0_{xx}}(x,t) - u_0(x,t)u_{0_{xx}}(x,t) = 0. \end{aligned} \tag{30}$$

Generally, we need few items only. Setting $p = 1$, we obtain the first order approximate solution which reads:

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) \\ &= \cos(0.5x + bt) - \frac{1}{32}\frac{1}{1 + 5b^2 + 4b^4}\Bigg\{e^{-1}\Bigg[\cos(x) + \cos(x)b^2 \\ &\quad +2\sin(x)b + 2\sin(x)b^3 - 104b^2\cos\left(\frac{1}{2}x\right) - 128b^2\cos\left(\frac{1}{2}x\right) \\ &\quad +14b\sin\left(\frac{1}{2}x\right) + 56b^3\sin\left(\frac{1}{2}x\right) - 18\cos\left(\frac{1}{2}x\right) - 1 - 5b^2 - 4b^2 \\ &\quad +8b\cos(x) + 8b^3\cos(x) - 4b^2\sin(x)\Bigg]\Bigg\} + \frac{1}{32}\Bigg\{\cos(x + 2bt) \\ &\quad +b^2\cos(x + 2bt) + 2b\sin(x + 2bt) + 2b^3\sin(x + 2bt) \\ &\quad -104b^2\cos(0.5x + bt) - 128b^4\cos(0.5x + bt) + 14b\sin(0.5x + bt) \end{aligned}$$

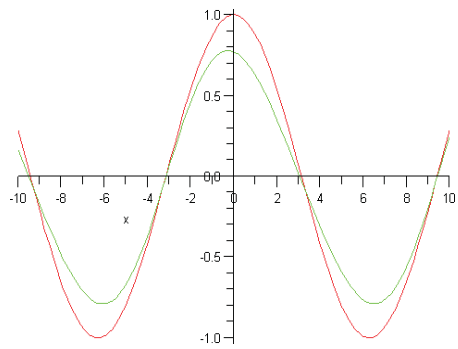

Figure 5. The graph of the approximate Solution 2 for Example 2 at $t = 0$ (red line) and $t = 0.5$ (green line) with $b = 0.25$.

$$+56b^3\sin(0.5x+bt)-18\cos(0.5x+bt)-1-5b^2-4b^4$$
$$\left.+8b\cos(x+2bt)+8b^3\cos(x+2bt)-4b^2\sin(x+2bt)\right\}\Big/(1+5b^2+4b^4) \quad (31)$$

There are many approaches to identification of the unknown parameter in the obtained solution. We suggest hereby the method of the weighted residuals, spatially the last squares method:

$$\int_0^1 R\frac{\partial R}{\partial b}dt = 0. \qquad (32)$$

Where R is the residual $R(u(x,t)) = Lu + Nu$.

As illustrated in Figure (5) and comparison with Figure (4), the initial solution can have intensive affection on the convergence of the process.

4. Conclusions

In this paper, our objective has been the investigation of nonlinear behavior of a prototypical partial-integral differential equation arising in fluid dynamics called Kuramoto-Sivashinsky equation using an effective and convenient method, called Homotopy perturbation method (HPM). We consider its special implementation in solidification of binary dilute alloy. The results obviously illustrate the axisymmetric behavior of this phenomenon that was predictable from the nature of spatiotemporal chaos of solidification.

Besides, this survey clearly demonstrated the capability of HPM to solve a large class of differential equations with rapid convergence. HPM is very intelligible, because it reduces the size of calculations. An interesting point about HPM is that with the fewest number of iterations or even in some cases, once, it can converge to correct results. The homotopy perturbation method, which has been used to solve the differential equations, seems to be very straightforward and accurate to approach reliable results. The obtained approximate results are only from three terms of evaluation which they are in perfect agreement with the Ref. [24].

References

1. K. R. Elder, J. D. Gunton and N. Goldenfeld, Transition to spatiotemporal chaos in the damped Kuramoto-Sivashinsky equation, *Physical review*, **56**(1997), no. 2, 1631-1634.

2. G. I. Sivashinsky, On cellular instability in the solidification of a dilute binary alloy, *Phys. D*, **8**(1983), 243-248.

3. V. G. Gertsberg and G. I. Sivashinsky, Large cells in nonlinear Rayleigh - Bénard convection, *Prog. Theor. Phys.*, **66**(1981), 1219.

4. M. Jardak and I. M. Navon, Particle filter and EnKF as data assimilation methods for the Kuramoto-Sivashinsky Equation. Preprint submitted to Elsevier, 2007.

5. Y. S. Smyrlis and D. T.Papageorgiou, Computational study of chaotic and ordered solutions of the Kuramoto-Sivashinsky equation, ICASE reports, 1996, ICASE Report No. 96-12.

6. Li. Huaming, Kuramoto-Sivashinsky weak turbulence, in the symmetry unrestricted space. School of Physics, Georgia Institute of Technology Atlanta, 2003, 30318.

7. S. Benammou and K. Omrani, A finite element method for the Sivashinsky equation, *J. Comput. Appl. Math.*, **16**(2002), 419-431.

8. K. Omrani, A second-order splitting method for a finite difference scheme for the Sivashinsky equation.}, *Appl. Math. Lett.*, **16**(2003), 441-445.

9. J. H. He and Xu-Hong Wu. Exp-function method for nonlinear wave equations, *Chaos Solitons Fractals*, **30**(2006), no. 3, 700-708.

10. I. Khatami, N. Tolou, J. Mahmoudi and M. Rezvani, Application of Homotopy Analysis Method and Variational Iteration Method for Shock Wave equation, *J. Appl. Sci.*, **8**(2008), no. 5, 848-853.

11. A. M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu-type equations, *Appl. Math. Comput.*, **166**(2005), 652-663.

12. S. Saha Ray, A numerical solution of the coupled sine-Gordon equation using the modified decomposition method, *Appl. Math. Comput.*, **175**(2006), 1046-1054.

13. J. H. He, Variational iteration method - a kind of non-linear analytical technique: Some examples, *Internat. J. Non-Linear Mech.*, **34**(1999), 699-708.

14. D. D. Ganji and A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, *J. Comput. Appl. Math.*, 2006, In press.

15. J. H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, *Internat. J. Non-Linear Mech.*, **35**(2000), no. 1, 37-43.

16. N. Tolou, I. Khatami, B. Jafari and D. D. Ganji, Analytical Solution of Nonlinear Vibrating Systems, *Amer. J. Appl. Sci.*, **5**(2008), no. 9, 1219-1224.

17. J. H. He, Some asymptotic methods for strongly nonlinear equations, *Internat. J. Modern Phys. B*, **20**(2006), no. 10, 1141-1199.

18. J. H. He, Non-perturbative methods for strongly nonlinear problems, Berlin: dissertation.de-Verlag Im. Internet GmbH, 2006.

19. J. H. He, Recent development of the homotopy perturbation method, *Topol. Methods Nonlinear Anal.*, **31**(2008), no. 2, 205-209.

20. J. H. He, An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, *Internat. J. Mod. Phys.*, **B**(2008), no. 22, 3487-3578.

21. A. M. Siddiqui, R. Mahmood and Q. K. Ghori, Homotopy-perturbation method fot thin film flow of a fourth grade fluid down a vertical cylinder, *Phys. Lett. A*, **352**(2006), 404-410.

22. A. Rajabi, D.D. Ganji and H. Taherian, Application of homotopy-perturbation method to nonlinear heat conduction and convection equations, *Phys. Lett. A*, **360**(2007), no. 4-5, 570-573.

23. D. D. Ganji and A. Rajabi, Assessment of homotopy-perturbation and perturbation method in heat radiation equations, *International Communications in Heat and Mass Transfer*, **33**(2006), 391-400.

24. S. Momani, A numerical scheme for the solution of Sivashinsky equation, *Appl. Math. Comput.*, **168**(2005), 1273-1280.

25. B. Boghosian, C. C. Chow and T. Hwa, Hydrodynamics of the Kuramoto-Sivashinsky Equation in Two Dimensions. arXiv:cond- mat/9911069v1 [cond-mat.soft] 4, Nov, 1999.

Department of Mechanical Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran.
E-mail: mehdi.kazeminia@yahoo.com

Department of Mechanical Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran.
E-mail: abolfazlzahedi@yahoo.com

Delft University of Technology, Department of Mechanical Engineering, Mekelweg 2, 2628 CD Delft, The Netherlands.
E-mail: n.tolou@tudelft.nl