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EIGENFUNCTIONS OF THE ADJOINT OPERATOR ASSOCIATED WITH A PULSE SOLUTION OF SOME REACTION-DIFFUSION SYSTEMS

BY

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Dedicated to Professor Masayasu Mimura on his 65th birthday

Abstract

In this paper, we consider the FitzHugh-Nagumo system and obtain by singular perturbation techniques a precise form of the eigenfunction of the adjoint operator associated with the linearization at a traveling pulse solution. We also show that some precise properties of the adjoint eigenfunction are useful to study the behavior of solutions such as interfacial dynamics, the interaction of traveling pulses and so on.

1. Introduction

The FitzHugh-Nagumo system is a simplified mathematical model that describes the generation and propagation of nerve impulses. This system can be written as

$$\begin{cases} \varepsilon u_t = \varepsilon^2 u_{xx} + f(u) - v, & x \in (-\infty, \infty), \\ v_t = u - \gamma v, & x \in (-\infty, \infty), \\ \lim\limits_{|x|\to\infty} u(x,t) = \lim\limits_{|x|\to\infty} v(x,t) = 0, & \end{cases} \tag{1.1}$$

where $f(u) = u(1-u)(u-a)$ with $0 < a < 1/2$, $\varepsilon > 0$ is a parameter and

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$\gamma > 0$ is a fixed constant. It is known that if $\varepsilon > 0$ is small enough, then (1.1) has two traveling pulse solutions with different speed. It was shown in [3] that the slower pulse solution is unstable, whereas it was proved independently by Jones [8] and Yanagida [12] that the fast traveling pulse solution is stable if $\varepsilon > 0$ is sufficiently small.

On the other hand, the following combustion model was proposed by Mimura and Ikeda [6];

$$\begin{cases} \varepsilon u_t = \varepsilon^2 u_{xx} + \gamma k(u)v - au, & x \in (-\infty, \infty), \\ v_t = -k(u)v, & x \in (-\infty, \infty), \\ \lim_{|x|\to\infty} u(x,t) = 0, \quad \lim_{x\to-\infty} v(x,t) = \overline{v}, \end{cases} \tag{1.2}$$

where $k(u)$ is Arrhenius Kinetics defined by

$$k(u) = \begin{cases} A\exp(-B/(u-\theta)), & u > \theta, \\ 0, & 0 \le u \le \theta \end{cases}$$

for some constants $A, B > 0$, $\theta \ge 0$, $\overline{v} > 0$, $a > 0$ and $\gamma > 0$. This system is a limiting equation of a 3-component system

$$\begin{cases} \varepsilon u_t = \varepsilon^2 u_{xx} + \gamma k(u)vw - au, & x \in (-\infty, \infty), \\ v_t = -k(u)vw, & x \in (-\infty, \infty), \\ w_t = w_{xx} - \lambda w_x - k(u)vw & x \in (-\infty, \infty), \\ \lim_{|x|\to\infty} u(x,t) = 0, \quad \lim_{x\to-\infty} v(x,t) = \overline{v}, \quad \lim_{x\to-\infty} w(x,t) = 1 \end{cases} \tag{1.3}$$

as $\lambda \to \infty$. In [5], it was shown that there is a linearly stable traveling wave solution in (1.2) if $\varepsilon > 0$ is sufficiently small and other parameters satisfy suitable conditions. Also, a linearly stable traveling wave solution in (1.3) has been obtained under the same conditions on the parameters as far as λ is sufficiently large.

In this paper, we focus on these two system:

$$\begin{cases} \varepsilon u_t = \varepsilon^2 u_{xx} + f(u,v), & x \in (-\infty, \infty), \\ v_t = g(u,v), & x \in (-\infty, \infty), \\ u(\pm\infty, t) = 0, \quad v(-\infty, t) = 0, \end{cases} \tag{1.4}$$

where the nonlinear terms $f(u,v)$ and $g(u,v)$ are either

$$\begin{pmatrix} f(u)-v \\ u-\gamma v \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \gamma k(u)(v+\overline{v})-au \\ -k(u)(v+\overline{v}) \end{pmatrix}.$$

(Later in this section, we shall give a remark on more general nonlinearities.) With a traveling coordinate system $(z,t)=(x+ct,t)$, (1.4) is written as

$$\begin{cases} \varepsilon u_t=\varepsilon^2 u_{zz}-\varepsilon c u_z+f(u,v) & z\in(-\infty,\infty), \\ v_t=-cv_z+g(u,v), & z\in(-\infty,\infty), \\ u(\pm\infty,t)=0, \quad v(-\infty,t)=0. \end{cases} \tag{1.5}$$

Any stationary solution of (1.5) corresponds to a traveling wave solution of (1.4) with the traveling speed c. Let $(u,v)=(u(z),v(z))$ be a traveling pulse solution with the propagation speed $c=c(\varepsilon)>0$. Then $(u(z),v(z))$ satisfies

$$\begin{cases} \varepsilon^2 u_{zz}-\varepsilon c u_z+f(u,v)=0, & z\in(-\infty,\infty), \\ -cv_z+g(u,v)=0, & z\in(-\infty,\infty), \\ u(\pm\infty)=0, \quad v(-\infty)=0. \end{cases} \tag{1.6}$$

We fix the translation of the pulse solution by demanding

$$u(0)=\alpha, \quad u_z(0)>0.$$

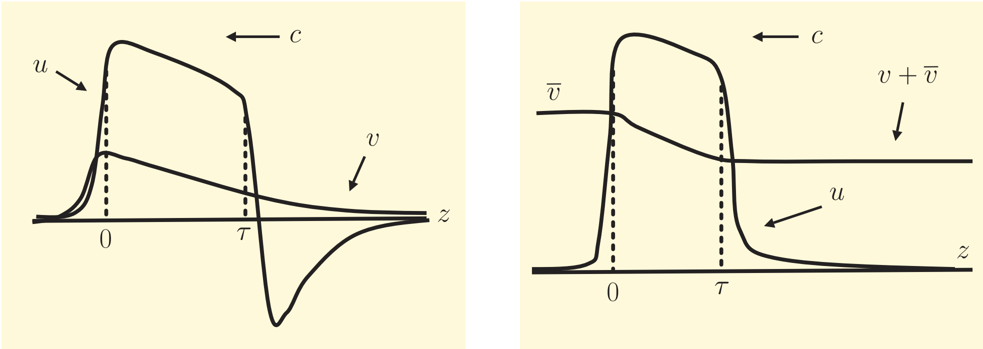
Let $\beta>0$ be chosen appropriately, and let $\tau>0$ be defined by

$$u(\tau)=\beta, \quad u_z(\tau)<0.$$

Then the profile of the solution is as shown in Figure 1. Note that it has two transition layers near $z=0$ and $z=\tau$.

Let us consider the linearized equation around (u,v), i.e.

$$\begin{cases} \varepsilon^2 P_{zz}-\varepsilon c P_z+f_u(u,v)P+f_v(u,v)Q=0, & z\in(-\infty,\infty), \\ -cQ_z+g_u(u,v)P+g_v(u,v)Q=0, & z\in(-\infty,\infty), \\ P(\pm\infty)=0, \quad Q(\pm\infty)=0 \end{cases} \tag{1.7}$$



(a) Pulse solution in the FitzHugh-Nagumo system.

(b) Pulse solution in the combustion model.

Figure 1. The profiles of pulse solutions.

and its adjoint equation

$$\begin{cases} \varepsilon^2 P_{zz} + \varepsilon c P_z + f_u(u,v)P + g_u(u,v)Q = 0, & z \in (-\infty,\infty), \\ cQ_z + f_v(u,v)P + g_v(u,v)Q = 0, & z \in (-\infty,\infty), \\ P(\pm\infty) = 0, \quad Q(\pm\infty) = 0. \end{cases} \tag{1.8}$$

Differentiating (1.6) by z, we see that (1.7) has a bounded solution $(P,Q) = (u_z, v_z)$. Hence the adjoint equation (1.8) also has a bounded solution. We will demonstrate that the properties of the bounded solution of the adjoint system (1.8) plays essential roles for several problems concerning the fast pulse solution, which will be described in the next section.

The main purpose of this paper is to construct the bounded solution of (1.8) when $\varepsilon > 0$ is sufficiently small, and obtain the asymptotic profile of the solution as $\varepsilon \to 0$. Although the existence of a pulse solution was proved by Hastings [4] and Langer [9] for the FitzHugh-Nagumo system and in [5] for the combustion model via geometrical methods, their results are insufficient for our purpose because more precise information about the pulse solution is needed to construct an eigenfunction of the adjoint operator and study the asymptotic behavior as $\varepsilon \to 0$. In this paper, we adapt a singular perturbation approach to get more precise information about the waveform of the fast pulse solution. Using this, we can construct the bounded solution of the adjoint equation and obtain useful properties of the eigenfunction.

This paper is organized as follows; In Section 2, we state main results of this paper and then give several applications of the results. In Section 3, we construct a fast pulse solution via a singular perturbation method. In Section 4, we construct a bounded solution of the adjoint equation. In Appendix, we give rigorous proofs of Theorems 3 and 4. Other theorems can be proved in the same way.

2. Main Result and Applications

We first describe several facts and notations. Under adequate assumptions, all conditions below hold true.

- There are $v_{min} < v_{max}$ such that the nullcline of f includes two smooth curves $u = h_-(v)$ and $u = h_+(v)$ defined on $[v_{min}, v_{max}]$ (see Figure 2).
- The problem

$$\begin{cases} \ddot{\Phi}_1 - c\dot{\Phi}_1 + f(\Phi_1, 0) = 0, & \xi \in (-\infty, \infty), \\ \Phi_1(-\infty) = 0, \quad \Phi_1(\infty) = h_+(0) \end{cases}$$

  has a monotone solution Φ_1 with a wave speed $c = c_0^* > 0$, where the dot " $\dot{}$ " represents a derivative of functions with respect to ξ. We fix the solution by $\Phi_1(0) = \alpha$, where $0 < \alpha < h_+(0)$ is arbitrarily fixed.
- The problem

$$\begin{cases} \ddot{\Phi}_2 - c_0^*\dot{\Phi}_2 + f(\Phi_2, v) = 0, & \xi \in (-\infty, \infty), \\ \Phi_2(-\infty) = h_+(v), \quad \Phi_2(\infty) = h_-(v) \end{cases}$$

  has a monotone solution Φ_2 for $v = v^* \neq 0$. We fix the solution by $\Phi_2(0) = \beta$, where $h_-(v^*) < \beta < h_+(v^*)$ is arbitrarily fixed.
- Let $V_0^{(2)}$ be a solution of

$$\begin{cases} c_0^* v' = g(h_+(v), v), \quad y > 0, \\ v(0) = 0 \end{cases}$$

  and attain v^* at $y = \tau_0^* > 0$. Set $U_0^{(2)} = h_+(V_0^{(2)})$.
- Let $V_0^{(3)}$ be a solution of

$$\begin{cases} c_0^* v' = g(h_-(v), v), \quad y > 0, \\ v(0) = v^* \end{cases}$$

  and set $U_0^{(3)} = h_-(V_0^{(3)})$.

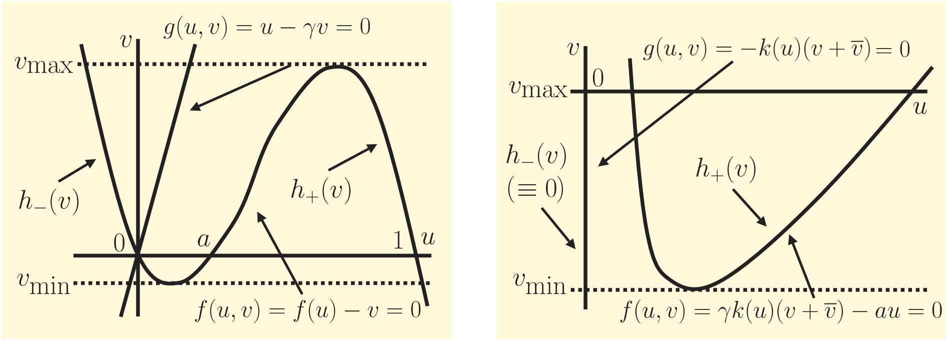


Figure 2. Nullclines of f and g. The left figure shows the null cline of the nonlinearity in the FitzHugh-Nagumo system and the right one does that in the combustion model. Although not only $u = 0$ but also $v = -\overline{v}$ should be included in the null cline of g for the combustion model, the line $v = -\overline{v}$ is omitted in this figure.

We may need more conditions for the parameters in order to show the existence of these functions.

Now we describe our main results in this paper, using these notations. In the following, we write $\phi = O(\delta)$ if a functions ϕ and a small parameter $\delta > 0$ satisfy $\|\phi\|_X \leq c\delta$ for a constant c independent of δ, where X is a Banach space endowed with the norm $\|\cdot\|_X$. Similarly, if ϕ satisfies $\|\phi\|_X/\delta \to 0$ as $\delta \to 0$, we write $\phi = o(\delta)$. The first theorem is concerning the existence of a pulse solution of (1.4).

Theorem 1. *The system* (1.6) *has a pulse solution with the following properties as* $\varepsilon \to 0$*:*

(a) *The propagation speed* c *satisfies*

$$c = c_0^* + O(\varepsilon).$$

(b) *The pulse width* τ *satisfies*

$$\tau = \tau_0^* + O(\varepsilon).$$

(c) *The waveform* $(u(z), v(z))$ *satisfies*

$$\begin{cases} u(z) = \Phi_1(\frac{z}{\varepsilon}) + O(\varepsilon), \\ v(z) = \frac{\varepsilon}{c_0^*} \int_{-\infty}^{\frac{z}{\varepsilon}} g(\Phi_1, 0) ds + o(\varepsilon), \end{cases} \qquad \text{in } X_\mu, z \in (-\infty, 0),$$

$$\begin{cases} u(z) = U_0^{(2)}(z) + \Phi_1(\frac{z}{\varepsilon}) - h_+(0) + \Phi_2(\frac{z-\tau}{\varepsilon}) - h_+(v^*) + O(\varepsilon), \\ v(z) = V_0^{(2)}(z) - \frac{\varepsilon}{c_0^*} \int_{\frac{z}{\varepsilon}}^{\infty} (g(\Phi_1, 0) - g(h_+(0), 0)) ds \\ \qquad + \frac{\varepsilon}{c_0^*} \int_{-\infty}^{\frac{z-\tau}{\varepsilon}} (g(\Phi_2, v^*) - g(h_+(v^*), v^*)) ds + o(\varepsilon), \end{cases} \qquad \text{in } X_\varepsilon, z \in [0, \tau],$$

$$\begin{cases} u(z) = U_0^{(3)}(z) + \Phi_2(\frac{z-\tau}{\varepsilon}) - h_-(v^*) + O(\varepsilon), \\ v(z) = V_0^{(3)}(z) + \frac{\varepsilon}{c_0^*} \int_{\frac{z-\tau}{\varepsilon}}^{\infty} (g(\Phi_2, v^*) - g(h_-(v^*), v^*)) ds \\ \qquad + o(\varepsilon), \end{cases} \qquad \text{in } X_{\mu,\varepsilon}, z \in (\tau, \infty),$$

where the functional spaces $X_\mu, X_\varepsilon, X_{\mu,\varepsilon}$ *will be defined in Section* 3.

In Theorem 1, we only give the information of the lowest order terms of the propagating speed, the pulse width and the waveform. But in its proof (see Section 3), we get more specific properties of higher order terms of traveling wave solution with respect to the small parameter ε, which are necessarily needed to show Theorem 2. Theorem 2 is applicable to many interesting problems as we shall give several applications below. To show only Theorem 1, it suffices for us to get lowest order approximations and use analytic or geometric singular perturbation theory (see [5], [8]).

Since the linearized equation (1.7) has a unique solution up to multiplication by constants, the solution of the adjoint equation (1.8) is also unique up to multiplication by constants by Fredholm's alternatives. In the following theorem, we normalize a solution P satisfying $P(0) = A/\varepsilon$.

Theorem 2. *Fix* $A \in (-\infty, \infty)$ *arbitrarily. The adjoint equation* (1.8) *has a bounded solution* (P, Q) *with the following properties as* $\varepsilon \downarrow 0$*:*

$$\begin{cases} P(z) = \varepsilon P_0^{(1)}(z) + A e^{-\frac{c_0^* z}{\varepsilon}} \dot{\Phi}_1(\frac{z}{\varepsilon}) + \varepsilon \zeta_1(\frac{z}{\varepsilon}) + o(\varepsilon), \\ Q(z) = \varepsilon Q_0^{(1)}(z) + \varepsilon \eta_0(\frac{z}{\varepsilon}) + o(\varepsilon), \end{cases} \qquad \text{in } X_{\nu,\varepsilon}, z \in (-\infty, 0),$$

$$\begin{cases} P(z)=\varepsilon P_0^{(2)}(z)+Ae^{-\frac{c_0^* z}{\varepsilon}}\dot{\Phi}_1(\frac{z}{\varepsilon})+B(A,\varepsilon)e^{-\frac{c_0^* z}{\varepsilon}}\dot{\Phi}_2(\frac{z-\tau}{\varepsilon})+o(\varepsilon), \\ Q(z)=\varepsilon Q_0^{(2)}(z)+\varepsilon\frac{A}{c_0^*}\int_{\frac{z}{\varepsilon}}^{\infty} f_v(\Phi_1,0)e^{-c_0^* s}\dot{\Phi}_1(s)ds \qquad in\ X_\varepsilon, z\in[0,\tau], \\ \qquad -\varepsilon\frac{B(A,\varepsilon)}{c_0^*}\int_{-\infty}^{\frac{z-\tau}{\varepsilon}} f_v(\Phi_2,v^*)e^{-c_0^* s}\dot{\Phi}_2(s)ds+o(\varepsilon), \end{cases}$$

$$\begin{cases} P(z)=B(A,\varepsilon)e^{-\frac{c_0^* z}{\varepsilon}}\dot{\Phi}_2(\frac{z-\tau}{\varepsilon})+o(\varepsilon), \\ Q(z)=\varepsilon\frac{B(A,\varepsilon)}{c_0^*}\int_{\frac{z-\tau}{\varepsilon}}^{\infty} f_v(\Phi_2,v^*)e^{-c_0^* s}\dot{\Phi}_2(s)ds+o(\varepsilon^2), \end{cases} \qquad in\ X_\nu, z\in(\tau,\infty).$$

Here $P_0^{(1)}, Q_0^{(1)}$ *are smooth and decay exponentially as* $z\to-\infty$, $P_0^{(2)}, Q_0^{(2)}$ *are smooth, and* $\zeta_1=\zeta_1(\xi)$ *and* $\eta_0=\eta_0(\xi)$ *are smooth and decay exponentially as* $\xi\to-\infty$. *All functional spaces and functions above will be defined in Section* 4. *The constant* $B(A,\varepsilon)$ *tends to* 0 *as* $\varepsilon\to 0$.

Here we remark the generalization of f and g. Although we consider only two nonlinearities in this paper, we can show the same results for more general nonlinear terms with a bistable condition. Then we obtain a different type of a homoclinic-heteroclinic solution, which means that the function u of a solution of (1.6) tends to 0 as $|z|\to\infty$ and the function v converges to different values as $|z|\to\infty$. Those types may not have been considered yet.

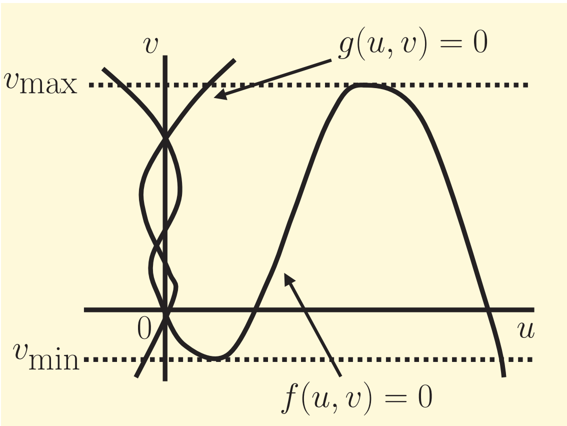


Figure 3. A nullcline of some nonlinearity for which there is other type pulse solution in (1.4).

In the following, we collect several applications of Theorem 2.

(A) Transversality of stable and unstable manifolds

In the case of FitzHugh-Nagumo nonlinearity, (1.6) can be rewritten as

$$\mathbf{u}_z = \mathbf{f}(\mathbf{u}; \varepsilon), \quad z \in (-\infty, \infty),$$

where we set $z \to \varepsilon z$ and put

$$u_z = p, \quad \mathbf{u} = (u, p, v, c)^t, \quad \mathbf{f}(\mathbf{u}, \varepsilon) = (p, cp - f(u) + v, \frac{\varepsilon}{c}u - \frac{\varepsilon}{c}\gamma v, 1)^t.$$

Let $(u^\varepsilon, v^\varepsilon, c^\varepsilon)$ be a traveling wave solution of (1.6) and W_u and W_s are a two-dimensional center-unstable manifold and a three-dimensional center-stable manifold of $(0,0,0)^t$ and c around c^ε. Since $(u^\varepsilon, u_z^\varepsilon, v^\varepsilon, c^\varepsilon)$ is on both these manifolds, $W_s \cap W_u \neq \phi$ at $c = c^\varepsilon$. It was shown by Evans [2] that if

$$\int_{-\infty}^{\infty} (u_z^\varepsilon P + v_z^\varepsilon Q) dz \neq 0,$$

then W_u and W_s intersect transversally as c exceeds c^*, where (P, Q) is an eigenfunction of the adjoint equation (1.8). Moreover, the sign of the above integral is closely related to the stability of the fast pulse solution. The transversality can be proved by Langer [9] using a geometric argument. By Theorems 1 and 2, we can show that the above integral is positive, which gives a necessary condition for the stability.

(B) Response to a disturbance

We introduce a traveling coordinate system $z = x + ct$ and do some suitable scaling. Then (1.1) is rewritten as

$$\begin{cases} u_t = u_{zz} - cu_z + f(u) - v, & z \in (-\infty, \infty), \\ v_t = -cu_z + \varepsilon(u - \gamma v), & z \in (-\infty, \infty). \end{cases} \tag{2.1}$$

Clearly, any traveling wave solution of (1.1) corresponds to a stationary solution of this equation.

It was proved independently by Jones [7] and Yanagida [12] that if ε is sufficiently small, the fast pulse solution, denoted by (u, v), is asymptotically stable in the sense of waveform stability. More precisely, if $(U(z), V(z))$ is bounded, then the solution of (2.1) with

$$u(0,t) = u(z) + \mu U(z), \quad v(0,t) = v(z) + \mu V(z),$$

where μ is a small parameter, satisfies

$$\lim_{t\to\infty} u(z,t) = u(z+\mu\theta), \quad \lim_{t\to\infty} v(z,t) = v(z+\mu\theta).$$

It was proved by Yanagida [13] that the phase shift θ satisfies

$$\theta = \frac{\int_{-\infty}^{\infty}(UP+VQ)dz}{\int_{-\infty}^{\infty}(u_z P + v_z Q)dz} + o(1) \quad \text{as } \mu \to 0.$$

Thus the bounded solution $(P(z), Q(z))$ of (1.8) gives a weight function.

(C) Stability of planar pulse solutions.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. We consider the equation

$$\begin{cases} \varepsilon u_t = \varepsilon^2 \Delta u - c\varepsilon u_z + f(u,v), & (z,y) \in (-\infty,\infty)\times\Omega, \\ v_t = -cv_z + g(u,v), & (z,y) \in (-\infty,\infty)\times\Omega. \end{cases} \tag{2.2}$$

It is clear that a traveling wave solution of (1.4) corresponds to a solution of (2.2) that is constant in y-direction. Such a solution is called a planar pulse solution. To consider the stability of the planar solution in the cylindrical domain, called the planar stability, we introduce a linearized eigenvalue problem with a new parameter $l > 0$;

$$\begin{cases} (\mu + l)\phi = \varepsilon^2\phi'' - \varepsilon c\phi' + f_u(u,v)\phi + f_v(u,v)\psi, \\ \mu\psi = -\varepsilon c\psi' + \varepsilon g_u(u,v)\phi + \varepsilon g_v(u,v)\psi. \end{cases} \tag{2.3}$$

By using a similar argument in [12] or [5], it was shown in [11] that there is no eigenvalue of (2.3) with a positive real part if $l > 0$ is independent of $\varepsilon > 0$.

Moreover, if l converges 0 as $\varepsilon \to 0$, eigenvalues μ of (2.3) staying in

$\{\lambda \in \mathbb{C} \mid Re\lambda > -\delta\}$ for a small $\delta > 0$ must satisfy

$$\mu \sim -l \frac{\int_{-\infty}^{\infty} u_z P dz}{\int_{-\infty}^{\infty} (u_z P + v_z Q) dz}$$

as $\varepsilon \to 0$. Hence the stability of the planar pulse solution is determined by the bounded solution of (1.8).

In the case of the combustion model, we need to study (2.3) in a weighted Sobolev space because the essential spectrum comes to the imaginary axis if we consider (2.3) in a usual Sobolev space or continuous functions space (see [10], [5]). We shall discuss this in detail in Section 4.

In fact, the authors in [11] also considered the planar stability of a traveling wave in the FitzHugh-Nagumo system. However, in that paper, the cylindrical domain depends on ε and becomes thinner and thinner as $\varepsilon \to 0$, and so it is insufficient to investigate eigenvalues of (2.3) for any given $l > 0$ independent of ε. On the other hand, our domain Ω is independent of ε in (2.2).

(D) Pulse interaction

In Section 3.2 of [1], the author considered the interaction of two stable 1-pulse solutions moving toward the same direction in (1.1) with $f(u) = u(1-u)(u-a)$ for $0 < a < \frac{1}{2}$, and demonstrated that the two pulses are repulsive. He claimed that the distance between two 1-pulses denoted by $h = h(t)$ is governed by a differential equation

$$\dot{h} \sim -M_\alpha e^{-\alpha h}$$

provided that h is sufficiently large, where $\dot{h}$ denotes the derivative of h with respect to t and $\alpha > 0, M_\alpha$ are some constants. Moreover we can calculate the sign of M_α by investigating the behaviors of the 1-pulse $(u(z), v(z))$ and the eigenfunction $(P(z), Q(z))$ as $|z| \to \infty$, and the sign determines whether those pulses interact repulsively or attractively. From our theorems, we see that

$$\begin{cases} {}^t(u(z), v(z)) \to e^{-\alpha z} a^+, & z \to \infty, \\ {}^t(P(z), Q(z)) \to e^{\alpha z} b^-, & z \to -\infty, \end{cases}$$

where α, a^+, b_- are defined by

$$\alpha = \frac{1}{c_0}\left(\gamma - \frac{1}{a}\right) + O(\varepsilon), \quad a^+ = -K_1\begin{pmatrix}1\\ a\end{pmatrix} + O(\varepsilon), \quad b^- = K_2\begin{pmatrix}1\\ -a\end{pmatrix} + O(\varepsilon)$$

for positive constants K_1, K_2. Then M_α can be calculated such as

$$M_\alpha = \varepsilon c_0 \left\langle a^+, b^-\right\rangle + O(\varepsilon^2) = -\varepsilon c_0 K_1 K_2 (1-a^2) + O(\varepsilon^2) < 0,$$

which implies that two 1-pulses interact repulsively. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^2$.

3. Construction of Pulse Solutions

We divide $(-\infty,\infty)$ into three parts

$$I_1 = (-\infty, 0), \quad I_2 = (0,\tau), \quad I_3 = (\tau,\infty)$$

for some $\tau > 0$, and consider the following three problems:

$$\begin{cases} \varepsilon^2 u^{(1)}_{zz} - \varepsilon c u^{(1)}_z + f(u^{(1)}, v^{(1)}) = 0, & z \in I_1, \\ -c v^{(1)}_z + g(u^{(1)}, v^{(1)}) = 0, & z \in I_1, \\ u^{(1)}(-\infty) = 0, \quad u^{(1)}(0) = \alpha, & \\ v^{(1)}(-\infty) = 0, & \end{cases} \tag{3.1}$$

$$\begin{cases} \varepsilon^2 u^{(2)}_{zz} - \varepsilon c u^{(2)}_z + f(u^{(2)}, v^{(2)}) = 0, & z \in I_2, \\ -c v^{(2)}_z + g(u^{(2)}, v^{(2)}) = 0, & z \in I_2, \\ u^{(2)}(0) = \alpha, \quad u^{(2)}(\tau) = \beta, & \\ v^{(2)}(0) = v^{(1)}(0), & \end{cases} \tag{3.2}$$

$$\begin{cases} \varepsilon^2 u^{(3)}_{zz} - \varepsilon c u^{(3)}_z + f(u^{(3)}, v^{(3)}) = 0, & z \in I_3, \\ -c v^{(3)}_z + g(u^{(3)}, v^{(3)}) = 0, & z \in I_3, \\ u^{(3)}(\tau) = \beta, \quad u^{(3)}(\infty) = 0, & \\ v^{(3)}(\tau) = v^{(2)}(\tau). & \end{cases} \tag{3.3}$$

The superscript $^{(k)}$ for $k = 1, 2, 3$ means that the functions are defined on the interval I_k. This notation shall be used throughout Sections 3 and 4.

3.1. The lowest order approximation

We first construct an *outer solution* of (1.6) that approximates (1.6) outside internal transition layers. Putting $\varepsilon = 0$ in (3.1), we formally get

$$\begin{cases} f(U_0^{(1)}, V_0^{(1)}) = 0, & z \in (-\infty, 0), \\ -c_0 V_0^{(1)\prime} + g(U_0^{(1)}, V_0^{(1)}) = 0, & z \in (-\infty, 0), \\ V_0^{(1)}(-\infty) = 0, \end{cases}$$

that is,

$$\begin{cases} U_0^{(1)} = h_-(V_0^{(1)}), & z \in (-\infty, 0), \\ c_0 V_0^{(1)\prime} = g(h_-(V_0^{(1)}), V_0^{(1)}), & z \in (-\infty, 0), \\ V_0^{(1)}(-\infty) = 0. \end{cases}$$

Due to $f_u(0,0)g_v(0,0) - f_v(0,0)g_u(0,0) \geq 0$, $V_0^{(1)}(z)$ must be identically 0 and then $U_0^{(1)}(z)$ is also 0. Next, putting $\varepsilon = 0$ in (3.2), we formally get

$$\begin{cases} f(U_0^{(2)}, V_0^{(2)}) = 0, & z \in (0, \tau_0), \\ -c_0 V_0^{(2)\prime} + g(U_0^{(2)}, V_0^{(2)}) = 0, & z \in (0, \tau_0), \\ V_0^{(2)}(0) = V_0^{(1)}(0), \end{cases}$$

that is,

$$\begin{cases} U_0^{(2)} = h_+(V_0^{(2)}), & z \in (0, \tau_0), \\ c_0 V_0^{(2)\prime} = g(h_+(V_0^{(2)}), V_0^{(2)}), & z \in (0, \tau_0), \\ V_0^{(2)}(0) = 0. \end{cases}$$

Finally, putting $\varepsilon = 0$ in (3.3), we formally get

$$\begin{cases} f(U_0^{(3)}, V_0^{(3)}) = 0, & z \in (\tau_0, \infty), \\ -c_0 V_0^{(3)\prime} + g(U_0^{(3)}, V_0^{(3)}) = 0, & z \in (\tau_0, \infty), \\ V_0^{(3)}(\tau_0) = V_0^{(2)}(\tau_0), \end{cases}$$

that is,

$$\begin{cases} U_0^{(3)} = h_-(V_0^{(2)}), & z \in (\tau_0, \infty), \\ c_0 V_0^{(3)\prime} = g(h_-(V_0^{(3)}), V_0^{(3)}), & z \in (\tau_0, \infty), \\ V_0^{(3)}(\tau_0) = V_0^{(2)}(\tau_0). \end{cases}$$

The outer solutions constructed as above do not satisfy (1.6) approximately in neighborhoods of $z = 0$ and $z = \tau$. So we will construct *inner solutions* of (1.6) that approximate (1.6) in the internal transition layers.

At $z = 0$, we rewrite (1.6) by the stretched variable $\xi = z/\varepsilon$ and put $\varepsilon = 0$. Then we formally get

$$\begin{cases} \ddot{\phi}_0 - c_0\dot{\phi}_0 + f(\phi_0, 0) = 0, & \xi \in (-\infty, \infty), \\ \phi_0(-\infty) = 0, \quad \phi_0(\infty) = h_+(0), \quad (\phi_0(0) = \alpha). \end{cases} \tag{3.4}$$

where the superscript "$\cdot$" denotes the differentiation with respect to ξ. As we state in Section 2, there is a unique wave speed, denoted by $c_0 = c_0^*$, such that (3.4) has a unique solution $\Phi_1(\xi)$.

At $z = \tau$, we introduce the stretched variable $\xi = (z - \tau)/\varepsilon$. Then, similar to the above, we formally get

$$\begin{cases} \ddot{\phi}_0 - c_0^*\dot{\phi}_0 + f(\phi_0, v) = 0, & \xi \in (-\infty, \infty), \\ \phi_0(-\infty) = h_+(v), \quad \phi_0(\infty) = h_-(v), \quad (\phi_0(0) = \beta). \end{cases} \tag{3.5}$$

As we state in Section 2, there is a unique value $v = v^* \in (v_{min}, v_{max})$ such that $\Phi_2(\xi)$ is a unique solution of (3.5).

From Section 2, we find that $V_0^{(2)}(z)$ is a solution of

$$\begin{cases} c_0^* V_0^{(2)\prime} = g(h_+(V_0^{(2)}), V_0^{(2)}), & z > 0, \\ V_0^{(2)}(0) = 0 \end{cases}$$

and satisfies the condition the condition

$$V_0^{(2)}(\tau_0^*) = v^*.$$

We set

$$U_0^{(2)}(z) = h_+(V_0^{(2)}(z)).$$

Moreover, $V^{(3)}(z)$ is a solution of

$$\begin{cases} c_0^* V_0^{(3)\prime} = g(h_-(V_0^{(3)}), V_0^{(3)}), & z \in (\tau_0^*, \infty), \\ V_0^{(3)}(\tau_0^*) = v^*, \end{cases}$$

and we set

$$U_0^{(3)}(z) = h_-(V_0^{(3)}(z)).$$

From this 0-th order approximation, we can construct the solution of (1.6) rigorously. But, to show Theorem 1, we must construct higher order approximations.

3.2. The first interval I_1

In this section, we consider the problem

$$\begin{cases} \varepsilon^2 u_{zz} - \varepsilon c u_z + f(u,v) = 0, & z \in I_1, \\ -c v_z + g(u,v) = 0, & z \in I_1, \\ u(-\infty) = 0, \quad u(0) = \alpha, \\ v(-\infty) = 0. \end{cases} \tag{3.6}$$

Outer Approximations

We expand u, v and c as

$$u = U_0 + \varepsilon U_1 + \cdots, \quad v = V_0 + \varepsilon V_1 + \cdots, \quad c = c_0^* + \varepsilon c_1 + \cdots.$$

Substituting these in (3.6) and equating the power of ε^0, we get

$$\begin{cases} f(U_0, V_0) = 0, & z \in I_1, \\ -c_0^* V_0' + g(U_0, V_0) = 0, & z \in I_1, \\ V_0(-\infty) = 0. \end{cases}$$

From the first equation, we have $U_0 = h_-(V_0)$. Then we obtain

$$\begin{cases} c_0^* V_0' = g(h_-(V_0), V_0), & z \in I_1, \\ V_0(-\infty) = 0, \end{cases}$$

which leads

$$U_0(z) = 0, \quad V_0(z) = 0$$

by the lowest order approximation in Section 3.1.

Similarly, equating the power ε^1, we get

$$\begin{cases} f_u(0,0)U_1 + f_v(0,0)V_1 = 0, & z \in I_1, \\ -c_0^* V_1' + g_u(0,0)U_1 + g_v(0,0)V_1 = 0, & z \in I_1, \\ V_1(-\infty) = 0, \end{cases}$$

which implies

$$U_1(z) = 0, \quad V_1(z) = 0.$$

Inner Approximations

In a neighborhood of $z = 0$, we expand u and v as

$$u(z) = \phi_0(\frac{z}{\varepsilon}) + \varepsilon\phi_1(\frac{z}{\varepsilon}) + \cdots, \quad v(z) = \varepsilon\psi_0(\frac{z}{\varepsilon}) + \cdots.$$

Substituting these in (3.6), putting $\xi = z/\varepsilon$, and equating the power of ε^0, we get

$$\begin{cases} \ddot{\phi}_0 - c_0^*\dot{\phi}_0 + f(\phi_0, 0) = 0, & \xi \in (-\infty, 0), \\ -c_0^*\dot{\psi}_0 + g(\phi_0, 0) = 0, & \xi \in (-\infty, 0), \\ \phi_0(-\infty) = 0, \quad \phi_0(0) = \alpha, \\ \psi_0(-\infty) = 0. \end{cases} \tag{3.7}$$

Hence we obtain

$$\phi_0(\xi) = \Phi_1(\xi), \quad \psi_0(\xi) = \frac{1}{c_0^*}\int_{-\infty}^{\xi} g(\Phi_1, 0)ds.$$

Similarly, from the power ε^1, we get

$$\begin{cases} \ddot{\phi}_1 - c_0^*\dot{\phi}_1 + f_u(\Phi_1, 0)\phi_1 = c_1\dot{\phi}_0 - f_v(\Phi_1, 0)\psi_0, & \xi \in (-\infty, 0), \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = 0. \end{cases}$$

Hence

$$\phi_1(\xi; c_1) = -\dot{\Phi}_1 \int_{\xi}^{0} \frac{e^{c_0^* s}}{(\dot{\Phi}_1)^2} \int_{-\infty}^{s} e^{-c_0^* \rho} \dot{\Phi}_1 (c_1 \dot{\Phi}_1 - f_v(\Phi_1, 0)\psi_0) d\rho ds.$$

We will find a solution of (3.6) such as

$$\begin{cases} u^{(1)}(z; \varepsilon, c_1) = \phi_0^{(1)}(\frac{z}{\varepsilon}) + \varepsilon \phi_1^{(1)}(\frac{z}{\varepsilon}; c_1) + \varepsilon R^{(1)}(\frac{z}{\varepsilon}; \varepsilon, c_1), \\ v^{(1)}(z; \varepsilon, c_1) = \varepsilon \psi_0^{(1)}(\frac{z}{\varepsilon}) + \varepsilon S^{(1)}(\frac{z}{\varepsilon}; \varepsilon, c_1). \end{cases} \tag{3.8}$$

Indeed, we can show the following theorem, which gives a solution of the form (3.8).

Theorem 3. *Fix $\delta > 0$ and $c_1^* \in (-\infty, \infty)$ arbitrarily, and put*

$$\Lambda_\delta \equiv \{c_1 \in (-\infty, \infty) \mid |c_1 - c_1^*| \le \delta\}.$$

Then, there is $\varepsilon_0 > 0$ such that the pair $(u^{(1)}, v^{(1)})$ given by (3.8) *for a function $(R^{(1)}, S^{(1)}) \in X_\mu$ is a solution of* (3.6) *for any $\varepsilon \in (0, \varepsilon_0)$, where the functional space X_μ is defined by*

$$\begin{cases} X_\mu \equiv X_\mu^2 \times X_\mu^1, \\ X_\mu^1(-\infty, 0) \equiv \{\varphi \in C(-\infty, 0) \mid \|\varphi\|_{X_\mu^0} + \|\dot{\varphi}\|_{X_\mu^0} < \infty\}, \\ X_\mu^2(-\infty, 0) \equiv \{\varphi \in C(-\infty, 0) \mid \|\varphi\|_{X_\mu^0} + \|\dot{\varphi}\|_{X_\mu^0} + \|\ddot{\varphi}\|_{X_\mu^0} < \infty, \varphi(0) = 0\}, \\ \|\varphi\|_{X_\mu^0} \equiv \sup_{-\infty < \xi < 0} e^{-\mu\xi} |\varphi(\xi)| \end{cases}$$

and μ is an arbitrary number satisfying

$$0 < \mu < \mu_0 \equiv \frac{1}{2}\left(-c_0^* + \sqrt{(c_0^*)^2 - 4 f_u(0,0)}\right).$$

In addition, $(R^{(1)}, S^{(1)}) = O(\varepsilon)$ in X_μ uniformly in $c_1 \in \Lambda_\delta$ as $\varepsilon \to 0$.

This theorem shall be proved in Appendix. Note that $R^{(1)}(0; \varepsilon, c_1) = 0$ and $R^{(1)}(-\infty; \varepsilon, c_1) = S^{(1)}(-\infty; \varepsilon, c_1) = 0$.

3.3. The second interval I_2

In this subsection, we consider the problem

$$\begin{cases} \varepsilon^2 u_{zz} - \varepsilon c u_z + f(u,v) = 0, & z \in I_2, \\ -c v_z + g(u,v) = 0, & z \in I_2, \\ u(0) = \alpha, \quad u(\tau) = \beta(\varepsilon), & \\ v(0) = \varepsilon \psi_0^{(1)}(0) + \varepsilon S^{(1)}(0; \varepsilon, c_1), & \end{cases} \tag{3.9}$$

where $\beta(\varepsilon) = \beta + o(\varepsilon)$ as $\varepsilon \to 0$. More precise definition of $\beta(\varepsilon)$ will be given after stating Theorem 4. By putting $y = \tau_0^* z/\tau$ $(\tau = \tau_0^* + \varepsilon\tau_1)$, the above problem is rewritten as

$$\begin{cases} \varepsilon^2 u_{yy} - \varepsilon c \left(1 + \varepsilon \dfrac{\tau_1}{\tau_0^*}\right) u_y + \left(1 + \varepsilon \dfrac{\tau_1}{\tau_0^*}\right)^2 f(u,v) = 0, & y \in (0, \tau_0^*), \\ -c v_y + \left(1 + \varepsilon \dfrac{\tau_1}{\tau_0^*}\right) g(u,v) = 0, & y \in (0, \tau_0^*), \\ u(0) = \alpha, \quad u(\tau_0^*) = \beta(\varepsilon), & \\ v(0) = \varepsilon \psi_0^{(1)}(0) + \varepsilon S^{(1)}(0; \varepsilon, c_1). & \end{cases} \tag{3.10}$$

Outer Approximations

We expand u, v and c as

$$u = U_0 + \varepsilon U_1 + \cdots, \quad v = V_0 + \varepsilon V_1 + \cdots, \quad c = c_0^* + \varepsilon c_1 + \cdots.$$

By substituting these in (3.10), the coefficient of ε^0 is calculated as

$$\begin{cases} f(U_0, V_0) = 0, & y \in (0, \tau_0^*), \\ -c_0^* V_0' + g(U_0, V_0) = 0, & y \in (0, \tau_0^*), \\ V_0(0) = 0. & \end{cases} \tag{3.11}$$

From Section 3.1, we find that $U_0 = h_+(V_0)(= h_+(V_0^{(2)}))$ and $V_0(y)(= V_0^{(2)}(y))$ is a solution of the equation

$$\begin{cases} c_0^* V_0' = g(h_+(V_0), V_0), & y \in (0, \tau_0^*), \\ V_0(0) = 0. & \end{cases}$$

Similarly, from the power ε^1, we have

$$\begin{cases} -c_0^* U_0' + f_u(U_0, V_0)U_1 + f_v(U_0, V_0)V_1 = 0, & y \in (0, \tau_0^*), \\ -c_0^* V_1' - c_1 V_0' + g_u(U_0, V_0)U_1 + g_v(U_0, V_0)V_1 + \frac{\tau_1}{\tau_0^*} g(U_0, V_0) = 0, & y \in (0, \tau_0^*), \\ V_1(0) = \psi_0^{(1)}(0) - \psi_0^{(2)}(0), \end{cases} \tag{3.12}$$

where $\psi_0^{(2)}$ is determined later. Integrating these, we get

$$\begin{aligned} U_1(y; c_1, \tau_1) &= \frac{c_0^* U_0' - f_v(U_0, V_0) V_1(y; c_1, \tau_1)}{f_u(U_0, V_0)}, \\ V_1(y; c_1, \tau_1) &= V_0' \left\{ \frac{1}{c_0^*} \int_0^y \left(\frac{c_0^* g_u(U_0, V_0) U_0'}{f_u(U_0, V_0)} - c_1 V_0' + \frac{\tau_1}{\tau_0^*} g(U_0, V_0) \right) \frac{1}{V_0'} dx \right. \\ &\quad \left. + \frac{1}{V_0'(0)} (\psi_0^{(1)}(0) - \psi_0^{(2)}(0)) \right\}. \end{aligned} \tag{3.13}$$

Inner Approximations at $y = 0$

In a neighborhood of $y = 0$, we expand u and v as

$$\begin{aligned} u(y) &= U_0(y) + \varepsilon U_1(y) + \phi_0(\frac{y}{\varepsilon}) + \varepsilon \phi_1(\frac{y}{\varepsilon}) + \cdots, \\ v(y) &= V_0(y) + \varepsilon V_1(y) + \varepsilon \psi_0(\frac{y}{\varepsilon}) + \varepsilon^2 \psi_1(\frac{y}{\varepsilon}) + \cdots. \end{aligned}$$

Here we note that the ε^1-th order term appears in the expression of the function u, while we obtain the ε^2-th order term for the function v. The ε^2-th order term will be just needed in the proof of Theorem 4. This may result from the influence by the underlying difference of the scaling between outer solutions and inner solutions with respect to ε. It is not that we would like to establish the existence of the traveling wave solution with ε^2-th order expansion.

Substituting these in (3.10), putting $\xi = y/\varepsilon$, and equating the power of ε^0, we get

$$\begin{cases} \ddot{\phi}_0 - c_0^* \dot{\phi}_0 + f(h_+(0) + \phi_0, 0) = 0, & \xi \in (0, \infty), \\ -c_0^* V_0'(0) - c_0^* \dot{\psi}_0 + g(h_+(0) + \phi_0, 0) = 0, & \xi \in (0, \infty), \\ \phi_0(0) = \alpha - h_+(0) = \alpha - U_0(0), \quad \phi_0(\infty) = 0, \\ \psi_0(\infty) = 0. \end{cases} \tag{3.14}$$

Hence we obtain

$$\phi_0(\xi) = \Phi_1(\xi) - h_+(0), \quad \psi_0(\xi) = -\frac{1}{c_0^*}\int_\xi^\infty (g(\Phi_1,0) - g(h_+(0),0))ds. \quad (3.15)$$

Similarly, from the power ε^1, we have

$$\begin{cases} \ddot{\phi}_1 - c_0^*\dot{\phi}_1 + f_u(\Phi_0,0)\phi_1 = c_0^* U_0'(0) + \left(c_1 + c_0^*\dfrac{\tau_1}{\tau_0^*}\right)\dot{\phi}_0 \\ -2\dfrac{\tau_1}{\tau_0^*} f(\Phi_1,0) - f_u(\Phi_1,0)(\xi U_0'(0) + U_1(0)) \\ -f_v(\Phi_1,0)(\xi V_0'(0) + V_1(0) + \psi_0), & \xi \in (0,\infty), \\ -c_0^*\dot{\psi}_1 - c_0^* V_1'(0) - c_1 V_0'(0) - c_0^* V_0''(0)\xi - c_1\dot{\psi}_0 + \dfrac{\tau_1}{\tau_0^*} g(\Phi_1,0) \\ +g_v(\Phi_1,0)(\xi V_0'(0) + V_1(0) + \psi_0) \\ +g_u(\Phi_1,0)(\xi U_0'(0) + U_1(0) + \phi_1) = 0, & \xi \in (0,\infty), \\ \phi_1(0) = -U_1(0), \quad \phi_1(\infty) = 0, \\ \psi_1(\infty) = 0. \end{cases} \quad (3.16)$$

Noting $f_u(U_0,V_0)U_0' + f_v(U_0,V_0)V_0' = 0$, we have

$$\begin{aligned} &\phi_1(\xi; c_1, \tau_1) \\ = \;& -U_1(0)\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)} - \dot{\Phi}_1 \int_0^\xi \frac{e^{c_0^* s}}{(\dot{\Phi}_1)^2}\int_s^\infty e^{-c_0^*\rho}\dot{\Phi}_1 \left[\left(c_1 + c_0^*\frac{\tau_1}{\tau_0^*}\right)\dot{\Phi}_1 - 2\frac{\tau_1}{\tau_0^*} f(\Phi_1,0)\right. \\ & -f_v(\Phi_1,0)\psi_0 - (f_u(\Phi_1,0) - f_u(h_+(0),0))(\rho U_0'(0) + U_1(0)) \\ & \left. -(f_v(\Phi_1,0) - f_v(h_+(0),0))(\rho V_0'(0) + V_1(0))\right] d\rho ds, \end{aligned}$$

$$\begin{aligned} &\psi_1(\xi; c_1, \tau_1) \\ = \;& \frac{1}{c_0^*}\int_\xi^\infty \{c_1\dot{\psi}_0 - g_u(\Phi_1,0)\phi_1 - g_v(\Phi_1,0)\psi_0 - \frac{\tau_1}{\tau_0^*}(g(\Phi_1,0) - g(h_+(0),0)) \\ & -(g_u(\Phi_1,0) - g_u(h_+(0),0))(sU_0'(0) + U_1(0)) \\ & -(g_v(\Phi_1,0) - g_v(h_+(0),0))(sV_0'(0) + V_1(0))\}ds. \end{aligned} \quad (3.17)$$

Here we note that we do not write c_1 and τ_1 for $U_1(0), V_1(0)$ in (3.17) explicitly. In fact, $U_1(0), V_1(0)$ are independent of c_1 and τ_1 though the functions U_1 and V_1 do depend on these parameters. This is easy to see by substituting $y = 0$ into U_1, V_1 directly.

Inner Approximations at $y = \tau_0^*$

In a neighborhood of $y = \tau_0^*$, we expand u and v as

$$\begin{aligned} u(y) &= U_0(y) + \varepsilon U_1(y) + \phi_0(\frac{y-\tau_0^*}{\varepsilon}) + \varepsilon\phi_1(\frac{y-\tau_0^*}{\varepsilon}) + \cdots, \\ v(y) &= V_0(y) + \varepsilon V_1(y) + \varepsilon\psi_0(\frac{y-\tau_0^*}{\varepsilon}) + \varepsilon^2\psi_1(\frac{y-\tau_0^*}{\varepsilon}) + \cdots. \end{aligned}$$

By substituting these in (3.10), putting $\xi = (y-\tau_0^*)/\varepsilon$, the coefficient of ε^0 is computed as

$$\begin{cases} \ddot{\phi}_0 - c_0^*\dot{\phi}_0 + f(h_+(v^*) + \phi_0, v^*) = 0, & \xi \in (-\infty, 0), \\ -c_0^* V_0'(\tau_0^*) - c_0^*\dot{\psi}_0 + g(h_+(v^*) + \phi_0, v^*) = 0, & \xi \in (-\infty, 0), \\ \phi_0(-\infty) = 0, \quad \phi_0(0) = \beta - U_0(\tau_0^*) = \beta - h_+(v^*), \\ \psi_0(-\infty) = 0. \end{cases} \tag{3.18}$$

Hence we obtain

$$\phi_0(\xi) = \Phi_2(\xi) - h_+(v^*), \quad \psi_0(\xi) = \frac{1}{c_0^*}\int_{-\infty}^{\xi} (g(\Phi_2, v^*) - g(h_+(v^*), v^*))ds. \tag{3.19}$$

Similarly, from the power ε^1, we have

$$\begin{cases} \ddot{\phi}_1 - c_0^*\dot{\phi}_1 + f_u(\Phi_2, v^*)\phi_1 = c_0^* U_0'(\tau_0^*) + \left(c_1 + c_0^*\frac{\tau_1}{\tau_0^*}\right)\dot{\phi}_0 \\ -2\frac{\tau_1}{\tau_0^*} f(\Phi_2, v^*) - f_u(\Phi_2, v^*)(\xi U_0'(\tau_0^*) + U_1(\tau_0^*)) \\ -f_v(\Phi_2, v^*)(\xi V_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0), & \xi \in (-\infty, 0), \\ -c_0^*\dot{\psi}_1 - c_0^* V_1'(\tau_0^*) - c_1 V_0'(\tau_0^*) - c_0^* V_0''(\tau_0^*)\xi - c_1\dot{\psi}_0 \\ +\frac{\tau_1}{\tau_0^*} g(\Phi_2, v^*) + g_u(\Phi_2, v^*)(\xi U_0'(\tau_0^*) \\ +U_1(\tau_0^*) + \phi_1) + g_v(\Phi_2, v^*)(\xi V_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0) = 0, & \xi \in (-\infty, 0), \\ \phi_1(-\infty) = 0, \quad \phi_1(0) = -U_1(\tau_0^*), \\ \psi_1(-\infty) = 0. \end{cases} \tag{3.20}$$

Noting $f_u(U_0, V_0)U_0' + f_v(U_0, V_0)V_0' = 0$ again, we have

$$\begin{aligned} &\phi_1(\xi; c_1, \tau_1) \\ &= -U_1(\tau_0^*; c_1, \tau_1)\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)} - \dot{\Phi}_2\int_\xi^0 \frac{e^{c_0^* s}}{(\dot{\Phi}_2)^2}\int_{-\infty}^s e^{-c_0^*\rho}\dot{\Phi}_2 \end{aligned}$$

$$\times\Big[\Big(c_1+c_0^*\frac{\tau_1}{\tau_0^*}\Big)\dot{\Phi}_2-2\frac{\tau_1}{\tau_0^*}f(\Phi_2,v^*)-f_v(\Phi_2,v^*)\psi_0$$
$$-(f_u(\Phi_2,v^*)-f_u(h_+(v^*),v^*))(\rho U_0'(\tau_0^*)+U_1(\tau_0^*;c_1,\tau_1))$$
$$-(f_v(\Phi_2,v^*)-f_v(h_+(v^*),v^*))(\rho V_0'(\tau_0^*)+V_1(\tau_0^*;c_1,\tau_1))\Big]d\rho ds, \quad (3.21)$$

$$\psi_1(\xi;c_1,\tau_1)$$
$$= -\frac{1}{c_0^*}\int_{-\infty}^{\xi}[c_1\dot{\psi}_0-(g_u(\Phi_2,v^*)-g_u(h_+(v^*),v^*))(sU_0'(\tau_0^*)+U_1(\tau_0^*;c_1,\tau_1))$$
$$-(g_v(\Phi_2,v^*)-g_v(h_+(v^*),v^*))(sV_0'(\tau_0^*)+V_1(\tau_0^*;c_1,\tau_1))-g_u(\Phi_2,v^*)\phi_1$$
$$-g_v(\Phi_2,v^*)\psi_0-\frac{\tau_1}{\tau_0^*}(g(\Phi_2,v^*)-g(h_+(v^*),v^*))]ds.$$

We will find a solution of (3.9) given by

$$u^{(2)}(y;\varepsilon,c_1,\tau_1)$$
$$= U_0^{(2)}(y)+\varepsilon U_1^{(2)}(y;c_1,\tau_1)+\theta(\frac{y}{\tau_0^*})(\phi_0^{(2),l}(\frac{y}{\varepsilon})+\varepsilon\phi_1^{(2),l}(\frac{y}{\varepsilon};c_1,\tau_1))$$
$$+\theta(1-\frac{y}{\tau_0^*})(\phi_0^{(2),r}(\frac{y-\tau_0^*}{\varepsilon})+\varepsilon\phi_1^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};c_1,\tau_1))$$
$$+\varepsilon R^{(2)}(y;\varepsilon,c_1,\tau_1)+\varepsilon h_+'(V_0^{(2)}(y))S^{(2)}(y;\varepsilon,c_1,\tau_1), \quad (3.22)$$

$$v^{(2)}(y;\varepsilon,c_1,\tau_1)$$
$$= V_0^{(2)}(y)+\varepsilon V_1^{(2)}(y;c_1,\tau_1)+\varepsilon\theta(\frac{y}{\tau_0^*})(\psi_0^{(2),l}(\frac{y}{\varepsilon})+\varepsilon\psi_1^{(2),l}(\frac{y}{\varepsilon};c_1,\tau_1))$$
$$+\varepsilon\theta(1-\frac{y}{\tau_0^*})(\psi_0^{(2),r}(\frac{y-\tau_0^*}{\varepsilon})+\varepsilon\psi_1^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};c_1,\tau_1))$$
$$+\varepsilon S^{(2)}(y;\varepsilon,c_1,\tau_1)+\varepsilon\theta(\frac{y}{\tau_0^*})(S^{(1)}(0;\varepsilon,c_1)-\varepsilon\psi_1^{(2),l}(0;c_1,\tau_1)), \quad (3.23)$$

where the superscript " l " means that the functions are given by (3.15) and (3.17), and " r " is also a superscript given for the functions of (3.19) and (3.21) as well as " l ". A smooth cut-off function $\theta(y)\in C^\infty[0,1]$ is supposed to satisfy

$$\begin{cases}\theta(y)=1 & \text{for } 0\le y\le\frac{1}{4},\\ 0\le\theta(y)\le 1 & \text{for } \frac{1}{4}\le y\le\frac{1}{2},\\ \theta(y)=0 & \text{for } \frac{1}{2}\le y\le 1.\end{cases}$$

Theorem 4. *Fix $\delta > 0$ and $c_1^*, \tau_1^* \in (-\infty, \infty)$ arbitrarily, and put*

$$\Xi_\delta = \{(c_1, \tau_1) \in \mathbb{R}^2 \mid |c_1 - c_1^*| + |\tau_1 - \tau_1^*| \leq \delta\}.$$

Then, there is $\varepsilon_0 > 0$ such that $(u^{(2)}, v^{(2)})$ given by (3.22) and (3.23) for a function $(R^{(2)}, S^{(2)}) \in X_\varepsilon$ is a solution of (3.10) for any $\varepsilon \in (0, \varepsilon_0)$, where the functional space X_ε is defined by

$$\begin{cases} X_\varepsilon = C^2_\varepsilon(0, \tau_0^*) \times C^2_{1,\varepsilon}(0, \tau_0^*), \\ C^2_\varepsilon(0, \tau_0^*) = \left\{\varphi \in C^2(0, \tau_0^*) \mid \varphi(0) = 0,\ \varphi(\tau_0^*) = 0, \quad \|\varphi\|_{C^2_\varepsilon} < \infty\right\}, \\ C^2_{1,\varepsilon}(0, \tau_0^*) = \{\varphi \in C^2(0, \tau_0^*) \mid \varphi(0) = 0, \quad \|\varphi\|_{C^2_{1,\varepsilon}} < \infty\}, \end{cases}$$

and the norms are defined by

$$\|\varphi\|_{C^2_\varepsilon} = \sum_{i=0}^{2} \max_{[0,\tau_0^*]} \left| \left(\varepsilon \frac{d}{dy}\right)^i \varphi \right|, \ \|\varphi\|_{C^2_{1,\varepsilon}} = \sum_{i=0}^{1} \max_{[0,\tau_0^*]} \left| \left(\frac{d}{dy}\right)^i \varphi \right| + \max_{[0,\tau_0^*]} \left| \varepsilon \frac{d^2}{dy^2} \varphi \right|.$$

In addition, $(R^{(2)}, S^{(2)}) = o(1)$ in X_ε uniformly for $(c_1, \tau_1) \in \Xi_\delta$ as $\varepsilon \to 0$.

This theorem shall be proved in Appendix. Note that $R^{(2)}(0; \varepsilon, c_1, \tau_1) = R^{(2)}(\tau_0^*; \varepsilon, c_1, \tau_1) = S^{(2)}(0; \varepsilon, c_1, \tau_1) = 0$. Therefore, we constructed a solution of (3.10) such as

$$\begin{aligned} u^{(2)}&(z; \varepsilon, c_1, \tau_1) \\ = {} & U_0^{(2)}(\frac{\tau_0^*}{\tau} z) + \varepsilon U_1^{(2)}(\frac{\tau_0^*}{\tau} z; c_1, \tau_1) + \theta(\frac{z}{\tau})(\phi_0^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau} z) + \varepsilon \phi_1^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau} z; c_1, \tau_1)) \\ & + \theta(1 - \frac{z}{\tau})(\phi_0^{(2),r}(\frac{\tau_0^*(z - \tau)}{\varepsilon\tau}) + \varepsilon \phi_1^{(2),r}(\frac{\tau_0^*(z - \tau)}{\varepsilon\tau}; c_1, \tau_1)) \\ & + \varepsilon R^{(2)}(\frac{\tau_0^*}{\tau} z; \varepsilon, c_1, \tau_1) + \varepsilon h_+'(V_0^{(2)}(\frac{\tau_0^*}{\tau} z)) S^{(2)}(\frac{\tau_0^*}{\tau} z; \varepsilon, c_1, \tau_1), \end{aligned} \tag{3.24}$$

$$\begin{aligned} v^{(2)}&(z; \varepsilon, c_1, \tau_1) \\ = {} & V_0^{(2)}(\frac{\tau_0^*}{\tau} z) + \varepsilon V_1^{(2)}(\frac{\tau_0^*}{\tau} z; c_1, \tau_1) + \varepsilon\theta(\frac{z}{\tau})(\psi_0^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau} z) + \varepsilon \psi_1^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau} z; c_1, \tau_1)) \\ & + \varepsilon\theta(1 - \frac{z}{\tau})(\psi_0^{(2),r}(\frac{\tau_0^*(z - \tau)}{\varepsilon\tau}) + \varepsilon \psi_1^{(2),r}(\frac{\tau_0^*(z - \tau)}{\varepsilon\tau}; c_1, \tau_1)) \\ & + \varepsilon S^{(2)}(\frac{\tau_0^*}{\tau} z; \varepsilon, c_1, \tau_1) + \varepsilon\theta(\frac{z}{\tau})(S^{(1)}(0; \varepsilon, c_1) - \varepsilon \psi_1^{(2),l}(0; c_1, \tau_1)). \end{aligned} \tag{3.25}$$

The constant $\beta(\varepsilon)$ is given by

$$\beta(\varepsilon) = \beta + \varepsilon h'_+(v^*)S^{(2)}(\tau_0^*; \varepsilon, c_1, \tau_1).$$

3.4. The third interval I_3

In this subsection, we find a bounded solution of

$$\begin{cases} \varepsilon^2 u_{zz} - \varepsilon c u_z + f(u,v) = 0, & z \in I_3, \\ -cv_z + g(u,v) = 0, & z \in I_3, \\ u(\tau) = \beta(\varepsilon), \quad u(+\infty) = 0, \\ v(\tau) = v^{(2)}(\tau), \end{cases} \tag{3.26}$$

where

$$v^{(2)}(\tau) = v^* + \varepsilon V_1^{(2)}(\tau_0^*; c_1, \tau_1) + \varepsilon \psi_0^{(2),r}(0) + \varepsilon^2 \psi_1^{(2),r}(0; c_1, \tau_1) + \varepsilon S^{(2)}(\tau_0^*; \varepsilon, c_1, \tau_1).$$

If we transform $y = z - \tau$, then this problem is rewritten as

$$\begin{cases} \varepsilon^2 u_{yy} - \varepsilon c u_y + f(u,v) = 0, & y \in (0,\infty), \\ -cv_y + g(u,v) = 0, & y \in (0,\infty), \\ u(0) = \beta(\varepsilon), \quad u(+\infty) = 0, \\ v(0) = v^{(2)}(\tau). \end{cases} \tag{3.27}$$

Outer Approximations

We expand u, v and c as

$$u = U_0 + \varepsilon U_1 + \cdots, \quad v = V_0 + \varepsilon V_1 + \cdots.$$

We substitute these in (3.27), and equate each power of ε. Then, from the order ε^0, we have

$$\begin{cases} f(U_0, V_0) = 0, & y \in (0,\infty), \\ -c_0^* V_0' + g(U_0, V_0) = 0, & y \in (0,\infty), \\ V_0(0) = v^*. \end{cases}$$

Hence $U_0 = h_-(V_0)$ and

$$\begin{cases} c_0^* V_0' = g(h_-(V_0), V_0), y \in (0, \infty), \\ V_0(0) = v^*. \end{cases}$$

From Section 3.1, we find that $V_0(y) = V_0^{(3)}(y)$ is a solution of the above problem and set $U_0(y) = h_-(V_0(y))(= h_-(V_0^{(3)}(y)))$.

Similarly, from ε^1-order term, we have

$$\begin{cases} -c_0^* U_0' + f_u(U_0, V_0)U_1 + f_v(U_0, V_0)V_1 = 0, & y \in (0, \infty), \\ -c_0^* V_1' - c_1 V_0' + g_u(U_0, V_0)U_1 + g_v(U_0, V_0)V_1 = 0, & y \in (0, \infty), \\ V_1(0) = -\psi_0^{(3)}(0) + V_1^{(2)}(\tau_0^*; c_1, \tau_1) + \psi_0^{(2),r}(0), \end{cases}$$

where the function $\psi_0^{(3)}$ is given later. Since $V_0' \neq 0$ for any $y > 0$ in case of the FitzHugh-Nagumo system,

$$\begin{aligned} U_1(y; c_1, \tau_1) &= \frac{-f_v(U_0, V_0)V_1 + c_0^* U_0'}{f_u(U_0, V_0)}, \\ V_1(y; c_1, \tau_1) &= V_0' \Big[\frac{1}{c_0^*} \int_0^y \frac{1}{V_0'} \left(-c_1 V_0' + c_0^* \frac{g_u(U_0, V_0)U_0'}{f_u(U_0, V_0)} \right) dx \\ &\quad + \frac{1}{V_0'(0)} (-\psi_0^{(3)}(0) + V_1^{(2)}(\tau_0^*; c_1, \tau_1) + \psi_0^{(2),r}(0)) \Big]. \end{aligned} \tag{3.28}$$

Since $V_0 \equiv v^*$ in case of the combustion model, U_1 and V_1 are identically constants given by

$$\begin{cases} U_1(y; c_1, \tau_1) = -\dfrac{f_v(h_-(v^*), v^*)}{f_u(h_-(v^*), v^*)} V_1, \\ V_1(y; c_1, \tau_1) = -\psi_0^{(3)}(0) + V_1^{(2)}(\tau_0^*; c_1, \tau_1) + \psi_0^{(2),r}(0). \end{cases}$$

Although U_1 and V_1 are constants, we use the variable y for these functions in order to correspond to (3.28).

Inner Approximations

In a neighborhood of $y = 0$, we expand u and v as

$$\begin{aligned} u(y) &= U_0(y) + \varepsilon U_1(y) + \phi_0(\frac{y}{\varepsilon}) + \varepsilon \phi_1(\frac{y}{\varepsilon}) + \cdots, \\ v(y) &= V_0(y) + \varepsilon V_1(y) + \varepsilon \psi_0(\frac{y}{\varepsilon}) + \varepsilon^2 \psi_1(\frac{y}{\varepsilon}) + \cdots. \end{aligned}$$

Substituting these in (3.27) and putting $\xi = y/\varepsilon$, we have from ε^0-order term

$$\begin{cases} \ddot{\phi}_0 - c_0^*\dot{\phi}_0 + f(U_0(0)+\phi_0, V_0(0)) = 0, & \xi \in (0,\infty), \\ -c_0^*V_0'(0) - c_0^*\dot{\psi}_0 + g(U_0(0)+\phi_0, V_0(0)) = 0, & \xi \in (0,\infty), \\ \phi_0(0) = \beta - U_0(0) = \beta - h_-(v^*), \quad \phi_0(\infty) = 0, \\ \psi_0(\infty) = 0. \end{cases} \tag{3.29}$$

Hence we obtain

$$\begin{aligned} \phi_0(\xi) &= \Phi_2(\xi) - h_-(v^*), \\ \psi_0(\xi) &= -\frac{1}{c_0^*}\int_\xi^\infty \{g(\Phi_2, v^*) - g(h_-(v^*), v^*)\}ds. \end{aligned}$$

Similarly, from the order of ε^1, we have

$$\begin{cases} \ddot{\phi}_1 - c_0^*\dot{\phi}_1 + f_u(\Phi_2, v^*)\phi_1 \\ \quad = c_0^*U_0'(0) + c_1\dot{\phi}_0 - f_u(\Phi_2, v^*)(\xi U_0'(0) + U_1(0)) \\ \qquad - f_v(\Phi_2, v^*)(\xi V_0'(0) + V_1(0) + \psi_0), & \xi \in (0,\infty), \\ -c_1V_0'(0) - c_0^*V_1'(0) - c_0^*\xi V_0''(0) - c_0^*\dot{\psi}_1 - c_1\dot{\psi}_0 \\ + g_u(\Phi_2, v^*)(\xi U_0'(0) + U_1(0) + \phi_1) \\ + g_v(\Phi_2, v^*)(\xi V_0'(0) + V_1(0) + \psi_0) = 0, & \xi \in (0,\infty), \\ \phi_1(0) = -U_1(0), \quad \phi_1(\infty) = 0, \\ \psi_1(\infty) = 0. \end{cases}$$

Hence we obtain

$$\begin{aligned} &\phi_1(\xi; c_1, \tau_1) \\ &\quad = -U_1(0; c_1, \tau_1)\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)} - \dot{\Phi}_2\int_0^\xi \frac{e^{c_0^* s}}{(\dot{\Phi}_2)^2}\int_s^\infty e^{-c_0^*\rho}\dot{\Phi}_2[c_1\dot{\Phi}_2 - f_v(\Phi_2, v^*)\psi_0 \\ &\qquad - (f_u(\Phi_2, v^*) - f_u(h_-(v^*), v^*))(\rho U_0'(0) + U_1(0; c_1, \tau_1)) \\ &\qquad - (f_v(\Phi_2, v^*) - f_v(h_-(v^*), v^*))(\rho V_0'(0) + V_1(0; c_1, \tau_1))]d\rho ds, \end{aligned}$$

$$\begin{aligned} &\psi_1(\xi; c_1, \tau_1) \\ &\quad = \frac{1}{c_0^*}\int_\xi^\infty [c_1\dot{\psi}_0 - g_u(\Phi_2, v^*)\phi_1 - g_v(\Phi_2, v^*)\psi_0 \\ &\qquad - (g_u(\Phi_2, v^*) - g_u(h_-(v^*), v^*))(sU_0'(0) + U_1(0; c_1, \tau_1)) \\ &\qquad - (g_v(\Phi_2, v^*) - g_v(h_-(v^*), v^*))(sV_0'(0) + V_1(0; c_1, \tau_1))]ds. \end{aligned}$$

The solution of (3.27) is given by

$$\begin{cases} u^{(3)}(y;\varepsilon,c_1,\tau_1) &= U_0^{(3)}(y)+\varepsilon U_1^{(3)}(y;c_1,\tau_1)+\phi_0^{(3)}(\frac{y}{\varepsilon})+\varepsilon\phi_1^{(3)}(\frac{y}{\varepsilon};c_1,\tau_1) \\ &\quad +\varepsilon h'_-(V_0^{(3)}(y))S^{(3)}(y;\varepsilon,c_1,\tau)+\varepsilon R^{(3)}(y;\varepsilon,c_1,\tau_1), \\ v^{(3)}(y;\varepsilon,c_1,\tau_1) &= V_0^{(3)}(y)+\varepsilon V_1^{(3)}(y;c_1,\tau_1)+\varepsilon\psi_0^{(3)}(\frac{y}{\varepsilon}) \\ &\quad +\varepsilon^2\{\psi_1^{(3)}(\frac{y}{\varepsilon};c_1,\tau_1)-\psi_1^{(3)}(0;c_1,\tau_1)e^{-\mu y}\} \\ &\quad +\{\varepsilon^2\psi_1^{(2),r}(0;c_1,\tau_1)+\varepsilon S^{(2)}(\tau_0^*;\varepsilon,c_1,\tau_1)\}e^{-\mu y} \\ &\quad +\varepsilon S^{(3)}(y;\varepsilon,c_1,\tau_1). \end{cases} \tag{3.30}$$

Actually, we can show the existence of a solution of (3.27) with the above form and the function (R,S) satisfies $\|R\|_{X^2_{\mu,\varepsilon}}=o(1)$ and $\|S\|_{\hat{X}^2_{\mu,\varepsilon}}=o(1)$ as $\varepsilon\to 0$ by a similar argument in Sections 3.2 and 3.3, where the functional spaces $X^2_{\mu,\varepsilon}$ and $\hat{X}^2_{\mu,\varepsilon}$ are given in the following theorem.

Theorem 5. *Fix $\delta>0$ and $c_1^*,\tau_1^*\in(-\infty,\infty)$ arbitrarily, and put*

$$\Xi_\delta=\{(c_1,\tau_1)\in\mathbb{R}^2 \mid |c_1-c_1^*|+|\tau_1-\tau_1^*|\le\delta\}.$$

Then, there is $\varepsilon_0>0$ such that the pair $(u^{(3)},v^{(3)})$ given by (3.30) for a function $(R^{(3)},S^{(3)})\in X_{\mu,\varepsilon}$ is a solution of (3.27) for any $\varepsilon\in(0,\varepsilon_0)$. The functional space $X_{\mu,\varepsilon}$ is defined by

$$\begin{cases} X_{\mu,\varepsilon}=X^2_{\mu,\varepsilon}\times\hat{X}^2_{\mu,\varepsilon}, \\ X_\mu\equiv\{\varphi\in C(0,\infty)\mid \|\varphi\|_{X_\mu}=\sup_{y\in(0,\infty)}|\varphi(y)e^{\mu y}|<\infty\}, \\ X^2_{\mu,\varepsilon}\equiv\{\varphi\in C^2(0,\infty)\mid \varphi(0)=0,\|\varphi\|_{X_\mu}+\varepsilon\|\varphi'\|_{X_\mu}+\varepsilon^2\|\varphi''\|_{X_\mu}<\infty\}, \\ \hat{X}^2_{\mu,\varepsilon}\equiv\{\varphi\in C^2(0,\infty)\mid \varphi(0)=0,\|\varphi\|_{X_\mu}+\|\varphi'\|_{X_\mu}+\varepsilon\|\varphi''\|_{X_\mu}<\infty\}, \end{cases}$$

and μ satisfies $0<\mu<\mu_0$ and is fixed, where μ_0 is given by

$$\mu_0=-\frac{f_u(0,0)g_v(0,0)-f_v(0,0)g_u(0,0)}{c_0^*f_u(0,0)}=\frac{1}{c_0^*}\left(\gamma+\frac{1}{a}\right)$$

in the FitzHugh-Nagumo case and it is any positive constant in the combustion model case. In addition, $(R^{(3)},S^{(3)})=o(1)$ in $X_{\mu,\varepsilon}$ uniformly for $(c_1,\tau_1)\in\Xi_\delta$ as $\varepsilon\to 0$.

This theorem can be shown by the same argument as in the proof of Theorem 4. So we omit the details.

Therefore, the solution of (3.26) is given by

$$\begin{cases} u^{(3)}(z;\varepsilon,c_1,\tau_1) = & U_0^{(3)}(z-\tau)+\varepsilon U_1^{(3)}(z-\tau;c_1,\tau_1)+\phi_0^{(3)}(\frac{z-\tau}{\varepsilon}) \\ & +\varepsilon\phi_1^{(3)}(\frac{z-\tau}{\varepsilon};c_1,\tau_1)+\varepsilon h'_-(V_0^{(3)}(z-\tau))S^{(3)}(z-\tau;\varepsilon,c_1,\tau_1) \\ & +\varepsilon R^{(3)}(z-\tau;\varepsilon,c_1,\tau_1), \\ v^{(3)}(z;\varepsilon,c_1,\tau_1) = & V_0^{(3)}(z-\tau)+\varepsilon V_1^{(3)}(z-\tau;c_1,\tau_1)+\varepsilon\psi_0^{(3)}(\frac{z-\tau}{\varepsilon}) \\ & +\varepsilon^2\{\psi_1^{(3)}(\frac{z-\tau}{\varepsilon};c_1,\tau_1)-\psi_1^{(3)}(0;c_1,\tau_1)e^{-\mu(z-\tau)}\} \\ & +\{\varepsilon^2\psi_1^{(2),r}(0;c_1,\tau_1)+\varepsilon S^{(2)}(\tau_0^*;\varepsilon,c_1,\tau_1)\}e^{-\mu(z-\tau)} \\ & +\varepsilon S^{(3)}(z-\tau;\varepsilon,c_1,\tau_1). \end{cases} \tag{3.31}$$

3.5. The whole interval

We have constructed solutions of (3.6), (3.9) and (3.26) on each intervals. From the boundary conditions we imposed, we know that

$$(u^{(1)}(0),v^{(1)}(0))=(u^{(2)}(0),v^{(2)}(0)),\quad (u^{(2)}(\tau),v^{(2)}(\tau))=(u^{(3)}(\tau),v^{(3)}(\tau)).$$

In order to obtain a smooth solution of (1.6), we match the solutions constructed in the previous subsections smoothly, that is, find (c_1,τ_1) for which

$$\frac{d}{dz}u^{(1)}(0)=\frac{d}{dz}u^{(2)}(0),\quad \frac{d}{dz}u^{(2)}(\tau)=\frac{d}{dz}u^{(3)}(\tau)$$

hold. Set

$$\begin{aligned} X(\varepsilon;c_1,\tau_1) &= \varepsilon\frac{d}{dz}u^{(1)}(0;\varepsilon,c_1)-\varepsilon\frac{d}{dz}u^{(2)}(0;\varepsilon,c_1,\tau_1) \\ &= \dot\phi_0^{(1)}(0)+\varepsilon\dot\phi_1^{(1)}(0;c_1) \\ &\quad -\frac{\tau_0^*}{\tau_0^*+\varepsilon\tau_1}(\varepsilon U_0^{(2)'}(0)+\dot\phi_0^{(2),l}(0)+\varepsilon\dot\phi_1^{(2),l}(0;c_1,\tau_1))+o(\varepsilon) \\ &= \dot\phi_0^{(1)}(0)-\dot\phi_0^{(2),l}(0) \\ &\quad +\varepsilon(\dot\phi_1^{(1)}(0;c_1)-\dot\phi_1^{(2),l}(0;c_1,\tau_1)-U_0^{(2)'}(0)+\frac{\tau_1}{\tau_0^*}\dot\phi_0^{(2),l}(0))+o(\varepsilon) \\ &= \varepsilon X_1(c_1,\tau_1)+o(\varepsilon). \end{aligned}$$

Here we put

$$X_1(c_1,\tau_1)\equiv\dot\phi_1^{(1)}(0;c_1)-\dot\phi_1^{(2),l}(0;c_1,\tau_1)+\frac{\tau_1}{\tau_0^*}\dot\Phi_1(0)-U_0^{(2)'}(0)$$

and recall $\dot{\Phi}_1 = \dot{\phi}_0^{(1)}$ for $\xi \le 0$ and $\dot{\Phi}_1 = \dot{\phi}_0^{(2),l}$ for $\xi \ge 0$. Note that

$$\begin{aligned}
\dot{\phi}_1^{(1)}(0;c_1) &= \int_{-\infty}^0 e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}(c_1\dot{\Phi}_1 - f_v(\Phi_1,0)\psi_0^{(1)})ds,\\
\dot{\phi}_1^{(2),l}(0;c_1,\tau_1) &= -U_1^{(2)}(0)\frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)} - \int_0^\infty e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}\Big\{\Big(c_1 + c_0^*\frac{\tau_1}{\tau_0^*}\Big)\dot{\Phi}_1\\
&\quad -2\frac{\tau_1}{\tau_0^*}f(\Phi_1,0) - (f_u(\Phi_1,0) - f_u(h_+(0),0))(sU_0^{(2)\prime}(0) + U_1^{(2)}(0))\\
&\quad -f_v(\Phi_1,0)\psi_0^{(2),l} - (f_v(\Phi_1,0) - f_v(h_+(0),0))(sV_0^{(2)\prime}(0) + V_1^{(2)}(0))\Big\}ds.
\end{aligned}$$

Recall that $U_1(0)$ and $V_1(0)$ are independent of c_1 and τ_1. The condition $X_1(c_1,\tau_1) = 0$ is written as

$$\begin{aligned}
&\left\{\int_{-\infty}^0 e^{-c_0^* s}\frac{(\dot{\Phi}_1)^2}{\dot{\Phi}_1(0)}ds + \int_0^\infty e^{-c_0^* s}\frac{(\dot{\Phi}_1)^2}{\dot{\Phi}_1(0)}ds\right\}c_1\\
&+\left\{\int_0^\infty e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}\left(\frac{c_0^*}{\tau_0^*}\dot{\Phi}_1 - \frac{2}{\tau_0^*}f(\Phi_1,0)\right)ds + \frac{\dot{\Phi}_1(0)}{\tau_0^*}\right\}\tau_1\\
&-\int_{-\infty}^0 e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}f_v(\Phi_1,0)\psi_0^{(1)}ds + U_1^{(2)}(0)\frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)}\\
&-\int_0^\infty e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}\Big[(f_u(\Phi_1,0) - f_u(h_+(0),0))(sU_0^{(2)\prime}(0) + U_1^{(2)}(0))\\
&+f_v(\Phi_1,0)\psi_0^{(2),l} + (f_v(\Phi_1,0) - f_v(h_+(0),0))(sV_0^{(2)\prime}(0) + V_1^{(2)}(0))\Big]ds\\
&\hspace{20em} -U_0^{(2)\prime}(0) = 0.
\end{aligned}$$

Here the coefficient of τ_1 vanishes by (3.14) so that $c_1 = c_1^*$ is determined as

$$\begin{aligned}
c_1^* &= \frac{1}{A}\Big\{\int_{-\infty}^0 e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}f_v(\Phi_1,0)\psi_0^{(1)}ds - U_1^{(2)}(0)\frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)}\\
&\quad + \int_0^\infty e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}[(f_u(\Phi_1,0) - f_u(h_+(0),0))(sU_0^{(2)\prime}(0) + U_1^{(2)}(0))\\
&\quad +f_v(\Phi_1,0)\psi_0^{(2),l} + (f_v(\Phi_1,0) - f_v(h_+(0),0))(sV_0^{(2)\prime}(0) + V_1^{(2)}(0))]ds\\
&\quad +U_0^{(2)\prime}(0)\Big\}.
\end{aligned}$$

Here we set $A = \int_{-\infty}^\infty e^{-c_0^* s}\ (\dot{\Phi}_1)^2/\dot{\Phi}_1(0)ds > 0$.

Next set

$$\begin{aligned}
Y(\varepsilon; c_1, \tau_1) &= \varepsilon \frac{d}{dz} u^{(2)}(\tau; \varepsilon, c_1, \tau_1) - \varepsilon \frac{d}{dz} u^{(3)}(\tau; \varepsilon, c_1, \tau_1) \\
&= \frac{\tau_0^*}{\tau_0^* + \varepsilon\tau_1} \{\varepsilon U_0^{(2)'}(\tau_0^*) + \dot{\phi}_0^{(2),r}(0) + \varepsilon \dot{\phi}_1^{(2),r}(0; c_1, \tau_1)\} \\
&\quad - \{\varepsilon U_0^{(3)'}(0) + \dot{\phi}_0^{(3)}(0) + \varepsilon \dot{\phi}_1^{(3)}(0; c_1, \tau_1)\} + o(\varepsilon) \\
&= \{\dot{\phi}_0^{(2),r}(0) - \dot{\phi}_0^{(3)}(0)\} + \varepsilon \{U_0^{(2)'}(\tau_0^*) + \dot{\phi}_1^{(2),r}(0; c_1, \tau_1) \\
&\quad - \frac{\tau_1}{\tau_0^*} \dot{\phi}_0^{(2),r}(0) - U_0^{(3)'}(0) - \dot{\phi}_1^{(3)}(0; c_1, \tau_1)\} + o(\varepsilon) \\
&= \varepsilon Y_1(c_1, \tau_1) + o(\varepsilon),
\end{aligned}$$

where

$$Y_1(c_1, \tau_1) = U_0^{(2)'}(\tau_0^*) + \dot{\phi}_1^{(2),r}(0; c_1, \tau_1) - \frac{\tau_1}{\tau_0^*} \dot{\Phi}_2(0) - U_0^{(3)'}(0) - \dot{\phi}_1^{(3)}(0; c_1, \tau_1).$$

Here we recall $\dot{\Phi}_2 = \dot{\phi}_0^{(2),r}$ for $\xi \leq 0$ and $\dot{\Phi}_2 = \dot{\phi}_0^{(3)}$ for $\xi \geq 0$. Note that

$$\begin{aligned}
&\dot{\phi}_1^{(2),r}(0; c_1, \tau_1) \\
&= -U_1^{(2)}(\tau_0^*; c_1, \tau_1) \frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)} + \int_{-\infty}^0 e^{-c_0^* s} \frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)} \Big[\Big(c_1 + c_0^* \frac{\tau_1}{\tau_0^*} \Big) \dot{\Phi}_2 - 2 \frac{\tau_1}{\tau_0^*} f(\Phi_2, v^*) \\
&\quad - (f_u(\Phi_2, v^*) - f_u(h_+(v^*), v^*))(s U_0^{(2)'}(\tau_0^*) + U_1^{(2)}(\tau_0^*; c_1, \tau_1)) \\
&\quad - (f_v(\Phi_2, v^*) - f_v(h_+(v^*), v^*))(s V_0^{(2)'}(\tau_0^*) + V_1^{(2)}(\tau_0^*; c_1, \tau_1)) \\
&\quad - f_v(\Phi_2, v^*) \psi_0^{(2),r} \Big] ds, \\
&\dot{\phi}_1^{(3)}(0; c_1, \tau_1) \\
&= -U_1^{(3)}(0; c_1, \tau_1) \frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)} - \int_0^\infty e^{-c_0^* s} \frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)} \Big[c_1 \dot{\Phi}_2 - f_v(\Phi_2, v^*) \psi_0^{(3)} \\
&\quad - (f_u(\Phi_2, v^*) - f_u(h_-(v^*), v^*))(s U_0^{(3)'}(0) + U_1^{(3)}(0; c_1, \tau_1)) \\
&\quad - (f_v(\Phi_2, v^*) - f_v(h_-(v^*), v^*))(s V_0^{(3)'}(0) + V_1^{(3)}(0; c_1, \tau_1)) \Big] ds.
\end{aligned}$$

The condition $Y_1(c_1^*, \tau_1) = 0$ is written as

$$U_1^{(2)}(\tau_0^*; c_1^*, \tau_1) \left\{ -\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)} - \int_{-\infty}^0 e^{-c_0^* s} \frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)} f_u(\Phi_2, v^*) ds \right\}$$

$$
\begin{aligned}
&+\tau_1\left\{\int_{-\infty}^{0} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}\left[\frac{c_0^*}{\tau_0^*}\dot{\Phi}_2-\frac{2}{\tau_0^*}f(\Phi_2,v^*)\right]ds-\frac{1}{\tau_0^*}\dot{\Phi}_2(0)\right\}\\
&+U_1^{(3)}(0;c_1^*,\tau_1)\left\{\frac{\ddot{\Phi}_2}{\dot{\Phi}_2(0)}-\int_0^{\infty} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_u(\Phi_2,v^*)ds\right\}\\
&-\int_{-\infty}^{0} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)V_1^{(2)}(\tau_0^*;c_1^*,\tau_1)ds\\
&-\int_0^{\infty} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)V_1^{(3)}(0;c_1^*,\tau_1)ds+Z=0,
\end{aligned}
$$

where Z consists of remaining terms independent of τ_1. Here, the first, second and third terms vanish by (3.18) and (3.29). In addition, from (3.13) and (3.28), we have

$$
\begin{aligned}
&\int_{-\infty}^{0} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)V_1^{(2)}(\tau_0^*;c_1^*,\tau_1)ds\\
&\qquad+\int_0^{\infty} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)V_1^{(3)}(0;c_1^*,\tau_1)ds\\
&=\int_{-\infty}^{\infty} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)V_1^{(2)}(\tau_0^*;c_1^*,\tau_1)ds+Z_1\\
&=B\tau_1+Z_2,
\end{aligned}
$$

where a constant B is independent of τ_1 and defined by

$$
B=\frac{V_0^{(2)\prime}(\tau_0^*)}{c_0^*\tau_0^*}\int_0^{\tau_0^*}\frac{g(U_0^{(2)},V_0^{(2)})}{V_0^{(2)\prime}}dx\int_{-\infty}^{\infty} e^{-c_0^* s}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}f_v(\Phi_2,v^*)ds,
$$

and Z_1 and Z_2 consist of remaining terms independent of τ_1. Note that $B\neq 0$. Therefore τ_1^* is determined as

$$
\tau_1^*=\frac{Z-Z_2}{B}.
$$

By the implicit function theorem, there exist a constant $\varepsilon_0>0$ and continuously differentiable functions $c_1=c_1(\varepsilon)$ and $\tau_1=\tau_1(\varepsilon)$ defined for $\varepsilon\in[0,\varepsilon_0)$ satisfying

$$
X(\varepsilon;c_1,\tau_1)=0,\quad Y(\varepsilon;c_1,\tau_1)=0
$$

and

$$\lim_{\varepsilon\downarrow 0} c_1(\varepsilon) = c_1^*, \quad \lim_{\varepsilon\downarrow 0} \tau_1(\varepsilon) = \tau_1^*.$$

Thus the proof of Theorem 1 is completed. □

4. Construction of Solutions of Adjoint Equations

Let us consider the linearized equation (1.7) and its adjoint equation (1.8). In the case of the combustion model, we should consider the problem (1.7) in a weighted Sobolev space $H^2_{-\kappa}(-\infty,\infty) \times H^1_{-\kappa}(-\infty,\infty)$ with a weighted function $e^{-\kappa z/\varepsilon}$ for a small $\kappa > 0$ because essential spectrums of (1.7) come to the imaginary axis if we consider usual Sobolev spaces and continuous functions spaces. This was pointed out in [10], and (1.7) is considered in [5]. The weighted Sobolev spaces $H^1_{-\kappa}(-\infty,\infty)$ and $H^2_{-\kappa}(-\infty,\infty)$ are defined by

$$H^1_{-\kappa}(-\infty,\infty) = \{\varphi \in H^1_{loc}(\mathbb{R}) \mid \|\varphi\|_{L^2_{-\kappa}} + \|\varphi'\|_{L^2_{-\kappa}} < \infty\},$$

$$H^2_{-\kappa}(-\infty,\infty) = \{\varphi \in H^2_{loc}(\mathbb{R}) \mid \|\varphi\|_{L^2_{-\kappa}} + \|\varphi'\|_{L^2_{-\kappa}} + \|\varphi''\|_{L^2_{-\kappa}} < \infty\}$$

and

$$\|\varphi\|_{L^2_{-\kappa}} \equiv \left(\int_{-\infty}^{\infty} |\varphi(z)|^2 e^{-2\kappa z/\varepsilon} dz\right)^{1/2}.$$

Differentiating (1.6) by z, we see that (1.7) has a solution $(P,Q) = (u_z, v_z)$ in $H^2_{-\kappa}(-\infty,\infty) \times H^1_{-\kappa}(-\infty,\infty)$. Hence the adjoint equation (1.8) also has a solution in $H^2_{\kappa}(-\infty,\infty) \times H^1_{\kappa}(-\infty,\infty)$. On the other hand, in the FitzHugh-Nagumo system, we consider both problems (1.7) and (1.8) in a usual Sobolev space $H^2(-\infty,\infty) \times H^1(-\infty,\infty)$, that is, we can set $\kappa = 0$. In the following, we do not distinguish these cases as far as readers are not confused.

We shall construct a solution of (1.8) by dividing $\mathbb{R}$ into three parts $I_1 = (-\infty, 0)$, $I_2 = (0, \tau)$, and $I_3 = (\tau, \infty)$ as in Section 3. By three given constants A, B, D, (1.8) is respectively written in each interval as

$$\begin{cases} \varepsilon^2 P_{zz} + \varepsilon c P_z + f_u(u,v)P + g_u(u,v)Q = 0, & z \in I_1, \\ cQ_z + f_v(u,v)P + g_v(u,v)Q = 0, & z \in I_1, \\ P(-\infty) = 0, \quad P(0) = \dfrac{A}{\varepsilon}, \\ Q(-\infty) = 0, \quad Q(0) = D, \end{cases} \tag{4.1}$$

$$\begin{cases} \varepsilon^2 P_{zz} + \varepsilon c P_z + f_u(u,v)P + g_u(u,v)Q = 0, & z \in I_2, \\ cQ_z + f_v(u,v)P + g_v(u,v)Q = 0, & z \in I_2, \\ P(0) = \dfrac{A}{\varepsilon}, \quad P(\tau) = \dfrac{B(\varepsilon)}{\varepsilon}, \\ Q(0) = D, \end{cases} \tag{4.2}$$

$$\begin{cases} \varepsilon^2 P_{zz} + \varepsilon c P_z + f_u(u,v)P + g_u(u,v)Q = 0, & z \in I_3, \\ cQ_z + f_v(u,v)P + g_v(u,v)Q = 0, & z \in I_3, \\ P(\tau) = \dfrac{B(\varepsilon)}{\varepsilon}, \quad P(\infty) = 0, \\ Q(\infty) = 0, \end{cases} \tag{4.3}$$

where it is supposed that $B(\varepsilon) = B + o(\varepsilon)$ as $\varepsilon \to 0$. We shall determine A, B, D and $B(\varepsilon)$ later.

In the following we expand P and Q in each interval and look for outer and inner solutions as well as in Section 3. Many functions obtained below may depend on the parameters A, B, D and so we should write these parameters for the functions explicitly as in Section 3. Recall that we keep writing c_1 and τ_1 explicitly for functions depending on these parameters in Section 3 in spite of cumbersome expressions. For it is important to see the parameter dependency of functions when we determine c_1^* and τ_1^* (see Section 3.5). On the other it is not much important in this section because eventual equations obtained in Section 4.4 to determine A, B, D seem simpler than in Section 3.5. So we omit writing the parameter dependency of functions in many cases.

4.1. The first interval I_1

Outer Approximations

We expand P, Q as

$$P(z) = P_0(z) + \cdots, \quad Q(z) = Q_0(z) + \cdots. \tag{4.4}$$

By substituting these in (4.1), the lowest approximation is obtained as

$$\begin{cases} f_u(0,0)P_0 + g_u(0,0)Q_0 = 0, & z \in I_1, \\ c_0^* Q_0' + f_v(0,0)P_0 + g_v(0,0)Q_0 = 0, & z \in I_1, \\ Q_0(-\infty) = 0, \quad Q_0(0) = D - \eta_0(0), \end{cases}$$

where the function η_0 is given later. Hence we have

$$P_0(z) = -\frac{g_u(0,0)}{f_u(0,0)} Q_0(z)$$

and

$$\begin{cases} c_0^* Q_0' = \left(\dfrac{f_v(0,0)g_u(0,0)}{f_u(0,0)} - g_v(0,0) \right) Q_0, \quad z \in I_1, \\ Q_0(-\infty) = 0, \quad Q_0(0) = D - \eta_0(0). \end{cases}$$

Thus we obtain

$$\begin{cases} Q_0(z) = (D - \eta_0(0)) \exp\left\{ \left(\dfrac{f_v(0,0)g_u(0,0)}{f_u(0,0)} - g_v(0,0) \right) \dfrac{z}{c_0^*} \right\}, \\ P_0(z) = -\dfrac{g_u(0,0)}{f_u(0,0)} Q_0(z). \end{cases}$$

Inner Approximations

Next, we consider the inner approximations. We expand P, Q as

$$P(z) = P_0(z) + \frac{1}{\varepsilon}\zeta_0(\frac{z}{\varepsilon}) + \zeta_1(\frac{z}{\varepsilon}) + \cdots, \quad Q(z) = Q_0(z) + \eta_0(\frac{z}{\varepsilon}) + \varepsilon\eta_1(\frac{z}{\varepsilon}) + \cdots.$$

By substituting this in (4.1) and putting $\xi = z/\varepsilon$, it follows from ε^{-1}-order terms that

$$\begin{cases} \ddot{\zeta}_0 + c_0^* \dot{\zeta}_0 + f_u(\Phi_1, 0)\zeta_0 = 0, & \xi \in (-\infty, 0), \\ c_0^* \dot{\eta}_0 + f_v(\phi_0, 0)\zeta_0 = 0, & \xi \in (-\infty, 0), \\ \zeta_0(-\infty) = 0, \quad \zeta_0(0) = A, \\ \eta_0(-\infty) = 0. \end{cases}$$

Hence we obtain

$$\zeta_0(\xi) = Ae^{-c_0^*\xi} \frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}, \quad \eta_0(\xi) = -\frac{1}{c_0^*} \int_{-\infty}^{\xi} f_v(\Phi_1, 0)\zeta_0 ds.$$

Similarly, from the order of ε^0, we get

$$\begin{cases} \ddot{\zeta}_1 + c_0^*\dot{\zeta}_1 + f_u(\Phi_1,0)\zeta_1 \\ \quad = -c_1^*\dot{\zeta}_0 - f_u(\Phi_1,0)P_0(0) - f_{uu}(\Phi_1,0)\phi_1\zeta_0 \\ \qquad -f_{uv}(\Phi_1,0)\psi_0\zeta_0 - g_u(\Phi_1,0)(Q_0(0)+\eta_0), & \xi\in(-\infty,0), \\ c_0^*\dot{\eta}_1 + c_1^*\dot{\eta}_0 + c_0^*Q_0'(0) + f_v(\Phi_1,0)(\zeta_1+P_0(0)) + f_{uv}(\Phi_1,0)\zeta_0\phi_1 \\ +f_{vv}(\Phi_1,0)\zeta_0\psi_0 + g_v(\Phi_1,0)(Q_0(0)+\eta_0) = 0, & \xi\in(-\infty,0), \\ \zeta_1(-\infty)=0, \quad \zeta_1(0) = -P_0(0), \\ \eta_1(-\infty) = 0. \end{cases}$$

Hence we have

$$\zeta_1(\xi) = -P_0(0)\frac{\hat{\zeta}_0}{\dot{\Phi}_1(0)} + \hat{\zeta}_0\int_\xi^0 \frac{e^{-c_0^*s}}{(\hat{\zeta}_0)^2}\int_{-\infty}^s e^{c_0^*\rho}\hat{\zeta}_0[c_1^*\dot{\zeta}_0 + f_u(\Phi_1,0)P_0(0)$$
$$+f_{uu}(\Phi_1,0)\phi_1\zeta_0 + f_{uv}(\Phi_1,0)\psi_0\zeta_0 + g_u(\Phi_1,0)(Q_0(0)+\eta_0)]d\rho ds,$$

$$\eta_1(\xi) = -\frac{1}{c_0^*}\int_{-\infty}^\xi \{c_1^*\dot{\eta}_0 + (f_v(\Phi_1,0)-f_v(0,0))P_0(0) + f_v(\Phi_1,0)\zeta_1 + f_{uv}(\Phi_1,0)\zeta_0\phi_1$$
$$+f_{vv}(\Phi_1,0)\zeta_0\psi_0 + (g_v(\Phi_1,0)-g_v(0,0))Q_0(0) + g_v(\Phi_1,0)\eta_0\}ds.$$

where

$$\hat{\zeta}_0(\xi) = e^{-c_0^*\xi}\dot{\Phi}_1.$$

The solution of (4.1) will be represented as

$$\begin{cases} P^{(1)}(z;\varepsilon,A,D) = P_0^{(1)}(z;A,D) + \frac{1}{\varepsilon}\zeta_0^{(1)}(\frac{z}{\varepsilon};A) + \zeta_1^{(1)}(\frac{z}{\varepsilon};A,D) \\ \qquad +Z^{(1)}(z;\varepsilon,A,D) - \frac{g_u(0,0)}{f_u(0,0)}W^{(1)}(z;\varepsilon,A,D), \\ Q^{(1)}(z;\varepsilon,A,D) = Q_0^{(1)}(z;A,D) + \eta_0^{(1)}(\frac{z}{\varepsilon};A) \\ \qquad +\varepsilon\left\{\eta_1^{(1)}(\frac{z}{\varepsilon};A,D) - \eta_1^{(1)}(0;A,D)e^{\nu z}\right\} \\ \qquad +W^{(1)}(z;\varepsilon,A,D). \end{cases} \tag{4.5}$$

Theorem 6. *Fix $\delta>0$ and $A^*, D^* \in (-\infty,\infty)$ arbitrarily, and put*

$$\Omega_\delta = \{(A,D)\in\mathbb{R}^2 \mid |A-A^*| + |D-D^*| \le \delta\}.$$

Then, there is $\varepsilon_0 > 0$ such that the pair $(P^{(1)}, Q^{(1)})$ given by (4.5) for a function $(Z^{(1)}, W^{(1)}) \in X_{\nu,\varepsilon}$ is a solution of (4.1) for any $\varepsilon \in (0, \varepsilon_0)$, where the functional space $X_{\nu,\varepsilon}$ is defined by

$$\begin{cases} X_{\nu,\varepsilon} = X^2_{\nu,\varepsilon} \times \hat{X}^2_{\nu,\varepsilon}, \\ X_\nu \equiv \{\varphi \in C(-\infty, 0) \mid \|\varphi\|_{X_\nu} \equiv \sup_{z\in(-\infty,0)} |\varphi(z)|e^{-\nu z} < \infty\}, \\ X^2_{\nu,\varepsilon} \equiv \{\varphi \in X_\nu \mid \varphi(0) = 0,\ \|\varphi\|_{X_\nu} + \varepsilon\|\varphi'\|_{X_\nu} + \varepsilon^2\|\varphi''\|_{X_\nu} < \infty\}, \\ \hat{X}^2_{\nu,\varepsilon} \equiv \{\varphi \in X_\nu \mid \varphi(0) = 0,\ \|\varphi\|_{X_\nu} + \|\varphi'\|_{X_\nu} + \varepsilon\|\varphi''\|_{X_\nu} < \infty\}, \end{cases}$$

and ν satisfies $0 < \nu < \mu_0$ and is fixed, where μ_0 was given in Theorem 5. In addition, $(Z^{(1)}, W^{(1)}) = o(1)$ in $X_{\nu,\varepsilon}$ uniformly for $(A, D) \in \Omega_\delta$ as $\varepsilon \to 0$.

This theorem can be shown by the same argument as in the proof of Theorem 4. So we omit the details. Note that $Z^{(1)}(-\infty; \varepsilon, A, D) = Z^{(1)}(0; \varepsilon, A, D) = W^{(1)}(-\infty; \varepsilon, A, D) = 0$.

4.2. The second interval I_2

In this subsection, we consider the problem (4.2). If we set $y = \tau_0^* z/\tau$, then this problem is rewritten as

$$\begin{cases} \varepsilon^2 P_{yy} + \varepsilon c(1+\varepsilon\frac{\tau_1}{\tau_0^*})P_y + (1+\varepsilon\frac{\tau_1}{\tau_0^*})^2(f_u(u,v)P + g_u(u,v)Q) = 0, & y \in (0, \tau_0^*), \\ cQ_y + \left(1+\varepsilon\frac{\tau_1}{\tau_0^*}\right)(f_v(u,v)P + g_v(u,v)Q) = 0, & y \in (0, \tau_0^*), \\ P(0) = \dfrac{A}{\varepsilon}, \quad P(\tau_0^*) = \dfrac{B(\varepsilon)}{\varepsilon}, \\ Q(0) = D, \end{cases} \tag{4.6}$$

where $B(\varepsilon) = B + o(1)$ as $\varepsilon \to 0$, as stated pervasively.

Outer Approximations

We expand P and Q as

$$P(y) = P_0(y) + \cdots, \quad Q(y) = Q_0(y) + \cdots.$$

By substituting these in (4.6), the lowest order approximation is obtained as

$$\begin{cases} f_u(U_0,V_0)P_0 + g_u(U_0,V_0)Q_0 = 0, & y \in (0,\tau_0^*), \\ c_0^* Q_0' + f_v(U_0,V_0)P_0 + g_v(U_0,V_0)Q_0 = 0, & y \in (0,\tau_0^*), \\ Q_0(0) = D - \eta_0(0), \end{cases}$$

where the function η_0 will be determined later. Hence we have

$$P_0(y) = -\frac{g_u(U_0,V_0)}{f_u(U_0,V_0)} Q_0(y)$$

and

$$\begin{cases} c_0^* Q_0' = \left(\dfrac{f_v(U_0,V_0) g_u(U_0,V_0)}{f_u(U_0,V_0)} - g_v(U_0,V_0) \right) Q_0, & y \in (0,\tau_0^*), \\ Q_0(0) = D - \eta_0(0). \end{cases}$$

Thus we obtain

$$\begin{cases} Q_0(y) = (D-\eta_0(0)) \exp \left\{ \frac{1}{c_0^*} \displaystyle\int_0^y \left(\frac{f_v(U_0,V_0) g_u(U_0,V_0)}{f_u(U_0,V_0)} - g_v(U_0,V_0) \right) dx \right\}, \\ P_0(y) = -\dfrac{g_u(U_0,V_0)}{f_u(U_0,V_0)} Q_0(y). \end{cases}$$

Inner Approximations at $y = 0$

In a neighborhood of $y = 0$, we expand P and Q as

$$P(y) = P_0(y) + \frac{1}{\varepsilon}\zeta_0(\frac{y}{\varepsilon}) + \zeta_1(\frac{y}{\varepsilon}) + \cdots, \quad Q(y) = Q_0(y) + \eta_0(\frac{y}{\varepsilon}) + \varepsilon\eta_1(\frac{y}{\varepsilon}) + \cdots.$$

By substituting this in (4.6) and putting $\xi = y/\varepsilon$, it follows from ε^{-1}-order terms that

$$\begin{cases} \ddot{\zeta}_0 + c_0^* \dot{\zeta}_0 + f_u(\Phi_1, 0)\zeta_0 = 0, & \xi \in (0,\infty), \\ c_0^* \dot{\eta}_0 + f_v(\Phi_1, 0)\zeta_0 = 0, & \xi \in (0,\infty), \\ \zeta_0(0) = A, \quad \zeta_0(\infty) = 0, \\ \eta_0(\infty) = 0. \end{cases}$$

Hence we obtain

$$\zeta_0(\xi) = A e^{-c_0^* \xi} \frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}, \quad \eta_0(\xi) = \frac{1}{c_0^*} \int_\xi^\infty f_v(\Phi_1, 0)\zeta_0 ds.$$

Similarly, from the order of ε^0, we have

$$\begin{cases} \ddot{\zeta}_1 + c_0^*\dot{\zeta}_1 + f_u(\Phi_1,0)\zeta_1 \\ \quad = -\left(c_1^* + c_0^*\dfrac{\tau_1^*}{\tau_0^*}\right)\dot{\zeta}_0 - 2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_1,0)\zeta_0 - f_u(\Phi_1,0)P_0(0) \\ \qquad -g_u(\Phi_1,0)(Q_0(0)+\eta_0) - f_{uu}(\Phi_1,0)(\xi U_0'(0)+U_1(0)+\phi_1)\zeta_0 \\ \qquad -f_{uv}(\Phi_1,0)(\xi V_0'(0)+V_1(0)+\psi_0)\zeta_0, & \xi\in(0,\infty), \\ c_0^*\dot{\eta}_1 = -c_0^*Q_0'(0) - c_1^*\dot{\eta}_0 - g_v(\Phi_1,0)(Q_0(0)+\eta_0) \\ \qquad -f_v(\Phi_1,0)(P_0(0)+\zeta_1) - \dfrac{\tau_1^*}{\tau_0^*}f_v(\Phi_1,0)\zeta_0 \\ \qquad -f_{uv}(\Phi_1,0)(\xi U_0'(0)+U_1(0)+\phi_1)\zeta_0 \\ \qquad -f_{vv}(\Phi_1,0)(\xi V_0'(0)+V_1(0)+\psi_0)\zeta_0, & \xi\in(0,\infty), \\ \zeta_1(0) = -P_0(0), \quad \zeta_1(\infty) = 0, \\ \eta_1(\infty) = 0. \end{cases}$$

Hence

$$\begin{aligned} \zeta_1(\xi) &= -P_0(0)\frac{\hat{\zeta}_0}{\dot{\Phi}_1(0)} + \hat{\zeta}_0\int_0^\xi \frac{e^{-c_0^*s}}{(\hat{\zeta}_0)^2}\int_s^\infty e^{c_0^*\rho}\hat{\zeta}_0\Big\{\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\dot{\zeta}_0 \\ &\quad +2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_1,0)\zeta_0 + f_u(\Phi_1,0)P_0(0) + g_u(\Phi_1,0)Q_0(0) \\ &\quad +g_u(\Phi_1,0)\eta_0 + f_{uu}(\Phi_1,0)(\rho U_0'(0)+U_1(0)+\phi_1)\zeta_0 \\ &\quad +f_{uv}(\Phi_1,0)(\rho V_0'(0)+V_1(0)+\psi_0)\zeta_0\Big\}d\rho ds, \\ \eta_1(\xi) &= \frac{1}{c_0^*}\int_\xi^\infty \Big\{c_1^*\dot{\eta}_0 + (g_v(\Phi_1,0) - g_v(h_+(0),0))Q_0(0) + g_v(\Phi_1,0)\eta_0 \\ &\quad +(f_v(\Phi_1,0) - f_v(h_+(0),0))P_0(0) + g_v(\Phi_1,0)\zeta_1 + \frac{\tau_1^*}{\tau_0^*}f_v(\Phi_1,0)\zeta_0 \\ &\quad +f_{uv}(\Phi_1,0)(sU_0'(0)+U_1(0)+\phi_1)\zeta_0 \\ &\quad +f_{vv}(\Phi_1,0)(sV_0'(0)+V_1(0)+\psi_0)\zeta_0\Big\}ds, \end{aligned}$$

where

$$\hat{\zeta}_0(\xi) = e^{-c_0^*\xi}\dot{\Phi}_1.$$

Inner Approximations at $y = \tau_0^*$

In a neighborhood of $y = \tau_0^*$, we expand P and Q as

$$P(y) = P_0(y) + \frac{1}{\varepsilon}\zeta_0\Big(\frac{y-\tau_0^*}{\varepsilon}\Big) + \zeta_1\Big(\frac{y-\tau_0^*}{\varepsilon}\Big) + \cdots,$$

$$Q(y) \;=\; Q_0(y) + \eta_0(\frac{y-\tau_0^*}{\varepsilon}) + \varepsilon\eta_1(\frac{y-\tau_0^*}{\varepsilon}) + \cdots.$$

By substituting these in (4.6), ε^{-1}-order terms satisfy

$$\begin{cases} \ddot{\zeta}_0 + c_0^*\dot{\zeta}_0 + f_u(\Phi_2, v^*)\zeta_0 = 0, & \xi \in (-\infty, 0), \\ c_0^*\dot{\eta}_0 + f_v(\Phi_2, v^*)\zeta_0 = 0, & \xi \in (-\infty, 0), \\ \zeta_0(0) = B, \quad \zeta_0(-\infty) = 0, \\ \eta_0(-\infty) = 0. \end{cases}$$

Hence we obtain

$$\zeta_0(\xi) = Be^{-c_0^*\xi}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}, \quad \eta_0(\xi) = -\frac{1}{c_0^*}\int_{-\infty}^{\xi} f_v(\Phi_2, v^*)\zeta_0 ds.$$

Similarly, from the order of ε^0, we have

$$\begin{cases} \ddot{\zeta}_1 + c_0^*\dot{\zeta}_1 + f_u(\Phi_2, v^*)\zeta_1 \\ \quad = -\left(c_1^* + c_0^*\dfrac{\tau_1^*}{\tau_0^*}\right)\dot{\zeta}_0 - 2\dfrac{\tau_1^*}{\tau_0^*}f_u(\Phi_2, v^*)\zeta_0 - f_u(\Phi_2, v^*)P_0(\tau_0^*) \\ \qquad -g_u(\Phi_2, v^*)(Q_0(\tau_0^*) + \eta_0) \\ \qquad -f_{uu}(\Phi_2, v^*)(\xi U_0'(\tau_0^*) + U_1(\tau_0^*) + \phi_1)\zeta_0 \\ \qquad -f_{uv}(\Phi_2, v^*)(\xi V_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0)\zeta_0, & \xi \in (-\infty, 0), \\ c_0^*\dot{\eta}_1 = -c_0^*Q_0'(\tau_0^*) - c_1^*\dot{\eta}_0 - g_v(\Phi_2, v^*)(Q_0(\tau_0^*) + \eta_0) \\ \qquad -f_v(\Phi_2, v^*)(P_0(\tau_0^*) + \xi_1) - \dfrac{\tau_1^*}{\tau_0^*}f_v(\Phi_2, v^*)\zeta_0 \\ \qquad -f_{uv}(\Phi_2, v^*)(\xi U_0'(\tau_0^*) + U_1(\tau_0^*) + \phi_1)\zeta_0 \\ \qquad -f_{vv}(\Phi_2, v^*)(\xi V_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0)\zeta_0, & \xi \in (-\infty, 0), \\ \zeta_1(0) = -P_0(\tau_0^*), \quad \zeta_1(-\infty) = 0, \\ \eta_1(-\infty) = 0. \end{cases}$$

Hence we have

$$\begin{aligned} \zeta_1(\xi) \;=\; & -P_0(\tau_0^*)\frac{\hat{\zeta}_0}{\dot{\Phi}_2(0)} + \hat{\zeta}_0\int_{\xi}^{0}\frac{e^{-c_0^*s}}{(\hat{\zeta}_0)^2}\int_{-\infty}^{s} e^{c_0^*\rho}\hat{\zeta}_0\Big\{\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\dot{\zeta}_0 \\ & +2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_2, v^*)\zeta_0 + f_u(\Phi_2, v^*)P_0(\tau_0^*) + g_u(\Phi_2, v^*)Q_0(\tau_0^*) \\ & +g_u(\Phi_2, v^*)\eta_0 + f_{uu}(\Phi_2, v^*)(\rho U_0'(\tau_0^*) + U_1(\tau_0^*) + \phi_1)\zeta_0 \\ & +f_{uv}(\Phi_2, v^*)(\rho V_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0)\zeta_0\Big\}d\rho ds, \end{aligned}$$

$$\begin{aligned}\eta_1(\xi) \;=\; & -\frac{1}{c_0^*}\int_{-\infty}^{\xi}\Big\{c_1^*\dot{\eta}_0 + (g_v(\Phi_2,v^*) - g_v(h_+(v^*),v^*))Q_0(\tau_0^*)\\ & + g_v(\Phi_2,v^*)\eta_0 + (f_v(\Phi_2,v^*) - f_v(h+(v^*),v^*))P_0(\tau_0^*) + f_v(\Phi_2,v^*)\zeta_1\\ & + \frac{\tau_1^*}{\tau_0^*}f_v(\Phi_2,v^*)\zeta_0 + f_{uv}(\Phi_2,v^*)(sU_0'(\tau_0^*) + U_1(\tau_0^*) + \phi_1)\zeta_0\\ & + f_{vv}(\Phi_2,v^*)(sV_0'(\tau_0^*) + V_1(\tau_0^*) + \psi_0)\zeta_0\Big\}ds,\end{aligned}$$

where

$$\hat{\zeta}_0(\xi) = e^{-c_0^*\xi}\dot{\Phi}_2.$$

The solution of (4.6) will be represented as

$$\begin{aligned}& P^{(2)}(y;\varepsilon,A,B,D)\\ & = P_0^{(2)}(y;A,D) + \theta(\frac{y}{\tau_0^*})\Big\{\frac{1}{\varepsilon}\zeta_0^{(2),l}(\frac{y}{\varepsilon};A) + \zeta_1^{(2),l}(\frac{y}{\varepsilon};A,D)\Big\}\\ & \quad + \theta(\frac{\tau_0^*-y}{\tau_0^*})\Big\{\frac{1}{\varepsilon}\zeta_0^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};B) + \zeta_1^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};A,B,D)\Big\}\\ & \quad + Z^{(2)}(y;\varepsilon,A,B,D) - h_+'(V_0^{(2)}(y))W^{(2)}(y;\varepsilon,A,B,D), \qquad (4.7)\end{aligned}$$

$$\begin{aligned}& Q^{(2)}(y;\varepsilon;A,B,D)\\ & = Q_0^{(2)}(y;A,D) + \theta(\frac{y}{\tau_0^*})\Big\{\eta_0^{(2),l}(\frac{y}{\varepsilon};A) + \varepsilon[\eta_1^{(2),l}(\frac{y}{\varepsilon};A,D) - \eta_1^{(2),l}(0;A,D)]\Big\}\\ & \quad + \theta(\frac{\tau_0^*-y}{\tau_0^*})\Big\{\eta_0^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};B) + \varepsilon\eta_1^{(2),r}(\frac{y-\tau_0^*}{\varepsilon};A,B,D)\Big\}\\ & \quad + W^{(2)}(y;\varepsilon,A,B,D), \qquad (4.8)\end{aligned}$$

where $\theta(y)$ is the same cut-off function as given in Section 3.3.

Theorem 7. *Fix $\delta > 0$ and $A^*, B^*, D^* \in (-\infty,\infty)$ arbitrarily, and put*

$$\Pi_\delta = \{(A,B,D) \mid |A-A^*| + |B-B^*| + |D-D^*| \le \delta\}.$$

Then, there is $\varepsilon_0 > 0$ such that the pair $(P^{(2)}, v^{(2)})$ given by (4.7) and (4.8) for a function $(Z^{(2)}, W^{(2)}) \in X_\varepsilon$ is a solution of (4.6) for any $\varepsilon_0 > 0$, where X_ε is the same one as defined in Section 3.3. In addition, $(Z^{(2)}, W^{(2)}) = o(1)$ in X_ε uniformly for $(A,B,D) \in \Pi_\delta$ as $\varepsilon \to 0$.

This theorem can be shown by the same argument as in the proof of Theorem 4. So we omit the details. Note that $Z^{(2)}(0;A,B,D) = Z^{(2)}(\tau_0^*;A,B,D) = W^{(2)}(0;A,B,D) = 0$. Therefore, we constructed a solution of (4.2) such as

$$\begin{aligned}
&P^{(2)}(z;\varepsilon,A,B,D)\\
&= P_0^{(2)}(\frac{\tau_0^*}{\tau}z;A,D) + \theta(\frac{z}{\tau})\Big\{\frac{1}{\varepsilon}\zeta_0^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau}z;A) + \zeta_1^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau}z;A,D)\Big\}\\
&\quad +\theta(\frac{\tau - z}{\tau})\Big\{\frac{1}{\varepsilon}\zeta_0^{(2),r}(\frac{\tau_0^*(z-\tau)}{\varepsilon\tau};B) + \zeta_1^{(2),r}(\frac{\tau_0^*(z-\tau)}{\varepsilon\tau};A,B,D)\Big\}\\
&\quad +Z^{(2)}(\frac{\tau_0^*}{\tau}z;\varepsilon,A,B,D) - h'_+(V_0^{(2)}(\frac{\tau_0^*}{\tau}z))W^{(2)}(\frac{\tau_0^*}{\tau}z;\varepsilon,A,B,D), \qquad (4.9)\\
&Q^{(2)}(z;\varepsilon,A,B,D)\\
&= Q_0^{(2)}(\frac{\tau_0^*}{\tau}z;A,D) + \theta(\frac{z}{\tau})\Big\{\eta_0^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau}z;A)\\
&\quad +\varepsilon\Big[\eta_1^{(2),l}(\frac{\tau_0^*}{\varepsilon\tau}z;A,D) - \eta_1^{(2),l}(0;A,D)\Big]\Big\}\\
&\quad +\theta(\frac{\tau - z}{\tau})\Big\{\eta_0^{(2),r}(\frac{\tau_0^*(z-\tau)}{\varepsilon\tau};A,B,D) + \varepsilon\eta_1^{(2),r}(\frac{\tau_0^*(z-\tau)}{\varepsilon\tau};A,B,D)\Big\}\\
&\quad +W^{(2)}(\frac{\tau_0^*}{\tau}z;\varepsilon,A,B,D). \qquad (4.10)
\end{aligned}$$

The constant $B(\varepsilon)$ is given by $B(\varepsilon) = B - h'_+(V_0^{(2)}(\tau_0^*))\, W^{(2)}(\tau_0^*;\varepsilon,A,B,D)$, which leads to $B(\varepsilon) = B + o(\varepsilon)$ as $\varepsilon \to 0$.

4.3. The third interval I_3

In this subsection, we consider the problem (4.3). If we put $y = z - \tau$, then the above problem is rewritten as

$$\begin{cases} \varepsilon^2 P_{zz} + \varepsilon c P_z + f_u(u,v)P + g_u(u,v)Q = 0, & y \in (0,\infty),\\ cQ_z + f_v(u,v)P + g_v(u,v)Q = 0, & y \in (0,\infty),\\ P(0) = \dfrac{B(\varepsilon)}{\varepsilon}, \quad P(\infty) = 0, & \\ Q(\infty) = 0. & \end{cases} \qquad (4.11)$$

Outer Approximations

We expand P and Q as

$$P(y) = P_0(y) + \cdots, \quad Q(y) = Q_0(y) + \cdots.$$

By substituting these in (4.11), the lowest order approximation is obtained as

$$\begin{cases} f_u(U_0,V_0)P_0 + g_u(U_0,V_0)Q_0 = 0, & y \in (0,\infty), \\ c_0^* Q_0' + f_v(U_0,V_0)P_0 + g_v(U_0,V_0)Q_0 = 0, & y \in (0,\infty), \\ Q_0(\infty) = 0. \end{cases}$$

Hence

$$P_0 = -\frac{f_u(U_0,V_0)}{g_u(U_0,V_0)} Q_0$$

and

$$\begin{cases} c_0^* Q_0' = \left(\dfrac{f_v(U_0,V_0) g_u(U_0,V_0)}{f_u(U_0,V_0)} - g_v(U_0,V_0) \right) Q_0, \quad y \in (0,\infty), \\ Q_0(\infty) = 0. \end{cases}$$

Thus we obtain

$$\begin{cases} Q_0(y) = Q_0(0) \exp \left\{ \left(\dfrac{f_v(U_0,V_0) g_u(U_0,V_0)}{f_u(U_0,V_0)} - g_v(U_0,V_0) \right) \dfrac{y}{c_0^*} \right\}, \\ P_0(y) = -\dfrac{g_u(U_0,V_0)}{f_u(U_0,V_0)} Q_0(y). \end{cases}$$

Since $f_u(U_0,V_0) < 0$ and $f_u(U_0,V_0)g_v(U_0,V_0) - g_u(U_0,V_0)f_v(U_0,V_0) \geq 0$ for sufficiently large $y > 0$, we have

$$P_0 \equiv 0, \quad Q_0 \equiv 0.$$

Inner Approximations at $y = 0$

In a neighborhood of $y = 0$, we expand P and Q as

$$P(y) = \frac{1}{\varepsilon}\zeta_0(\frac{y}{\varepsilon}) + \zeta_1(\frac{y}{\varepsilon}) + \cdots, \quad Q(y) = \eta_0(\frac{y}{\varepsilon}) + \varepsilon\eta_1(\frac{y}{\varepsilon}) + \cdots.$$

By substituting these in (4.11) and putting $\xi = y/\varepsilon$, it follows from the order of ε^{-1} that

$$\begin{cases} \ddot{\zeta}_0 + c_0^* \dot{\zeta}_0 + f_u(\Phi_2, v^*)\zeta_0 = 0, & \xi \in (0,\infty), \\ c_0^* \dot{\eta}_0 + f_v(\Phi_2, v^*)\zeta_0 = 0, & \xi \in (0,\infty), \\ \zeta_0(0) = B, \quad \zeta_0(\infty) = 0, \\ \eta_0(\infty) = 0. \end{cases}$$

Hence we obtain

$$\zeta_0(\xi) = Be^{-c_0^*\xi}\frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}, \quad \eta_0(\xi) = \frac{1}{c_0^*}\int_\xi^\infty f_v(\Phi_2, v^*)\zeta_0 ds.$$

Similarly, from the order of ε^0, we have

$$\begin{cases} \ddot{\zeta}_1 + c_0^*\dot{\zeta}_1 + f_u(\Phi_2, v^*)\zeta_1 = -c_1^*\dot{\zeta}_0 \\ \quad -g_u(\Phi_2, v^*)\eta_0 - f_{uu}(\Phi_2, v^*)(\xi U_0'(0) + U_1(0) + \phi_1)\zeta_0 \\ \qquad -f_{uv}(\Phi_2, v^*)(\xi V_0'(0) + V_1(0) + \psi_0)\zeta_0, & \xi \in (0, \infty), \\ c_0^*\dot{\eta}_1 = -c_1^*\dot{\eta}_0 - g_v(\Phi_2, v^*)\eta_0 - f_v(\Phi_2, v^*)\zeta_1 \\ \qquad -f_{uv}(\Phi_2, v^*)(\xi U_0'(0) + U_1(0) + \phi_1)\zeta_0 \\ \qquad -f_{vv}(\Phi_2, v^*)(\xi V_0'(0) + V_1(0) + \psi_0)\zeta_0, & \xi \in (0, \infty), \\ \zeta_1(0) = 0, \quad \zeta_1(\infty) = 0, \\ \eta_1(\infty) = 0. \end{cases}$$

Hence we have

$$\begin{aligned} \zeta_1(\xi) &= -P_0(0)\frac{\hat{\zeta}_0}{\dot{\Phi}_2(0)} + \hat{\zeta}_0\int_0^\xi \frac{e^{-c_0^*s}}{(\hat{\zeta}_0)^2}\int_s^\infty e^{c_0^*\rho}\hat{\zeta}_0\Big\{c_1^*\dot{\zeta}_0 + g_u(\Phi_2, v^*)\eta_0 \\ &\quad + f_{uu}(\Phi_2, v^*)(\rho U_0'(0) + U_1(0) + \phi_1)\zeta_0 \\ &\quad + f_{uv}(\Phi_2, v^*)(\rho V_0'(0) + V_1(0) + \psi_0)\zeta_0\Big\}d\rho ds, \\ \eta_1(\xi) &= \frac{1}{c_0^*}\int_\xi^\infty \Big\{c_1^*\dot{\eta}_0 + g_v(\Phi_2, v^*)\eta_0 + f_v(\Phi_2, v^*)\zeta_1 \\ &\qquad + f_{uv}(\Phi_2, v^*)(sU_0'(0) + U_1(0) + \phi_1)\zeta_0 \\ &\qquad + f_{vv}(\Phi_2, v^*)(sV_0'(0) + V_1(0) + \psi_0)\zeta_0\Big\}ds, \end{aligned}$$

where

$$\hat{\zeta}_0(\xi) = e^{-c_0^*\xi}\dot{\Phi}_2.$$

The solution of (4.11) will be represented as

$$\begin{cases} P^{(3)}(y; \varepsilon, A, B, D) = \dfrac{1}{\varepsilon}\zeta_0^{(3)}(\dfrac{y}{\varepsilon}; B) + \zeta_1^{(3)}(\dfrac{y}{\varepsilon}; B) \\ \quad -h_+'(V_0^{(2)}(\tau_0^*))W^{(2)}(\tau_0^*; \varepsilon, A, B, D)e^{-\nu y/\varepsilon} + Z^{(3)}(\dfrac{y}{\varepsilon}; \varepsilon, A, B, D), \\ Q^{(3)}(y; \varepsilon, A, B, D) = \eta_0^{(3)}(\dfrac{y}{\varepsilon}; B) + \varepsilon\eta_1^{(3)}(\dfrac{y}{\varepsilon}; B) + \varepsilon W^{(3)}(\dfrac{y}{\varepsilon}; \varepsilon, A, B, D), \end{cases} \tag{4.12}$$

where ν is a constant independent of ε, given as well as μ in Theorem 3.

Theorem 8. *Fix $\delta > 0$ and $A^*, B^*, D^* \in (-\infty, \infty)$ arbitrarily, and put*

$$\Pi_\delta = \{(A, B, D) \mid |A - A^*| + |B - B^*| + |D - D^*| \leq \delta\}.$$

Then, there is $\varepsilon_0 > 0$ such that the pair $(P^{(3)}, Q^{(3)})$ given by (4.12) for a function $(Z^{(3)}, W^{(3)}) \in X_\nu$ is a solution of (4.11) for any $\varepsilon \in (0, \varepsilon_0)$. The functional space X_ν is defined as well as in Section 3.2 and ν satisfies $0 < \nu < \mu_0$, where μ_0 was given in Theorem 3. In addition, $(Z^{(3)}, W^{(3)}) = o(1)$ in X_ν uniformly for $(A, B, D) \in \Pi_\delta$ as $\varepsilon \to 0$.

The solution of (4.3) is represented as

$$\begin{cases} P^{(3)}(z; \varepsilon, A, B, D) = \frac{1}{\varepsilon}\zeta_0^{(3)}(\frac{z-\tau}{\varepsilon}; B) + \zeta_1^{(3)}(\frac{z-\tau}{\varepsilon}; B) \\ \quad -h'_+(V_0^{(2)}(\tau_0^*))W^{(2)}(\tau_0^*; \varepsilon, A, B, D)e^{-\nu(z-\tau)/\varepsilon} + Z^{(3)}(\frac{z-\tau}{\varepsilon}; \varepsilon, A, B, D), \\ Q^{(3)}(z; \varepsilon, A, B, D) = \eta_0^{(3)}(\frac{z-\tau}{\varepsilon}; B) + \varepsilon\eta_1^{(3)}(\frac{z-\tau}{\varepsilon}; B) + \varepsilon W^{(3)}(\frac{z-\tau}{\varepsilon}; \varepsilon, A, B, D). \end{cases}$$

4.4. The whole interval

In summary, we have

$$\begin{cases} P^{(1)}(z) = P_0^{(1)}(z) + \frac{1}{\varepsilon}\zeta_0^{(1)}(\frac{z}{\varepsilon}) + \zeta_1^{(1)}(\frac{z}{\varepsilon}) + \cdots, & z \in I_1, \\ Q^{(1)}(z) = Q_0^{(1)}(z) + \eta_0^{(1)}(\frac{z}{\varepsilon}) + \varepsilon\eta_1^{(1)}(\frac{z}{\varepsilon}) + \cdots, & z \in I_1, \end{cases}$$

$$\begin{cases} P^{(2)}(z) = P_0^{(2)}(\frac{\tau_0^*}{\tau}z) + \theta(\frac{z}{\tau})\left\{\frac{1}{\varepsilon}\zeta_0^{(2),l}(\frac{\tau_0^* z}{\varepsilon\tau}) + \zeta_1^{(2),l}(\frac{\tau_0^* z}{\epsilon\tau})\right\} \\ \qquad +\theta(\frac{\tau-z}{\tau})\left\{\frac{1}{\varepsilon}\zeta_0^{(2),r}(\frac{\tau_0^*(z-\tau)}{\epsilon\tau}) + \zeta_1^{(2),r}(\frac{\tau_0^*(z-\tau)}{\epsilon\tau})\right\} + \cdots, & z \in I_2, \\ Q^{(2)}(z) = Q_0^{(2)}(\frac{\tau_0^*}{\tau}z) + \theta(\frac{z}{\tau})\left\{\eta_0^{(2),l}(\frac{\tau_0^* z}{\epsilon\tau}) + \varepsilon\eta_1^{(2),l}(\frac{\tau_0^* z}{\epsilon\tau})\right\} \\ \qquad +\theta(\frac{\tau-z}{\tau})\left\{\eta_0^{(2),r}(\frac{\tau_0^*(z-\tau)}{\epsilon\tau}) + \varepsilon\eta_1^{(2),r}(\frac{\tau_0^*(z-\tau)}{\epsilon\tau})\right\} + \cdots, & z \in I_2, \end{cases}$$

$$\begin{cases} P^{(3)}(z) = \frac{1}{\varepsilon}\zeta_0^{(3)}(\frac{z-\tau}{\varepsilon}) + \zeta_1^{(3)}(\frac{z-\tau}{\varepsilon}) + \cdots & z \in I_3, \\ Q^{(3)}(z) = \eta_0^{(3)}(\frac{z-\tau}{\varepsilon}) + \varepsilon\eta_1^{(3)}(\frac{z-\tau}{\varepsilon}) + \cdots, & z \in I_3. \end{cases}$$

It is clear from the construction of the above solutions that

$$\begin{cases} P^{(1)}(0) = P^{(2)}(0), \\ Q^{(1)}(0) = Q^{(2)}(0), \\ P^{(2)}(\tau) = P^{(3)}(\tau). \end{cases}$$

Hence we will determine A, B, D by the following three conditions:

$$\begin{cases} X(A,B,D;\varepsilon) \equiv \varepsilon P_z^{(1)}(0) - \varepsilon P_z^{(2)}(0) = 0, \\ Y(A,B,D;\varepsilon) \equiv \varepsilon P_z^{(2)}(\tau) - \varepsilon P_z^{(3)}(\tau) = 0, \\ Z(A,B,D;\varepsilon) \equiv Q^{(2)}(\tau) - Q^{(3)}(\tau) = 0. \end{cases}$$

Since $\dot\zeta_0^{(1)}(0) = \dot\zeta_0^{(2),l}(0)$, we have

$$\begin{aligned} X(A,B,D;\varepsilon) &= \frac{1}{\varepsilon}\{\dot\zeta_0^{(1)}(0) - \dot\zeta_0^{(2),l}(0)\} + \{\dot\zeta_1^{(1)}(0) - \dot\zeta_1^{(2),l}(0) + \frac{\tau_1^*}{\tau_0^*}\dot\zeta_0^{(2),l}(0)\} + O(\varepsilon) \\ &= X_0(A,B,D) + O(\varepsilon), \end{aligned}$$

where

$$X_0(A,B,D) \equiv \dot\zeta_1^{(1)}(0) - \dot\zeta_1^{(2),l}(0) + \frac{\tau_1^*}{\tau_0^*}\dot\zeta_0^{(2),l}(0).$$

Here we calculate

$$\begin{aligned} \dot\zeta_1^{(1)}(0) &= -P_0^{(1)}(0)\left(-c_0^* + \frac{\ddot\Phi_1(0)}{\dot\Phi_1(0)}\right) - \int_{-\infty}^0 \frac{\dot\Phi_1}{\dot\Phi_1(0)}\{c_1^*\dot\zeta_0^{(1)} \\ &\quad + f_u(\Phi_1,0)P_0^{(1)}(0) + f_{uu}(\Phi_1,0)\phi_1^{(1)}\zeta_0^{(1)} + f_{uv}(\Phi_1,0)\psi_0^{(1)}\zeta_0^{(1)} \\ &\quad + g_u(\Phi_1,0)(Q_0^{(1)}(0) + \eta_0^{(1)})\}ds \\ &= \frac{P_0^{(1)}(0)}{\dot\Phi_1(0)}\{c_0^*\dot\Phi_1(0) - \ddot\Phi_1(0) + \ddot\Phi_1(0) - c_0^*\dot\Phi_1(0)\} \\ &\quad - Q_0^{(1)}(0)\frac{g(\Phi_1(0),0)}{\dot\Phi_1(0)} - \int_{-\infty}^0 \frac{\dot\Phi_1}{\dot\Phi_1(0)}\{c_1^*\dot\zeta_0^{(1)} + f_{uu}(\Phi_1,0)\phi_1^{(1)}\zeta_0^{(1)} \\ &\quad + f_{uv}(\Phi_1,0)\psi_0^{(1)}\zeta_0^{(1)} + g_u(\Phi_1,0)\eta_0^{(1)}\}ds \\ &= -Q_0^{(1)}(0)\frac{g(\Phi_1(0),0)}{\dot\Phi_1(0)} - \int_{-\infty}^0 \frac{\dot\Phi_1}{\dot\Phi_1(0)}\{c_1^*\dot\zeta_0^{(1)} + f_{uu}(\Phi_1,0)\phi_1^{(1)}\zeta_0^{(1)} \\ &\quad + f_{uv}(\Phi_1,0)\psi_0^{(1)}\zeta_0^{(1)} + g_u(\Phi_1,0)\eta_0^{(1)}\}ds. \end{aligned}$$

Here we note that

$$Q_0^{(1)}(0) = D - \eta_0^{(1)}(0),$$
$$\int_{-\infty}^{0} \dot{\Phi}_1 \dot{\zeta}_0^{(1)} ds = \dot{\Phi}_1(0)\zeta_0^{(1)}(0) - \int_{-\infty}^{0} \ddot{\Phi}_1 \zeta_0^{(1)} ds,$$
$$\int_{-\infty}^{0} \dot{\Phi}_1 g_u(\Phi_1, 0)\eta_0^{(1)} ds = g(\Phi_1(0), 0)\eta_0^{(1)}(0) + \frac{1}{c_0^*}\int_{-\infty}^{0} g(\Phi_1, 0) f_v(\Phi_1, 0)\zeta_0^{(1)} ds.$$

On the other hand, $\phi_1^{(1)}$ satisfies

$$\begin{aligned}
&\dddot{\phi}_1^{(1)} - c_0^*\ddot{\phi}_1^{(1)} + f_u(\Phi_1, 0)\dot{\phi}_1^{(1)} \\
&\quad = c_1^*\ddot{\Phi}_1 - f_{uu}(\Phi_1, 0)\phi_1^{(1)}\dot{\Phi}_1 - f_v(\Phi_1, 0)\dot{\psi}_0^{(1)} - f_{uv}(\Phi_1, 0)\dot{\Phi}_1\psi_0^{(1)}.
\end{aligned}$$

Multiplying this by $\zeta_0^{(1)}$, integrating it on $(-\infty, 0)$ and using $\dot{\psi}_0^{(1)} = g(\Phi_1, 0)/c_0^*$, we get

$$\begin{aligned}
A\ddot{\phi}_1^{(1)}(0) - A\dot{\phi}_1^{(1)}(0)\frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)} = \int_{-\infty}^{0} &\zeta_0^{(1)}\{c_1^*\ddot{\Phi}_1 - f_{uu}(\Phi_1, 0)\phi_1^{(1)}\dot{\Phi}_1 \\
&- \frac{1}{c_0^*} f_v(\Phi_1, 0) g(\Phi_1, 0) - f_{uv}(\Phi_1, 0)\dot{\Phi}_1\psi_0^{(1)}\} ds.
\end{aligned}$$

Using these relations, we obtain

$$\begin{aligned}
\dot{\zeta}_1^{(1)}(0) &= -(D - \eta_0^{(1)}(0))\frac{g(\Phi_1(0), 0)}{\dot{\Phi}_1(0)} - c_1^*\zeta_0^{(1)}(0) - \frac{g(\Phi_1(0), 0)\eta_0^{(1)}(0)}{\dot{\Phi}_1(0)} \\
&\quad + A\frac{\ddot{\phi}_1^{(1)}(0)}{\dot{\Phi}_1(0)} - A\frac{\ddot{\Phi}_1(0)\dot{\phi}_1^{(1)}(0)}{(\dot{\Phi}_1(0))^2} \\
&= -D\frac{g(\Phi_1(0), 0)}{\dot{\Phi}_1(0)} - c_1^*\zeta_0^{(1)}(0) + A\frac{\ddot{\phi}_1^{(1)}(0)}{\dot{\Phi}_1(0)} - A\frac{\ddot{\Phi}_1(0)\dot{\phi}_1^{(1)}(0)}{(\dot{\Phi}_1(0))^2}.
\end{aligned}$$

Next we calculate $\dot{\zeta}_1^{(2),l}(0)$.

$$\begin{aligned}
\dot{\zeta}_1^{(2),l}(0) &= -P_0^{(2)}(0)\Big(-c_0^* + \frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)}\Big) \\
&\quad + \int_0^{\infty} \frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)}\Big\{\left(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\right)\dot{\zeta}_0^{(2),l} + 2\frac{\tau_1^*}{\tau_0^*} f_u(\Phi_1, 0)\zeta_0^{(2),l} \\
&\quad + f_u(\Phi_1, 0)P_0^{(2)}(0) + g_u(\Phi_1, 0)(Q_0^{(2)}(0) + \eta_0^{(2),l})
\end{aligned}$$

$$
\begin{aligned}
&+f_{uu}(\Phi_1,0)(sU_0^{(2)'}(0)+U_1^{(2)}(0)+\phi_1^{(2),l})\zeta_0^{(2),l}\\
&+f_{uv}(\Phi_1,0)(sV_0^{(2)'}(0)+V_1^{(2)}(0)+\psi_0^{(2),l})\zeta_0^{(2),l}\Big\}ds\\
=\ &\frac{Q_0^{(2)}(0)}{\dot\Phi_1(0)}(g(h_+(0),0)-g(\Phi_1(0),0))\\
&+\int_0^\infty\frac{\dot\Phi_1}{\dot\Phi_1(0)}\Big\{\Big(c_1^*+c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\dot\zeta_0^{(2),l}+2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_1,0)\zeta_0^{(2),l}\\
&+g_u(\Phi_1,0)\eta_0^{(2),l}+f_{uu}(\Phi_1,0)(sU_0^{(2)'}(0)+U_1^{(2)}(0)+\phi_1^{(2),l})\zeta_0^{(2),l}\\
&+f_{uv}(\Phi_1,0)(sV_0^{(2)'}(0)+V_1^{(2)}(0)+\psi_0^{(2),l})\zeta_0^{(2),l}\Big\}ds.
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
\int_0^\infty\dot\Phi_1\dot\zeta_0^{(2),l}ds\ &=\ -\dot\Phi_1(0)\zeta_0^{(2),l}(0)-\int_0^\infty\ddot\Phi_1\zeta_0^{(2),l}ds,\\
\int_0^\infty\dot\Phi_1 g_u(\Phi_1,0)\eta_0^{(2),l}ds\ &=\ -g(\Phi_1(0),0)\eta_0^{(2),l}(0)\\
&\quad+\frac{1}{c_0^*}\int_0^\infty g(\Phi_1,0)f_v(\Phi_1,0)\zeta_0^{(2),l}ds.
\end{aligned}
$$

Also, $\phi_1^{(2),l}$ satisfies

$$
\begin{aligned}
&\dddot\phi_1^{(2),l}-c_0^*\ddot\phi_1^{(2),l}+f_u(\Phi_1,0)\dot\phi_1^{(2),l}\\
&=(c_1^*+c_0^*\frac{\tau_1^*}{\tau_0^*})\ddot\Phi_1-2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_1,0)\dot\Phi_1-f_{uu}(\Phi_1,0)(\xi U_0^{(2)'}(0)+U_1^{(2)}(0)+\phi_1^{(2),l})\dot\Phi_1\\
&\quad-f_u(\Phi_1,0)U_0^{(2)'}(0)-f_{uv}(\Phi_1,0)(\xi V_0^{(2)'}(0)+V_1^{(2)}(0)+\psi_0^{(2),l})\dot\Phi_1\\
&\quad-f_v(\Phi_1,0)(V_0^{(2)'}(0)+\dot\psi_0^{(2),l}).
\end{aligned}
$$

Multiplying this by $\zeta_0^{(2),l}$, integrating it on $(0,\infty)$ and using $c_0^*V_0^{(2)'}(0)+c_0^*\dot\psi_0^{(2),l}=g(\Phi_1,0)$, we get

$$
\begin{aligned}
&-A\ddot\phi_1^{(2),l}(0)+A\dot\phi_1^{(2),l}(0)\frac{\ddot\Phi_1(0)}{\dot\Phi_1(0)}\\
&\quad=\ \int_0^\infty\zeta_0^{(2),l}\Big\{\Big(c_1^*+c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\ddot\Phi_1-2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_1,0)\dot\Phi_1\\
&\qquad-f_{uu}(\Phi_1,0)(sU_0^{(2)'}(0)+U_1^{(2)}(0)+\phi_1^{(2),l})\dot\Phi_1-f_u(\Phi_1,0)U_0^{(2)'}(0)\\
&\qquad-f_{uv}(\Phi_1,0)(sV_0^{(2)'}(0)+V_1^{(2)}(0)+\psi_0^{(2),l})\dot\Phi_1-\frac{1}{c_0^*}f_v(\Phi_1,0)g(\Phi_1,0)\Big\}ds.
\end{aligned}
$$

Using these relations and noting

$$A\frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)} = \int_0^\infty f_u(\Phi_1, 0)\zeta_0^{(2),l} ds,$$

we obtain

$$\begin{aligned}\dot{\zeta}_1^{(2),l}(0) &= Q_0^{(2)}(0)\frac{g(h_+(0),0) - g(\Phi_1(0),0)}{\dot{\Phi}_1(0)} - \left(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\right)\zeta_0^{(2),l}(0) \\ &\quad -\eta_0^{(2),l}(0)\frac{g(\Phi_1(0),0)}{\dot{\Phi}_1(0)} + A\frac{\ddot{\phi}_1^{(2),l}(0)}{\dot{\Phi}_1(0)} - A\frac{\ddot{\Phi}_1(0)\dot{\phi}_1^{(2),l}(0)}{(\dot{\Phi}_1(0))^2} \\ &\quad -A\frac{\ddot{\Phi}_1(0)}{(\dot{\Phi}_1(0))^2}U_0^{(2)\prime}(0).\end{aligned}$$

Substituting these into $X_0(A, B, D)$ and using $\Phi_1(0) = \alpha - h_+(0)$ and $Q_0^{(2)}(0) = D - \eta_0^{(2),l}(0)$, we get

$$\begin{aligned}&X_0(A, B, D) \\ &= -D\frac{g(\Phi_1(0),0)}{\dot{\Phi}_1(0)} - c_1^*\zeta_0^{(1)}(0) + A\frac{\ddot{\phi}_1^{(1)}(0)}{\dot{\Phi}_1(0)} - A\frac{\ddot{\Phi}_1(0)\dot{\phi}_1^{(1)}(0)}{(\dot{\Phi}_1(0))^2} \\ &\quad -\Big[Q_0^{(2)}(0)\frac{g(h_+(0),0) - g(\Phi_1(0),0)}{\dot{\Phi}_1(0)} - \eta_0^{(2),l}(0)\frac{g(\Phi_1(0),0)}{\dot{\Phi}_1(0)} \\ &\quad -\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\zeta_0^{(2),l}(0) + A\frac{\ddot{\phi}_1^{(2),l}(0)}{\dot{\Phi}_1(0)} - A\frac{\ddot{\Phi}_1(0)\dot{\phi}_1^{(2),l}(0)}{(\dot{\Phi}_1(0))^2} - A\frac{\ddot{\Phi}_1(0)}{(\dot{\Phi}_1(0))^2}U_0^{(2)\prime}(0)\Big] \\ &\quad +A\frac{\tau_1^*}{\tau_0^*}\left(-c_0^* + \frac{\ddot{\Phi}_1(0)}{\dot{\Phi}_1(0)}\right) \\ &= A\frac{1}{\dot{\Phi}_1(0)}\{\ddot{\phi}_1^{(1)}(0) - \ddot{\phi}_1^{(2),l}(0) + \frac{\tau_1^*}{\tau_0^*}\ddot{\Phi}_1(0)\} \\ &\quad -A\frac{\ddot{\Phi}_1(0)}{(\dot{\Phi}_1(0))^2}(\dot{\phi}_1^{(1)}(0) - \dot{\phi}_1^{(2),l}(0) - U_0^{(2)\prime}(0)) - (D - \eta_0^{(2),l}(0))\frac{g(h_+(0),0)}{\dot{\Phi}_1(0)}.\end{aligned}$$

Here, by noting

$$\begin{aligned}&\ddot{\phi}_1^{(1)}(0) - \ddot{\phi}_1^{(2),l}(0) + 2\frac{\tau_1^*}{\tau_0^*}\ddot{\Phi}_1(0) = 0, \\ &\dot{\phi}_1^{(1)}(0) - \dot{\phi}_1^{(2),l}(0) - U_0^{(2)\prime}(0) + \frac{\tau_1^*}{\tau_0^*}\dot{\Phi}_1(0) = 0,\end{aligned}$$

we obtain

$$X_0(A,B,D) = -(D - \eta_0^{(2),l}(0))\frac{g(h_+(0),0)}{\dot{\Phi}_1(0)}.$$

Thus we obtain

$$D = \eta_0^{(2),l}(0) = \frac{A}{c_0^*}\int_0^\infty e^{-c_0^* s}\frac{\dot{\Phi}_1}{\dot{\Phi}_1(0)} f_v(\Phi_1,0)ds.$$

Secondly, we calculate $Y(A,B,D;\varepsilon)$. Since $\dot{\zeta}_0^{(2),r}(0) = \dot{\zeta}_0^{(3)}(0)$, we have

$$\begin{aligned} Y(A,B,D;\varepsilon) &= \frac{1}{\varepsilon}\{\dot{\zeta}_0^{(2),r}(0) - \dot{\zeta}_0^{(3)}(0)\} + \{\dot{\zeta}_1^{(2),r}(0) - \dot{\zeta}_1^{(3)}(0) \\ &\quad -\frac{\tau_1^*}{\tau_0^*}\dot{\zeta}_0^{(2),r}(0)\} + O(\varepsilon) \\ &\equiv Y_0(A,B,D) + O(\varepsilon), \end{aligned}$$

where

$$Y_0(A,B,D) = \dot{\zeta}_1^{(2),r}(0) - \dot{\zeta}_1^{(3)}(0) - \frac{\tau_1^*}{\tau_0^*}\dot{\zeta}_0^{(2),r}(0).$$

From the similar argument, we have

$$\begin{aligned} \dot{\zeta}_1^{(2),r}(0) &= \frac{Q_0^{(2)}(\tau_0^*)}{\dot{\Phi}_2(0)}(g(h_+(v^*),v^*) - g(\Phi_2(0),v^*)) \\ &\quad -\int_{-\infty}^0 \frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}\Big\{\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\dot{\zeta}_0^{(2),r} + 2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_2,v^*)\zeta_0^{(2),r} \\ &\quad +g_u(\Phi_2,v^*)\eta_0^{(2),r} + f_{uu}(\Phi_2,v^*)(sU_0^{(2)\prime}(\tau_0^*) + U_1^{(2)}(\tau_0^*) + \phi_1^{(2),r})\zeta_0^{(2),r} \\ &\quad +f_{uv}(\Phi_2,v^*)(sV_0^{(2)\prime}(\tau_0^*) + V_1^{(2)}(\tau_0^*) + \psi_0^{(2),r})\zeta_0^{(2),r}\Big\}ds. \end{aligned}$$

Here we note that

$$\begin{aligned} \int_{-\infty}^0 \dot{\Phi}_2\dot{\zeta}_0^{(2),r}ds &= \dot{\Phi}_2(0)\zeta_0^{(2),r}(0) - \int_{-\infty}^0 \ddot{\Phi}_2\zeta_0^{(2),r}ds, \\ \int_{-\infty}^0 \dot{\Phi}_2 g_u(\Phi_2,v^*)\eta_0^{(2),r}ds &= g(\Phi_2(0),v^*)\eta_0^{(2),r}(0) \\ &\quad +\frac{1}{c_0^*}\int_{-\infty}^0 g(\Phi_2,v^*)f_v(\Phi_2,v^*)\zeta_0^{(2),r}ds. \end{aligned}$$

Also, $\phi_1^{(2),r}$ satisfies

$$\begin{aligned}
&\dddot{\phi}_1^{(2),r} - c_0^*\ddot{\phi}_1^{(2),r} + f_u(\Phi_2, v^*)\dot{\phi}_1^{(2),r} \\
&= \left(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\right)\ddot{\Phi}_2 - 2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_2, v^*)\dot{\Phi}_2 \\
&\quad -f_{uu}(\Phi_2, v^*)(\xi U_0^{(2)'}(\tau_0^*) + U_1^{(2)}(\tau_0^*) + \phi_1^{(2),r})\dot{\Phi}_2 \\
&\quad -f_u(\Phi_2, v^*)U_0^{(2)'}(\tau_0^*) - f_{uv}(\Phi_2, v^*)(\xi V_0^{(2)'}(\tau_0^*) + V_1^{(2)}(\tau_0^*) \\
&\quad +\psi_0^{(2),r})\dot{\Phi}_2 - f_v(\Phi_2, v^*)(V_0^{(2)'}(\tau_0^*) + \dot{\psi}_0^{(2),r}).
\end{aligned}$$

Multiplying this by $\zeta_0^{(2),r}$, integrating it on $(-\infty, 0)$ and using $c_0^* V_0^{(2)'}(\tau_0^*) + c_0^*\dot{\psi}_0^{(2),r} = g(\Phi_2, v^*)$, we get

$$\begin{aligned}
&B\ddot{\phi}_1^{(2),r}(0) - B\dot{\phi}_1^{(2),r}(0)\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)} \\
&= \int_{-\infty}^0 \zeta_0^{(2),r}\Big\{\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big)\ddot{\Phi}_2 - 2\frac{\tau_1^*}{\tau_0^*}f_u(\Phi_2, v^*)\dot{\Phi}_2 \\
&-f_{uu}(\Phi_2, v^*)(sU_0^{(2)'}(\tau_0^*) + U_1^{(2)}(\tau_0^*) + \phi_1^{(2),r})\dot{\Phi}_2 - f_u(\Phi_2, v^*)U_0^{(2)'}(\tau_0^*) \\
&-f_{uv}(\Phi_2, v^*)(sV_0^{(2)'}(\tau_0^*) + V_1^{(2)}(\tau_0^*) + \psi_0^{(2),r})\dot{\Phi}_2 - \frac{1}{c_0^*}f_v(\Phi_2, v^*)g(\Phi_2, v^*)\Big\}ds.
\end{aligned}$$

Noting

$$B\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)} = -\int_{-\infty}^0 \zeta_0^{(2),r} f_u(\Phi_2, v^*)ds$$

and using the relations above, we get

$$\begin{aligned}
\dot{\zeta}_1^{(2),r}(0) \;=\; & Q_0^{(2)}(\tau_0^*)\frac{g(h_+(v^*), v^*) - g(\Phi_2(0), v^*)}{\dot{\Phi}_2(0)} - \left(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\right)\zeta_0^{(2),r}(0) \\
& -\eta_0^{(2),r}(0)\frac{g(\Phi_2(0), v^*)}{\dot{\Phi}_2(0)} + B\frac{\ddot{\phi}_1^{(2),r}(0)}{\dot{\Phi}_2(0)} - B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(2),r}(0)}{(\dot{\Phi}_2(0))^2} \\
& -B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(2)'}(\tau_0^*).
\end{aligned}$$

Next we calculate $\dot{\zeta}_1^{(3)}(0)$.

$$\dot{\zeta}_1^{(3)}(0) \;=\; \int_0^\infty \frac{\dot{\Phi}_2}{\dot{\Phi}_2(0)}(c_1^*\dot{\zeta}_0^{(3)} + g_u(\Phi_2, v^*)\eta_0^{(3)}$$

$$+f_{uu}(\Phi_2,v^*)(sU_0^{(3)'}(0)+U_1^{(3)}(0)+\phi_1^{(3)})\zeta_0^{(3)}$$
$$+f_{uv}(\Phi_2,v^*)(sV_0^{(3)'}(0)+V_1^{(3)}(0)+\psi_0^{(3)})\zeta_0^{(3)})ds.$$

Here we note that

$$\int_0^\infty \dot{\Phi}_2\dot{\zeta}_0^{(3)}ds=-\dot{\Phi}_2(0)\zeta_0^{(3)}(0)-\int_0^\infty \ddot{\Phi}_2\zeta_0^{(3)}ds,$$
$$\int_0^\infty \dot{\Phi}_2 g_u(\Phi_2,v^*)\eta_0^{(3)}ds=-g(\Phi_2(0),v^*)\eta_0^{(3)}(0)+\frac{1}{c_0^*}\int_0^\infty g(\Phi_2,v^*)f_v(\Phi_2,v^*)\zeta_0^{(3)}ds.$$

Here we note

$$\begin{aligned}&\dddot{\phi}_1^{(3)}-c_0^*\ddot{\phi}_1^{(3)}+f_u(\Phi_2,v^*)\dot{\phi}_1^{(3)}\\ &=\ c_1^*\ddot{\Phi}_2-f_{uu}(\Phi_2,v^*)(\xi U_0^{(3)'}(0)+U_1^{(3)}(0)+\phi_1^{(3)})\dot{\Phi}_2-f_u(\Phi_2,v^*)U_0^{(3)'}(0)\\ &\quad -f_{uv}(\Phi_2,v^*)(\xi V_0^{(3)'}(0)+V_1^{(3)}(0)+\psi_0^{(3)})\dot{\Phi}_2-f_v(\Phi_2,v^*)(V_0^{(3)'}(0)+\dot{\psi}_0^{(3)}).\end{aligned}$$

Multiplying this by $\zeta_0^{(3)}$, integrating it on $(0,\infty)$ and using $c_0^*V_0^{(3)'}(0)+c_0^*\dot{\psi}_0^{(3)} = g(\Phi_2,v^*)$, we get

$$\begin{aligned}&-B\ddot{\phi}_1^{(3)}(0)+B\dot{\phi}_1^{(3)}(0)\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)}\\ &=\int_0^\infty \zeta_0^{(3)}\{c_1^*\ddot{\Phi}_2-f_{uu}(\Phi_2,v^*)(sU_0^{(3)'}(0)+U_1^{(3)}(0)+\phi_1^{(3)})\dot{\Phi}_2\\ &\quad -f_u(\Phi_2,v^*)U_0^{(3)'}(0)-f_{uv}(\Phi_2,v^*)(sV_0^{(3)'}(0)+V_1^{(3)}(0)+\psi_0^{(3)})\dot{\Phi}_2\\ &\quad -\frac{1}{c_0^*}f_v(\Phi_2,v^*)g(\Phi_2,v^*)\}ds.\end{aligned}$$

Using these relations, we obtain

$$\begin{aligned}\dot{\zeta}_1^{(3)}(0)\ =\ &-c_1^*\zeta_0^{(3)}(0)-g(\Phi_2(0),v^*)\frac{\eta_0^{(3)}(0)}{\dot{\Phi}_2(0)}+B\frac{\ddot{\phi}_1^{(3)}(0)}{\dot{\Phi}_2(0)}-B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(3)}(0)}{(\dot{\Phi}_2(0))^2}\\ &-\int_0^\infty \frac{\zeta_0^{(3)}}{\dot{\Phi}_2(0)}f_u(\Phi_2,v^*)U_0^{(3)'}(0)ds.\end{aligned}$$

Moreover, by noting

$$B\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)}=\int_0^\infty \zeta_0^{(3)}f_u(\Phi_2,v^*)d\zeta,$$

we get

$$\begin{aligned}\dot{\zeta}_1^{(3)}(0) &= -c_1^*\zeta_0^{(3)}(0) - \eta_0^{(3)}(0)\frac{g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} \\ &\quad +B\frac{\ddot{\phi}_1^{(3)}(0)}{\dot{\Phi}_2(0)} - B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(3)}(0)}{(\dot{\Phi}_2(0))^2} - B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(3)\prime}(0).\end{aligned}$$

Thus

$$\begin{aligned}&Y_0(A,B,D) \\ &= Q_0^{(2)}(\tau_0^*)\frac{g(h_+(v^*),v^*) - g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} - B\Big(c_1^* + c_0^*\frac{\tau_1^*}{\tau_0^*}\Big) \\ &\quad -\eta_0^{(2),r}(0)\frac{g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} + B\frac{\ddot{\phi}_1^{(2),r}(0)}{\dot{\Phi}_2(0)} - B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(2),r}(0)}{(\dot{\Phi}_2(0))^2} \\ &\quad -B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(2)\prime}(\tau_0^*) - \Big[-Bc_1^* - \eta_0^{(3)}(0)\frac{g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} + B\frac{\ddot{\phi}_1^{(3)}(0)}{\dot{\Phi}_2(0)} \\ &\quad -B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(3)}(0)}{(\dot{\Phi}_2(0))^2} - B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(3)\prime}(0)\Big] - B\frac{\tau_1^*}{\tau_0^*}\Big(-c_0^* + \frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)}\Big) \\ &= Q_0^{(2)}(\tau_0^*)\frac{g(h_+(v^*),v^*) - g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} - (\eta_0^{(2),r}(0) - \eta_0^{(3)}(0))\frac{g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} \\ &\quad +B\frac{\ddot{\phi}_1^{(2),r}(0)}{\dot{\Phi}_2(0)} - B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(2),r}(0)}{(\dot{\Phi}_2(0))^2} - B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(2)\prime}(\tau_0^*) \\ &\quad -\left[B\frac{\ddot{\phi}_1^{(3)}(0)}{\dot{\Phi}_2(0)} - B\frac{\ddot{\Phi}_2(0)\dot{\phi}_1^{(3)}(0)}{(\dot{\Phi}_2(0))^2} - B\frac{\ddot{\Phi}_2(0)}{(\dot{\Phi}_2(0))^2}U_0^{(3)\prime}(0)\right] - B\frac{\tau_1^*}{\tau_0^*}\frac{\ddot{\Phi}_2(0)}{\dot{\Phi}_2(0)}.\end{aligned}$$

Furthermore, by noting

$$\begin{aligned}&\ddot{\phi}_1^{(2),r}(0) - \ddot{\phi}_1^{(3)}(0) - 2\frac{\tau_1^*}{\tau_0^*}\ddot{\Phi}_2(0) = 0, \\ &\dot{\phi}_1^{(2),r}(0) + U_0^{(2)\prime}(\tau_0^*) - \frac{\tau_1^*}{\tau_0^*}\dot{\Phi}_2(0) - U_0^{(3)\prime}(0) - \dot{\phi}_1^{(3)}(0) = 0,\end{aligned}$$

we obtain

$$\begin{aligned}Y_0(A,B,D) &= Q_0^{(2)}(\tau_0^*)\frac{g(h_+(v^*),v^*) - g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)} \\ &\quad -(\eta_0^{(2),r}(0) - \eta_0^{(3)}(0))\frac{g(\Phi_2(0),v^*)}{\dot{\Phi}_2(0)}. \qquad (4.13)\end{aligned}$$

Thirdly, we calculate $Z(A,B,D;\varepsilon)$.

$$\begin{aligned} Z(A,B,D;\varepsilon) &= Q_0^{(2)}(\tau_0^*)+\eta_0^{(2),r}(0)-\eta_0^{(3)}(0)+O(\varepsilon)\\ &= Z_0(A,B,D)+O(\varepsilon), \end{aligned}$$

where

$$Z_0(A,B,D)\equiv Q_0^{(2)}(\tau_0^*)+\eta_0^{(2),r}(0)-\eta_0^{(3)}(0).$$

Easy calculations give us

$$\begin{aligned} Q_0^{(2)}(\tau_0^*) &= (D-\eta_0^{(2),l}(0))\exp\Big\{\frac{1}{c_0^*}\int_0^{\tau_0^*}\Big(\frac{f_v(U_0^{(2)},V_0^{(2)})g_u(U_0^{(2)},V_0^{(2)})}{f_u(U_0^{(2)},V_0^{(2)})}\\ &\quad -g_v(U_0^{(2)},V_0^{(2)})\Big)dx\Big\},\\ \eta_0^{(2),r}(0) &= -\frac{B}{c_0^*}\int_{-\infty}^0 e^{-c_0^*s}\frac{f_v(\Phi_2,v^*)\dot\Phi_2}{\dot\Phi_2(0)}ds,\\ \eta_0^{(3)}(0) &= \frac{B}{c_0^*}\int_0^\infty e^{-c_0^*s}\frac{f_v(\Phi_2,v^*)\dot\Phi_2}{\dot\Phi_2(0)}ds. \end{aligned}$$

Therefore

$$\begin{aligned} Z_0(A,B,D) &= (D-\eta_0^{(2),l}(0))\exp\Big\{\frac{1}{c_0^*}\int_0^{\tau_0^*}\Big(\frac{f_v(U_0^{(2)},V_0^{(2)})g_u(U_0^{(2)},V_0^{(2)})}{f_u(U_0^{(2)},V_0^{(2)})}\\ &\quad -g_v(U_0^{(2)},V_0^{(2)})\Big)dx\Big\}-\frac{B}{c_0^*}\int_{-\infty}^\infty e^{-c_0^*s}\frac{f_v(\Phi_2,v^*)\dot\Phi_2}{\dot\Phi_2(0)}ds. \end{aligned}$$

From the equations

$$\begin{cases} X_0(A,B,D)=0,\\ Z_0(A,B,D)=0, \end{cases} \tag{4.14}$$

we obtain

$$D^*=\eta_0^{(2),l}(0)=\frac{A}{c_0^*}\int_0^\infty e^{-c_0^*s}f_v(\Phi_1,0)\frac{\dot\Phi_1}{\dot\Phi_1(0)}ds,\quad B^*=0.$$

Also, (4.13) implies that

$$Y_0(A,B^*,D^*)=0.$$

Now we apply the implicit function theorem to the equations

$$\begin{cases} X(A,B,D;\varepsilon)=0, \\ Z(A,B,D;\varepsilon)=0. \end{cases}$$

Then, for any constant A, there exist

$$\begin{aligned} B&=B(\varepsilon;A) \qquad (B^*=B(0;A)), \\ D&=D(\varepsilon;A) \qquad (D^*=D(0;A)), \end{aligned}$$

such that

$$\begin{cases} X(A,B(\varepsilon;A),D(\varepsilon;A);\varepsilon)=0, \\ Z(A,B(\varepsilon;A),D(\varepsilon;A);\varepsilon)=0. \end{cases}$$

Since the adjoint equation must have a bounded solution that is unique (up to multiplications by constants), the equality

$$Y(A,B(\varepsilon;A),D(\varepsilon;A);\varepsilon)=0$$

must automatically hold. Thus the proof is complete.

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Appendix

In Appendix, we shall show Theorems 3 and 4.

First we show Theorem 3. Putting $\xi = z/\epsilon$, we set

$$u(\xi;\varepsilon,c_1)=\phi_0(\xi)+\varepsilon\phi_1(\xi;c_1)+\varepsilon R(\xi;\varepsilon,c_1), \quad v(\xi;\varepsilon,c_1)=\varepsilon\psi_0(\xi)+\varepsilon S(\xi;\varepsilon,c_1).$$

Substituting these into (3.6), we put

$$\begin{cases} T_1(t;\varepsilon,c_1) \equiv \dfrac{1}{\varepsilon}(\ddot{\phi}_0+\varepsilon\ddot{\phi}_1+\varepsilon\ddot{R})-\dfrac{1}{\varepsilon}(c_0^*+\varepsilon c_1)(\dot{\phi}_0+\varepsilon\dot{\phi}_1+\varepsilon\dot{R}) \\ \qquad\qquad\qquad +\dfrac{1}{\varepsilon}f(\phi_0+\varepsilon\phi_1+\varepsilon R,\varepsilon\psi_0+\varepsilon S), \\ T_2(t;\varepsilon,c_1) \equiv (c_0^*+\varepsilon c_1)(\dot{S}+\dot{\psi}_0)-g(\phi_0+\varepsilon\phi_1+\varepsilon R,\varepsilon\psi_0+\varepsilon S) \end{cases}$$

and consider the boundary conditions

$$R(-\infty)=0, \quad R(0)=0, \quad S(-\infty)=0.$$

Here we put $t=(R,S)$ and $T(t;\varepsilon,c_1)=(T_1,T_2)$ for simplicity of notations. Then $T(t;\varepsilon,c_1)$ is a continuous mapping from $X_\mu\times(0,\infty)\times\Lambda_\delta$ to Y_μ and continuously differentiable with respect to t, where X_μ and Λ_δ were given in Section 3.2, and $Y_\mu=X_\mu^0\times X_\mu^0$.

Now we outline the proof of Theorem 3. From Lemma 1 given below, we have the invertibility of $T_t(0;\varepsilon,c_1)$ and set

$$\begin{aligned} F(t;\varepsilon,c_1)&=T(t;\varepsilon,c_1)-T(0;\varepsilon,c_1)-T_t(0;\varepsilon,c_1)t, \\ G(t;\varepsilon,c_1)&=T_t^{-1}(0;\varepsilon,c_1)[T(0;\varepsilon,c_1)+F(t;\varepsilon,c_1)]. \end{aligned}$$

Then $G:B_\gamma(0)\to B_\gamma(0)$ is a contraction map with respect to t, where $B_\gamma(0)$ is a closed ball in X_μ with a radius $\gamma>0$ and a center at 0 for a small $\gamma>0$. Indeed, if ε is sufficiently small,

$$\|G(t_0;\varepsilon,c_1)\|_{X_\mu}\le 2K\varepsilon, \quad \|G(t_1;\varepsilon,c_1)-G(t_2;\varepsilon,c_1)\|_{X_\mu}\le\frac{1}{2}\|t_1-t_2\|_{X_\mu}$$

for any $\|t_i\|_{X_\mu}\le\gamma$ for $i=0,1,2$, $c_1\in\Lambda_\delta$, where $K>0$ is a constant independent of ε, given in Lemma 1. Therefore $G(t;\varepsilon,c_1)=t$, namely, $T(t;\varepsilon,c_1)$ has a unique solution $t(\varepsilon,c_1)$ in $B_\gamma(0)$ for small $\varepsilon>0$ by contraction mapping theorem. In the following lemma, we shall use $\|\cdot\|_{X_\mu}$, a norm for X_μ defined by $\|t\|_{X_\mu}=\|R\|_{X_\mu^2}+\|S\|_{X_\mu^1}$ for $t=(R,S)$.

We follow the argument above and show Theorem 3. At first, we compute the Fréchet derivative of T_1 and T_2 with respect to t and have

$$\begin{aligned} T_{1t}(0;\varepsilon,c_1)[R,S]&=\ddot{R}-(c_0^*+\varepsilon c_1)\dot{R}+f_u(\phi_0+\varepsilon\phi_1,\varepsilon\psi_0)R+f_v(\phi_0+\varepsilon\phi_1,\varepsilon\psi_0)S, \\ T_{2t}(0;\varepsilon,c_1)[R,S]&=(c_0^*+\varepsilon c_1)\dot{S}-\varepsilon g_u(\phi_0+\varepsilon\phi_1,\varepsilon\psi_0)R-\varepsilon g_v(\phi_0+\varepsilon\phi_1,\varepsilon\psi_0)S. \end{aligned}$$

We see the invertibility of $T_t(0;\varepsilon,c_1)$ by the next lemma.

Lemma 1. *$T_t(0;\varepsilon,c_1) : X \to Y$ is invertible. Moreover, there is a constant $K > 0$ independent of $\varepsilon > 0$ such that $\|T_t^{-1}(0;\varepsilon,c_1)\| \leq K$ uniformly in $c_1 \in \Lambda_\delta$ and small $\varepsilon > 0$, where $\|\cdot\|$ represents the usual operator norm.*

Proof. Since $\|T_t(0;0,c_1) - T_t(0;\varepsilon,c_1)\|$ is small, it is sufficient to see that $T_t(0;0,c_1)$ is invertible. Set $T_t(0;0,c_1)[R,S] = 0$. Then We have

$$\begin{cases} \ddot{R} - c_0^*\dot{R} + f_u(\phi_0,0)R + f_v(\phi_0,0)S = 0, \\ \hfill c_0^*\dot{S} = 0. \end{cases}$$

The second equation and boundary condition imply that $S = 0$. Any solution of the first equation must be a multiplicity of $\dot{\Phi}_1$ so that $R \equiv 0$ because $\dot{\Phi}_1 > 0$ and we impose the boundary condition of $R(0) = 0$. Hence $T_t(0;0,c_1)$ is one-to-one. Thus it is clear that $T_t(0;0,c_1)$ is onto. □

The following lemma is obvious from the definition of T.

Lemma 2. *Fix $\delta > 0$ independent of ε and c_1. Then, $\|T(0;\varepsilon,c_1)\|_{X_\mu} = O(\varepsilon)$ uniformly in $c_1 \in \Lambda_\delta$ as $\varepsilon \to 0$.*

Finally, we obtain the following lemma, which complete the proof of Theorem 3.

Lemma 3. *Fix $\delta > 0$. Then, there exists $\varepsilon_0 > 0$ such that the equation*

$$T(t;\varepsilon,c_1) = 0$$

has a unique solution $t(\varepsilon,c_1) \in B_\gamma(0)$ for any $\varepsilon \in (0,\varepsilon_0)$ and $c_1 \in \Lambda_\delta$, Moreover,

$$\|t(\varepsilon,c_1)\|_{X_\mu} = O(\varepsilon)$$

uniformly in $c_1 \in \Lambda_\delta$ as $\varepsilon \to 0$.

Next we prove Theorem 4. We denote $\theta^l(y) = \theta(y/\tau_0^*)$ and $\theta^r(y) = \theta((1-y)/\tau_0^*)$. Substituting these into (3.10), we get

$T_1(t;\varepsilon,c_1,\tau_1)$

$$
\begin{aligned}
&= \varepsilon\{U_0+\varepsilon U_1+\theta^l\left(\phi_0^l+\varepsilon\phi_1^l\right)+\theta^r\left(\phi_0^r+\varepsilon\phi_1^r\right)+\varepsilon R+\varepsilon h_+'(V_0)S\}_{yy} \\
&\quad -(c_0^*+\varepsilon c_1)\left(1+\varepsilon\frac{\tau_1}{\tau_0^*}\right)\{U_0+\varepsilon U_1+\theta^l\left(\phi_0^l+\varepsilon\phi_1^l\right)+\theta^r\left(\phi_0^r+\varepsilon\phi_1^r\right) \\
&\quad +\varepsilon R+\varepsilon h_+'(V_0)S\}_y+\frac{1}{\varepsilon}\left(1+\varepsilon\frac{\tau_1}{\tau_0^*}\right)^2 f(U_0+\varepsilon U_1+\theta^l(\phi_0^l+\varepsilon\phi_1^l) \\
&\quad +\theta^r(\phi_0^r+\varepsilon\phi_1^r)+\varepsilon R+\varepsilon h_+'(V_0)S, V_0+\varepsilon V_1+\varepsilon\theta^l(\psi_0^l+\varepsilon\psi_1^l) \\
&\quad +\varepsilon\theta^r(\psi_0^r+\varepsilon\psi_1^r)+\varepsilon S+\varepsilon\theta^l(S^{(1)}(0)-\varepsilon\psi_1^l(0))),
\end{aligned}
$$

$$
\begin{aligned}
&T_2(t;\varepsilon,c_1,\tau_1) \\
&= \frac{1}{\varepsilon}(c_0^*+\varepsilon c_1)\{V_0+\varepsilon V_1+\varepsilon\theta^l\left(\psi_0^l+\varepsilon\psi_1^l\right)+\varepsilon\theta^r\left(\psi_0^r+\varepsilon\psi_1^r\right) \\
&\quad +\varepsilon S+\varepsilon\theta^l(S^{(1)}(0)-\varepsilon\psi_1^l(0))\}_y-\frac{1}{\varepsilon}\left(1+\varepsilon\frac{\tau_1}{\tau_0^*}\right)g(U_0+\varepsilon U_1 \\
&\quad +\theta^l(\phi_0^l+\varepsilon\phi_1^l)+\theta^r(\phi_0^r+\varepsilon\phi_1^r)+\varepsilon R+\varepsilon h_+'(V_0)S, V_0+\varepsilon V_1 \\
&\quad +\varepsilon\theta^l(\psi_0^l+\varepsilon\psi_1^l)+\varepsilon\theta^r(\psi_0^r+\varepsilon\psi_1^r)+\varepsilon S+\varepsilon\theta^l(S^{(1)}(0)-\varepsilon\psi_1^l(0)))
\end{aligned}
$$

with the boundary conditions

$$R(0)=R(\tau_0^*)=S(0)=0.$$

Here we put $t=(R,S)$ and $T=(T_1,T_2)$ for simplicity. We regard $T(t;\varepsilon,c_1,\tau_1)$ as an operator from $X_\varepsilon\times(0,\infty)\times\Xi_\delta$ to Y_ε, where X_ε and Ξ_δ are given in Section 3.3, and $Y_\varepsilon=C(0,\tau_0^*)\times C_\varepsilon^1(0,\tau_0^*)$ and

$$C_\varepsilon^1(0,\tau_0^*)=\left\{\varphi\in C^1(0,\tau_0^*)\;\middle|\;\|\varphi\|_{C_\varepsilon^1}=\sum_{i=0}^{1}\max_{[0,\tau_0^*]}\left|\left(\varepsilon\frac{d}{dy}\right)^i\varphi\right|<\infty\right\}.$$

We prove Theorem 4 by the same argument as in the proof of Theorem 3. We first show the invertibility of $T_t(0;\varepsilon,c_1,\tau_1)$.

Lemma 4. *Fix $\delta>0$, and let $\alpha-h_+(0)$ and $\beta-h_+(v^*)$ be sufficiently small and fixed. Then there exists a constant $\varepsilon_0>0$ such that $T_t(0;\varepsilon,c_1,\tau_1):X_\varepsilon\to Y_\varepsilon$ is invertible for any $\varepsilon\in(0,\varepsilon_0)$ and $(c_1,\tau_1)\in\Xi_\delta$. Also, there exists a constant $M>0$ independent of ε,c_1 and τ_1 such that*

$$\|T_t^{-1}(0;\varepsilon,c_1,\tau_1)\|\le M.$$

We divide $T_t(0;\varepsilon,c_1,\tau_1)$ into two parts such as

$$T_t(0;\varepsilon,c_1,\tau_1) = L + K,$$

where L is a main part of $T_t(0;\varepsilon,c_1,\tau_1)$ defined by

$$L\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} L_1 R & -\varepsilon c_0^* h_+'(V_0) S' \\ -g_u R & L_2 S \end{pmatrix}$$

and K is a small part of $T_t(0;\varepsilon,c_1,\tau_1)$ and satisfies

$$\|K\| \leq c(\varepsilon + |\phi_0^l| + |\phi_0^r|).$$

The differential operators L_1 and L_2 are defined by

$$L_1 R = \varepsilon^2 R'' - \varepsilon c_0^* R' + f_u R, \quad L_2 S = c_0^* S' - (g_u h_+'(V_0) + g_v) S$$

and f_u, g_u, g_v are defined by

$$\begin{aligned} f_u &= f_u(U_0 + \theta^l \phi_0^l + \theta^r \phi_0^r, V_0), \\ g_u &= g_u(U_0 + \theta^l \phi_0 + \theta^r \phi_0, V_0), \\ g_v &= g_v(U_0 + \theta^l \phi_0 + \theta^r \phi_0, V_0). \end{aligned}$$

If both $\alpha - h_+(0)$ and $\beta - h_+(v^*)$ are small, we know that $\|K\|$ is small for small $\varepsilon > 0$. From the above calculations, it suffices to show that L is invertible and $\|L^{-1}\| \leq K$. We first consider L_1.

Proposition 1. $L_1(\varepsilon,c_1,\tau_1) : C_\varepsilon^2(0,\tau_0^*) \to C(0,\tau_0^*)$ *is invertible. Moreover, there is a constant* $M > 0$ *independent of* ε *such that* $\|L_1^{-1}\| \leq M$.

Proof. Note that

$$f_u < 0 \tag{5.1}$$

in $[0,\tau_0^*]$ because $\alpha - h_+(0)$ and $\beta - h_+(v^*)$ are small. Suppose $L_1 R = 0$. Multiplying R to the both sides of $L_1 R = 0$ and integrating it by parts, we obtain

$$\varepsilon^2 \int_0^{\tau_0^*} |R'|^2 dy - \int_0^{\tau_0^*} f_u |R|^2 dy = 0,$$

which implies that $R \equiv 0$. From the Fredholm's alternative, L_1 is invertible.

Next we assume that for $F \in C_\varepsilon$, there is a solution of $L_1 R = F$ in $R \in C^2_\varepsilon$. If R has a maximum at $\tau \in (0, \tau_0^*)$, we have

$$f_u R \geq F$$

because of $R'' \leq 0$ and $R' = 0$ at τ. From (5.1), $R \leq C\|F\|_{L^\infty}$ for a constant $C > 0$ independent of ε. Similarly, if R has a minimum in $\tau \in (0, \tau_0^*)$, $R \geq -C\|F\|_{L^\infty}$ holds. Therefore we have $|R| \leq C\|F\|_{L^\infty}$ in $[0, \tau_0^*]$. From the same argument and using $L_1 R = F$, we prove

$$|\varepsilon R'| \leq C\|F\|_{L^\infty}, \quad |\varepsilon^2 R''| \leq C\|F\|_{L^\infty}.$$

This completes the proof. □

Proposition 2. $L_2(\varepsilon, c_1, \tau_1) : C^2_{1,\varepsilon}(0, \tau_0) \to C^1_\varepsilon(0, \tau_0)$ *is invertible. Moreover, there is a constant* $M > 0$ *independent of* ε *such that* $\|L_2^{-1}\| \leq M$.

Proof. The invertibility of L_2 is obvious. We suppose that for $h \in C^1_\varepsilon$, there is a solution $S \in C^2_{1,\varepsilon}$ such that

$$c_0^* S' - (g_u h'_+(V_0) + g_v)S = h.$$

This equation can be written as an integral form of

$$S(y) = \int_0^y \exp\left\{\int_s^y (g_u h'_+(V_0) + g_v)dx\right\} \frac{h}{c_0^*} ds.$$

Hence

$$|S(y)| \leq C\|h\|_{L^\infty}.$$

Similarly, we have

$$|S'(y)| \leq C\|h\|_{L^\infty}, \quad |\varepsilon S''(y)| \leq C\|h\|_{C^1_\varepsilon}.$$

This completes the proof. □

From the above two propositions, we can prove Lemma 4.

Proof. Consider $L[R, S] = (F, G)$ for any given $F \in C(0, \tau_0^*)$ and $G \in$

$C^1_\varepsilon(0,\tau_0^*)$. Then,

$$\begin{cases} \varepsilon^2 R'' - \varepsilon c_0^* R' + f_u R - \varepsilon c_0^* h_+'(V_0) S' = F, \\ c_0^* S' - (g_u h_+'(V_0) + g_v) S - g_u R = G. \end{cases}$$

From the second equation,

$$S' = \frac{1}{c_0^*} \left\{ (g_u h_+'(V_0) + g_v) S + g_u R + G \right\}.$$

Substituting this into the first equation, we have

$$L_1 R - \frac{\varepsilon h_+'(V_0)}{c_0^*} \{ (g_u h_+'(V_0) + g_v) S + g_u R + G \} = F.$$

Since L_1 is invertible,

$$\tilde{L}_1 \equiv L_1 - \frac{\varepsilon h_+'(V_0)}{c_0^*} g_u$$

is so. We solve the first equation with respect to R, having

$$R = \tilde{L}_1^{-1} \left[\frac{\varepsilon h_+'(V_0)}{c_0^*} \{ (g_u h_+'(V_0) + g_v) S + G \} + F \right].$$

We substitute this into the second equation. The result equation is

$$L_2 S - \varepsilon g_u \tilde{L}_1^{-1} \frac{h_+'(V_0)}{c_0^*} (g_u h_+'(V_0) + g_v) S = G + g_u \tilde{L}_1^{-1} \left(\frac{\varepsilon h_+'(V_0)}{c_0^*} G + F \right).$$

Since L_2 is invertible, this equation can be solved and the solution (R, S) satisfies

$$\begin{aligned} \|R\|_{C^2_\varepsilon(0,\tau_0^*)} &\le C(\|F\|_{L^\infty} + \|G\|_{C^1_\varepsilon(0,\tau_0^*)}), \\ \|S\|_{C^2_{1,\varepsilon}(0,\tau_0^*)} &\le C(\|F\|_{L^\infty} + \|G\|_{C^1_\varepsilon(0,\tau_0^*)}). \end{aligned}$$ □

From Lemma 4, we know that $T_t(0;\varepsilon, c_1, \tau_1)$ is invertible. The following lemma is obvious from the definition of T.

Lemma 5. *Fix $\delta > 0$, and let $\alpha - h_+(0)$ and $\beta - h_+(v^*)$ be sufficiently small and fixed. Then, $\|T(0;\varepsilon, c_1, \tau_1)\|_{Y_\varepsilon} = o(1)$ uniformly in $(c_1, \tau_1) \in \Xi_\delta$ as $\varepsilon \to 0$.*

The proof of this lemma contains only simple calculations and shall be given below.

Thus we obtain the following lemma by the same argument as in the proof of Lemma 3.

Lemma 6. *Fix $\delta > 0$, and let $\alpha - h_+(0)$ and $\beta - h_+(v^*)$ be sufficiently small and fixed. Then, there exists $\varepsilon_0 > 0$ such that , the equation*

$$T(t;\varepsilon, c_1, \tau_1) = 0$$

has a unique solution $t(\varepsilon, c_1, \tau_1) \in X_\varepsilon$ for any $\varepsilon \in (0, \varepsilon_0)$ and $(c_1, \tau_1) \in \Xi_\delta$. Moreover,

$$\|t(\varepsilon, c_1, \tau_1)\|_{X_\varepsilon} = o(1)$$

uniformly in $(c_1, \tau_1) \in \Xi_\delta$ as $\varepsilon \to 0$.

We prove Lemma 5.

Proof. We first estimate $T_1(0;\varepsilon, c_1, \tau_1)$. Since f is continuously differentiable,

$$\begin{aligned}
&T_1(0;\varepsilon, c_1, \tau_1)\\
&= \frac{1}{\varepsilon}(\ddot{\phi}_0 - c_0^*\dot{\phi}_0) - c_0^* U_0' + \ddot{\phi}_1 - c_0^*\dot{\phi}_1 - \left(c_1 + \frac{\tau_1}{\tau_0^*}c_0^*\right)\dot{\phi}_0\\
&\quad + \frac{1}{\varepsilon} f(U_0 + \varepsilon U_1 + \phi_0 + \varepsilon\phi_1, V_0 + \varepsilon V_1 + \varepsilon\psi_0) + 2\frac{\tau_1}{\tau_0^*} f(U_0 + \phi_0, V_0) + O(\varepsilon)
\end{aligned}$$

in $y \in [0, \tau_0^*/4]$. Recall that $\theta^l = 1$ and $\theta^r = 0$ in $y \in [0, \tau_0^*/4]$. The superscript “ l ” is omitted for notational convenience. We expand f around $(U_0 + \phi_0, V_0)$ as

$$\begin{aligned}
&f(U_0 + \varepsilon U_1 + \phi_0 + \varepsilon\phi_1, V_0 + \varepsilon V_1 + \varepsilon\psi_0)\\
&= f(U_0+\phi_0, V_0) + \varepsilon f_u(U_0+\phi_0, V_0)(U_1+\phi_1) + \varepsilon f_v(U_0+\phi_0, V_0)(V_1+\psi_0) + o(\varepsilon).
\end{aligned}$$

From the equations (3.14) and (3.16),

$$
\begin{aligned}
&T_1(0;\varepsilon,c_1,\tau_1)\\
&=\frac{1}{\varepsilon}\left(1+2\varepsilon\frac{\tau_1}{\tau_0^*}\right)\{f(U_0+\phi_0,V_0)-f(U_0(0)+\phi_0,0)\}\\
&\quad+\{f_v(U_0+\phi_0,V_0)-f_v(U_0,V_0)\}V_1+\{f_v(U_0(0)+\phi_0,0)-f_v(U_0(0),0)\}V_1(0)\\
&\quad+\{f_v(U_0(0)+\phi_0,0)-f_v(U_0(0)+\phi_0,0)\}\psi_0+\{f_u(U_0+\phi_0,V_0)\\
&\quad-f_u(U_0,V_0)\}U_1+\{f_u(U_0+\phi_0,V_0)V_0-f_u(U_0(0)+\phi_0,0)V_0(0)\}\phi_1\\
&\quad-\{f_u(U_0(0)+\phi_0,0)-f_u(U_0(0),V_0)\}U_1(0)+O(\varepsilon).
\end{aligned}
$$

We estimate each terms of the right-hand side of the above equality. Expanding $f(U_0+\phi_0,V_0)$ around U_0 as

$$
f(U_0+\phi_0,V_0)=f_u(U_0,V_0)\phi_0+\int_0^{\phi_0}(\phi_0-t)f_{uu}(U_0+t,V_0)dt,
$$

we have

$$
\begin{aligned}
&f(U_0+\phi_0,V_0)-f(U_0(0)+\phi_0,V_0)=(f_u(U_0,V_0)-f_u(U_0(0),0))\phi_0\\
&\quad+\int_0^{\phi_0}(\phi_0-t)(f_{uu}(U_0+t,V_0)-f_{uu}(U_0(0)+t,0))dt.
\end{aligned}
$$

If $\alpha-h_+(0)$ is small, there is sufficiently small $\sigma>0$ such that $|\phi_0|\le\sigma e^{-\kappa y/\varepsilon}$ for a constant $\kappa>0$ independent of ε. Moreover, there is a constant $C>0$ independent of ε such that $|f_u(U_0,V_0)-f_u(U_0(0),0)|\le Cy$. Hence we readily see that

$$
\frac{1}{\varepsilon}|f_u(U_0,V_0)-f_u(U_0(0),0)||\phi_0|\le C\sigma\frac{y}{\varepsilon}e^{-\kappa\frac{y}{\varepsilon}}\le C\sigma.
$$

Similarly, it holds that

$$
\frac{1}{\varepsilon}\int_0^{\phi_0}|\phi_0-t||f_{uu}(U_0+t,V_0)-f_{uu}(U_0(0)+t,0)|dt\le C\sigma.
$$

We readily see that $T_1(0;\varepsilon,c_1,\tau_1)=O(\varepsilon)$ in $y\in[\tau_0^*/4,\tau_0^*/2]$ by using (3.11), (3.12) and the similar arguments above. Additionally, it is shown that $T_1(0;\varepsilon,c_1,\tau_0^*)$ is small in $y\in[\tau_0^*/2,\tau_0^*]$ from the same argument as in $y\in[0,\tau_0^*/2]$.

Next we claim that $T_2(0;\varepsilon,c_1,\tau_1)$ is small in $C^2_{1,\varepsilon}$. Since g is continuously differentiable,

$$\begin{aligned} T_2(0;\varepsilon,c_1,\tau_1) &= \frac{1}{\varepsilon}c_0^*(V_0'+\dot\psi_0)+(c_1V_0'+c_0^*V_1'+c_1\dot\psi_0) \\ &\quad -\frac{1}{\varepsilon}g(U_0+\varepsilon U_1+\phi_0+\varepsilon\phi_1,V_0+\varepsilon V_1+\varepsilon\psi_0) \\ &\quad -\frac{\tau_1}{\tau_0^*}g(U_0+\phi_0,V_0)+O(\varepsilon), \end{aligned}$$

in $y\in(0,\tau_0^*/4]$ where " l" of functions is omitted for notational convenience. From the similar argument to the above, it is easy to see that $|T_2(0;\varepsilon,c_1,\tau_1)|$ is small for sufficiently small $\varepsilon>0$. On the other hand,

$$\begin{aligned} \varepsilon(T_2(0;\varepsilon,c_1,\tau_1))' &= c_0^*V_0''+\frac{1}{\varepsilon}(c_0^*+\varepsilon c_1)\ddot\psi_0+c_0^*\ddot\psi_1-g_u(U_0+\phi_0,V_0)(U_0'+\dot\phi_1) \\ &\quad -\frac{1}{\varepsilon}\left(1+\varepsilon\frac{\tau_1}{\tau_0^*}\right)g_u(U_0+\varepsilon U_1+\phi_0+\varepsilon\phi_1,V_0+\varepsilon V_1+\varepsilon\psi_0)\dot\phi_0 \\ &\quad -g_v(U_0+\phi_0,V_0)(V_0'+\dot\psi_0)+O(\varepsilon). \end{aligned}$$

Differentiating the both sides of (3.11), (3.12), (3.14) and (3.16), we have

$$\begin{aligned} &\varepsilon(T_2(0;\varepsilon,c_1,\tau_1))' \\ &= \frac{1}{\varepsilon}\{g_u(U_0(0)+\phi_0,0)-g_u(U_0+\phi_0,V_0)\}\dot\phi_0 \\ &\quad +\{g_{uu}(U_0(0)+\phi_0,0)(U_1(0)+\phi_1)-g_{uu}(U_0+\phi_0,V_0)(U_1+\phi_1)\}\dot\phi_0 \\ &\quad +\{g_{uv}(U_0(0)+\phi_0,0)(V_1(0)+\psi_0)-g_{uv}(U_0+\phi_0,V_0)(V_1+\psi_0)\}\dot\phi_0 \\ &\quad +\frac{\tau_1}{\tau_0^*}\{g_u(U_0(0)+\phi_0,0)-g_u(U_0+\phi_0,V_0)\}\dot\phi_0 \\ &\quad +\{g_v(U_0(0)+\phi_0,0)-g_v(U_0+\phi_0,V_0)\}\dot\psi_0 \\ &\quad +\{g_u(U_0(0)+\phi_0,0)-g_u(U_0+\phi_0,V_0)\}\dot\phi_1 \\ &\quad +\{g_u(U_0,V_0)-g_u(U_0+\phi_0,V_0)\}U_0'+\{g_v(U_0,V_0)-g_v(U_0+\phi_0,V_0)\}V_0' \\ &\quad +\{g_u(U_0(0)+\phi_0,0)-g_u(U_0(0),V_0(0))\}U_0'(0) \\ &\quad +\{g_v(U_0(0)+\phi_0,0)-g_v(U_0(0),V_0(0))\}V_0'(0) \\ &\quad +\{g_{uu}(U_0(0)+\phi_0,0)U_0'(0)g_{uv}(U_0(0)+\phi_0,0)V_0'(0)\}\frac{y}{\varepsilon}\dot\phi_0+O(\varepsilon). \end{aligned}$$

From the similar argument to the above, it is easy to see that $\varepsilon(T_2(0;\varepsilon,c_1,\tau_1))'$ is small for sufficiently small $\varepsilon>0$. This completes the proof. □

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