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THE LOGARITHMIC COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS

BY

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Abstract

Abstract. We prove that if $n \geq 2$ for each close-to-convex functions in S whose n-th logarithmic coefficients γ_n satisfies $|\gamma_n| \leq A \log n/n$, where A is an absolute constant.

1. Introduction and Statement of Result

Let S be the class of functions f analytic and univalent in the unit disk $D = \{z \in C : |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$. Let S^* denote the subset of S consisting of those functions f $\in$S for which $f(D)$ is starlike with respect to 0. It is well known that if $f \in S^*$, then $Re\{zf'(z)/f(z)\} > 0$, for all $z \in D$. Finally, we let S_c denote the set of those functions f$\in S$ for which there exists a function $g \in S^*$ such that $Re\{zf'(z)/g(z)\} > 0$, for all $z \in D$. The elements of S_c are called close-to-convex functions. Clearly, $S^* \subset S_c$.

Associated with each $f \in S$ is well defined logarithmic function

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \tag{1}$$

z$\in D$. The numbers γ_n are called the logarithmic coefficients of f. Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = 1/n$. It is clear that $|\gamma_1| \leq 1$ for each $f \in S$. The estimate of the logarithmic

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coefficients is a important problem in the theory of univalent functions. The inequality $|\gamma_n| \leq 1/n$ holds for functions $f \in S^*$, but is false for the full class S, even in order of magnitude. Indeed, there exists a bounded function $f \in S$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [1] p.242). In a resent paper [2], it is presented that inequality $|\gamma_n| \leq 1/n$ holds also for close-to-convex functions. However, it is pointed out in [3] that there are some errors in the proof and , hence, the result is not substantiated. It is proved in [4] that there exists a function $f \in S_c$ such that $|\gamma_n| > 1/n$. In this paper, we will prove the following theorem.

Theorem 1. *Suppose* $f \in S_c$ *and that* f *has logarithmic coefficients* $\{\gamma_n\}_{n=1}^{\infty}$. *Then for* $n = 2, 3, \ldots$

$$|\gamma_n| \leq A\frac{\log n}{n}$$

where A *is an absolute constant.*

2. Preliminary Lemmas

First, we prove some lemmas for the proof of Theorem.

Lemma 1. *Let* $f \in S$, $z = re^{i\theta}$, $\frac{1}{2} \leq r < 1$. *Then*

$$\begin{aligned} J_r &= \frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{zf'(z)}{f(z)}\Big|^2 d\theta \leq 1 + 4\frac{1}{1-r}\log\frac{1}{1-\sqrt{r}}, \\ I_r &= \frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\Big|\frac{zf'(z)}{f(z)}\Big|^2 d\theta dr \leq 1 + 2\log\frac{1}{1-r}. \end{aligned}$$

Proof. It is clear that

$$\frac{zf'(z)}{f(z)} = 1 + z(\log\frac{f(z)}{z})' = 1 + \sum_{k=1}^{\infty} 2k\gamma_k z^k.$$

Lebedev proved (see [5]) that if $f \in S$ then

$$\sum_{k=1}^{\infty} k|\gamma_k|^2 r^{2k} \leq \log\frac{1}{1-r}.$$

Since $kr^k < 1/(1-r)$, we obtain that

$$J_r = 1 + 4\sum_{k=1}^{\infty} k^2|\gamma_k|^2 r^{2k} \le 1 + \frac{4}{1-r}\log\frac{1}{1-\sqrt{r}},$$

$$I_r = \int_{\frac{1}{2}}^{r}\Big(1 + 4\sum_{k=1}^{\infty} k^2|\gamma_k|^2 r^{2k}\Big)dr < 1 + 4\sum_{k=1}^{\infty}\frac{k}{2k+1}k|\gamma_k|^2 r^{2k+1}$$
$$\le 1 + 2\log\frac{1}{1-r}.$$

□

Lemma 2. *Let $f \in S_c$ and $g \in S^*$ such that $Re\{zf'(z)/g(z)\} > 0$. Let $z = re^{i\theta}$, $0 \le r < 1$. Write*

$$\frac{zf'(z)}{f(z)} = u(re^{i\theta}) + iv(re^{i\theta}). \tag{2}$$

Then

$$I_1 = \frac{1}{2\pi}\Big|\int_0^{2\pi} u(re^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| \le 3.$$

Proof. It is clear that

$$\frac{zf'(z)}{f(z)} = \frac{1}{i}\frac{\partial}{\partial\theta}\log\frac{f(z)}{z} + 1. \tag{3}$$

It follows that

$$u(re^{i\theta}) = Im\{\frac{\partial}{\partial\theta}\log\frac{f(z)}{z}\} + 1 = \frac{\partial}{\partial\theta}\arg\frac{f(z)}{z} + 1. \tag{4}$$

We obtain from (4) that

$$I_1 \le \frac{1}{2\pi}\Big|\int_0^{2\pi} e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| + \frac{1}{2\pi}\Big|\int_0^{2\pi}\frac{\partial}{\partial\theta}\arg\frac{f(z)}{z}e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| = I_{11} + I_{12}. \tag{5}$$

It is clear that

$$I_{11} \le \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1. \tag{6}$$

By the part of integration, we obtain that

$$\begin{aligned} I_{12} &= \frac{1}{2\pi}\Big|\int_0^{2\pi} \frac{\partial}{\partial\theta}(e^{i\arg\frac{f(z)}{z}})e^{-i\arg\frac{g(z)}{z}}d\theta\Big| \\ &= \frac{1}{2\pi}\Big|\int_0^{2\pi} e^{i\arg\frac{f(z)}{g(z)}}\frac{\partial}{\partial\theta}(\arg\frac{g(z)}{z})d\theta\Big| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\Big(|\frac{\partial}{\partial\theta}\arg g(z)| + |\frac{\partial z}{\partial\theta}|\Big)d\theta. \end{aligned} \tag{7}$$

Since g$\in S^*$, it follows that $\frac{\partial \arg g(z)}{\partial\theta} > 0$. The right-hand of (7) is

$$= \frac{1}{2\pi}\int_0^{2\pi} d_\theta \arg g(z) + \frac{1}{2\pi}\int_0^{2\pi} r d\theta = 1 + r \leq 2$$

Thus, we have proved Lemma. □

Lemma 3. *Let $f \in S_c$ and $g \in S^*$ such that $Re\{zf'(z)/g(z)\} > 0$. Let $z = re^{i\theta}$, $\frac{1}{2} \leq r < 1$. The function $v(re^{i\theta})$ is defined in* (2). *Then*

$$I_2 = \frac{1}{2\pi}\left|\int_0^{2\pi} v(re^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}d\theta\right| \leq 7 + 8\log\frac{1}{1-r}.$$

Proof. By the Cauchy-Riemann condition, we obtain for $0 < r_0 < r < 1$ that

$$v(re^{i\theta}) - v(r_0e^{i\theta}) = \int_{r_0}^{r} \frac{\partial v(re^{i\theta})}{\partial r}dr = -\int_{r_0}^{r}\frac{1}{r}\frac{\partial u(re^{i\theta})}{\partial\theta}dr. \tag{8}$$

By (8), it follows that

$$\begin{aligned} I_2 &\leq \frac{1}{2\pi}\left|\int_0^{2\pi} v(r_0e^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}d\theta\right| + \frac{1}{2\pi}\left|\int_0^{2\pi}\int_{r_0}^{r}\frac{1}{r}\frac{\partial u(re^{i\theta})}{\partial\theta}e^{i\arg\frac{f(z)}{g(z)}}drd\theta\right| \\ &= I_{21} + I_{22}. \end{aligned} \tag{9}$$

Taking $r_0 = \frac{1}{2}$, it follows that

$$I_{21} \leq \max_{\theta\in[0,2\pi]}|v(r_0e^{i\theta})| \leq \max_{\theta\in[0,2\pi]}|\frac{r_0f'(r_0e^{i\theta})}{f'(r_0e^{i\theta})}| \leq \frac{1+r_0}{1-r_0} = 3. \tag{10}$$

Now, we estimate I_{22}. By the part of integration, it follows that

$$I_{22}=\frac{1}{2\pi}\Big|\int_{r_0}^{r}\int_0^{2\pi}\frac{1}{r}u(re^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}(\frac{\partial\arg\frac{f(z)}{z}}{\partial\theta}-\frac{\partial\arg\frac{g(z)}{z}}{\partial\theta})d\theta dr\Big|.$$

By (4), it follows that

$$\Big|\frac{\partial\arg\frac{f(z)}{z}}{\partial\theta}-\frac{\partial\arg\frac{g(z)}{z}}{\partial\theta}\Big|=\Big|Re\frac{zf'(z)}{f(z)}-Re\frac{zg'(z)}{g(z)}\Big|\le\Big|\frac{zf'(z)}{f(z)}\Big|+\Big|\frac{zg'(z)}{g(z)}\Big|.$$

By Schwartz inequality and Lemma 2.1, we obtain that

$$\begin{aligned}I_{22}&\le\frac{2}{2\pi}\int_{r_0}^{r}\int_0^{2\pi}\Big[|\frac{zf'(z)}{f(z)}|^2+|\frac{zf'(z)}{f(z)}||\frac{zg'(z)}{g(z)}|\Big]d\theta dr\\&\le\frac{1}{\pi}\Big[\int_{r_0}^{r}\int_0^{2\pi}\Big|\frac{zf'(z)}{f(z)}\Big|^2d\theta dr\\&\quad+\Big(\int_{r_0}^{r}\int_0^{2\pi}\Big|\frac{zf'(z)}{f(z)}\Big|^2d\theta dr\int_{r_0}^{r}\int_0^{2\pi}\Big|\frac{zg'(z)}{g(z)}\Big|^2d\theta dr\Big)^{\frac{1}{2}}\Big]\\&\le4(1+2\log\frac{1}{1-r}).\end{aligned}\tag{11}$$

Thus, we have proved Lemma by (9), (10) and (11). □

Lemma 4. *Let $f\in S_c$ and $g\in S^*$ such that $Re\{zf'(z)/g(z)\}>0$. Let $z=re^{i\theta}$, $0\le r<1$. Then for $n=2,3,\dots$*

$$\begin{aligned}I_3&=\frac{1}{2\pi}\Big|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{i2\arg\frac{f(z)}{g(z)}}e^{in\theta}d\theta\Big|\\&\le4\Big(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\Big)^{\frac{1}{2}}\Big(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Proof. From (1) we have

$$\begin{aligned}\frac{zf(z)}{f(z)}e^{in\theta}&=e^{in\theta}(1+\sum_{k=1}^{\infty}2k\gamma_kz^k)=e^{in\theta}+\sum_{k=1}^{\infty}2k\gamma_kr^ke^{i(n+k)\theta}\\&=\frac{1}{i}\frac{\partial}{\partial\theta}\Big(\frac{e^{in\theta}}{n}+\sum_{k=1}^{\infty}\frac{2k\gamma_kr^ke^{i(n+k)\theta}}{n+k}\Big)=\frac{\partial}{\partial\theta}F(z).\end{aligned}\tag{12}$$

By the part of integration, we obtain that

$$I_3 = \frac{1}{\pi}\Big|\int_0^{2\pi} F(z)e^{i2\arg\frac{f(z)}{g(z)}}\Big(\frac{\partial \arg\frac{f(z)}{z}}{\partial\theta} - \frac{\partial \arg\frac{g(z)}{z}}{\partial\theta}\Big)d\theta\Big|. \tag{13}$$

By (4) and Schwartz inequality, it follows from (13) that

$$I_3 \le 2\Big(\frac{1}{2\pi}\int_0^{2\pi}|F(z)|^2 d\theta\Big)^{\frac{1}{2}}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(|\frac{zf'(z)}{f(z)}| + |\frac{zg'(z)}{g(z)}|\Big)^2 d\theta\Big)^{\frac{1}{2}} = 2(J_1J_2)^{\frac{1}{2}}. \tag{14}$$

By the definition of F(z) in (12), we obtain from Lebedev inequality that

$$\begin{aligned} J_1 &= \frac{1}{n^2} + 4\sum_{k=1}^{\infty}\frac{k^2|\gamma_k|^2 r^{2k}}{(n+k)^2} \\ &\le \frac{1}{n^2} + \frac{4}{n}\sum_{k=1}^{\infty}k|\gamma_k|^2 r^{2k} \le \frac{1}{n^2} + \frac{4}{n}\log\frac{1}{(1-r)}. \end{aligned} \tag{15}$$

By Lemma 2.1, it follows that

$$J_2 \le 4\Big(1 + \frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\Big). \tag{16}$$

Combining (15), (16) and (14), we have proved Lemma 2.4. □

3. Proof of Theorem

Proof. If $f \in S_c$ then there exists $g \in S^*$ such that $Re\{zf'(z)/g(z)\} > 0$. Write $zf'(z)/g(z) = h(z)$, then $Reh(z) > 0$. It is clear that

$$h(z) = 2Reh(z) - \overline{h(z)}.$$

From (1), we obtain for $z = re^{i\theta}$ $(0 < r < 1)$ and $n = 2, 3, \ldots$ that

$$2n\gamma_n = \frac{1}{2i\pi}\int_{|z|=r}\frac{zf'(z)}{f(z)}z^{-n-1}dz.$$

Hence, we obtain that

$$|2n\gamma_n r^n| = \frac{1}{2\pi}\Big|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{-in\theta}d\theta\Big| = \frac{1}{2\pi}\Big|\int_0^{2\pi}\frac{h(z)g(z)}{f(z)}e^{-in\theta}d\theta\Big|$$

$$\begin{aligned} &\le \frac{1}{2\pi}\Big|\int_0^{2\pi} 2Reh(z)\frac{g(z)}{f(z)}e^{-in\theta}d\theta\Big| + \frac{1}{2\pi}\Big|\int_0^{2\pi} \overline{h(z)}\frac{g(z)}{f(z)}e^{-in\theta}d\theta\Big| \\ &= P_1 + P_2. \end{aligned} \tag{17}$$

By Lemma 2.2 and Lemma 2.3, we obtain that

$$\begin{aligned} P_1 &\le \frac{1}{\pi}\int_0^{2\pi} Reh(z)\Big|\frac{g(z)}{f(z)}\Big|d\theta \le \frac{1}{\pi}\Big|\int_0^{2\pi} h(z)\Big|\frac{g(z)}{f(z)}\Big|d\theta\Big| \\ &= \frac{1}{\pi}\Big|\int_0^{2\pi} \frac{zf'(z)}{f(z)}e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| \\ &\le \frac{1}{\pi}\Big|\int_0^{2\pi} u(re^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| + \frac{1}{\pi}\Big|\int_0^{2\pi} v(re^{i\theta})e^{i\arg\frac{f(z)}{g(z)}}d\theta\Big| \\ &\le 20 + 16\log\frac{1}{1-r}. \end{aligned} \tag{18}$$

By Lemma 2.4, we obtain that

$$\begin{aligned} P_2 &= \frac{1}{2\pi}\Big|\int_0^{2\pi} h(z)\overline{(\frac{g(z)}{f(z)})}e^{in\theta}d\theta\Big| = \frac{1}{2\pi}\Big|\int_0^{2\pi} \frac{zf'(z)}{f(z)}e^{i2\arg\frac{f(z)}{g(z)}}e^{in\theta}d\theta\Big| \\ &\le 4\Big(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\Big)^{\frac{1}{2}}\Big(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\Big)^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Set $r = 1-1/n$ $(n = 2,3,\ldots)$. We obtain from (17), (18) and (19) that for $n = 2,3,\ldots$

$$\begin{aligned} |\gamma_n| &\le \frac{1}{2n}\Big(1-\frac{1}{n}\Big)^{-n}\Big[\Big(20+16\log\frac{1}{1-r}\Big)+4\Big(\frac{1}{n^2}+\frac{4\log n}{n}\Big)^{\frac{1}{2}}\Big(1+8n\log n\Big)^{\frac{1}{2}}\Big] \\ &\le A\frac{\log n}{n}. \end{aligned}$$

Thus, we have proved Theorem. □

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