Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **3** (2008), No. 3, pp. 433-444

EXPLICIT EXPRESSIONS FOR MOMENTS OF χ^2 ORDER STATISTICS

BY

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Abstract

Explicit closed form expressions are derived for moments of order statistics from the chi square distribution. The expressions involve Lauricella functions of type A and type B. Some numerical tabulations are provided for selected parameter values.

1. Introduction

Order statistics of the chi square distribution and their moments arise in many areas of probability and statistics. Some examples are: approximations to convolutions of random variables, multivariate test statistics, Pearson's X^2 statistics and other goodness–of–fit statistics, ranking and selection procedures, tests of homogeneity of variances, and transformations. For details we refer the readers to Tiku (1965), Jensen (1973), Manoukian (1982), Hall (1983), Holtzman and Good (1986), Ko and Yum (1991), Wang (1994), Mathai and Pederzoli (1996), Fujikoshi (1997), Fujisawa (1997), Fujikoshi (2000) and Garcia–Perez and Nunez–Anton (2004).

The need for moments of order statistics also arises in applied areas such as quality control testing and reliability. For example, if the reliability of an item is high, the duration of an "all items fail" life–test can be too expensive in both time and money. This fact prevents a practitioner from knowing enough about the product in a relatively short time. Therefore, a

Received January 1, 2007 and in revised form October 30, 2007.

AMS Subject Classification: 33C90; 62E99.

Key words and phrases: Chi square distribution, Lauricella function of type A, Lauricella function of type B, Order statistics.

practitioner needs to predict the failure of future items based on the times of a few early failures. These predictions are often based on moments of order statistics.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the chi square distribution with degrees of freedom 2α and the probability density function (pdf) given by:

$$f(x) = \frac{x^{\alpha-1} \exp(-x/2)}{2^{\alpha} \Gamma(\alpha)} \tag{1}$$

for $x > 0$ and $\alpha > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. The moments of the chi square order statistics are $E(X_{r:n}^k)$ for $k = 1, 2, \ldots$.

There has been a large amount of work relating to moments of the chi square order statistics. As far as we know, there are eight significant papers on the calculation of $E(X_{r:n}^k)$. In the earliest paper, Gupta (1960) derived a recurrence relation for $E(X_{r:n}^k)$ for integer values of the shape parameter α. Gupta used this relation to tabulate values of $E(X_{r:n}^k)$ for various combinations of k, n and α. Gupta also discussed some illustrative applications to life–testing and reliability problems. Joshi (1979) re–derived the recurrence relation of Gupta (1960) and showed that if $E(X_{1:n}^k)$ for $k = -(r-1), \ldots, -1$ are known then one can obtain expressions for all of $E(X_{r:n}^k)$. Krishnaiah *et al.* (1967) extended the work of Gupta (1960) for the case that α is any positive real number. Breiter and Krishnaiah (1968) tabulated the values of $E(X_{r:n}^k)$, $k = 1, 2, 3, 4$ for various α obtained by using the recurrence relations in Krishnaiah *et al.* (1967). A Gauss–Legendre quadrature formula was used for the computations. Khan and Khan (1983) derived some recurrence relations for $E(X_{r:n}^k)$ when $f(\cdot)$ is the generalized chi square pdf given by

$$f(x) = \frac{c x^{c\alpha-1} \exp(-x^c)}{\Gamma(\alpha)} \tag{2}$$

for $x > 0$, $\alpha > 0$ and $c > 0$. Based on the available recurrence relations, Walter and Stitt (1988) constructed extensive tabulations of $E(X_{r:n}^k)$ for the chi square distribution. Sobel and Wells (1990) showed that $E(X_{r:n}^k)$ can be expressed in terms of Dirichlet integrals (integrals involving gamma functions) and provided a table for reading the Dirichlet integrals. Most recently, Abdelkader (2004) derived some recurrence relations for $E(X_{r:n}^k)$ when $X_1, X_2, \ldots, X_n$ are independent but not identically distributed chi square random variables. Abdelkader also discussed some applications in reliability.

As seen above, all of the work except for Sobel and Wells (1990) express $E(X_{r:n}^k)$ in terms of recurrence relations and/or numerical tables. That is, no explicit expressions are available for $E(X_{r:n}^k)$ except for the one given by Sobel and Wells (1990). The representation given in Sobel and Wells (1990) involves the Dirichlet integrals which are not well known and for which no standard routines are available. The use of the various numerical tables can be limited and highly inaccurate.

In this note, we derive explicit expressions for $E(X_{r:n}^k)$ that are finite sums of well known special functions – namely, the Lauricella function of type A (Exton, 1978) defined by

$$
\begin{aligned}
& F_A^{(n)}\left(a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right) \\
& \quad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{a_{m_1+\cdots+m_n}\left(b_1\right)_{m_1} \cdots \left(b_n\right)_{m_n}}{\left(c_1\right)_{m_1} \cdots \left(c_n\right)_{m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}
\end{aligned} \tag{3}
$$

and, the Lauricella function of type B (Exton, 1978) defined by

$$
\begin{aligned}
& F_B^{(n)}\left(a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n\right) \\
& \quad = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{\left(a_1\right)_{m_1} \cdots \left(a_n\right)_{m_n}\left(b_1\right)_{m_1} \cdots \left(b_n\right)_{m_n}}{c_{m_1+\cdots+m_n}} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!},
\end{aligned} \tag{4}
$$

where $(f)_p = f(f+1)\cdots(f+p-1)$ denotes the ascending factorial. Numerical routines for the direct computation of (3) and (4) are available, see, for example, Exton (1978, 2007).

This note is outlined as follows. Section 2 derives explicit expressions for $E(X_{r:n}^k)$ when $X_1, X_2, \ldots, X_n$ is a random sample from (1). The extension of this result to non–identically distributed (INID) chi square random variables is considered in Section 3. Some further extensions when $X_1, X_2, \ldots, X_n$ is a sample from (2) are considered in Section 4. Finally, some numerical tabulations of the explicit expressions for $E(X_{r:n}^k)$ are provided in Section 5.

2. IID Case

If $X_1, X_2, \ldots, X_n$ is a random sample from (1) then it is well known that the pdf of $Y = X_{r:n}$ is given by

$$f_Y(y) = \frac{n!}{(r-1)!(n-r)!} \{F(y)\}^{r-1} \{1 - F(y)\}^{n-r} f(y)$$

for $r = 1, 2, \ldots, n$, where $F(\cdot)$ is the cumulative distribution function (cdf) corresponding to (1) given by

$$F(y) = \frac{\gamma(\alpha, y/2)}{\Gamma(\alpha)},$$

where $\gamma(\cdot, \cdot)$ denotes the incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \exp(-t)\, dt.$$

Thus, the kth moment of $X_{r:n}$ can be expressed as

$$\begin{aligned} E\left(X_{r:n}^k\right) &= \frac{n!}{2^\alpha (r-1)!(n-r)! \{\Gamma(\alpha)\}^n} \\ &\quad \times \int_0^\infty y^{k+\alpha-1} \exp(-y/2) \{\gamma(\alpha, y/2)\}^{r-1} \{\Gamma(\alpha) - \gamma(\alpha, y/2)\}^{n-r} dy \\ &= \frac{n!}{2^\alpha (r-1)!(n-r)! \{\Gamma(\alpha)\}^n} \int_0^\infty y^{k+\alpha-1} \exp(-y/2) \\ &\quad \times \sum_{l=0}^{n-r} \binom{n-r}{l} \{\Gamma(\alpha)\}^{n-r-l} (-1)^l \{\gamma(\alpha, y/2)\}^{r+l-1} dy \\ &= \frac{n!}{2^\alpha (r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} \\ &\quad \times \int_0^\infty y^{k+\alpha-1} \exp(-y/2) \{\gamma(\alpha, y/2)\}^{r+l-1} dy \\ &= \frac{n!}{2^\alpha (r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} I(l). \end{aligned} \tag{5}$$

Using the series expansion

$$\gamma(\alpha, x/2) = (x/2)^\alpha \sum_{m=0}^{\infty} \frac{(-x/2)^m}{(\alpha+m)m!}$$

(see Gradshteyn and Ryzhik (2000)), the integral $I(l)$ in (5) can be expressed as

$$I(l) = \int_0^\infty y^{k+\alpha-1}\exp(-y/2)\left\{(y/2)^\alpha\sum_{m=0}^\infty\frac{(-y/2)^m}{(\alpha+m)m!}\right\}^{r+l-1}dy$$
$$=\int_0^\infty\sum_{m_1=0}^\infty\cdots\sum_{m_{r+l-1}=0}^\infty\frac{(-1)^{m_1+\cdots+m_{r+l-1}}(y/2)^{\alpha(r+l-1)+m_1+\cdots+m_{r+l-1}}}{(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!}$$
$$\times y^{k+\alpha-1}\exp(-y/2)dy$$
$$=\sum_{m_1=0}^\infty\cdots$$
$$\sum_{m_{r+l-1}=0}^\infty\frac{(-1)^{m_1+\cdots+m_{r+l-1}}}{2^{\alpha(r+l-1)+m_1+\cdots+m_{r+l-1}}(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!}$$
$$\times\int_0^\infty y^{k+\alpha(r+l)+m_1+\cdots+m_{r+l-1}-1}\exp(-y/2)dy$$
$$=2^{k+\alpha}\sum_{m_1=0}^\infty\cdots\sum_{m_{r+l-1}=0}^\infty\frac{(-1)^{m_1+\cdots+m_{r+l-1}}\Gamma\left(k+\alpha(r+l)+m_1+\cdots+m_{r+l-1}\right)}{(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!}. \tag{6}$$

Using the fact $(f)_k = \Gamma(f+k)/\Gamma(f)$ and the definition in (3), one can reexpress (6) as

$$I(l) = 2^{k+\alpha}\alpha^{1-r-l}\Gamma\left(k+\alpha(r+l)\right)$$
$$\times F_A^{(r+l-1)}\left(k+\alpha(r+l),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right). \tag{7}$$

Combining (5) and (7), we obtain the expression

$$E\left(X_{r:n}^k\right)=\frac{2^k n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}(-1)^l\binom{n-r}{l}\left\{\Gamma(\alpha)\right\}^{-r-l}\alpha^{1-r-l}\Gamma\left(k+\alpha(r+l)\right)$$
$$\times F_A^{(r+l-1)}\left(k+\alpha(r+l),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right). \tag{8}$$

Note that (8) is a finite sum of the Lauricella function of type A, a function that can be computed directly by using a software supplied by Exton (2007). It is easy to show that the infinite sum in (6) converges and hence (8) exists. Consider the sum in (6) with respect to the variable m_{r+l-1}. One can write

$$\sum_{m_1=0}^\infty\cdots\sum_{m_{r+l-1}=0}^\infty\frac{(-1)^{m_1+\cdots+m_{r+l-1}}\Gamma\left(k+\alpha(r+l)+m_1+\cdots+m_{r+l-1}\right)}{(\alpha+m_1)\cdots(\alpha+m_{r+l-1})m_1!\cdots m_{r+l-1}!}$$

$$= \frac{1}{\alpha} \sum_{m_1=0}^{\infty} \cdots \sum_{m_{r+l-2}=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_{r+l-2}} \Gamma\left(k+\alpha(r+l)+m_1+\cdots+m_{r+l-2}\right)}{(\alpha+m_1)\cdots(\alpha+m_{r+l-2}) m_1! \cdots m_{r+l-2}!}$$
$$\times\ {}_2F_1\left(\alpha, k+\alpha(r+l)+m_1+\cdots+m_{r+l-2}; 1+\alpha; -1\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by

$${}_2F_1\left(a, b; c; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}.$$

Clearly, ${}_2F_1(\alpha, k+\alpha(r+l)+m_1+\cdots+m_{r+l-2}; 1+\alpha; -1) < 1$ for all m_1, $\ldots$, m_{r+l-2} because ${}_2F_1(\alpha, k+\alpha(r+l)+m_1+\cdots+m_{r+l-2}; 1+\alpha; -1) = 2^{-\alpha}\ {}_2F_1(\alpha, 1+\alpha-k-\alpha(r+l)-m_1-\cdots-m_{r+l-2}; 1+\alpha; 1/2)$ is a convergent series. Now apply the same argument as above for the variable m_{r+l-2}. Repeating this process for all of the variables, one will find that the infinite sum in (6) converges.

3. INID Case

Suppose now that $X_1, X_2, \ldots, X_n$ are independent chi square random variables with the probability density functions (pdfs) given by

$$f_i(x) = \frac{x^{\alpha_i-1}\exp(-x/2)}{2^{\alpha_i}\Gamma\left(\alpha_i\right)}$$

for $x > 0$ and $\alpha_i > 0$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. To find the moment of $X_{r:n}$, we use the following result due to Barakat and Abdelkader (2004):

$$E\left(X_{r:n}^k\right) = \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} I_j(k), \tag{9}$$

where

$$I_j(k) = k \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} \left\{1 - F_{i_t}(x)\right\} dx, \tag{10}$$

where $F_{i_t}(\cdot)$ is the cdf of X_{i_t} given by

$$F_{i_t}(x) \;=\; \frac{\gamma(\alpha_{i_t}, x/2)}{\Gamma(\alpha_{i_t})}.$$

Using the series expansion

$$1 - \frac{\gamma(\alpha, x/2)}{\Gamma(\alpha)} \;=\; (x/2)^{\alpha-1}\exp(-x/2)\sum_{m=0}^{\infty}\frac{(x/2)^{-m}}{\Gamma(\alpha-m)}$$

(see Gradshteyn and Ryzhik (2000)), one can express $I_j(k)$ in (10) as

$$\begin{aligned}
&I_j(k)\\
&= 2^{k-1}k\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\int_0^{\infty}(x/2)^{a-1}\exp(-jx/2)\prod_{t=1}^{j}\sum_{m=0}^{\infty}\frac{(x/2)^{-m}}{\Gamma(\alpha_{i_t}-m)}dx\\
&= 2^{k-1}k\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\int_0^{\infty}(x/2)^{a-1}\exp(-jx/2)\\
&\qquad\times\sum_{m_1=0}^{\infty}\cdots\sum_{m_j=0}^{\infty}\frac{(x/2)^{-m_1-\cdots-m_j}}{\Gamma(\alpha_{i_1}-m_1)\cdots\Gamma(\alpha_{i_j}-m_j)}dx\\
&= 2^{k-1}k\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\sum_{m_1=0}^{\infty}\cdots\sum_{m_j=0}^{\infty}\frac{1}{\Gamma(\alpha_{i_1}-m_1)\cdots\Gamma(\alpha_{i_j}-m_j)}\\
&\qquad\times\int_0^{\infty}(x/2)^{a-m_1-\cdots-m_j-1}\exp(-jx/2)dx\\
&= 2^{k}k\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\sum_{m_1=0}^{\infty}\cdots\sum_{m_j=0}^{\infty}\frac{\Gamma\left(a-m_1-\cdots-m_j\right)j^{m_1+\cdots+m_j-a}}{\Gamma(\alpha_{i_1}-m_1)\cdots\Gamma(\alpha_{i_j}-m_j)},
\end{aligned}\tag{11}$$

where $a = k + (\alpha_{i_1} - 1) + \cdots + (\alpha_{i_j} - 1)$. Noting that $\Gamma(\alpha_{i_t} - m_t) = (-1)^{m_t}\Gamma(\alpha_{i_t})/(1-\alpha_{i_t})_{m_t}$, $\Gamma(a - m_1 - \cdots - m_j) = (-1)^{m_1+\cdots+m_j}\Gamma(a)/(1-a)_{m_1+\cdots+m_j}$ and the definition in (4), one can reexpress (11) as

$$I_j(k) \;=\; 2^k k\frac{\Gamma(a)\,F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right)j^a}$$

and hence (9) can be rewritten as

$$E\left(X_{r:n}^k\right) = 2^k k \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \frac{\Gamma(a)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right) j^a} \times F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right). \quad (12)$$

Note that (12) is a finite sum of the Lauricella function of type B, a function that can be computed directly by using a software supplied by Exton (2007). The fact that the infinite sum in (11) converges and hence that (12) exists can be proved similarly to Section 2. An alternative representation for (8) can be obtained by setting all of the αs in (12) equal and so yielding the form

$$E\left(X_{r:n}^k\right) = 2^k k \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \frac{\Gamma(a)}{\Gamma^j(\alpha) j^a} \times F_B^{(j)}(1,\ldots,1,1-\alpha,\ldots,1-\alpha;1-a;j,\ldots,j). \quad (13)$$

The fact that (8) and (13) are the same can be proved by using known relationships between the Lauricella functions of type A and type B. At the moment, the author is unable to establish that (8) and (13) are equal.

4. Generalizations

A natural extension of the results in Sections 2 and 3 is to consider the moments of order statistics for the generalized chi square distribution in (2). Similar calculations show that (8) generalizes to

$$E\left(X_{r:n}^k\right) = \frac{n!}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \{\Gamma(\alpha)\}^{-r-l} \alpha^{1-r-l} \Gamma\left(\frac{k}{c}+\alpha(r+l)\right) \times F_A^{(r+l-1)}\left(\frac{k}{c}+\alpha(r+l),\alpha,\ldots,\alpha;\alpha+1,\ldots,\alpha+1;-1,\ldots,-1\right)$$

and that (12) generalizes to

$$E\left(X_{r:n}^k\right) = \frac{k}{c} \sum_{j=n-r+1}^{n} (-1)^{j-n+r-1} \binom{j-1}{n-r} \frac{\Gamma(a)}{\Gamma\left(\alpha_{i_1}\right)\cdots\Gamma\left(\alpha_{i_j}\right) j^a} \times F_B^{(j)}\left(1,\ldots,1,1-\alpha_{i_1},\ldots,1-\alpha_{i_j};1-a;j,\ldots,j\right),$$

where $a = k/c + (\alpha_{i_1} - 1) + \cdots + (\alpha_{i_j} - 1)$. For the derivation of the second expression, we have assumed that $X_1, X_2, \ldots, X_n$ are independent chi square random variables from (2) with non–identical α_i, $i = 1, 2, \ldots, n$ and common c. The derivation of an explicit expression for $E(X_{r:n}^k)$ for the case of non–identical α_i, $i = 1, 2, \ldots, n$ and non–identical c_i, $i = 1, 2, \ldots, n$ is an open problem.

5. Computational Issues

Here, we provide some numerical tabulations of (8). The purpose for the tabulations is two folded: to illustrate the calculation of $E(X_{r:n}^k)$ using (8); and, to verify the results with existing tables such as the ones in Breiter and Krishnaiah (1968). We used a software supplied by Exton (2007) to compute (8). The numerical values so computed for $n = 10$, $r = 1, 2, \ldots, 10$, $k = 1, 2, 3, 4$ and $2\alpha = 1, 2, \ldots, 10$ are shown in Tables 1 to 4. The values in Tables 1 to 4 match those given by Breiter and Krishnaiah (1968) for the three decimal places.

Table 1. Values of $E(X_{r:10})$ for $2\alpha = 1, 2, \ldots, 10$.

2α	r									
	1	2	3	4	5	6	7	8	9	10
1	0.025	0.076	0.158	0.275	0.438	0.660	0.971	1.426	2.171	3.800
2	0.200	0.422	0.672	0.958	1.291	1.691	2.191	2.858	3.858	5.858
3	0.518	0.928	1.329	1.749	2.210	2.737	3.369	4.182	5.358	7.621
4	0.932	1.519	2.051	2.584	3.150	3.780	4.519	5.449	6.769	9.246
5	1.413	2.164	2.813	3.444	4.101	4.820	5.651	6.682	8.125	10.788
6	1.942	2.846	3.601	4.321	5.059	5.856	6.769	7.891	9.443	12.273
7	2.509	3.556	4.408	5.209	6.020	6.890	7.877	9.081	10.733	13.716
8	3.107	4.287	5.229	6.106	6.986	7.922	8.978	10.257	12.002	15.127
9	3.729	5.036	6.063	7.010	7.953	8.952	10.072	11.422	13.254	16.510
10	4.371	5.799	6.907	7.919	8.923	9.980	11.160	12.577	14.491	17.872

Table 2. Values of $E(X_{r:10}^2)$ for $2\alpha = 1, 2, \ldots, 10$.

2α	r									
	1	2	3	4	5	6	7	8	9	10
1	0.003	0.015	0.050	0.129	0.293	0.621	1.275	2.657	6.051	18.907
2	0.080	0.268	0.604	1.151	2.012	3.365	5.556	9.367	17.083	40.514
3	0.411	1.098	2.104	3.518	5.498	8.319	12.504	19.199	31.645	65.705
4	1.173	2.737	4.762	7.382	10.815	15.440	21.959	31.891	49.429	94.412
5	2.510	5.333	8.702	12.823	17.993	24.700	33.840	47.310	70.266	126.524
6	4.534	8.991	14.002	19.890	27.047	36.085	48.097	65.372	94.046	161.936
7	7.339	13.792	20.722	28.619	37.991	49.581	64.692	86.016	120.686	200.562
8	10.998	19.797	28.907	39.038	50.832	65.179	83.599	109.197	150.124	242.329
9	15.576	27.057	38.591	51.166	65.578	82.872	104.797	134.879	182.309	287.174
10	21.126	35.613	49.805	65.023	82.235	102.655	128.266	163.032	217.198	335.046

Table 3. Values of $E(X_{r:10}^3)$ for $2\alpha = 1, 2, \ldots, 10$.

2α	r									
	1	2	3	4	5	6	7	8	9	10
1	0.001	0.005	0.025	0.087	0.269	0.765	2.129	6.168	20.872	119.679
2	0.048	0.226	0.679	1.666	3.678	7.716	16.050	34.784	86.032	329.119
3	0.430	1.572	3.858	7.993	15.209	27.830	50.765	96.274	205.262	640.806
4	1.822	5.672	12.320	23.078	40.194	67.793	114.289	199.869	388.768	1066.194
5	5.278	14.673	29.313	51.268	84.084	134.162	214.277	354.315	646.359	1616.270
6	12.182	31.099	58.402	97.122	152.403	233.416	358.174	568.042	987.494	2301.667
7	24.197	57.761	103.381	165.339	250.721	371.984	553.280	849.255	1421.355	3132.728
8	43.225	97.696	168.220	260.729	384.646	556.261	806.784	1205.983	1956.900	4119.555
9	71.374	154.127	257.025	388.186	559.813	792.609	1125.795	1646.121	2602.907	5272.044
10	110.929	230.429	374.016	552.673	781.881	1087.371	1517.348	2177.446	3367.996	6599.911

Table 4. Values of $E(X_{r:10}^4)$ for $2\alpha = 1, 2, \ldots, 10$.

2α	r									
	1	2	3	4	5	6	7	8	9	10
1	0.000	0.003	0.017	0.079	0.315	1.176	4.345	17.287	86.833	939.945
2	0.038	0.240	0.919	2.823	7.727	20.072	52.173	144.931	489.061	3122.016
3	0.555	2.631	8.033	20.244	46.325	101.705	224.180	524.910	1457.689	7063.727
4	3.334	13.241	35.086	78.334	160.767	318.643	635.055	1338.126	3288.630	13328.784
5	12.700	44.436	106.657	219.002	416.987	770.184	1431.549	2802.571	6319.666	22526.248
6	36.686	116.536	259.751	501.177	902.590	1581.962	2791.386	5171.795	10928.674	35309.443
7	88.092	259.105	544.926	1001.551	1727.107	2905.679	4922.523	8733.795	17532.310	52374.913
8	185.422	511.970	1027.136	1815.125	3022.181	4918.925	8062.609	13810.171	26584.969	74461.492
9	353.708	926.069	1786.401	3055.634	4941.716	7825.033	12478.558	20755.457	38577.943	102349.480
10	625.250	1564.155	2918.356	4855.887	7661.988	11852.958	18466.209	29956.572	54038.710	136859.914

6. Conclusions

We have derived expressions for moments of chi square order statistics as finite sums of well known special functions. These expressions are more efficient than previously known work.

Acknowledgments

The author would like to thank the Editor (Professor Helena Holovska) and the referee for carefully reading the paper and for their comments which greatly improved the paper.

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