

ON THE LAW OF THE ITERATED LOGARITHM FOR L-STATISTICS WITHOUT VARIANCE

BY

DELI LI, DONG LIU AND ANDREW ROSALSKY

Abstract

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with distribution function $F(x)$. For each positive integer n , let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of X_1, X_2, \dots, X_n . Let $H(\cdot)$ be a real Borel-measurable function defined on \mathcal{R} such that $\mathbb{E}|H(X)| < \infty$ and let $J(\cdot)$ be a Lipschitz function of order one defined on $[0, 1]$. Write $\mu = \mu(F, J, H) = \mathbb{E}(J(U)H(F^{\leftarrow}(U)))$ and $\mathbb{L}_n(F, J, H) = \frac{1}{n} \sum_{i=1}^n J\left(\frac{i}{n}\right) H(X_{i:n})$, $n \geq 1$, where U is a random variable with the uniform $(0, 1)$ distribution and $F^{\leftarrow}(t) = \inf\{x; F(x) \geq t\}$, $0 < t < 1$. In this note, the Chung-Smirnov LIL for empirical processes and the Einmahl-Li LIL for partial sums of i.i.d. random variables without variance are used to establish necessary and sufficient conditions for having with probability 1: $0 < \limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} |\mathbb{L}_n(F, J, H) - \mu| < \infty$, where $\varphi(\cdot)$ is from a suitable subclass of the positive, non-decreasing, and slowly varying functions defined on $[0, \infty)$. The almost sure value of the limsup is identified under suitable conditions. Specializing our result to $\varphi(x) = 2(\log \log x)^p$, $p > 1$ and to $\varphi(x) = 2(\log x)^r$, $r > 0$, we obtain an analog of the Hartman-Wintner-Strassen LIL for L-statistics in the infinite variance case. A stability result for L-statistics in the infinite variance case is also obtained.

Received October 11, 2007 and in revised form January 8, 2008.

AMS Subject Classification: Primary 60F15; Secondary 62G30.

Key words and phrases: Law of the iterated logarithm; strong law of large numbers; L-statistics.

1. Introduction

Let $\{X, X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) real random variables with distribution function $F(x) = P(X \leq x)$, $x \in \mathcal{R} = (-\infty, \infty)$ and let $\{U, U_n; n \geq 1\}$ be a sequence of i.i.d. random variables with the uniform $(0, 1)$ distribution. For each positive integer n , let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of X_1, X_2, \dots, X_n . Let $H(\cdot)$ be a real-valued Borel-measurable function defined on \mathcal{R} . A linear combination of order statistics (in short, an L-statistic) is a statistic of the form

$$\mathbb{L}_n = \frac{1}{n} \sum_{i=1}^n c_{i,n} H(X_{i:n})$$

where the weights $c_{i,n}$, $1 \leq i \leq n$ are real numbers and $n \geq 1$. Define $Lt = \log_e \max\{e, t\}$ and $LLt = L(Lt)$ for $t \in \mathcal{R}$. The classical Hartman-Wintner-Strassen law of the iterated logarithm (LIL) states that

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \frac{\sum_{i=1}^n H(X_i)}{\sqrt{2nLLn}} =_{(-)}^{(+)} \sigma \text{ almost surely (a.s.)} \quad (1.1)$$

if and only if

$$\mathbb{E}H(X) = 0 \text{ and } \sigma^2 = \mathbb{E}H^2(X) < \infty. \quad (1.2)$$

Moreover, if (1.2) holds, then

$$C \left(\left\{ \sum_{i=1}^n H(X_i) / \sqrt{2nLLn}; n \geq 1 \right\} \right) = [-\sigma, \sigma] \text{ a.s.}, \quad (1.3)$$

where $C(\{x_n; n \geq 1\})$ stands for the cluster set (i.e., the set of limit points) of the numerical sequence $\{x_n; n \geq 1\}$. See Hartman and Wintner (1941) for the ‘‘if’’ part and Strassen (1966) for the converse. The conclusion (1.3) is due to Strassen (1964).

Alternative proofs of the Hartman-Wintner (1941) LIL were discovered by Strassen (1964), Heyde (1969), Egorov (1971), Teicher (1974), Csörgő and Révész (1981, p. 119), de Acosta (1983), Li (1987), and Einmahl and Li (2005). Substantially simpler proofs of Strassen’s (1966) converse were obtained by Feller (1968), Heyde (1968), and Steiger and Zaremba (1972). Martikainen (1980), Rosalsky (1980), and Pruitt (1981) simultaneously and inde-

pendently obtained a “one-sided” converse to the Hartman-Wintner (1941) LIL. Specifically, they proved that each part of (1.1) *individually* implies (1.2).

Many authors, including Helmers (1977), Helmers, Janssen, and Serfling (1988), Li, Rao, and Tomkins (2001), Mason (1982), Sen (1978), van Zwet (1980), and Wellner (1977a,b), have investigated the strong limiting behavior for a class of L-statistics of the form

$$\mathbb{L}_n(F, J, H) \equiv \frac{1}{n} \sum_{i=1}^n J\left(\frac{i}{n}\right) H(X_{i:n}), \quad n \geq 1 \quad (1.4)$$

where $J(\cdot)$ is a real-valued function, often called a *score function*, defined on $[0, 1]$. Helmers (1977), Mason (1982), Sen (1978), van Zwet (1980), and Wellner (1977a) have studied the strong law of large numbers (SLLN) for \mathbb{L}_n , $n \geq 1$ and have shown that under a variety of conditions on $J(\cdot)$ and $H(\cdot)$

$$\lim_{n \rightarrow \infty} \mathbb{L}_n(F_U, J, H) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n J\left(\frac{i}{n}\right) H(U_{i:n}) = \int_0^1 J(t)H(t)dt \quad (\text{finite}) \text{ a.s.},$$

where F_U is the distribution function of the random variable U and $U_{i:n}$, $1 \leq i \leq n$, are the order statistics of the U_i , $1 \leq i \leq n$, $n \geq 1$.

If $J(\cdot)$ is a Lipschitz function of order one defined on $[0, 1]$ and

$$\mathbb{E}|H(X)| < \infty, \quad (1.5)$$

let us write

$$\begin{cases} Z = J(U)H(F^{\leftarrow}(U)), \\ Y = -Z + \mu - \int_0^1 (I(U \leq t) - t) J'(t)H(F^{\leftarrow}(t))dt, \end{cases} \quad (1.6)$$

where $F^{\leftarrow}(t)$ is the quantile function

$$F^{\leftarrow}(t) = \inf \{s; F(s) \geq t\}, \quad 0 < t < 1,$$

and

$$\mu = \mu(F, J, H) = \mathbb{E}Z = \mathbb{E}(J(U)H(F^{\leftarrow}(U))). \quad (1.7)$$

Then μ exists and is finite and Y and Z are both well-defined random variables under (1.5). Moreover,

$$\sigma_Y^2 = \text{Var}(Y) = \mathbb{E}Y^2. \quad (1.8)$$

To see this, note that (1.8) is equivalent to

$$\mathbb{E}Y = -\mathbb{E} \left(\int_0^1 (I(U \leq t) - t) J'(t) H(F^{\leftarrow}(t)) dt \right) = 0. \quad (1.9)$$

Since $\mathbb{E}(I(U \leq t) - t) = 0$, (1.9) follows from an application of Fubini's theorem, subject to the existence of the integral

$$I = \int_0^1 J'(t) H(F^{\leftarrow}(t)) dt = \int_0^1 H(F^{\leftarrow}(t)) dJ(t) = \mathbb{E}(J'(U) H(F^{\leftarrow}(U))).$$

But the score function $J(\cdot)$ has an almost everywhere (with respect to Lebesgue measure) bounded derivative $J'(\cdot)$. From this fact and the equality $\mathbb{E}|H(X)| = \mathbb{E}|H(G(U))| < \infty$, it follows that I exists and is finite; clearly, $\sigma_Y^2 < \infty$ if and only if

$$\mathbb{E}Z^2 < \infty. \quad (1.10)$$

Recall that a sequence of random variables $\{\xi_n; n \geq 1\}$ is said to be *bounded in probability* if

$$\lim_{x \rightarrow \infty} \sup_{n \geq 1} P(|\xi_n| \geq x) = 0.$$

Combining the Chung-Smirnov LIL (see Chung (1949) and Smirnov (1944)) and the Finkelstein functional LIL (see Finkelstein (1971)) for empirical processes, Li, Rao, and Tomkins (2001, Theorem 2.1) proved that the following three statements are equivalent:

(1.10) holds;

$$\limsup_{n \rightarrow \infty} \sqrt{n/(2LLn)} |\mathbb{L}_n(F, J, H) - \mu| < \infty \text{ a.s.};$$

$\{\sqrt{n}(\mathbb{L}_n(F, J, H) - \mu); n \geq 1\}$ is bounded in probability.

Moreover, if any of the three statements above holds, then

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \sqrt{n/(2LLn)} (\mathbb{L}_n(F, J, H) - \mu) = \binom{+}{-} \sigma_Y \quad \text{a.s.}, \quad (1.11)$$

$$C \left(\left\{ \sqrt{n/(2LLn)} (\mathbb{L}_n(F, J, H) - \mu); n \geq 1 \right\} \right) = [-\sigma_Y, \sigma_Y] \quad \text{a.s.}, \quad (1.12)$$

and

$$\sqrt{n} (\mathbb{L}_n(F, J, H) - \mu) \xrightarrow{d} N(0, \sigma_Y^2), \quad (1.13)$$

where “ \xrightarrow{d} ” denotes convergence in distribution. This powerful result contains many previous results obtained under more restrictive conditions, although it is still not the last word as the authors mention in an open problem (to weaken the conditions on $J(\cdot)$). The authors illustrate with examples that their result can handle some cases that previous results could not; for example, the Gini mean-difference statistic.

The main purpose of the present note is to find necessary and sufficient conditions for

$$0 < \limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} |\mathbb{L}_n(F, J, H) - \mu| < \infty \quad \text{a.s.},$$

where $\varphi(\cdot)$ is from a suitable subclass of the positive, nondecreasing, and slowly varying functions. But we also treat the case where the limit is 0 a.s. We emphasize that we are not assuming that $\mathbb{E}Z^2 < \infty$ where Z is as in (1.6).

The plan of this note is as follows. Our main results, Theorems 2.1 and 2.2, and their proofs and corollaries are presented in Section 2. The proofs are obtained via a nice application of the Chung-Smirnov LIL (see Chung (1949) and Smirnov (1944)) for empirical processes and the Einmahl-Li LIL (see Einmahl and Li (2005)) for partial sums of i.i.d. random variables without variance. This work of Einmahl and Li (2005) is related to previous work of Martikainen (1984); it appears that slightly different proofs of Theorems 2.1 and 2.2 can be given using this work of Martikainen (1984) instead of

that of Einmahl and Li (2005). In Section 3, we provide an example to illustrate our results.

2. Main Results

Let \mathcal{H} be the set of continuous, nondecreasing functions $\varphi(\cdot) : [0, \infty) \rightarrow (0, \infty)$, which are slowly varying at infinity. By monotonicity, the slow variation of $\varphi(\cdot)$ is equivalent to $\lim_{t \rightarrow \infty} \varphi(et)/\varphi(t) = 1$. Very often one can even show that $\lim_{t \rightarrow \infty} \varphi(tf(t))/\varphi(t) = 1$, where $f(\cdot)$ is a nondecreasing function such that $\lim_{t \rightarrow \infty} f(t) = \infty$. Set $f_\tau(t) = \exp((Lt)^\tau), 0 \leq \tau < 1$. Given $0 \leq q < 1$, let $\mathcal{H}_q \subset \mathcal{H}$ be the class of functions $\varphi(\cdot)$ such that

$$\lim_{t \rightarrow \infty} \frac{\varphi(tf_\tau(t))}{\varphi(t)} = 1, \quad 0 \leq \tau < 1 - q$$

and set $\mathcal{H}_1 = \mathcal{H}$.

We consider q to be a measure for how slow the slow variation is. So functions in \mathcal{H}_0 are the “slowest” and it will turn out that this class is particularly interesting for LIL type results (see Theorem 2.2 below). Examples of functions in \mathcal{H}_0 are $\varphi(t) = (Lt)^r, r \geq 0$ and $\varphi(t) = (LLt)^p, p \geq 0$.

The following Theorem 2.1 gives LIL type results when $\lambda > 0$ and stability results when $\lambda = 0$ with respect to a large class of normalizing sequences, without assuming that $\mathbb{E}Z^2 < \infty$, where Z is defined in (1.6).

Theorem 2.1. *Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with distribution function $F(x) = P(X \leq x), x \in \mathcal{R}$ and let $H(\cdot)$ be a real-valued Borel-measurable function defined on \mathcal{R} satisfying (1.5). Let $J(\cdot)$ be a Lipschitz function of order one defined on $[0, 1]$ and let $\mathbb{L}_n(F, J, H), n \geq 1, Z,$ and μ be defined by (1.4), (1.6), and (1.7), respectively. Given a function $\varphi(\cdot) \in \mathcal{H}_q$ where $0 \leq q \leq 1$, set $\Psi(x) = \sqrt{x\varphi(x)}, x \in \mathcal{R}$. If*

$$\lim_{x \rightarrow \infty} \frac{\log \log x}{\varphi(x)} = \infty \tag{2.1}$$

and

$$\mathbb{E}\Psi^{-1}(|Z|) < \infty \quad \text{and} \quad \lambda = \sqrt{2 \limsup_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} \mathbb{E}(Z^2 I\{|Z| \leq x\})}, \tag{2.2}$$

then when $\lambda < \infty$ we have

$$(1 - q)^{1/2} \lambda \leq \limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} |\mathbb{L}_n(F, J, H) - \mu| \leq \lambda \quad a.s. \quad (2.3)$$

Conversely, if $q < 1$, then the relation

$$\limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} |\mathbb{L}_n(F, J, H) - \mu| < \infty \quad a.s. \quad (2.4)$$

implies that (2.2) holds with $\lambda < \infty$.

Moreover, the limsup in (2.3) is positive and finite if and only if (2.2) holds with $0 < \lambda < \infty$.

For slowly varying functions $\varphi(\cdot) \in \mathcal{H}_0$ and λ as in (2.2) with $0 < \lambda < \infty$, we obtain for L-statistics $\{\mathbb{L}_n(F, J, H), n \geq 1\}$ the following complete analogue of the Hartman-Wintner-Strassen LIL. Of course, (2.5) follows from (2.6) but nevertheless it is worthwhile to label them separately.

Theorem 2.2. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on \mathcal{R} satisfying (1.5) and that $\varphi(\cdot) \in \mathcal{H}_0$ satisfies (2.1). Assume that $\{X, X_n; n \geq 1\}$, $J(\cdot)$, $\mathbb{L}_n(F, J, H)$, Z , μ , and $\Psi(\cdot)$ are as in Theorem 2.1. Let $0 \leq \lambda < \infty$. Then*

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \sqrt{n/\varphi(n)} (\mathbb{L}_n(F, J, H) - \mu) = \begin{matrix} + \\ - \end{matrix} \lambda \quad a.s. \quad (2.5)$$

and

$$C \left(\left\{ \sqrt{n/\varphi(n)} (\mathbb{L}_n(F, J, H) - \mu); n \geq 1 \right\} \right) = [-\lambda, \lambda] \quad a.s. \quad (2.6)$$

if and only if condition (2.2) holds.

Remark 2.1. Due to condition (2.1), Theorems 2.1 and 2.2 do not include as a special case the classical Hartman-Wintner-Strassen LIL for L-statistics obtained by Li, Rao, and Tomkins (2001, Theorem 2.1). It is interesting to note that, under the condition (2.1), the limiting behavior in Theorems 2.1 and 2.2 is determined by the distribution of Z , whereas, by contrast, the three conclusions (1.11), (1.12), and (1.13) depend on the distribution of Y , where Y is defined in (1.6).

Remark 2.2. In general, it is not easy to find $F^{\leftarrow}(t), 0 < t < 1$. However, if the distribution function $F(\cdot)$ of the random variable X is continuous, then

$$Z = J(U)H(F^{\leftarrow}(U)) \stackrel{d}{=} J(F(X))H(X)$$

where “ $\stackrel{d}{=}$ ” denotes “equal in distribution”.

Remark 2.3. We conjecture that Theorems 2.1 and 2.2 are still true without condition (1.5).

Remark 2.4. We note that if $\mathbb{E}Z^2 < \infty$, then the constant λ in Theorems 2.1 and 2.2 is simply

$$\lambda = \sqrt{2(\mathbb{E}Z^2) \limsup_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx}}.$$

We shall illustrate Theorem 2.2 by considering the following two special cases:

Case I. Choose $\varphi(x) = 2(LLx)^p$ where $p > 1$. Then one can check that

$$\lim_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{1/(2(LLx)^{p-1})} = 1.$$

Case II. Take $\varphi(x) = 2(Lx)^r$ where $r > 0$. Then one also easily sees that

$$\lim_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)/(x^2LLx)}{LLx/(Lx)^r} = 1.$$

Thus, Theorem 2.2 yields the following two results.

Corollary 2.1. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on \mathcal{R} satisfying (1.5). Let $p > 1$. For any constant $0 \leq \lambda < \infty$, we have:*

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \sqrt{\frac{n}{2(LLn)^p}} (\mathbb{L}_n(F, J, H) - \mu) = \binom{+}{-} \lambda \quad a.s.$$

and

$$C \left(\left\{ \sqrt{\frac{n}{2(LLn)^p}} (\mathbb{L}_n(F, J, H) - \mu); n \geq 1 \right\} \right) = [-\lambda, \lambda] \quad a.s.$$

if and only if

$$\mathbb{E} \left(\frac{Z^2}{(LL|Z|)^p} \right) < \infty \quad \text{and} \quad \lambda = \sqrt{\limsup_{x \rightarrow \infty} (LLx)^{1-p} \mathbb{E}(Z^2 I\{|Z| \leq x\})}.$$

Corollary 2.2. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on \mathcal{R} satisfying (1.5). Let $r > 0$. For any constant $0 \leq \lambda < \infty$, we have:*

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \sqrt{\frac{n}{2(Ln)^r}} (\mathbb{L}_n(F, J, H) - \mu) = \begin{matrix} + \\ - \end{matrix} \lambda \quad a.s.$$

and

$$C \left(\left\{ \sqrt{\frac{n}{2(Ln)^r}} (\mathbb{L}_n(F, J, H) - \mu); n \geq 1 \right\} \right) = [-\lambda, \lambda] \quad a.s.$$

if and only if

$$\mathbb{E} \left(\frac{Z^2}{(L|Z|)^r} \right) < \infty \quad \text{and} \quad \lambda = \sqrt{2^{-r} \limsup_{x \rightarrow \infty} \frac{LLx}{(Lx)^r} \mathbb{E}(Z^2 I\{|Z| \leq x\})}.$$

If condition (2.2) is satisfied with $\lambda = 0$, we obtain the following stability result for L-statistics.

Corollary 2.3. *Assume that $H(\cdot)$ is a real-valued Borel-measurable function defined on \mathcal{R} satisfying (1.5). Let $\varphi(\cdot) \in \mathcal{H}$ and let $\Psi(\cdot)$ be as in Theorem 2.1. If*

$$\mathbb{E}\Psi^{-1}(|Z|) < \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} \mathbb{E}(Z^2 I\{|Z| \leq x\}) = 0, \quad (2.7)$$

then

$$\lim_{n \rightarrow \infty} \sqrt{n/\varphi(n)} (\mathbb{L}_n(F, J, H) - \mu) = 0 \quad a.s. \quad (2.8)$$

Moreover, if $\varphi(\cdot) \in \mathcal{H}_q$ for some $0 \leq q < 1$, then condition (2.7) is necessary and sufficient for (2.8) to hold.

Proof of Theorems 2.1 and 2.2. Let $\{U, U_n; n \geq 1\}$ represent a sequence of i.i.d. random variables with the uniform $(0, 1)$ distribution. Then it is well known that

$$\{X, X_n; n \geq 1\} \stackrel{d}{=} \{F^{\leftarrow}(U), F^{\leftarrow}(U_n); n \geq 1\}.$$

It now follows that

$$\{X_{i:n}; 1 \leq i \leq n, n \geq 1\} \stackrel{d}{=} \{F^{\leftarrow}(U_{i:n}); 1 \leq i \leq n, n \geq 1\},$$

where $U_{i:n}, 1 \leq i \leq n$, are the order statistics of $U_i, 1 \leq i \leq n$. Thus, one may set without loss of generality $X_n = F^{\leftarrow}(U_n)$ and $X_{i:n} = F^{\leftarrow}(U_{i:n})$ for $1 \leq i \leq n$ and $n \geq 1$. Note that

$$P(U_i \neq U_j \text{ for all } 1 \leq i < j < \infty) = 1.$$

So we have that

$$\begin{aligned} & \sum_{i=1}^n J\left(\frac{i}{n}\right)H(X_{i:n}) \\ &= \sum_{i=1}^n J\left(\frac{i}{n}\right)H(F^{\leftarrow}(U_{i:n})) \\ &= \sum_{i=1}^n J(U_{i:n})H(F^{\leftarrow}(U_{i:n})) \\ & \quad + \sum_{i=1}^n \left(J\left(\frac{i}{n}\right) - J(U_{i:n})\right)H(F^{\leftarrow}(U_{i:n})) \tag{2.9} \\ &\stackrel{\text{a.s.}}{=} \sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i)) \\ & \quad + \sum_{i=1}^n (J(\mathbb{D}_n(U_{i:n})) - J(U_{i:n}))H(F^{\leftarrow}(U_{i:n})) \\ &= \sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i)) + R_n \text{ (say), } n \geq 1 \end{aligned}$$

where $\mathbb{D}_n(t) \equiv n^{-1} \sum_{i=1}^n I\{U_i \leq t\}$ is the empirical distribution function of U_1, U_2, \dots, U_n . Since $J(\cdot)$ is a Lipschitz function of order one defined on

$[0, 1]$, there exists a constant $0 \leq C < \infty$, depending on $J(\cdot)$ only, such that $|J(t_1) - J(t_2)| \leq C|t_1 - t_2|$ uniformly for $t_1, t_2 \in [0, 1]$. Hence

$$\begin{aligned} |R_n| &\leq C \max_{1 \leq i \leq n} |\mathbb{D}_n(U_{i:n}) - U_{i:n}| \sum_{i=1}^n |H(F^{\leftarrow}(U_{i:n}))| \\ &\leq C \sup_{0 \leq t \leq 1} |\mathbb{D}_n(t) - t| \sum_{i=1}^n |H(F^{\leftarrow}(U_i))|, \quad n \geq 1. \end{aligned}$$

Since (1.5) holds, i.e., $\mathbb{E}|H(F^{\leftarrow}(U))| = \mathbb{E}|H(X)| < \infty$, by the Kolmogorov SLLN, we have

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n |H(F^{\leftarrow}(U_i))|}{n} = \mathbb{E}|H(X)| \quad \text{a.s.}$$

Note that the Chung-Smirnov LIL for empirical processes (see Chung (1949) and Smirnov (1944)) states that

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{n}{2LLn}} \sup_{0 \leq t \leq 1} |\mathbb{D}_n(t) - t| = \frac{1}{2} \quad \text{a.s.}$$

Thus, for any given $\varphi(\cdot) \in \mathcal{H}$ satisfying (2.1), since

$$\frac{|R_n|}{\sqrt{n\varphi(n)}} \leq C \sqrt{\frac{n}{2LLn}} \sup_{0 \leq t \leq 1} |\mathbb{D}_n(t) - t| \times \sqrt{\frac{2LLn}{\varphi(n)} \frac{\sum_{i=1}^n |H(F^{\leftarrow}(U_i))|}{n}}, \quad n \geq 1,$$

we have

$$\lim_{n \rightarrow \infty} \frac{R_n}{\sqrt{n\varphi(n)}} = 0 \quad \text{a.s.} \tag{2.10}$$

It then follows from (2.9) and (2.10) that, for any given $\varphi(\cdot) \in \mathcal{H}$ satisfying (2.1),

$$\lim_{n \rightarrow \infty} \sqrt{n/\varphi(n)} \left(\mathbb{L}_n(F, J, H) - n^{-1} \sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i)) \right) = 0 \quad \text{a.s.} \tag{2.11}$$

It is easy to see that $\varphi(\cdot) \in \mathcal{H}_q$ and (2.1) imply that $\frac{\Psi^{-1}(xLLx)}{x^2LLx}$ is slowly varying at infinity with

$$\lim_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} = 0.$$

Thus (2.2) with $\lambda < \infty$ is equivalent to

$$\mathbb{E}\Psi^{-1}(|V|) < \infty \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} \mathbb{E}(V^2 I\{|Z| \leq x\}) = \frac{\lambda^2}{2} \quad (2.12)$$

where $V = J(U)H(F^{\leftarrow}(U)) - \mu = Z - \mu$. Then, by Theorem 1 of Einmahl and Li (2005), (2.12) implies that

$$(1 - q)^{1/2} \lambda \leq \limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} \left| \frac{\sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i))}{n} - \mu \right| \leq \lambda \quad \text{a.s.} \quad (2.13)$$

and (2.3) follows from (2.11) and (2.13).

Conversely, by Theorem 1 of Einmahl and Li (2005), if $0 \leq q < 1$, then the relation

$$\limsup_{n \rightarrow \infty} \sqrt{n/\varphi(n)} \left| \frac{\sum_{i=1}^n J(U_i)H(F^{\leftarrow}(U_i))}{n} - \mu \right| < \infty \quad \text{a.s.} \quad (2.14)$$

implies that (2.12) holds with $\lambda < \infty$ and, moreover, the lim sup in (2.14) is positive if and only if (2.12) holds with $0 < \lambda < \infty$. Thus combining (2.4) and (2.11) yields (2.14) and hence (2.12) holds with $\lambda < \infty$. As was noted above, (2.2) with $\lambda < \infty$ is equivalent to (2.12), and the last assertion in Theorem 2.1 is now immediate.

Similarly, if $\varphi(\cdot) \in \mathcal{H}_0$ and (2.1) holds then by combining Theorem 2 of Einmahl and Li (2005) and (2.11), the proof of Theorem 2.2 follows. \square

3. An Interesting Example

In this section, we shall provide an example to illustrate our results.

Example. Take $J(t) = 4t - 2$, $0 \leq t \leq 1$, and $H(x) = x$, $x \in \mathcal{R}$. Then the L-statistic

$$\mathbb{L}_n(F, J, H) = \frac{1}{n} \sum_{i=1}^n \left(4 \cdot \frac{i}{n} - 2 \right) X_{i:n}$$

is related to Gini’s mean difference,

$$\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| = \frac{1}{n} \sum_{i=1}^n \left(4 \cdot \frac{i-1}{n-1} - 2 \right) X_{i:n},$$

which is a well-known U -statistic for unbiased estimation of the dispersion parameter

$$\theta = E(|X_1 - X_2|);$$

see, e.g., Serfling (1980, p. 263) or Shorack and Wellner (1986, p. 676). Li, Rao, and Tomkins (2001, Theorem 3.3) established analogues of the classical SLLN, LIL, and central limit theorem for Gini’s mean difference. Given a function $\varphi(\cdot) \in \mathcal{H}_q$ satisfying (2.1) where $0 \leq q \leq 1$, let $\Psi(\cdot)$ be as in Theorem 2.1. Then it is easy to check that, for any constant $0 \leq \lambda < \infty$, (2.2) holding with $Z = (4U - 2)F^{\leftarrow}(U)$ is equivalent to

$$\mathbb{E}\Psi^{-1}(|X|) < \infty \quad \text{and} \quad \limsup_{x \rightarrow \infty} \frac{\Psi^{-1}(xLLx)}{x^2LLx} \mathbb{E}(Z^2 I\{|Z| \leq x\}) = \frac{\lambda^2}{2}. \quad (3.1)$$

Note that $\mu = \mathbb{E}Z = \mathbb{E}|X_1 - X_2| = \theta$ and that

$$\limsup_{n \rightarrow \infty} \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| < \infty \quad \text{a.s.}$$

if and only if

$$\mathbb{E}|X| < \infty;$$

see Li, Rao, and Tomkins (2001, Theorem 3.3(i)). So, under (3.1), we have

$$(1-q)^{1/2}\lambda \leq \limsup_{n \rightarrow \infty} \sqrt{\frac{n}{\varphi(n)}} \left| \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| - \theta \right| \leq \lambda \quad \text{a.s.}$$

Conversely, if $q < 1$, then the relation

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{n}{\varphi(n)}} \left| \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| - \theta \right| < \infty \quad \text{a.s.} \quad (3.2)$$

implies that (3.1) holds with $\lambda < \infty$. Moreover, the lim sup in (3.2) is

positive if and only if (3.1) holds with $0 < \lambda < \infty$. If $q = 0$, then for any constant $0 \leq \lambda < \infty$,

$$\limsup_{n \rightarrow \infty} (\liminf_{n \rightarrow \infty}) \sqrt{\frac{n}{\varphi(n)}} \left(\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| - \theta \right) = \binom{+}{-} \lambda \text{ a.s.}$$

and

$$C \left(\left\{ \sqrt{\frac{n}{\varphi(n)}} \left(\frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |X_i - X_j| - \theta \right); n \geq 1 \right\} \right) = [-\lambda, \lambda] \text{ a.s.}$$

if and only if condition (3.1) holds.

Acknowledgments

The research of the first author was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

References

1. A. de Acosta, A new proof of the Hartman-Wintner law of the iterated logarithm, *Ann. Probab.*, **11**(1983), 270-276.
2. K. L. Chung, An estimate concerning the Kolmogoroff limit distribution, *Trans. Amer. Math. Soc.*, **67**(1949), 36-50.
3. M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistics*, Academic Press, New York, 1981.
4. V. A. Egorov, A generalization of the Hartman-Wintner theorem on the law of the iterated logarithm, *Vestnik Leningrad. Univ.*, **1971** (1971), 22-28 (in Russian). English translation in *Vestnik Leningrad Univ. Math.*, **4** (1977), 117-124.
5. U. Einmahl and D. Li, Some results on two-sided LIL behavior, *Ann. Probab.*, **33**(2005), 1601-1624.
6. W. Feller, An extension of the law of the iterated logarithm to variables without variance, *J. Math. Mech.*, **18**(1968), 343-356.
7. H. Finkelstein, The law of the iterated logarithm for empirical distributions, *Ann. Math. Statist.*, **42**(1971), 607-615.
8. P. Hartman and A. Wintner, On the law of the iterated logarithm, *Amer. J. Math.*, **63**(1941), 169-176.

9. R. Helmers, A strong law of large numbers for linear combinations of order statistics, Report SW50/77 (1977), Mathematisch Centrum, Amsterdam, The Netherlands.
10. R. Helmers, P. Janssen, and R. Serfling, Glivenko-Cantelli properties of some generalized empirical DF's and strong convergence of generalized L-statistics, *Probab. Theory Related Fields*, **79** (1988), 75-93.
11. C. C. Heyde, On the converse to the iterated logarithm law, *J. Appl. Probab.*, **5**(1968), 210-215.
12. C. C. Heyde, *Some properties of metrics in a study on convergence to normality*, *Z. Wahrsch. Verw. Gebiete*, **11**(1969), 181-192.
13. D. L. Li, A simple proof of the Hartman-Wintner law of the iterated logarithm, *Acta Sci. Natur. Univ. Jilin*, **1987**(1987), 43-45 (in Chinese).
14. D. Li, M. B. Rao, and R. J. Tomkins, The law of the iterated logarithm and central limit theorem for L-statistics, *J. Multivariate Anal.*, **78**(2001), 191-217.
15. A. I. Martikainen, A converse to the law of the iterated logarithm for a random walk, *Teor. Veroyatnost. i Primenen.*, **25**(1980), 364-366 (in Russian). English translation in *Theory Probab. Appl.*, **25** (1981), 361-362.
16. A. I. Martikainen, Criteria for strong convergence of normalized sums of independent random variables and their applications, *Teor. Veroyatnost. i Primenen.*, **29** (1984), 502-516 (in Russian). English translation in *Theory Probab. Appl.*, **29**(1985), 519-533.
17. D. M. Mason, Some characterizations of strong laws for linear functions of order statistics, *Ann. Probab.*, **10**(1982), 1051-1057.
18. W. E. Pruitt, *General one-sided laws of the iterated logarithm*, *Ann. Probab.*, **9** (1981), 1-48.
19. A. Rosalsky, On the converse to the iterated logarithm law, *Sankhyā Ser. A*, **42**(1980), 103-108.
20. P. K. Sen, An invariance principle for linear combinations of order statistics, *Z. Wahrsch. Verw. Gebiete*, **42** (1978), 327-340.
21. R. J. Serfling, *Approximation Theorems of Mathematical Statistics*, John Wiley, New York, 1980.
22. G. R. Shorack and J. A. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, New York, 1986.
23. N. V. Smirnov, An approximation to distribution laws of random quantities determined by empirical data, *Uspehi Matem. Nauk.*, **10**(1944), 179-206 (in Russian).
24. W. L. Steiger and S. K. Zaremba, *The converse of the Hartman-Wintner theorem*, *Z. Wahrsch. Verw. Gebiete*, **22** (1972), 193-194.
25. V. Strassen, An invariance principle for the law of the iterated logarithm, *Z. Wahrsch. Verw. Gebiete*, **3**(1964), 211-226.
26. V. Strassen, A converse to the law of the iterated logarithm, *Z. Wahrsch. Verw. Gebiete*, **4**(1966), 265-268.

27. H. Teicher, On the law of the iterated logarithm, *Ann. Probab.*, **2** (1974), 714-728.
28. W. R. van Zwet, A strong law for linear functions of order statistics, *Ann. Probab.*, **8** (1980), 986-990.
29. J. A. Wellner, A Glivenko-Cantelli theorem and strong laws of large numbers for functions of order statistics, *Ann. Statist.*, **5** (1977a), 473-480.
30. J. A. Wellner, A law of the iterated logarithm for functions of order statistics, *Ann. Statist.*, **5** (1977b), 481-494.

Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario, P7B 5E1, Canada.

E-mail: dli@lakeheadu.ca

Department of Mathematical Sciences, Lakehead University, Thunder Bay, Ontario, P7B 5E1, Canada.

Department of Statistics, University of Florida, Gainesville, FL 32611, U.S.A.

E-mail: rosalsky@stat.ufl.edu