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FACTORS FOR $|\bar{N}, p_n, \theta_n|_k$ SUMMABILITY OF FOURIER SERIES

BY

HÜSEYİN BOR

Abstract

In the present paper, the author presents a generalization of some known results on the $|\bar{N}, p_n|_k$ summability factors for the $|\bar{N}, p_n, \theta_n|_k$ summability factors. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n the n-th $(C,1)$ mean of the sequence (na_n). A series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [4, 6])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

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defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|\Delta\sigma_{n-1}|^k < \infty, \tag{4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n \geq 1. \tag{5}$$

In the special case $p_n = 1$ for all values of n $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

Let (θ_n) be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n, \theta_n|_k, k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \theta_n^{k-1}|\Delta\sigma_{n-1}|^k < \infty. \tag{6}$$

In the special case if we take $\theta_n = \frac{P_n}{p_n}$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. Also if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C, 1|_k$ summability. Furthermore if we take $\theta_n = n$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ (see [3]) summability.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\ dt = 0 \tag{7}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n\ \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{8}$$

We write

$$\varphi(t) = \frac{1}{2}\left\{f(x+t) + f(x-t)\right\}, \quad \varphi_1(t) = \frac{1}{2}\int_0^t \varphi(u)\ du.$$

2. Known Results

In [7] Mishra has proved two theorems for $|\bar{N}, p_n|$ summability factors. Later on, Bor [2] has generalized these theorems for $|\bar{N}, p_n|_k$ summability factors in the following forms.

Theorem A. *Let (p_n) be a sequence such that*

$$P_n = O(np_n) \tag{9}$$

$$P_n \Delta p_n = O(p_n\, p_{n+1}). \tag{10}$$

If $\varphi_1(t)$ is of bounded variation in $(0, \pi)$ and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} \frac{1}{n} |\lambda_n|^k < \infty \tag{11}$$

and

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| < \infty, \tag{12}$$

then the series $\sum A_n(t) \frac{P_n \lambda_n}{np_n}$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem B. *If the sequences (p_n) and (λ_n) satisfy the conditions* (9)$-$(12) *of Theorem* A *and*

$$B_n \equiv \sum_{v=1}^{n} v a_v = O(n), \tag{13}$$

then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. Main results.

The aim of this paper is to generalize Theorem A and Theorem B for $|\bar{N}, p_n, \theta_n|_k$ summability methods.

Now we shall prove the following theorems.

Theorem 1. *Let $(\frac{\theta_n p_n}{P_n})$ be a non-increasing sequence. If all conditions* *of Theorem* A *are satisfied with the condition* (11) *replaced by;*

$$\sum_{n=1}^{\infty} \theta_n^{k-1} n^{-k} |\lambda_n|^k < \infty, \tag{14}$$

then the series $\sum A_n(t) \frac{P_n \lambda_n}{n p_n}$ *is summable* $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$.

Theorem 2. *If the conditions* (9)$-$(10) *and* (12)$-$(14) *are satisfied and* $(\frac{\theta_n p_n}{P_n})$ *is a non-increasing sequence, then the series* $\sum a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$.

Remark. It should be noted that if we take $\theta_n = \frac{P_n}{p_n}$ in Theorem 1 and Theorem 2, then we get Theorem A and Theorem B, respectively. In this case the condition $(\frac{\theta_n p_n}{P_n})$ which is a non-increasing sequence is automatically satisfied and condition (14) reduces to condition (11).

We need the following lemmas for the proof of our Theorems.

Lemma 1([7]). *If* $\varphi_1(t)$ *is of bounded variation in* $(0, \pi)$, *then*

$$\sum v A_v(x) = O(n) \quad as \quad n \to \infty.$$

Lemma 2([2]). *If the sequence* (p_n) *such that conditions* (9) *and* (10) *of Theorem* A *are satisfied, then*

$$\Delta \Big\{ \frac{P_n}{p_n n^2} \Big\} = O\Big(\frac{1}{n^2}\Big).$$

4. Proof of Theorem 2.

Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\sum a_n P_n \lambda_n (n p_n)^{-1}$. Then, by definition, we have

$$\begin{aligned} T_n &= \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r P_r \lambda_r (r p_r)^{-1} \\ &= \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v P_v \lambda_v (v p_v)^{-1}. \end{aligned}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v P_v \lambda_v (v p_v)^{-1}.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= B_n \ \lambda_n n^{-2} - p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} p_v P_v B_v \lambda_v (v^2 p_v)^{-1} \\ &\quad + p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v B_v (v^2 p_v)^{-1} \\ &\quad + p_n (P_n P_{n-1})^{-1} \sum_{v=1}^{n-1} P_v B_v \lambda_{v+1} \Delta \Big\{ \frac{P_v}{v^2 p_v} \Big\} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

Firstly, we have that

$$\begin{aligned} \sum_{n=1}^{m} \theta_n^{k-1} |T_{n,1}|^k &= \sum_{n=1}^{m} \theta_n^{k-1} |\lambda_n|^k |B_n|^k n^{-2k} \\ &= O(1) \sum_{n=1}^{m} \theta_n^{k-1} |\lambda_n|^k n^k \ n^{-2k} \\ &= O(1) \sum_{n=1}^{m} \theta_n^{k-1} n^{-k} |\lambda_n|^k \\ &= O(1) \quad \text{as} \quad m \to \infty, \end{aligned}$$

by (13) and (14).

Now, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \Big(\frac{p_n}{P_n} \Big)^k \frac{1}{P_{n-1}^k} \Big| \sum_{v=1}^{n-1} \frac{p_v P_v B_v \lambda_v}{v^2 p_v} \Big|^k$$

$$
\begin{aligned}
&\leq \sum_{n=2}^{m+1} \theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \Big\{\frac{P_v|B_v||\lambda_v|}{v^2 p_v}\Big\}^k \\
&\quad \times\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\Big\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} p_v|\lambda_v|^k\Big\{\frac{P_v}{p_v}\Big\}^k v^k v^{-2k} \sum_{n=v+1}^{m+1}\Big(\frac{\theta_n p_n}{P_n}\Big)^{k-1}\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} p_v|\lambda_v|^k\Big\{\frac{P_v}{p_v}\Big\}^k v^{-k}\Big(\frac{\theta_v p_v}{P_v}\Big)^{k-1} \sum_{n=v+1}^{m+1}\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} |\lambda_v|^k\Big\{\frac{P_v}{p_v}\Big\}^{k-1} v^{-k}\Big(\frac{\theta_v p_v}{P_v}\Big)^{k-1} \\
&= O(1)\sum_{v=1}^{m} \theta_v^{k-1} v^{-k}|\lambda_v|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by (13) and (14). On the other hand, since

$$
\sum_{v=1}^{n-1} P_v|\Delta\lambda_v| \leq P_{n-1}\sum_{v=1}^{n-1}|\Delta\lambda_v| \Rightarrow \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v|\Delta\lambda_v| \leq \sum_{v=1}^{n-1}|\Delta\lambda_v| = O(1),
$$

by (12), we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,3}|^k &\leq \sum_{n=2}^{m+1} \theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k \frac{1}{P_{n-1}^k}\Big|\sum_{v=1}^{n-1}\frac{P_v P_v B_v \Delta\lambda_v}{v^2 p_v}\Big|^k \\
&\leq \sum_{n=2}^{m+1} \theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k \frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v\Delta\lambda_v\Big\{\frac{P_v|B_v|}{v^2 p_v}\Big\}^k \\
&\quad \times\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v|\Delta\lambda_v|\Big\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}|B_v|^k v^{-2k}\Big\{\frac{P_v}{p_v}\Big\}^k P_v|\Delta\lambda_v| \sum_{n=v+1}^{m+1}\Big(\frac{\theta_n p_n}{P_n}\Big)^{k-1} \\
&\quad \times\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} v^k v^{-2k} v^k|\Delta\lambda_v|\Big(\frac{\theta_v p_v}{P_v}\Big)^{k-1}
\end{aligned}
$$

$$\begin{aligned}
&= O(1)\Big(\frac{\theta_1 p_1}{P_1}\Big)^{k-1}\sum_{v=1}^{m}|\Delta\lambda_v| \\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v| = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

in view of (9), (12) and (13).

Finally, using the fact that $\Delta\{\frac{P_v}{v^2 p_v}\} = O(\frac{1}{v^2})$ by Lemma 2, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}|T_{n,4}|^k &\le \sum_{n=2}^{m+1}\theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k\frac{1}{P_{n-1}^k}\Big\{\sum_{v=1}^{n-1}P_v|B_v|\lambda_{v+1}\Delta\Big\{\frac{P_v}{v^2 p_v}\Big\}\Big\}^k \\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k\frac{1}{P_{n-1}^k}\Big\{\sum_{v=1}^{n-1}p_v\frac{P_v}{v^2 p_v}|B_v||\lambda_{v+1}|\Big\}^k \\
&= O(1)\sum_{n=2}^{m-1}\theta_n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\Big(\frac{P_v}{p_v}\Big)^k p_v|\lambda_{v+1}|^k v^{-2k} \\
&\quad \times|B_v|^k\Big\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\Big\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^k p_v|\lambda_{v+1}|^k v^{-2k}v^k\sum_{n=v+1}^{m+1}\Big(\frac{\theta_n p_n}{P_n}\Big)^{k-1} \\
&\quad \times\frac{p_n}{P_n P_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^{k-1}v^{-k}|\lambda_{v+1}|^k\Big(\frac{\theta_v p_v}{P_v}\Big)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\theta_v^{k-1}v^{-k}|\lambda_{v+1}|^k = O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}$$

by (13) and (14). Therefore, we get that

$$\sum_{n=1}^{m}\theta_n^{k-1}|T_{n,r}|^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r=1,2,3,4.$$

This completes the proof of Theorem 2.

Proof of Theorem 1. Theorem 1 is a direct consequence of Theorem 2 and Lemma 1. If we take $p_n = 1$ and $\theta_n = n$ in Theorem 1 and Theorem

2, then we get the following corollaries. It should be noted that, in this case condidtion (14) reduces to condition (11).

Corollary 1. *If* $\varphi_1(t)$ *is of bounded variation in* $(0,\pi)$ *and* (λ_n) *is a sequence such that conditions* (11) *and* (12) *are satisfied, then the series* $\sum A_n(t)\lambda_n$*, at* $t = x$ *is summable* $|C,1|_k$*,* $k \geq 1$*.*

Corollary 2. *If the conditions* (11)$-$(13) *are satisfied, then the series* $\sum a_n\lambda_n$ *is summable* $|C,1|_k$*,* $k \geq 1$*.*

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Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey.

E-mail: bor@erciyes.edu.tr, hbor33@gmail.com