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DIFFUSION UNDER GRAVITATIONAL AND BOUNDARY EFFECTS

BY

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1. Introduction

Consider the initial-boundary value problem for the Boltzmann equation

$$\begin{cases} \partial_t \mathsf{f} + \xi \cdot \nabla_{\vec{\boldsymbol{x}}} \mathsf{f} - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{f} = Q(\mathsf{f}),\ \vec{\boldsymbol{x}} \equiv (x,y,z),\ (\vec{\boldsymbol{x}}, t, \xi) \in \mathbb{H}^+ \times \mathbb{R}^+ \times \mathbb{R}^3, \\ \left.\begin{cases} \mathsf{f}(x,y,0,t,\xi)|_{\xi^3>0} = \rho(x,y,t)\frac{\sqrt{2\pi}}{\sqrt{\theta}}\mathsf{M}(\xi)|_{\xi^3>0}, \\ \rho(x,y,t) = \displaystyle\int_{\substack{\xi\in\mathbb{R}^3 \\ \xi^3<0,}} -\xi^3 \mathsf{f}(x,y,0,\xi)d\xi, \end{cases}\right\} \begin{matrix}\text{(Maxwell diffuse} \\ \text{boundary condition)}\end{matrix} \\ 0 \le \mathsf{f}(\vec{\boldsymbol{x}},0,\xi) \le \epsilon e^{-\sqrt{x^2+y^2}} e^{-\frac{gz}{\theta}} \mathsf{M}(\xi) \text{ for some small } \epsilon > 0, \end{cases} \tag{1.1}$$

where

$$\begin{cases} \mathbb{H}^+ \equiv \{(x,y,z) \in \mathbb{R}^3 | z \ge 0\}, \\ \mathbb{R}^+ \equiv \{t \in \mathbb{R} | t \ge 0\}, \\ \mathsf{M}(\xi) \equiv \dfrac{e^{-\frac{|\xi|^2}{2\theta}}}{(2\pi\theta)^{3/2}},\ \text{(a global Maxwellian distribution in velocities} \\ \qquad\qquad \xi \in \mathbb{R}^3 \text{ with boundary temperature } \theta > 0), \\ \vec{\boldsymbol{g}} \equiv (0,0,g),\ g > 0 \text{ (the constant gravitational force).} \end{cases}$$

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This models the propagation of gas of finite total mass in the upper half space $(x, y, z),\ z > 0$ around the solid at $(x, y, 0)$. We study this phenomena on the level of Boltzmann equation in the kinetic theory. The boundary condition is the classical diffuse reflection condition: The gas is reflected off the solid with the Maxwellian distribution of the given boundary temperature and satisfying the conservation of mass. The gravitational force pulls the gas back to the solid surface. The main purpose of the present paper is to study the diffusion of the gas under these two combined effects. We show that the propagation of the gas toward vacuum is governed by a process similar to the two-dimensional surface heat diffusion, Main Theorem A and Main Theorem B.

For gas near vacuum, thermodynamics, in particular fluid equations such as the Euler and Navier-Stokes equations do not apply, see Sone [13] and Aoki [1]. In our study based on the kinetic theory, there are the following two main analytical points. The first is that the diffusion reflection boundary condition is modeled by a stochastic process. This approach, initiated by the second author, [14], is physically natural as the diffuse reflection boundary condition and the Boltzmann equation were conceived with the probabilistic thinking. It is only through the analytical setting in [14] that the limiting theory, such as the central limit theorem and the law of large number, can be used to bear on the quantitative estimate of the effect of the boundary. In the present paper, we encounter stochastic process with random variables not independent. This causes some analytical difficulties, see (1.6) and Lemma 3.9. The second main analytical point is that we do not, and cannot, assume the gravitational force as a perturbation with small total effect. Instead, the gravitational force is treated as a main part; the other main part is the boundary condition. Instead, because we are interested in the behavior near vacuum, the collision term is viewed as a perturbation. Thus we will carry out the main analytical steps for the free transport equation with gravitational force and the thermal diffuse reflection boundary condition,

$$\begin{cases} \partial_t \mathsf{t} + \xi \cdot \mathsf{t} - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{t} = 0, \\ \begin{cases} \mathsf{t}(x, y, 0, t, \xi)|_{\xi^3>0} = \rho(x, y, t)\frac{\sqrt{2\pi}}{\sqrt{\theta}}\mathsf{M}(\xi)|_{\xi^3>0}, \\ \rho(x, y, t) = \displaystyle\int_{\substack{\xi\in\mathbb{R}^3 \\ \xi^3<0,}} -\xi^3 \mathsf{t}(x, y, 0, \xi)d\xi, \end{cases} \\ |\mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi)| \le e^{-|x|-|y|}e^{-\frac{gz}{\theta}}\mathsf{M}(\xi). \end{cases} \tag{1.2}$$

With the pointwise structure of $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$, we then build and justify the ansatz for the full nonlinear Boltzmann equation. The new tool to obtain the pointwise structure of $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ is a probability theory to represent the solution of $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$. The probabilistic representation generalizes that of [14] to take into account of the gravitational force. The stochastic processes in [14] are generated in a bounded gas region by the infinitely many consecutive particle collisions with the diffuse boundary. One views the reflected velocity as a random variable. Similarly, in this paper we have a probabilistic representation of the solution $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ due to the gravitational force, which pulls particle towards the surface continuously to generate infinitely many consecutively collisions with the diffuse surface of a planet. These consecutive collisions with the random reflected velocities generate the stochastic processes $\{\vec{\boldsymbol{x}}_t\}_{t\in\mathbb{R}^+}$ and $\{\vec{\boldsymbol{v}}_t\}_{t\in\mathbb{R}^+}$. We also denote the stochastic processes by their components

$$\begin{cases}\vec{\boldsymbol{x}}_t \equiv (x_t, y_t, z_t),\\ \vec{\boldsymbol{v}}_t \equiv (v_t^1, v_t^2, v_t^3).\end{cases}$$

The probability space $(\Omega, \boldsymbol{P}_0)$ for the stochastic processes $\{\vec{\boldsymbol{x}}_t\}_{t\in\mathbb{R}^+}$ and $\{\vec{\boldsymbol{v}}_t\}_{t\in\mathbb{R}^+}$ is defined as follows

$$\begin{cases}\omega \equiv \{\vec{\boldsymbol{V}}_j\}_{j\in\mathbb{N}} \in \Omega \equiv \displaystyle\prod_{j=1}^{\infty}(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+),\ \vec{\boldsymbol{V}}_j \equiv (V_j^1, V_j^2, V_j^3),\\ \boldsymbol{P}_0(\vec{\boldsymbol{V}}_{n_k}\in A_k,\ 1\le k\le l) = \displaystyle\prod_{k=1}^{l}\int_{A_k}\mathsf{G}(v^1)\mathsf{G}(v^2)\mathsf{H}(v^3)dv^1dv^2dv^3,\end{cases}$$

where

$$\mathsf{G}(v)\equiv\frac{e^{-\frac{v^2}{2\theta}}}{\sqrt{2\pi\theta}},\quad \mathsf{H}(v)\equiv\begin{cases}\dfrac{ve^{-\frac{v^2}{2\theta}}}{\theta} \text{ for } v\ge 0,\\ 0 \text{ else.}\end{cases}$$

For any $\omega\equiv\{\vec{\boldsymbol{V}}_j\}_{j\in\mathbb{N}}\in\Omega$ the stochastic processes $\{\vec{\boldsymbol{x}}_t(\vec{\boldsymbol{x}}_0,\xi_0,\omega)\}_{t\in\mathbb{R}^+}$ and $\{\vec{\boldsymbol{v}}_t(\vec{\boldsymbol{x}}_0,\xi_0,\omega)\}_{t\in\mathbb{R}^+}$ can be realized as a random particle with a trajectory $\vec{\boldsymbol{x}} = \vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = (X^1(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega), X^2(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega), X^3(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega))$, $(\vec{\boldsymbol{x}}_t(\vec{\boldsymbol{x}}_0,\xi_0,\omega)\equiv\vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega),\ \vec{\boldsymbol{v}}_t(\vec{\boldsymbol{x}}_0,\xi_0,\omega)\equiv\dot{\vec{X}}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega))$, with the law of motion in the free space by

$$\begin{cases} \ddot{\vec{X}}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = -\vec{\boldsymbol{g}} \text{ when } X^3(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega) > 0, \\ \vec{X}(0;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \vec{\boldsymbol{x}}_0, \\ \dot{\vec{X}}(0;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \xi_0, \end{cases}$$

with $\{\vec{\boldsymbol{V}}_j\}_{j\in\mathbb{N}}$ the sequence of random reflective velocities at the boundary:

$$\begin{cases} X^3(\tau_n;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = 0, \\ \dot{\vec{X}}(\tau_n+;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \vec{\boldsymbol{V}}_{n+1}, \\ \tau_0 < \tau_1 < \tau_2 < \cdots, \end{cases}$$

where $\{\tau_n\}_{n\geq 0}$ is the sequence of collision time at the diffuse boundary. From this sequence of random reflected velocities one has a sequence of random collision times $\{\tau_j\}_{j\in\mathbb{N}}$, i.e.,

$$\begin{cases} z_{\tau_j} = 0 \text{ for } j \in \{0\}\cup\mathbb{N}, \\ 0 < \tau_1 < \tau_2 < \cdots < \tau_j < \cdots, \\ \tau_j - \tau_{j-1} \equiv 2V_j^3/g \text{ for } j \geq 1, \end{cases}$$

where $\tau_0 = (\sqrt{2gz_0 + |\xi_0^3|^2} + \xi_0^3)/g$ is the initial deterministic collision time.

In Corollary 3.4 one construct a probability measure $\boldsymbol{P}$ from the probability measure $\boldsymbol{P}_0$, the stochastic processes $\{\vec{\boldsymbol{x}}_t,\ \vec{\boldsymbol{v}}_t\}_{t\in\mathbb{R}^+}$, and the non negative-valued initial data $\mathtt{t}(\vec{\boldsymbol{x}},0,\xi)$ to represent $\mathtt{t}(\vec{\boldsymbol{x}},t,\xi)$ if $\int_{\mathbb{H}^+\times\mathbb{R}^3} \mathtt{t}(\vec{\boldsymbol{x}},0,\xi)\, d\xi d\vec{\boldsymbol{x}} = 1$. The probability representation of $\mathtt{t}(\vec{\boldsymbol{x}},t,\xi)$ is

$$\mathtt{t}(\vec{\boldsymbol{x}},t,\xi)d\vec{\boldsymbol{x}}d\xi = \boldsymbol{P}\left(\vec{\boldsymbol{x}}_t \in \vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}_t \in \xi + d\xi\right),$$

and the probabilistic representation of the boundary flux function $\rho(x,y,t)$ is

$$\rho(x,y,t)dxdydt = \sum_{j=0}^{\infty} \boldsymbol{P}\left((x_{\tau_n}, y_{\tau_n}) \in (x,x+dx)\times(y,y+dy), \tau_n \in [t,t+dt]\right). \tag{1.3}$$

The stochastic process $\{(x_{\tau_n}, y_{\tau_n})\}_{n\in\mathbb{N}}$ generates a 2-dimensional diffuse phenomenon whose wave propagation structure is closely related to heat diffuse in 2-D but its physical generating mechanism is irrelevant to the thermal diffusion for the gas dynamics. The stochastic process $\{\vec{\boldsymbol{x}}_{\tau_n}\}_{n\in\mathbb{N}}$ is a Markovian process. The increment $\{\vec{\boldsymbol{x}}_{\tau_n} - \vec{\boldsymbol{x}}_{\tau_{n-1}}\}_{n\geq 0}$ is an i.i.d.. This process is related to the random reflected velocity $\{\vec{\boldsymbol{V}}_n\}_{n\in\mathbb{N}}$ as follows: The three components $\{V_n^1\}_{n\in\mathbb{N}}$, $\{V_n^2\}_{n\in\mathbb{N}}$, and $\{V_n^3\}_{n\in\mathbb{N}}$ are three i.i.d. with distribution functions G , G, and H. The time lapse σ_n and the horizontal displacement $\mathsf{d}_n = (x_{\tau_n} - x_{\tau_{n-1}}, y_{\tau_n} - y_{\tau_{n-1}})$ between two consecutive collisions under the constant gravitational field $-\vec{\boldsymbol{g}}$ is

$$\begin{cases} \sigma_n \equiv \tau_n - \tau_{n-1} = 2V_n^3/g, \\ \mathsf{d}_n \equiv (x_{\tau_n} - x_{\tau_{n-1}}, y_{\tau_n} - y_{\tau_{n-1}}) = (\sigma_n V_n^1, \sigma_n V_n^2). \end{cases} \tag{1.4}$$

By a strong central limit theorem, (which can be obtained through Fourier transformation and complex contour integral), one can relate the time lapse

$$\mathsf{T}_n \equiv \sum_{j=1}^{n} \sigma_j \text{ (random time for first consecutive n-collision)}$$

and the horizontal displacement

$$\mathsf{D}_n \equiv \sum_{j=1}^{n} \mathsf{d}_j \text{ (random walk for first consecutive n-collision)}$$

to the number n of collisions as follows:

$$\begin{cases} \boldsymbol{P}\left(T_n \in (t, t+dt)\right) \leq C\left(\dfrac{e^{-\frac{|t-n\mathbb{E}[\sigma_1]|^2}{C\mathrm{Var}[\sigma_1]}}}{\sqrt{n}} + e^{-\frac{|t-n\mathbb{E}[\sigma_1]|+t}{C}}\right) dt, \\ \boldsymbol{P}\left(D_n \in (x, x+dx)\times(y, y+dy)\right) \leq C\left(\dfrac{e^{-\frac{x^2+y^2}{C\mathrm{Var}[\sigma_1]}}}{n} + e^{-\frac{|x|+|y|+n}{C}}\right) dxdy \end{cases} \tag{1.5}$$

for some constant $C > 0$. From this, we see that the decaying rates of $\boldsymbol{P}(T_n \in (t, t+dt))$ and $\boldsymbol{P}(\mathsf{D}_n \in (x, x+dx) \times (y, y+dy))$ are exponentially fast at infinity. This yields the weak, exponentially decaying connection between the space and time variables outside the region $|x| > O(1)(1+t)^{1/2}$. One needs to study the strong connection between the space-time variable in the diffusion region $|x| \leq O(1)(1+t)^{1/2}$. In the neighborhood of the diffusion region, one needs to estimate the joint probability distribution of T_n and D_n when $|\mathsf{D}_n| \leq O(1)n^{\frac{1}{2}+\gamma}$ for some $\gamma \in (0, 1/6)$. The joint probability distribution function is almost a product of the marginal probability distribution functions in the diffusion region $|x| \leq n^{\frac{1}{2}+\gamma}$

$$\begin{aligned}&\boldsymbol{P}\left(\mathsf{D}_n \in (x, x+dx) \times (y, y+dy), \mathsf{T}_n \in (t, t+dt)\right)\\&\leq\; C\left(\frac{e^{-\frac{(x^2+y^2)+(t-n\mathbb{E}[\sigma_1])^2}{Cn}}}{n^{3/2}} + e^{-\frac{\sqrt{x^2+y^2}+t}{C}}\right) dxdydt.\end{aligned} \tag{1.6}$$

To obtain (1.6) one needs rather detailed analysis than that for (1.5), since the random variables T_n and D_n are not independent. An auxiliary stochastic process $\sum_{k=0}^{n}(\sigma_k)^2$ is introduced to analyze the fluctuation of T_n for the purpose of estimating the joint probability of T_n and D_n in a diffusion region, see Lemma 3.9

By the marginal probability distributions $\mathsf{P}(\mathsf{T}_n \in (t, t+dt))$ and $\mathsf{P}(\mathsf{D}_n \in (x, x+dx) \times (y, y+dy))$ in (1.5), the estimate in (1.6) for the joint probability distributions of T_n and D_n within the region $|\mathsf{D}_n| \leq n^{1/2+\gamma}$, and the expression of $\rho(x, y, t)$ in (1.3), one will obtain a global diffusion property on the boundary flux function $\rho(x, y, t)$

$$\rho(x, y, t) \leq C\left(\frac{e^{-\frac{(x^2+y^2)}{C(1+t)}}}{(1+t)} + e^{-\frac{\sqrt{x^2+y^2}+t}{C}}\right).$$

This diffusion property of the boundary flux function implies the Main Theorem A, Section 3:

Theorem 1.1.(Main Theorem A) *Suppose that the initial configuration* $\mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi)$ *of* (1.2) *satisfies*

$$|\mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi)| \leq e^{-\sqrt{x^2+y^2}}\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{\alpha}, \tag{1.7}$$

for a given $\alpha \in (0, 1)$. *Then, there exists* $C_* > 0$ *such that the solution* $\mathsf{t}(\vec{\boldsymbol{x}}, t, \xi)$ *of* (1.2) *satisfies*

$$|\mathsf{t}(\vec{\boldsymbol{x}}, t, \xi)| \leq O(1)\left(\frac{e^{-\frac{x^2+y^2}{C_*(1+t)}}}{(1+t)} + e^{-(|x|+|y|+t)/C_*}\right)\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{\alpha}. \tag{1.8}$$

Furthermore, if the initial data $\mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi)$ *satisfies a zero total excess mass condition (or zero total fluctuation condition)*

$$\int_{\mathbb{H}\times\mathbb{R}^3} \mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi) d\xi d\vec{\boldsymbol{x}} = 0, \tag{1.9}$$

then the solution $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ *satisfies*

$$|\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)|\le O(1)\left(\frac{e^{-\frac{x^2+y^2}{C_*(1+t)}}}{(1+t)^{1+\gamma'}}+e^{-(|x|+|y|+t)/C_*}\right)\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{\alpha}$$
$$\text{for some } \gamma'\in(0,1/6). \quad (1.10)$$

The equation (1.1) is treated as the perturbation of (1.2) when the gas is sufficiently rarefied. The main complications in dealing with the nonlinear collision term $Q(\mathsf{f})$ are:

(1) Main Theorem A needs not impose that the solution $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ is a positive-valued function, though the notion of a probability measure would require it. Any initial data $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)$ can be decomposed into a difference of two positive-valued functions $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)=(\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)\wedge 0)+(\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)\vee 0)$ so that Main Theorem A can be applied to both initial data $-(\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)\wedge 0)$ and $(\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)\vee 0)$. From the linear superposition property of (1.2), one will have the estimates for the solution of $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ without the condition $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)\ge 0$. In contrast to this, for the nonlinear problem (1.1), a non-negative valued function,

$$\mathsf{f}(\vec{\boldsymbol{x}},0,\xi)\ge 0$$

is needed. When this condition is false, the damping effect in the collision operator Q could become an amplifying effect. In particular, for a hard potential model the amplifying effect could become un-manageable for constructing a solution local in time.

(2) Compared to (1.8), in (1.10) there is an extra decaying property in the time variable. This extra decaying factor is necessary in order to obtain a global in time existence theory. Such an extra time decaying property is common in much existing research on the nonlinear problems for viscous conservation laws.

(3) One can not have sharp estimates in the (x,y)-(z,ξ) variables at the same time. One can not have sharp information in (x,y)-variable from those particles with high velocity or at a high altitude, since these particles can travel from far away in the space. One shall not expect any fluid-like structure for those particles such as an extra exponentially decaying factor in the (x,y)-variable. We introduce a norm to reflect this fact and

use it to analyze this nonlinear problem:

$$|||\mathsf{g}|||_T \equiv \sup_{t\in[0,T]}\left(\sup_{\substack{\vec{\boldsymbol{x}}\in\mathbb{H},\\ \xi\in\mathbb{R}^3}} \frac{|\mathsf{g}(\vec{\boldsymbol{x}},t,\xi)|}{\left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{(1+t)}+e^{-(|x|+|y|+t)/C_0}\right)\sqrt{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)}} + \sup_{\substack{\vec{\boldsymbol{x}}\in\mathbb{H}\\ \xi\in\mathbb{R}^3}} \frac{|\mathsf{g}(\vec{\boldsymbol{x}},t,\xi)|}{\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{3/4}}\right) \tag{1.11}$$

with $C_0=\frac{3}{2}C_*$ where C_* is given in Main Theorem A.

We use the method of continuity to show that

$\{T\in\mathbb{R}^+ : |||\mathsf{f}|||_T<\epsilon^{2/3}\}=\mathbb{R}^+$ where $\mathsf{f}(\vec{\boldsymbol{x}},t,\xi)$ is the solution of (1.1).

Finally in Section 4, we prove the following theorem.

Theorem 1.2.(Main Theorem B) *There exist $\epsilon_0>0$ and $C_0>0$ such that when $\epsilon<\epsilon_0$ the solution of the initial value problem* (1.1) *satisfies*

$$0\le \mathsf{f}(\vec{\boldsymbol{x}},t,\xi)\le C_0\epsilon\begin{cases}\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{3/4}\\[2ex] \left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{1+t}+e^{-\frac{\sqrt{x^2+y^2}+t}{C_0}}\right)\sqrt{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)}.\end{cases} \tag{1.12}$$

As in [14], our approach is based on the stochastic formulation of the free transport motion (1.2). The stochastic formulation yields explicit solution representation. In this, we are following the classical approach of using the Green's function. For the Boltzmann equation, the Green's function approach has been initiated in [9], [10] in the case without the boundary effects. The study of Green's function for initial-boundary value problem in [11] is for the perturbation around a Maxwellian and is therefore not applicable here. For the study of gases near vacuum, it is natural to consider the free transport equation, as is the case in the present consideration. For the study on the well-posedness of initial value problem for gases near vacuum based on the free transport motion, see [8] and references therein. For the study of the Boltzmann equation with a source, it is common to consider the Boltzmann equation without the source first and then use the Duhamel's principle to study the effect of the source. Such an approach works either for local in time, [2], or for the case when the source is small, see [5] and references therein. In our case here, the source is the gravitational force and thus is not small. We need to consider the source as an essential part of our stochastic formulation for the explicit representation. For analytical study of the boundary thermal effects, see [12], [13] [4], and references therein.

2. Preliminaries

The collision operator Q with Grad's angular cutoff potential with an inverse power force law r^{-s} is defined in the form:

$$\begin{cases} Q(f) \equiv Q_+(f) - Q_-(f) \text{ (gain-loss decomposition)} \\ Q_+(f) \equiv \mathsf{B}_+(f,f), \text{ (gain operator)} \\ Q_-(f) \equiv \mathsf{B}_-(f,f), \text{ (loss operator)} \\ \mathsf{B}_+(f,g) \equiv \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3\times S^2} q(V,\theta)(\mathsf{f}(\xi')\mathsf{g}(\xi'_*) + \mathsf{g}(\xi')\mathsf{h}(\xi'_*))d\xi_* d\omega, \\ \mathsf{B}_-(f,g) \equiv \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3\times S^2} q(V,\theta)(\mathsf{f}(\xi)\mathsf{g}(\xi_*) + \mathsf{g}(\xi)\mathsf{h}(\xi_*))d\xi_* d\omega, \end{cases} \tag{2.1}$$

where

$$\begin{cases} \xi' = \xi - ((\xi-\xi_*)\cdot\omega)\omega, \\ \xi'_* = \xi_* + ((\xi-\xi_*)\cdot\omega)\omega, \end{cases}$$
$$V = |\xi-\xi_*|,\ \theta = \cos^{-1}\left(\frac{(\xi-\xi_*)}{|\xi-\xi_*|}\cdot\omega\right).$$

Here, the condition for Grad's angular cutoff potential for an inverse power force law imposes the following on $q(V,\theta)$:

$$q_0 V^\gamma|\cos\theta| < q(V,\theta) < q_1 V^\gamma|\cos\theta| \text{ for some } 0 < q_0 < q_1,$$

$$\gamma \equiv \frac{(s-4)}{s}.$$

Those collision operators are classified as follows:

	$s<4$	$s=4$	$s>4$	$s=\infty$
Collision model: Q	soft potential	Maxwell molecule	hard potential	hard sphere

Maxwellians and Collision Invariants

Theorem 2.1.(Boltzmann (1872)) *If* $Q(M) \equiv 0$, *then the state* M *is called a Maxwellian distribution and it is in the form*

$$M(\xi) = \mathsf{M}_{[\rho,\vec{\boldsymbol{u}},\theta]} \equiv \rho \frac{e^{-\frac{|\xi-\vec{\boldsymbol{u}}|^2}{2\theta}}}{\sqrt{(2\pi\theta)^3}},\ \rho > 0,\ \theta > 0,\ \xi \in \mathbb{R}^3.$$

The collision invariants χ_i, $i = 0, 1, 2, 3, 4$, *are the five polynomials in* $\xi \in \mathbb{R}^3$

$$\begin{cases} \chi_0(\xi) = 1, \\ \chi_i(\xi) = \xi^i \textit{ for } i = 1, 2, 3, \\ \chi_4(\xi) = |\xi|^2. \end{cases}$$

Those collision invariants are orthogonal to the collision operator Q*:*

$$\int_{\mathbb{R}^3} \chi_i(\xi) Q(\mathsf{h}) d\xi = 0 \text{ for } i = 0, \ldots, 4,\ \textit{for any } \mathsf{h} \textit{ under consideration.}$$

Lemma 2.2.(Positive-valued operators) *Suppose that* h *is a non-negative valued function in* ξ. *Then,*

$$q_-[\mathsf{h}](\xi) \equiv \frac{Q_-(\mathsf{h})(\xi)}{\mathsf{h}(\xi)} \geq 0 \textit{ for all } \xi \in \mathbb{R}^3,$$
$$Q_+(\mathsf{h})(\xi) \geq 0.$$

The equal sign holds only when the non-negative valued function h *is identical to* 0.

This lemma follows directly from the definition of Q_- and Q_+ in (2.1).

Grad's Estimates on Collision operator

Theorem 2.3.(Grad [6]) *Suppose that the collision operator* Q *satisfies the Grad's angular cutoff condition for a potential with an inverse power force law* r^{-s}. *Then, for given* $M = \mathsf{M}_{[1,\vec{\boldsymbol{u}},\theta]}$ *and* $\alpha > 0$ *there exists* C_0 *such that*

$$\|(1 + |\xi|)^{1-\gamma} M^{-\alpha} \mathsf{B}_+(M^\alpha \mathsf{g}, M^\alpha \mathsf{h})\|_{L^\infty_\xi} \leq C_0 \|\mathsf{g}\|_{L^\infty_\xi} \cdot \|\mathsf{h}\|_{L^\infty_\xi}, \tag{2.2}$$

where $\gamma = \frac{(s-4)}{s}$, $\|\mathsf{g}\|_{L^\infty_\xi} \equiv \sup_{\xi\in\mathbb{R}^3} |\mathsf{g}(\xi)|$.

3. Stochastic Representations of Solutions of the Transport Equation

3.1. Probabilistic representation of a transport equation with diffuse reflection boundary condition

One can derive a probabilistic representation for the solution of the transport equation

$$\begin{cases} \partial_t \mathsf{t} + \xi \cdot \nabla_{\vec{\boldsymbol{x}}} \mathsf{t} - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{t} = 0, \\ \mathsf{t}(x, y, 0, t, \xi)|_{\xi^3 > 0} = \rho(x, y, t) \sqrt{\frac{2\pi}{\theta}} \mathsf{M}(\xi)|_{\xi^3>0}, \\ \rho(x, y, t) = \int_{\substack{\xi \in \mathbb{R}^3, \\ \xi^3 < 0}} -\xi^3 \mathsf{t}(x, y, 0, t, \xi) d\xi \\ \int_{\mathbb{R}^3 \times \mathbb{H}^+} \mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi) d\xi d\vec{\boldsymbol{x}} = 1, \ \ \mathsf{t}(\vec{\boldsymbol{x}}, 0, \xi) \geq 0. \end{cases} \tag{3.1}$$

The sample space and the probability measure

The probability space $(\Omega, \boldsymbol{P}_0)$ is defined by the sample space

$$\Omega \equiv \prod_{j=1}^{\infty} \mathbb{H}^+$$

with the probability measure $\boldsymbol{P}_0$ given by

$$\boldsymbol{P}_0\left((\pi_j^{-1} A_j) \cap (\pi_i^{-1} B_i)\right) = \boldsymbol{P}_0\left(\pi_j^{-1} A_j\right) \boldsymbol{P}_0\left(\pi_i^{-1} B_i\right) \text{ for } A_i, B_j \subset \mathbb{H}^+, i \neq j,$$

and

$$\boldsymbol{P}_0\left(\pi_i^{-1} A_i\right) \equiv \int_{A_j} \mathsf{G}(v^1)\mathsf{G}(v^2)\mathsf{H}(v^3) dv^1 dv^2 dv^3,$$

where $\pi_i : \{\vec{\boldsymbol{V}}_j\}_{j \in \mathbb{N}} \in \Omega \longmapsto \vec{\boldsymbol{V}}_i \in \mathbb{H}^+$ and

$$\mathsf{G}(v) \equiv \frac{e^{-\frac{v^2}{2\theta}}}{\sqrt{2\pi\theta}}, \ \ \mathsf{H}(v) \equiv \begin{cases} \dfrac{v}{\theta} e^{-\frac{v^2}{2\theta}} \text{ for } v > 0, \\ 0 \text{ for } v < 0. \end{cases}$$

For any $(\vec{\boldsymbol{x}}_0, \xi_0, \omega) \in \mathbb{H}^+ \times \mathbb{R}^3 \times \Omega$, we introduce the following stochastic processes

Basic parameters

notion	$\vec{\boldsymbol{x}}_0 \in \mathbb{H}^+$	$\xi_0 \in \mathbb{R}^3$	$\omega \in \Omega$
definition	$\vec{\boldsymbol{x}}_0 = (x_0, y_0, z_0)$	$\xi_0 = (\xi_0^1, \xi_0^2, \xi_0^3)$	$\omega = (\vec{\boldsymbol{V}}_1, \vec{\boldsymbol{V}}_2, \vec{\boldsymbol{V}}_3, \vec{\boldsymbol{V}}_4, \cdots)$, $\vec{\boldsymbol{V}}_j = (V_j^1, V_j^2, V_j^3) \in \mathbb{H}^+$

Deterministic variables

	$\tau_0 = ((\lvert\xi_0^3\rvert^2 + 2gz_0)^{\frac{1}{2}} + \xi_0^3)/g$	$\mathsf{d}_0 = \tau_0(\xi_0^1, \xi_0^2)$	$W_0 = (x_0, y_0) + \mathsf{d}_0$
description	first collision time	horizontal displacement up to the first collision	location where the first collision taking place

mutual independent i.i.d.'s

	$\{V_j^1\}_{j\in\mathbb{N}} \in \prod_{j=1}^{\infty} \mathbb{R}$	$\{V_j^2\}_{j\in\mathbb{N}} \in \prod_{j=1}^{\infty} \mathbb{R}$	$\{V_j^3\}_{j\in\mathbb{N}} \in \prod_{j=1}^{\infty} \mathbb{R}^+$
Distribution function	$\mathsf{G}(v) = \frac{\boldsymbol{P}_0(V_j^1 \in (v, v+dv))}{dv}$	$\mathsf{G}(v) = \frac{\boldsymbol{P}_0(V_j^2 \in (v, v+dv))}{dv}$	$\mathsf{H}(v) = \frac{\boldsymbol{P}_0(V_j^3 \in (v, v+dv))}{dv}$

random variables between two collision

$\sigma_n = 2V_n^3/g$	the time lapse between the n-th and $n+1$-th collision, $n \geq 1$
$\mathsf{d}_n = (\sigma_n V_n^1, \sigma_n V_n^2)$	the horizontal displacement between the n-th and $n+1$-th collision, $n \geq 1$

random walks

$\mathsf{T}_n = \sum_{l=1}^n \sigma_l$	the time lapse between $n+1$ and the first collision
$\tau_n = \mathsf{T}_n + \tau_0$	$n+1$-th collision time
$\mathsf{D}_n = \sum_{l=1}^n \mathsf{d}_l$	the horizontal displacement between $n+1$ and the first collision
$W_n = (x_0, y_0) + \mathsf{d}_0 + \mathsf{D}_n$	the location where $n+1$-th collision taking place
$\mathsf{S}_n = \sum_{l=1}^n (\sigma_l)^2$	an auxiliary process

(3.2)

The continuous processes (random path): $\vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega) \equiv (X^1(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega),$ $X^2(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega),\ X^3(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega))$

the dynamics	the initial data	the random reflected velocity
$\ddot{\vec{X}}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = -\vec{\boldsymbol{g}}$ for $t \notin \{\tau_0,\tau_1,\ldots,\}$	$\vec{X}(0;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \vec{\boldsymbol{x}}_0$ $\dot{\vec{X}}(0;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \xi_0$	$\dot{\vec{X}}(\tau_{n-1}+;\vec{\boldsymbol{x}}_0,\xi_0,\omega) = \vec{\boldsymbol{V}}_n$ $X^3(\tau_{n-1};\vec{\boldsymbol{x}}_0,\xi_0,\omega) = 0$ for $n = 1,\ldots$

The stochastic process

notions	definition
$\vec{\boldsymbol{x}}_t \equiv (x_t, y_t, z_t)$	$(x_t, y_t, z_t) \equiv \vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega)$
$\vec{\boldsymbol{v}}_t \equiv (v_t^1, v_t^2, v_t^3)$	$(v_t^1, v_t^2, v_t^3) \equiv \dot{\vec{X}}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega)$

(3.3)

Remark 3.1. The random variables d_n and σ_n are not independent.

Lemma 3.2.

$$\boldsymbol{P}_0\left\{\lim_{n\to\infty}\sum_{j=0}^{n}\sigma_j = \infty\right\} = 1. \tag{3.4}$$

Proof. The random variables $\{\sigma_i\}_{i\geq 1}$ are i.i.d. Define the event $E_i \equiv \{\sigma_i > 1\}$. Thus, one has that

$$\sum_{i=1}^{\infty}\boldsymbol{P}_0(E_i) = \lim_{n\to\infty} n\boldsymbol{P}_0(E_1) = \infty.$$

Then, by the Borel-Cantelli Lemma one has

$$\boldsymbol{P}_0\left\{E_i \text{ infinite often}\right\} = 1.$$

This and $\sigma_i \geq 0$ yield this lemma. □

From (3.4), one has $\lim_{n\to\infty}\tau_n = \infty$. This limit yields that for almost all $\omega \in \Omega$ for any $t \in \mathbb{R}^+$ there exists n such that

$$\tau_n(\omega) \leq t \leq \tau_{n+1}(\omega) \text{ or } 0 < t < \tau_1(\omega).$$

With this, one can define the random trajectory $\vec{\boldsymbol{x}} = \vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega)(\equiv \vec{\boldsymbol{x}}_t \equiv$

$(x_t, y_t, z_t))$ is defined by

$$
\begin{aligned}
&\text{for} \quad t \in (0, \tau_0) \\
&\qquad \vec{\boldsymbol{x}}_t = \vec{\boldsymbol{x}}_0 + t\xi_0 - \tfrac{t^2}{2}\vec{\boldsymbol{g}}, \\
&\text{for} \quad t \in (\tau_{n-1}, \tau_n) \\
&\qquad \begin{cases} (x_t, y_t) \equiv W_{n-1} + (t - \tau_{n-1})\,(V_n^1, V_n^2), \\ z_t = V_n^3(t - \tau_{n-1}) - \dfrac{g}{2}\,(t - \tau_{n-1})^2 . \end{cases}
\end{aligned}
$$

The trajectory of the random particle $\vec{\boldsymbol{x}} = \vec{X}(t, \vec{\boldsymbol{x}}_0, \xi_0, \omega)$ is illustrated in Figure 1.

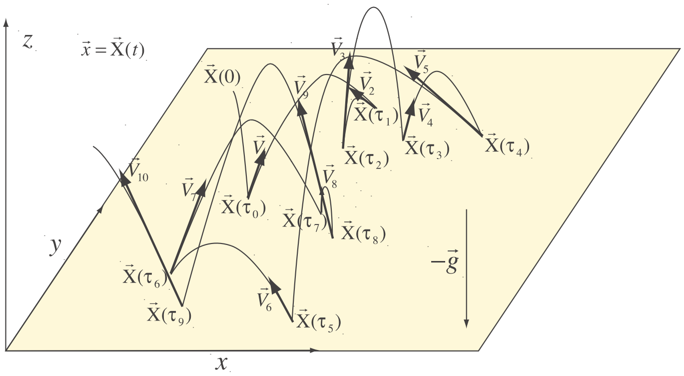

Figure 1.

Theorem 3.3. *The probabilistic representation of the solution* $\mathsf{t}(\vec{\boldsymbol{x}}, t, \xi)$ *of* (3.1) *is*

$$
\begin{aligned}
\mathsf{t}(\vec{\boldsymbol{x}}, t, \xi)d\vec{\boldsymbol{x}}d\xi = {} & \chi(z-\xi^3 t-\frac{g}{2}t^2)\mathsf{t}(x-\xi^1 t, y-\xi^2 t, z-\xi^3 t-\frac{g}{2}t^2, 0, \xi+\vec{\boldsymbol{g}}t)d\vec{\boldsymbol{x}}d\xi \\
& + \int_{(\vec{\boldsymbol{x}}_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^3} \chi(t-\tau_0)\boldsymbol{P}_0\left(\vec{\boldsymbol{x}}_t \in \vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}_t \in \xi + d\xi\right) \\
& \qquad \times \mathsf{t}(\vec{\boldsymbol{x}}_0, 0, \xi_0)d\vec{\boldsymbol{x}}_0 d\xi_0, \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
& \rho(x, y, t)dxdydt \\
& = \int_{\substack{\xi_0^3 < \frac{gt}{2} \\ \xi_0 \in \mathbb{R}^3}} -(\xi_0^3 - gt)\mathsf{t}(x - \xi_0^1 t, y - \xi_0^2 t, -\xi_0^3 t + \frac{g}{2}t^2, 0, \xi_0)d\xi_0 dxdydt \\
& \quad + \int_{\mathbb{H}^+\times\mathbb{R}^3} \chi(t-\tau_0)\sum_{n=1}^{\infty}\boldsymbol{P}_0(\tau_n \in (t, t+dt),
\end{aligned}
$$

$$W_n \in (x, x+dx) \times (y, y+dy))\mathsf{t}(\vec{\boldsymbol{x}}_0, 0, \xi_0)d\vec{\boldsymbol{x}}_0 d\xi_0, \tag{3.6}$$

where $\chi(x)$ is a characteristic function for the set $\{x|\ x > 0\}$, and both τ_n and W_n are functions of $(\vec{\boldsymbol{x}}_0, \xi_0, \omega) \in \mathbb{H}^+ \times \mathbb{R}^3 \times \Omega$.

Proof. Let $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}}, t, \xi)$ denote the probability density function of

$$\begin{aligned}
&\tilde{\mathsf{t}}(\vec{\boldsymbol{x}}, t, \xi)d\vec{\boldsymbol{x}}d\xi \\
&= \chi(z - \xi^3 t - \frac{g}{2}t^2)\mathsf{t}(x - \xi^1 t, y - \xi^2 t, z - \xi^3 t - \frac{g}{2}t^2, 0, \xi + \vec{\boldsymbol{g}}t)d\vec{\boldsymbol{x}}d\xi \\
&\quad + \int_{(\vec{\boldsymbol{x}}_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^3} \chi(t - \tau_0)\boldsymbol{P}_0\left(\vec{\boldsymbol{x}}_t \in \vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}_t \in \xi + d\xi\right)\mathsf{t}(\vec{\boldsymbol{x}}_0, 0, \xi_0)d\vec{\boldsymbol{x}}_0 d\xi_0.
\end{aligned} \tag{3.7}$$

From the definition of the random path $\vec{X}(t)$, for any regular set $\Gamma_0 \subset \mathbb{H}^+ \times \mathbb{R}^3$ and its trajectory Γ_t under the flow $\dot{\vec{\boldsymbol{x}}} = \xi$, $\dot{\xi} = -\vec{\boldsymbol{g}}$, the probability enclosed in the region Γ_t is invariant in time whenever Γ_t stays away from the boundary $z = 0$. Thus,

$$\frac{d}{dt}\int_{\Gamma_t}\left(\int_{\mathbb{H}^+\times\mathbb{R}^3}\boldsymbol{P}_0\big(\vec{\boldsymbol{x}}_t \in \vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, \vec{\boldsymbol{v}}_t \in \xi + d\xi\big)\mathsf{t}(\vec{\boldsymbol{x}}_0, 0, \xi_0)d\vec{\boldsymbol{x}}_0 d\xi_0\right)d\vec{\boldsymbol{x}}d\xi = 0$$

for any $\Gamma_0 \subset \mathbb{H}^+ \times \mathbb{R}^3$. By letting $|\Gamma_0| \to 0$ and the above yields that the probability density function $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}}, t, \xi)$ satisfies

$$\partial_t\tilde{\mathsf{t}} + \xi \cdot \nabla_{\vec{\boldsymbol{x}}}\tilde{\mathsf{t}} - \vec{\boldsymbol{g}} \cdot \nabla_\xi\tilde{\mathsf{t}} = 0. \tag{3.8}$$

The function

$$\int_{\substack{\xi_0^3 < gt/2 \\ \xi_0\in\mathbb{R}^3}} -(\xi_0^3 - gt)\mathsf{t}(x - \xi_0^1 t, y - \xi_0^2 t, -\xi_0^3 t + \frac{g}{2}t^2, 0, \xi_0)dxdydtd\xi_0$$

represents the number of the first collision particles (the deterministic collision) in the region given by $x \in x + dx$, $y \in y + dy$, $z = 0$, $\tau_0 \in (t, t + dt)$. Thus, the function $\tilde{\rho}(x, y, t)dxdydt$ denotes the number of particles with all possible number of collisions (with random and deterministic collisions) at $x \in (x, x + dx)$, $y \in (y, y + dy)$, $z = 0$, $t \in (t, t + dt)$

$$\begin{aligned}
&\tilde{\rho}(x, y, t)dxdydt \\
&= \int_{\substack{\xi_0^3 < gt/2 \\ \xi_0\in\mathbb{R}^3}} -(\xi_0^3 - gt)\mathsf{t}(x - \xi_0^1 t, y - \xi_0^2 t, -\xi_0^3 t + \frac{g}{2}t^2, 0, \xi_0)dxdyd\xi dt
\end{aligned}$$

$$+\int_{\mathbb{H}^+\times\mathbb{R}^3}\chi(\tau_0-t)\sum_{n=1}^{\infty}\boldsymbol{P}_0(\tau_n\in(t,t+dt),$$
$$W_n\in(x,x+dx)\times(y,y+dy))\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0d\xi_0. \tag{3.9}$$

Here, the random variables τ_n and W_n are functions of $\vec{\boldsymbol{x}}_0$, ξ_0, and ω. This function can be estimated directly from the stochastic process given in (3.2). Next, we will need to relate $\tilde{\rho}(x,y,t)$ to $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}},t,\xi)$. Since $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}},t,\xi)$ is time-reversible in the interior domain through the flow generated by (3.8), the only non time-reversible part appears at the boundary $z=0$. Thus, we just need to relate $\tilde{\rho}(x,y,t)dxdydt$ and $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}},t,\xi)d\xi d\vec{\boldsymbol{x}}|_{z=0,\xi^3>0}$. Let

$$dz=-\xi^3dt \tag{3.10}$$

the infinitesimal line element in the z-component for particle arriving boundary $z=0$ with impinging velocity ξ. This yields, for $n\geq 1$,

$$\int_{\substack{\xi^3<0\\ \xi\in\mathbb{R}^3}}\bigg(\int_{(\vec{\boldsymbol{x}}_0,\vec{\boldsymbol{v}}_0)\in\mathbb{H}^+\times\mathbb{R}^3}\boldsymbol{P}_0(x_t\in x+dx,y_t\in y+dy,z_t\in z+dz,t=\tau_n,$$
$$\xi\in\xi+d\xi)d\vec{\boldsymbol{x}}_0d\xi_0\bigg)d\xi$$
$$=\int_{(\vec{\boldsymbol{x}}_0,\vec{\boldsymbol{v}}_0)\in\mathbb{H}^+\times\mathbb{R}^3}\boldsymbol{P}_0(x_{\tau_n}\in x+dx,y_{\tau_n}\in y+dy,\tau_n\in(t,t+dt))\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0d\xi_0$$
$$=\int_{(\vec{\boldsymbol{x}}_0,\vec{\boldsymbol{v}}_0)\in\mathbb{H}^+\times\mathbb{R}^3}\boldsymbol{P}_0(W_n\in(x,x+dx)\times(y,y+dy),\tau_n\in(t,t+dt))$$
$$\times\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0d\xi_0. \tag{3.11}$$

Again by (3.10), one has

$$\int_{\substack{\xi^3<0\\ \xi\in\mathbb{R}^3}}\left(-\xi^3\tilde{\mathsf{t}}(x,y,0,t,\xi)dxdydt\right)d\xi$$
$$=\bigg(\int_{\substack{\xi_0^3<\frac{gt}{2}\\ \xi\in\mathbb{R}^3}}-(\xi_0^3-gt)\mathsf{t}(x-\xi_0^1t,y-\xi_0^2t,-\xi_0^3t+\frac{1}{2}gt^2,0,\xi_0)d\xi_0\bigg)dxdydt$$
$$+\int_{\substack{\xi^3<0\\ \xi\in\mathbb{R}^3}}\bigg(\int_{(\vec{\boldsymbol{x}}_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^3}\chi(t-\tau_0)\boldsymbol{P}_0(x_t\in x+dx,y_t\in y+dy,z_t\in z+dz,$$
$$\vec{\boldsymbol{v}}_t\in\xi+d\xi)\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0d\xi_0\bigg)d\xi\bigg|_{z=0}. \tag{3.12}$$

One can partition the event $\{z_t=0\}$ into $\cup_{n=0}^{\infty}\{\tau_n=t\}$ and use (3.11) to result in that

$$\begin{aligned}
&\int_{\substack{\xi^3<0\\ \xi\in\mathbb{R}^3}}(-\xi^3\mathsf{t}(x,y,0,t,\xi)dxdydt)d\xi \\
&= \left(\int_{\substack{\xi_0^3<\frac{gt}{2}\\ \xi\in\mathbb{R}^3}} -(\xi_0^3-gt)\mathsf{t}(x-\xi_0^1 t, y-\xi_0^2 t, -\xi_0^3 t+\frac{1}{2}gt^2, 0, \xi_0)d\xi\right)dxdydt \\
&\quad +\sum_{n=1}^{\infty}\int_{(\vec{\boldsymbol{x}}_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^3}\chi(t-\tau_0)\boldsymbol{P}_0\left(W_n\in(x,x+dx)\times(y,y+dy),\tau_n\in(t,t+dt)\right) \\
&\qquad\qquad \mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0 d\xi_0 \\
&= \tilde{\rho}(x,y,t)dxdydt. \qquad (3.13)
\end{aligned}$$

This proves (3.9).

Similar to the first identity in (3.13) one has that

$$\tilde{\rho}(x,y,t)dxdydt=\int_{\xi^3>0}\xi^3\tilde{\mathsf{t}}(x,y,0,t,\xi)dxdyd\xi dt=\int_{\xi^3>0}\tilde{\mathsf{t}}(x,y,0,t,\xi)d\vec{\boldsymbol{x}}d\xi$$

On the other hand from the random velocity, one has

$$\begin{aligned}
\tilde{\rho}(x,y,t)dxdydt &= \int_{\xi^3>0}\frac{\sqrt{2\pi}}{\sqrt{\theta}}\xi^3\tilde{\rho}(x,y,t)\mathsf{M}(\xi)d\xi^1 d\xi^2 d\xi^3 dxdydt \\
&= \int_{\xi^3>0}\frac{\sqrt{2\pi}}{\sqrt{\theta}}\tilde{\rho}(x,y,t)\mathsf{M}(\xi)d\xi^1 d\xi^2 d\xi^3 dxdydz.
\end{aligned}$$

By comparing the above two, we conclude that

$$\tilde{\mathsf{t}}(x,y,0,t,\xi)|_{\xi^3>0}=\frac{\sqrt{2\pi}}{\sqrt{\theta}}\tilde{\rho}(x,y,t)\mathsf{M}(\xi)|_{\xi^3>0}.$$

From this and (3.8), the probabilistic function $\tilde{\mathsf{t}}(\vec{\boldsymbol{x}},t,\xi)$ solves (3.1) with the Maxwell diffuse reflection boundary condition. (3.6) from (3.13). □

Corollary 3.4. *There exists a probability measure $\boldsymbol{P}$ and two vector-valued stochastic processes $\{\vec{\boldsymbol{x}}_t\}_{t\in\mathbb{R}^+}$ and $\{\vec{\boldsymbol{v}}_t\}_{t\in\mathbb{R}^+}$ such that the solution $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ of (3.1) can be represented as follows*

$$\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)d\vec{\boldsymbol{x}}d\xi=\boldsymbol{P}(\vec{\boldsymbol{x}}_t\in\vec{\boldsymbol{x}}+d\vec{\boldsymbol{x}},\vec{\boldsymbol{v}}_t\in\xi+d\xi). \qquad (3.14)$$

Furthermore, the boundary flux function $\rho(x,y,t)$ can be represented as

$$\rho(x,y,t)dtdxdy=\sum_{n=0}^{\infty}\boldsymbol{P}\left(\vec{\boldsymbol{x}}_{\tau_n}\in[x,x+dx]\times[y,y+dy],\tau_n\in[t,t+dt]\right). \tag{3.15}$$

Proof. For each $(\vec{\boldsymbol{x}}_0,\xi_0,\omega)\in\mathbb{H}^+\times\mathbb{R}^3\times\Omega$ the stochastic processes $\{\vec{\boldsymbol{x}}_t\}_{t\in\mathbb{R}^+}$ and $\{\vec{\boldsymbol{v}}_t\}_{t\in\mathbb{R}^+}$ are defined by

$$\begin{cases}\vec{\boldsymbol{x}}_t(\omega)\equiv\vec{X}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega),\\ \vec{\boldsymbol{v}}_t(\omega)\equiv\dot{\vec{X}}(t;\vec{\boldsymbol{x}}_0,\xi_0,\omega).\end{cases}$$

The probability measure $\boldsymbol{P}(\vec{\boldsymbol{x}}_t\in\vec{\boldsymbol{x}}+d\vec{\boldsymbol{x}},\vec{\boldsymbol{v}}_t\in\xi+d\xi)$ is defined by

$$\begin{aligned}&\boldsymbol{P}(\vec{\boldsymbol{x}}_t\in\vec{\boldsymbol{x}}+d\vec{\boldsymbol{x}},\vec{\boldsymbol{v}}_t\in\xi+d\xi)\\&\quad=\chi(z-\xi^3t-\frac{g}{2}t^2)\mathsf{t}(x-\xi^1t,y-\xi^2t,z-\xi^3t-\frac{g}{2}t^2,0,\xi+\vec{\boldsymbol{g}}t)d\vec{\boldsymbol{x}}d\xi\\&\qquad+\int_{(\vec{\boldsymbol{x}}_0,\xi_0)\in\mathbb{H}\times\mathbb{R}^3}\chi(t-\tau_0)\boldsymbol{P}_0\left(\vec{\boldsymbol{x}}_t\in\vec{\boldsymbol{x}}+d\vec{\boldsymbol{x}},\vec{\boldsymbol{v}}_t\in\xi+d\xi\right)\mathsf{t}(\vec{\boldsymbol{x}}_0,0,\xi_0)d\vec{\boldsymbol{x}}_0d\xi_0.\end{aligned} \tag{3.16}$$

From this definition of $\boldsymbol{P}$ and (3.6), one has (3.15). □

One can re-assembly $\boldsymbol{P}(\vec{\boldsymbol{x}}_{\tau_n}\in[x,x+dx]\times[y,y+dy],\tau_n\in(t,t+dt))$ as follows $\boldsymbol{P}(\vec{\boldsymbol{x}}_{\tau_{n-1}}\in[\bar{x},\bar{x}+dx]\times[\bar{y},\bar{y}+dy],\tau_{n-1}\in[\bar{t},\bar{t}+dt])$ and projectile trajectories from $(\bar{x},\bar{y})$ at $\bar{t}$ to (x,y) at t according to the probability assigned to the projectile trajectories. Then, it follows

$$\begin{aligned}&\boldsymbol{P}(\vec{\boldsymbol{x}}_{\tau_n}\in[x,x+dx]\times[y,y+dy],\tau_n\in(t,t+dt))\\&\quad=\int_{\mathbb{R}^3,0<v^3<\frac{g}{2}t}\mathsf{G}(v^1)\mathsf{G}(v^2)\mathsf{H}(v^3)\\&\qquad\cdot\boldsymbol{P}\bigg(\vec{\boldsymbol{x}}_{\tau_{n-1}}\in\left[x-\frac{2v^1v^3}{g},x-\frac{2v^1v^3}{g}+dx\right]\times\left[y-\frac{2v^2v^3}{g},y-\frac{2v^2v^3}{g}+dy\right],\\&\qquad\tau_{n-1}\in\left(t-\frac{2v^3}{g},t-\frac{2v^3}{g}+dt\right)\bigg)dv^1dv^2dv^3.\end{aligned} \tag{3.17}$$

This and (3.15) gives rise to the representation

$$\begin{aligned}\rho(x,y,t)\;=\;&\int_0^t\int_{\mathbb{R}^2}\frac{g}{2}\mathsf{G}(v^1)\mathsf{G}(v^2)\mathsf{H}\left(\frac{g(t-\tau)}{2}\right)\\&\times\rho(x-v^1(t-\tau),y-v^2(t-\tau),\tau)dv^1dv^2d\tau\end{aligned}$$

$$+\int_{\substack{\mathbb{R}^3\\ v^3<gt/2}} |v^3-gt|\mathsf{t}(x-v^1t, y-v^2t, -v^3t+\frac{g}{2}t^2, 0, v^1, v^2, v^3)\, dv^1dv^2dv^3. \quad (3.18)$$

We next examine this from consideration of the characteristic curve method. At any given point $(x, y, 0, \xi) \in \partial\mathbb{H}^+ \times (-\mathbb{H}^+)$, the characteristic curve reaches $(x, y, 0, \xi)$ at time t can be classified into two classes from $t = 0$ or from $\partial\mathbb{H}^+$:

	time interval	end points of the characteristic curve in $\mathbb{H}^+ \times \mathbb{R}^3$	Conditions
Class 1	$(0, t)$	$(x-\xi^1t, y-\xi^2t, -\xi^3t-\frac{1}{2}t^2g, \xi+\vec{\boldsymbol{g}}t)$, $(\vec{\boldsymbol{x}}, \xi)$	$\xi^3 < -\frac{t}{2}g$
Class 2	$(t-\sigma_1, t)$	$(x-\xi^1\sigma_1, y-\xi^2\sigma_1, 0, \xi+\vec{\boldsymbol{g}}\sigma_1)$, $(x, y, 0, \xi)$	$-2\xi^3=\sigma_1 g$, $0<\sigma_1<t$

From this table, we can have that from the diffuse reflection boundary condition given in (3.1)

$$\begin{aligned}
&\rho(x,y,t)\\
&= \int_{\substack{\xi^3<0\\ \xi\in\mathbb{R}^3}} -\xi^3\mathsf{t}(x,y,0,t,\xi)d\xi\\
&= \int_{\substack{\xi^3<0,\xi^3<-\frac{g}{2}t\\ \xi\in\mathbb{R}^3}} -\xi^3\mathsf{t}(x-\xi^1t, y-\xi^2t, -\xi^3t-\frac{g}{2}t^2, 0, \xi-\vec{\boldsymbol{g}}t)d\xi\\
&\quad+\int_{\substack{0<\sigma^1<t,\xi\in\mathbb{R}^3\\ 0<\sigma_1\equiv-\frac{2\xi^3}{g}}} -\xi^3\rho(x-\xi^1\sigma_1, y-\xi^2\sigma_1, t-\sigma_1)\frac{\sqrt{2\pi}}{\sqrt{\theta}}\mathsf{M}(\xi)d\xi\\
&= \int_{\substack{\xi_0^3<\frac{g}{2}t\\ \xi_0\in\mathbb{R}^3}} -(\xi_0^3-gt)\mathsf{t}(x-\xi_0^1t, y-\xi_0^2t, -\xi_0^3t+\frac{g}{2}t^2, 0, \xi_0)d\xi_0\\
&\quad+\int_{\substack{0<\sigma^1<t,\\ (\xi^1,\xi^2)\in\mathbb{R}^2}} \frac{g}{2}\rho(x-\xi^1\sigma_1, y-\xi^2\sigma_1, t-\sigma_1)\mathsf{G}(\xi^1)\mathsf{G}(\xi^2)\mathsf{H}(\frac{g\sigma_1}{2})d\xi^1d\xi^2d\sigma_1. \quad (3.19)
\end{aligned}$$

This gives an alternative derivation of (3.18). Rewrite (3.18) as an integral equation

$$\rho(x,y,t) = \mathsf{Q}(x,y,t) * \rho(x,y,t) + \mathsf{t}^+(x,y,t), \quad (3.20)$$

whence

$$\rho(x,y,t)=\left(1+\sum_{l=1}^{\infty}\mathsf{Q}_l(x,y,t)\right)*\mathsf{t}^+(x,y,t). \tag{3.21}$$

Here,

$$\begin{aligned}
&\mathsf{Q}(x,y,t)=\frac{g}{2}\mathsf{G}\left(\frac{x}{t}\right)\mathsf{G}\left(\frac{y}{t}\right)\mathsf{H}\left(\frac{gt}{2}\right)=\frac{g^2te^{-\frac{x^2+y^2}{2\theta t^2}-\frac{t^2g^2}{8\theta}}}{8\pi\theta^2},\\
&\mathsf{Q}_l(x,y,t)=\underbrace{\mathsf{Q}(x,y,t)*\mathsf{Q}(x,y,t)*\cdots*\mathsf{Q}(x,y,t)}_{l\text{ convolutions}},\\
&\mathsf{h}(x,y,t)*\mathsf{g}(x,y,t)\equiv\int_0^t\int_{\mathbb{R}^2}\mathsf{h}(x-u,y-v,t-\tau)\mathsf{g}(u,v,\tau)dudvd\tau,\\
&\mathsf{t}^+(x,y,t)\equiv\int_{\substack{\mathbb{R}^3\\ v^3<\frac{gt}{2}}}|v^3-gt|\mathsf{t}(x-v^1t,y-v^2t,-v^3t+\frac{g}{2}t^2,0,v^1,v^2,v^3)dv^1dv^2dv^3.
\end{aligned} \tag{3.22}$$

Remark 3.5. From the representation (3.21), we just need to obtain a refined structure of $\mathsf{Q}_l(x,y,t)$, then the structure of the boundary flux function $\rho(x,y,t)$ will follow.

Here, the functions $\mathsf{Q}(x,y,t)$ and $\mathsf{Q}_l(x,y,t)$ are the joint probability density functions of the horizontal displacement random variables $\sum_{j=1}^{l}\mathsf{d}_j$ and the hitting time variables $\sum_{n=1}^{l}\mathsf{d}_n$ with respect to the total number l of collisions:

$$\begin{cases}\mathsf{Q}(x,y,t)dxdydt=\boldsymbol{P}(\mathsf{d}_1\in(x,x+dx)\times(y,y+dy),\sigma_1\in(t,t+dt)),\\ \mathsf{Q}_l(x,y,t)dxdydt=\boldsymbol{P}\left(\mathsf{D}_l\in(x,x+dx)\times(y,y+dy),\mathsf{T}_l\in(t,t+dt)\right).\end{cases} \tag{3.23}$$

The probability density function $\mathscr{F}(t)$ of the random variable σ_n is

$$\mathscr{T}(t)\equiv\begin{cases}\dfrac{g^2t}{4\theta}e^{-\frac{g^2t^2}{8\theta}}\text{ for }t>0,\\ 0\text{ for }t\le 0.\end{cases} \tag{3.24}$$

Lemma 3.6. *The function* $\mathsf{t}^+(x,y,t)$ *defined in* (3.22) *satisfies*

$$\int_0^\infty\left(\int_{\mathbb{R}^2}\mathsf{t}^+(x,y,t)dxdy\right)dt=\int_{\mathbb{H}^+\times\mathbb{R}^3}\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)d\vec{\boldsymbol{x}}d\xi. \tag{3.25}$$

Proof. Suppose that $\mathsf{t}(x,0,\xi) \geq 0$ and $\int_{\mathbb{H}\times\mathbb{R}^3} \mathsf{t}(\vec{\boldsymbol{x}},0,\xi)d\vec{\boldsymbol{x}}d\xi = 1$. The measure $\mathsf{t}^+(x,y,t)dtdxdy$ can be identified with the probability measure

$$\mathsf{t}^+(x,y,t)dxdydt = \boldsymbol{P}((x_{\tau_0},y_{\tau_0}) \in (x,x+dx)\times(y,y+dy), \tau_0 \in (t,t+dt)).$$

Due to the constant gravitational field $-\vec{\boldsymbol{g}}$, every particle will be pulled back to the surface $z=0$ in finite time. This yields

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^2} \boldsymbol{P}((x_{\tau_0},y_{\tau_0}) \in (x,x+dx)\times(y,y+dy), \tau_0 \in (t,t+dt))\\ &\quad = 1 = \int_{\mathbb{H}\times\mathbb{R}^3} \mathsf{t}(\vec{\boldsymbol{x}},0,\xi)d\vec{\boldsymbol{x}}d\xi.\end{aligned}$$

Thus,

$$\int_0^\infty \left(\int_{\mathbb{R}^2} \mathsf{t}^+(x,y,t)dxdy\right)dt = 1 = \int_{\mathbb{H}\times\mathbb{R}^3} \mathsf{t}(\vec{\boldsymbol{x}},0,\xi)d\vec{\boldsymbol{x}}d\xi.$$

For any positive-valued initial value $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)$ this lemma also holds by multiplying it with a normalizing constant. A general real-valued function $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)$ can be decomposed into the difference of two positive-valued functions. Apply the above to the two positive-valued functions, and so the lemma holds for the difference as well. □

Lemma 3.7.(Delay Estimates) *For a given $\alpha \in (0,1)$ suppose the initial data $\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)$ satisfies*

$$|\mathsf{t}(\vec{\boldsymbol{x}},0,\xi)| \leq \left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^\alpha e^{-\sqrt{x^2+y^2}}. \tag{3.26}$$

Then, there exists $C_1 > 0$ such that the function $\mathsf{t}^+(x,y,t)$ satisfies

$$|\mathsf{t}^+(x,y,t)| \leq O(1)e^{-\left(\frac{|gt|^2}{\theta}+\sqrt{x^2+y^2}\right)/C_1}. \tag{3.27}$$

Proof. From the definition of t^+ in (3.21), one has that

$$\begin{aligned}\mathsf{t}^+(x,y,t) \;&\leq\; \left(\int_{v^3<gt/4} + \int_{gt/4<v^3<gt/2}\right)\\ &\quad \times|v^3-gt|e^{-\sqrt{(x-v^1t)^2+(y-v^2t)^2}-\frac{2g(-v^3t+\frac{g}{2}t^2)+\sum_{j=1}^3(v^j)^2}{2\theta}\alpha}dv^1dv^2dv^3\\ &= \left(\int_{\substack{v^3<gt/4\\ |v^1|^2+|v^2|^2<\frac{|x|^2+|y|^2}{16t^2}}} + \int_{\substack{v^3<gt/4\\ |v^1|^2+|v^2|^2\geq\frac{|x|^2+|y|^2}{16t^2}}}\right)\end{aligned}$$

$$
\begin{aligned}
&\times|v^3-gt|e^{-\sqrt{(x-v^1t)^2+(y-v^2t)^2}-\frac{\frac{g^2}{2}t^2+\sum_{j=1}^3(v^j)^2}{2\theta}\alpha}dv^1dv^2dv^3\\
&+\Bigg(\int_{\substack{gt/4<v^3<gt/2\\|v^1|^2+|v^2|^2<\frac{|x|^2+|y|^2}{16t^2}}}+\int_{\substack{gt/4<v^3<gt/2\\|v^1|^2+|v^2|^2\geq\frac{|x|^2+|y|^2}{16t^2}}}\Bigg)\\
&\times|v^3-gt|e^{-\sqrt{(x-v^1t)^2+(y-v^2t)^2}-\frac{\sum_{j=1}^3(v^j)^2}{2\theta}\alpha}dv^1dv^2dv^3\\
\leq\ & O(1)\left(e^{-\frac{3\sqrt{x^2+y^2}}{4}-\frac{g^2t^2}{32\theta}\alpha}+e^{-\alpha\frac{g^2t^2}{8\theta}-\alpha\frac{|x|^2+|y|^2}{32t^2\theta}}\right)\\
\leq\ & O(1)\left(e^{-\frac{3\sqrt{x^2+y^2}}{4}-\alpha\frac{g^2t^2}{C_1\theta}}+e^{-\alpha\frac{g\sqrt{x^2+y^2}}{64\theta}-\alpha\frac{g^2t^2}{C_0\theta}}\right)\\
\leq\ & O(1)e^{-\left(\frac{g^2t^2}{\theta}+\sqrt{x^2+y^2}\right)/C_1}\quad\text{for some } C_0,\ C_1>0. \qquad (3.28)
\end{aligned}
$$

□

3.2. Central limit theorems for space-time displacements

In this subsection we derive the central limit theorems for the 2-dimensional random walk D_n and the 1-dimensional random walks T_n and S_n defined in (3.2).

From a direct calculation, one has that

$$
\begin{cases}
\mathbb{E}[\sigma_i]=\displaystyle\int_0^\infty\frac{2v^2e^{-\frac{v^2}{2\theta}}}{g\theta}dv=\frac{\sqrt{2\pi\theta}}{g},\\
\operatorname{Var}(\sigma_i)=E[(\sigma_i)^2]-E[\sigma_i]^2=\displaystyle\int_0^\infty\left(\frac{2v}{g}\right)^2\frac{ve^{-\frac{v^2}{2\theta}}}{\theta}dv-\left(\frac{\sqrt{2\pi\theta}}{g}\right)^2=\frac{\theta(8-2\pi)}{g^2}.
\end{cases}
\qquad (3.29)
$$

Lemma 3.8.(Central Limit Theorem) *There exists $C>0$ such that the stochastic processes T_n, S_n, and D_n given in* (3.2) *satisfy*

$$
\left|\partial_t^k\left(\frac{\boldsymbol{P}\{\mathsf{T}_n\in(t,t+dt)\}}{dt}\right)\right|\leq C\left(e^{-\frac{(|t-n\mathbb{E}[\sigma_1]|+n)}{C}}+\frac{e^{-\frac{(t-n\mathbb{E}[\sigma_1])^2}{Cn}}}{n^{\frac{1+k}{2}}}\right)\ \text{for } k=0,1, \qquad (3.30)
$$

$$
\frac{\boldsymbol{P}\{\mathsf{S}_n\in(\sigma,\sigma+d\sigma)\}}{d\sigma}\leq C\left(e^{-\frac{(|\sigma-n\mathbb{E}[\sigma_1^2]|+n)}{C}}+\frac{e^{-\frac{(\sigma-n\mathbb{E}[\sigma_1])^2}{Cn}}}{\sqrt{n}}\right), \qquad (3.31)
$$

$$\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x, x+dx)\times(y, y+dy)\}}{dxdy} \le C\Bigg(e^{-\frac{(|x|+|y|+n)}{C}} + \frac{e^{-\frac{(x^2+y^2)}{Cn}}}{n}\Bigg). \quad (3.32)$$

Proof. Take the Fourier transformation of the probability density function $\mathscr{T}(t)$ of the random variable σ_j i.e.

$$\mathbb{E}[e^{-i\eta\sigma_j}] \equiv \int_{\mathbb{R}} e^{-it\eta}\mathscr{T}(t)dt.$$

By the i.i.d. property of $\{\sigma_j\}_{j\in\mathbb{N}}$ one has that

$$\begin{aligned}
\mathbb{E}[e^{-i\eta\mathsf{T}_n}] &= \mathbb{E}\left[e^{-i\eta\sum_{j=1}^{n}\sigma_j}\right] = \left(\mathbb{E}[e^{-it\eta\sigma_1}]\right)^n \\
&= \left(\mathbb{E}\left[1 - i\eta\sigma_1 - \frac{\sigma_1^2}{2}\eta^2 + O(\eta^3)\right]\right)^n \\
&= \left(1 - i\eta\mathbb{E}[\sigma_1] - \frac{\text{Var}\,[\sigma_1]}{2}\eta^2 + O(\eta^3)\right)^n \\
&= e^{-in\eta\mathbb{E}[\sigma_1] - \frac{\text{Var}\,[\sigma_1]}{2}n\eta^2 + nO(\eta^3)} \text{ for } |\eta| \ll 1. \qquad (3.33)
\end{aligned}$$

The exponentially decaying structure in $\mathscr{T}(t)$ results the generating function $\mathbb{E}[e^{-i\eta\sigma_1}]$ is an analytic function in η when $\eta \in \{\eta \in \mathbb{C}|\ |\eta| \ll 1\}$ and the function $O(\eta)$ is also analytic around $\eta = 0$.

For $|t - n\mathbb{E}[\sigma_1]| \le n/C$ (in the hyperbolic scale region) by inverse Fourier transformation of $\mathbb{E}[e^{-i\eta\mathsf{T}_n}]$, using the complex contour integral method,

$$\begin{aligned}
&\partial_t^k \frac{\boldsymbol{P}(\mathsf{T}_n \in (t, t+dt))}{dt} = \int_{\mathbb{R}} (i\eta)^k e^{it\eta}\mathbb{E}[e^{-i\eta\mathsf{T}_n}]d\eta \\
&= \int_{\mathbb{R}} (i\eta)^k e^{i(t-n\mathbb{E}[\sigma_1])\eta} e^{in\mathbb{E}[\sigma_1]\eta}\mathbb{E}[e^{-i\eta\mathsf{T}_n}]d\eta \\
&= \int_{\mathbb{R}} (i\eta)^k e^{i(t-n\mathbb{E}[\sigma_1])\eta} e^{-\frac{\text{Var}\,[\sigma_1]}{2}n\eta^2 + O(1)n\eta^3}\, d\eta \\
&= \frac{1}{n^{\frac{k+1}{2}}}\int_{\mathbb{R}} (i\bar\eta)^k e^{i\frac{(t-n\mathbb{E}[\sigma_1])}{\sqrt{n}}\bar\eta} e^{-\frac{\text{Var}\,[\sigma_1]}{2}\bar\eta^2 + O(1)\frac{\bar\eta^3}{\sqrt{n}}}\, d\bar\eta \\
&= \frac{e^{-\frac{(t-n\mathbb{E}[\sigma_1])^2}{2\text{Var}[\sigma_1]n}}}{n^{\frac{k+1}{2}}}\Bigg(\int_{|\bar\eta|\le\kappa_0\sqrt{n}} + \int_{|\bar\eta|\ge\kappa_0\sqrt{n}}\Bigg)(i\eta)^k e^{-\frac{\text{Var}\,[\sigma_1]\left(\bar\eta - i\frac{(t-n\mathbb{E}[\sigma_1])}{\sqrt{n\text{Var}[\sigma_1]}}\right)^2}{2} + O(1)\frac{\bar\eta^3}{\sqrt{n}}}\, d\bar\eta \\
&= \frac{e^{-\frac{(t-n\mathbb{E}[\sigma_1])^2}{2\text{Var}[\sigma_1]n}}}{n^{\frac{k+1}{2}}}\left(\int_{\Gamma} + \int_{|\bar\eta|\ge\kappa_0\sqrt{n}}\right)(i\bar\eta)^k e^{-\frac{\text{Var}[\sigma_1]\left(\bar\eta - i\frac{(t-n\mathbb{E}[\sigma_1])}{\sqrt{n\text{Var}\,[\sigma_1]}}\right)^2}{2} + O(1)\frac{\bar\eta^3}{\sqrt{n}}}\, d\bar\eta
\end{aligned}$$

$$\leq O(1)\frac{e^{-\frac{(t-n\mathbb{E}[\sigma_1])^2}{4\mathrm{Var}\,[\sigma_1]n}}}{\sqrt{\mathrm{Var}\,[\sigma_1]}\;n^{\frac{k+1}{2}}}+O(1)\frac{e^{-(\kappa_0)^2n/C}}{\sqrt{n\mathrm{Var}\,[\sigma_1]}} \tag{3.34}$$

where

$$\Gamma\ \subset\ \left\{|Re(z)|=\kappa_0\sqrt{n}, Im(z)\in\left(0,\frac{1}{2}\frac{t-n\mathbb{E}[\sigma_1]}{\sqrt{n\mathrm{Var}\,[\sigma_1]}}\right)\right\}$$
$$\cup\left\{|Re(z)|\leq\kappa_0\sqrt{n}, Im(z)=\frac{1}{2}\frac{t-n\mathbb{E}[\sigma_1]}{\sqrt{n\mathrm{Var}\,[\sigma_1]}}\right\}.$$

Here, we have used the condition that $O(1)\eta^3$ is an analytic function in η in order to apply the complex contour integral to yield the exponentially sharp estimates.

We use weighted energy estimates to prove the exponential decaying structure in t-variable. Denote by $\mathscr{T}_n(t)$ the probability density function of T_n,

$$\mathscr{T}_n(t)=\frac{\boldsymbol{P}(\mathsf{T}_n\in(t,t+dt))}{dt}=\underbrace{\mathscr{T}(t)*\mathscr{T}(t)*\cdots*\mathscr{T}(t)}_{n\text{ convolutions in }t.}.$$

We consider

$$\left(\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1]n)/C}\mathscr{T}_n(t)\right|^2dt\right)^{\frac{1}{2}}.$$

By Hölder inequality,

$$\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1]n)/C}\mathscr{T}_n(t)\right|^2dt$$
$$=\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1]n)/C}\int_{\mathbb{R}}\mathscr{T}(t-\tau)\mathscr{T}_{n-1}(\tau)d\tau\right|^2dt$$
$$=\int_{-\infty}^{\infty}\left|e^{-\frac{3}{2}\mathbb{E}[\sigma_1]/C}\int_{\mathbb{R}}\mathscr{T}(t-\tau)e^{(t-\tau)/C}\mathscr{T}_{n-1}(\tau)e^{(\tau-\frac{3}{2}\mathbb{E}[\sigma_1](n-1))/C}d\tau\right|^2dt$$
$$\leq\ e^{-3\mathbb{E}[\sigma_1]/C}\left(\int_{\mathbb{R}}\mathscr{T}(t)e^{t/C}dt\right)^2\int_{-\infty}^{\infty}\left|\mathscr{T}_{n-1}(\tau)e^{(\tau-\frac{3}{2}\mathbb{E}[\sigma_1](n-1))/C}\right|^2d\tau. \tag{3.35}$$

Now one can expand $e^{t/C}=1+t/C+(t/C)^2$ and substitute it into

$\int_{\mathbb{R}} \mathscr{T}(t)\ e^{t/C}dt$ to get that

$$\left(\int_{\mathbb{R}} \mathscr{T}(t)e^{t/C}dt\right)^2 = 1+2\mathbb{E}[\sigma_1]/C + O(1)1/C^2 \le e^{\frac{5}{2}\mathbb{E}[\sigma_1]/C}$$

for sufficiently large $C>0$. This and (3.35) result in that

$$\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1]n)/C}\mathscr{T}_n(t)\right|^2 dt \le e^{-\frac{1}{2}\mathbb{E}[\sigma_1]/C}\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1](n-1))/C}\mathscr{T}_{n-1}(t)\right|^2 dt, \tag{3.36}$$

and so

$$\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1]n)/C}\mathscr{T}_n(t)\right|^2 dt \le e^{-\frac{1}{2}(n-1)\mathbb{E}[\sigma_1]/C}\int_{-\infty}^{\infty}\left|e^{(t-\frac{3}{2}\mathbb{E}[\sigma_1])/C}\mathscr{T}(t)\right|^2 dt. \tag{3.37}$$

Similarly we have that

$$\int_{-\infty}^{\infty}\left|e^{-(t-\frac{1}{2}\mathbb{E}[\sigma_1]n)/C}\mathscr{T}_n(t)\right|^2 dt \le e^{-\frac{1}{2}(n-1)\mathbb{E}[\sigma_1]/C}\int_{-\infty}^{\infty}\left|e^{-(t-\frac{1}{2}\mathbb{E}[\sigma_1])/C}\mathscr{T}(t)\right|^2 dt. \tag{3.38}$$

The exponential decay estimates in (3.36) and (3.38) give the exponential decaying structure in t and n outside a hyperbolic region $|t-n\mathbb{E}[\sigma_1]| > \frac{1}{2}\mathbb{E}[\sigma_1]n$. Thus the estimate (3.30) follows. By same argument, one can prove (3.32) and (3.31). □

Lemma 3.9.(Estimates on Joint Probabilities) *There exists* $\gamma \in (0,1/6)$ *and* $C>0$ *such that, for* $|x| \le n^{\frac{1}{2}+\gamma}$,

$$\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x,x+dx)\times(y,y+dy),\ \ \mathsf{T}_n \in (t,t+dt)\}}{dxdydt} \le C\left(\frac{e^{-\frac{(x^2+y^2)}{Cn}-\frac{(t-n\mathbb{E}[\sigma_1])^2}{Cn}}}{n^{3/2}} + e^{-\frac{(|t-n\mathbb{E}[\sigma_1]|+n)}{C}}\right), \tag{3.39}$$

$$\left|\nabla_x\left(\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x,x+dx)\times(y,y+dy),\ \ \mathsf{T}_n \in (t,t+dt)\}}{dxdydt}\right)\right| \le C\left(\frac{e^{-\frac{(x^2+y^2)}{Cn}-\frac{(t-n\mathbb{E}[\sigma_1])^2}{Cn}}}{n^{2}} + e^{-\frac{(|t-n\mathbb{E}[\sigma_1]|+n)}{C}}\right), \tag{3.40}$$

and

$$\partial_t\left(\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x,x+dx)\times(y,y+dy),\ \ \mathsf{T}_n \in (t,t+dt)\}}{dxdydt}\right)$$

$$\leq C \left(\frac{e^{-\frac{(x^2+y^2)}{Cn} - \frac{(t-n\mathbb{E}[\sigma_1])^2}{Cn}}}{n^{\frac{3}{2}+(\frac{1}{2}-3\gamma)}} + e^{-\frac{(|t-n\mathbb{E}[\sigma_1]|+n)}{C}} \right). \tag{3.41}$$

Proof. We only prove (3.40) and (3.41), the estimate for (3.39) follows by similar argument.

Let $\{\vec{\boldsymbol{V}}_j\}_{j\in\mathbb{N}}$ denote the stochastic process representing the random reflected velocities, where $\vec{\boldsymbol{V}}_j \equiv (V_j^1, V_j^2, V_j^3)$. Since V_j^1, V_j^2, and V_j^3 are independent random variables, the probability measure $\boldsymbol{P}\{\mathsf{D}_n \in (x, x+dx) \times (y, y+dy), \mathsf{T}_n \in (t, t+dt)\}/(dxdydt)$ can be represented as the following iterated convolution integral

$$\begin{aligned}
&\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x, x+dx) \times (y, y+dy), \mathsf{T}_n \in (t, t+dt)\}}{dxdydt} \\
&= \frac{\boldsymbol{P}(\sum_{j=1}^n \sigma_j(V_j^1, V_j^2) \in (x, x+dx) \times (y, y+dy), \mathsf{T}_n \in (t, t+dt))}{dxdydt} \\
&= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{\boldsymbol{P}(\sum_{j=1}^n \delta_j(V_j^1, V_j^2) \in (x, x+dx) \times (y, y+dy))}{dxdy} \\
&\qquad\qquad \times \prod_{j=1}^n \mathscr{T}(\delta_j) dt_1 \cdots dt_{n-1},
\end{aligned} \tag{3.42}$$

where

$$\begin{cases} \delta_1 = t_1, \\ \delta_i = t_i - t_{i-1} \text{ for } i \in \{2, \ldots, n-1\}, \\ \delta_n = t - t_{n-1}. \end{cases}$$

Since $\{V_j^1\}_{j\in\mathbb{N}}$ and $\{V_j^2\}_{j\in\mathbb{N}}$ are mutual independent i.i.d. with the Gaussian distribution function $\mathsf{G}(\mathsf{v})$, it follows that

$$\frac{\boldsymbol{P}(\sum_{j=1}^n \delta_j(V_j^1, V_j^2) \in (x, x+dx) \times (y, y+dy))}{dxdy} = \frac{e^{-\dfrac{x^2+y^2}{2\theta \sum_{j=1}^n |\delta_j|^2}}}{2\theta\pi \sum_{j=1}^n |\delta_j|^2}. \tag{3.43}$$

First, we introduce the stochastic processes $\{\Delta_n\}_{n\in\mathbb{N}}$:

$$\Delta_n \equiv \sum_{j=1}^n \left(|\sigma_j|^2 - \mathbb{E}[\sigma_1^2] \right). \tag{3.44}$$

One can expand the factor $\frac{x^2+y^2}{\sum_{j=1}\delta_j^2} = \frac{x^2+y^2}{n\mathbb{E}[\sigma_1^2]+\Delta_n} = \left(\frac{x^2+y^2}{n\mathbb{E}[\sigma_1^2]}\right)(1+O(1)\frac{\Delta_n}{n})$, and split the integration domain in terms of Δ_n, i.e. $\{|\Delta_n| \le O(1)n^{\frac{1}{2}+\gamma}\} \cup \{|\Delta_n| \in [n^{\frac{1}{2}+\gamma}, \frac{1}{2}n\mathbb{E}[\sigma_1^2]]\} \cup \{|\Delta_n| > \frac{1}{2}n\mathbb{E}[\sigma_1^2]\}$. We have

$$\begin{aligned}
&\left|\partial_x\left(\frac{\boldsymbol{P}\{\mathsf{D}_n \in (x,x+dx)\times(y,y+dy), \mathsf{T}_n \in (t,t+dt)\}}{dxdydt}\right)\right| \\
&= \left|\partial_x\left(\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \frac{e^{-\frac{x^2+y^2}{2\theta\sum_{j=1}^n|\delta_j|^2}}}{2\theta\pi\sum_{j=1}^n|\delta_j|^2}\prod_{j=1}^n \mathscr{T}(\delta_j)dt_n\cdots dt_1\right)\right| \\
&\le O(1)\left|\left(\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \frac{1}{\sqrt{\sum_{j=1}^n|\delta_j|^2}}\frac{e^{-\frac{x^2+y^2}{8\theta\sum_{j=1}^n|\delta_j|^2}}}{4\theta\pi\sum_{j=1}^n|\delta_j|^2}\prod_{j=1}^n \mathscr{T}(\delta_j)dt_n\cdots dt_1\right)\right| \\
&= O(1)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \frac{e^{-\frac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{\left(4\theta\pi\mathbb{E}[\sigma_1^2]n\right)^{3/2}}\mathscr{T}(\delta_n)\prod_{j=1}^{n-1}\mathscr{T}(\delta_j)dt_n\cdots dt_1 \\
&\quad +O(1)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} \frac{e^{-\frac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{\left(4\theta\pi\mathbb{E}[\sigma_1^2]n\right)^{3/2}}\left(\frac{(x^2+y^2)}{n^{\frac{3}{2}-\gamma}}+n^{\gamma-\frac{1}{2}}\right)\mathscr{T}(\delta_n) \\
&\qquad\qquad\qquad\times\prod_{j=1}^{n-1}\mathscr{T}(\delta_j)dt_n\cdots dt_1 \\
&\quad +O(1)\frac{e^{-\frac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{4\theta\pi\mathbb{E}[\sigma_1^2]n}\boldsymbol{P}\left\{|\Delta_n| \in \left(\sqrt{\mathrm{Var}\,[\sigma_1^2]}\,n^{\frac{1}{2}+\gamma}, \frac{1}{2}n\mathbb{E}[\sigma_1^2]\right)\right\} \\
&\quad +O(1)\boldsymbol{P}(|\Delta_n| > \frac{1}{2}n\mathbb{E}[\sigma_1^2]) \\
&= \frac{e^{-\frac{x^2+y^2}{\left(8\theta\mathbb{E}[\sigma_1^2]\right)n^{3/2}}}}{\left(4\theta\pi\mathbb{E}[\sigma_1^2]n\right)^{3/2}}\frac{\boldsymbol{P}(\mathsf{T}_n \in (t,t+dt))}{dt} \\
&\quad +O(1)n^{-\frac{1}{2}+3\gamma}\frac{e^{-\frac{x^2+y^2}{4\theta\mathbb{E}[\sigma_1^2]n}}}{\left(4\theta\pi\mathbb{E}[\sigma_1^2]n\right)^{3/2}}\frac{\boldsymbol{P}(\mathsf{T}_n \in (t,t+dt))}{dt}
\end{aligned}$$

$$
\begin{aligned}
&+O(1)\frac{e^{-\frac{n^{2\gamma}}{C}}e^{-\dfrac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{4\theta\pi\mathbb{E}[\sigma_1^2]n}+O(1)e^{-n/C}\\
=\;&O(1)\frac{e^{-\dfrac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}-\dfrac{(t-\mathbb{E}[\sigma_1]n)^2}{4\mathrm{Var}\,[\sigma_1]n}}}{n^2}+O(1)e^{-n/C}.
\end{aligned}
\tag{3.45}
$$

This concludes (3.40). For (3.41) one has

$$
\begin{aligned}
&\partial_t\left(\frac{\boldsymbol{P}\{\mathsf{D}_n\in(x,x+dx)\times(y,y+dy),\mathsf{T}_n\in(t,t+dt)\}}{dxdydt}\right)\\
&=\partial_t\Bigg(\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\frac{e^{-\dfrac{x^2+y^2}{2\theta\sum_{j=1}^n|\delta_j|^2}}}{2\theta\pi\sum_{j=1}^n|\delta_j|^2}\prod_{j=1}^n\mathscr{T}(\delta_j)dt_n\cdots dt_1\Bigg)\\
&=\partial_t\Bigg(\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\frac{e^{-\dfrac{x^2+y^2}{2\theta\mathbb{E}[\sigma_1^2]n}}}{2\theta\pi\mathbb{E}[\sigma_1^2]n}\prod_{j=1}^n\mathscr{T}(\delta_j)dt_n\cdots dt_1\Bigg)\\
&\quad+O(1)\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\frac{e^{-\dfrac{x^2+y^2}{2\theta\mathbb{E}[\sigma_1^2]n}}}{2\theta\pi\mathbb{E}[\sigma_1^2]n}\frac{(x^2+y^2)}{n^{\frac{3}{2}-\gamma}}\mathscr{T}'(\delta_n)\prod_{j=1}^{n-1}\mathscr{T}(\delta_j)dt_n\cdots dt_1\\
&\quad+O(1)\frac{e^{-\dfrac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{4\theta\pi\mathbb{E}[\sigma_1^2]n}\boldsymbol{P}\left(|\Delta_n|\in\left(\sqrt{\mathrm{Var}\,[\sigma_1^2]}\,n^{\frac{1}{2}+\gamma},\frac{1}{2}n\mathbb{E}[\sigma_1^2]\right)\right)\\
&\quad+O(1)\boldsymbol{P}(|\Delta_n|>\frac{1}{2}n\mathbb{E}[\sigma_1^2])\\
&=\frac{e^{-\dfrac{x^2+y^2}{2\theta\mathbb{E}[\sigma_1^2]n}}}{2\theta\pi\mathbb{E}[\sigma_1^2]n}\partial_t\left(\frac{\boldsymbol{P}(\mathsf{T}_n\in(t,t+dt))}{dt}\right)\\
&\quad+O(1)n^{(-\frac{1}{2}+3\gamma)}\frac{e^{-\dfrac{x^2+y^2}{4\theta\mathbb{E}[\sigma_1^2]n}}}{4\theta\pi\mathbb{E}[\sigma_1^2]n}\frac{\boldsymbol{P}(\mathsf{T}_n\in(t,t+dt))}{dt}\\
&\quad+O(1)\frac{e^{-\frac{n^{2\gamma}}{C}}e^{-\dfrac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}}}{4\theta\pi\mathbb{E}[\sigma_1^2]n}+O(1)e^{-n/C}
\end{aligned}
$$

$$= O(1)\left(n^{-2}+n^{-\frac{3}{2}-\frac{1}{2}+3\gamma}\right) e^{-\dfrac{x^2+y^2}{8\theta\mathbb{E}[\sigma_1^2]n}-\dfrac{(t-\mathbb{E}[\sigma_1]n)^2}{4\mathrm{Var}\,[\sigma_1]n}} + O(1)e^{-n/C}, \quad (3.46)$$

and (3.41) follows. □

Lemma 3.10. *For any given* (x, y, t) *there exists* $C > 0$ *such that the infinite sum* $\sum_{l=1}^{\infty} \mathsf{Q}_l(x, y, t)$ *satisfies, for* $t \geq 1$,

$$\sum_{l=1}^{\infty} \mathsf{Q}_l(x, y, t) \leq O(1)\left(\frac{e^{-\frac{x^2+y^2}{C(1+t)}}}{(1+t)} + e^{-(|x|+|y|+t)/C}\right), \quad (3.47)$$

and there exists $\gamma \in (0, 1/6)$ *such that*

$$\sum_{l=1}^{\infty} |\partial_t \mathsf{Q}_l(x, y, t)| \leq O(1)\left(\frac{e^{-\frac{x^2+y^2}{C(1+t)}}}{(1+t)^{1+\gamma}} + e^{-(|x|+|y|+t)/C}\right), \quad (3.48)$$

$$\sum_{l=1}^{\infty} |\nabla_x \mathsf{Q}_l(x, y, t)| \leq O(1)\left(\frac{e^{-\frac{x^2+y^2}{C(1+t)}}}{(1+t)^2} + e^{-(|x|+|y|+t)/C}\right). \quad (3.49)$$

Proof. It is sufficient to prove (3.47), since (3.48) and (3.49) will follow by the same argument. By (3.23) one can related $\mathsf{Q}_l(x, y, t)dxdydt$ to $\boldsymbol{P}(\mathsf{D}_l \in (x, x+dx) \times (y, y+dy), \mathsf{T}_n \in (t, t+dt))$ estimated in Lemma 3.9.

The proof of (3.47) is done in the four cases, according to the value of

$$\boldsymbol{r} \equiv \sqrt{x^2+y^2}.$$

Case 1. $\boldsymbol{r} > t$.

We use (3.32) for $l \in (0, 2\boldsymbol{r}/\mathbb{E}[\sigma_1])$ and (3.30) with $k = 0$ for $l \in (2\boldsymbol{r}/\mathbb{E}[\sigma_1], \infty)$ in the following summation of Q_l:

$$\begin{aligned}
\sum_{l=1}^{\infty} \mathsf{Q}_l(x, y, t) &= \sum_{l=1}^{2\boldsymbol{r}/\mathbb{E}[\sigma_1]} \mathsf{Q}_l(x, y, t) + \sum_{l=|2\vec{\boldsymbol{x}}|/\mathbb{E}[\sigma_1]}^{\infty} \mathsf{Q}_l(x, y, t) \\
&= O(1)\left(\sum_{l=1}^{2\boldsymbol{r}/\mathbb{E}[\sigma_1]} \frac{e^{-\frac{\boldsymbol{r}^2}{4\theta\mathbb{E}[\sigma_1^2]l}}}{l} + \sum_{l=2\boldsymbol{r}/\mathbb{E}[\sigma_1]}^{\infty} \frac{e^{-\frac{|t-\mathbb{E}[\sigma_1]l|^2}{\mathrm{Var}\,[\sigma_1]l}}}{\sqrt{l}}\right) \\
&= O(1)e^{-\boldsymbol{r}/C} \text{ for some } C > 0. \qquad (3.50)
\end{aligned}$$

Case 2. $\boldsymbol{r} \in (t^{\frac{1}{2}+\gamma}, t)$.

We use (3.32) for $l \in (0, 2\mathbb{E}[\sigma_1]t)$ and (3.30) with $k = 0$ for $l \in (2\mathbb{E}[\sigma_1]t, \infty)$ in the following summation of Q_l:

$$
\begin{aligned}
\sum_{l=1}^{\infty} \mathsf{Q}_l(x,y,t) &= \sum_{l=1}^{2\mathbb{E}[\sigma_1]t} \mathsf{Q}_l(x,y,t) + \sum_{l=2\mathbb{E}[\sigma_1]t}^{\infty} \mathsf{Q}_l(x,y,t) \\
&= O(1)\left(\sum_{l=1}^{2\mathbb{E}[\sigma_1]t} \frac{e^{-\frac{\boldsymbol{r}^2}{4\theta\mathbb{E}[\sigma_1^2]l}}}{l} + \sum_{l=2\mathbb{E}[\sigma_1]t}^{\infty} \frac{e^{-\frac{|t-\mathbb{E}[\sigma_1]l|^2}{\mathrm{Var}\,[\sigma_1]l}}}{\sqrt{l}} \right) \\
&= O(1)\left(e^{-t^{2\gamma}-\frac{\boldsymbol{r}^2}{Ct}} + e^{-t/C} \right) \text{ for some } C > 0. \qquad (3.51)
\end{aligned}
$$

Case 3. $\boldsymbol{r} \in (0, t^{\frac{1}{2}+\gamma})$.

We use (3.39) for $l \in (\frac{1}{2}t/\mathbb{E}[\sigma_1], 2t/\mathbb{E}[\sigma_1])$, and (3.30) with $k = 0$ for $l \in (0, \frac{1}{2}t/\mathbb{E}[\sigma_1]) \cup (2t/\mathbb{E}[\sigma_1], \infty)$ in the following summation of Q_l:

$$
\begin{aligned}
&\sum_{l=1}^{\infty} \mathsf{Q}_l(x,y,t) = \sum_{l=1}^{\frac{1}{2}t/\mathbb{E}[\sigma_1]} \mathsf{Q}_l(x,y,t) + \sum_{l=\frac{1}{2}t/\mathbb{E}[\sigma_1]}^{2t/\mathbb{E}[\sigma_1]} \mathsf{Q}_l(x,y,t) + \sum_{l=2t/\mathbb{E}[\sigma_1]}^{\infty} \mathsf{Q}_l(x,y,t) \\
&= O(1)\left(\sum_{l=1}^{\frac{1}{2}t/\mathbb{E}[\sigma_1]} \frac{e^{-\frac{|t-\mathbb{E}[\sigma_1]l|^2}{\mathrm{Var}\,[\sigma_1]l}}}{\sqrt{l}} + \sum_{l=\frac{1}{2}t/\mathbb{E}[\sigma_1]}^{2t/\mathbb{E}[\sigma_1]} \frac{e^{-\frac{\boldsymbol{r}^2}{4\theta\mathbb{E}[\sigma_1^2]l}}}{l} \frac{e^{-\frac{|t-\mathbb{E}[\sigma_1]l|^2}{\mathrm{Var}\,[\sigma_1]l}}}{\sqrt{l}} \right. \\
&\quad \left. + \sum_{l=2t/\mathbb{E}[\sigma_1]}^{\infty} \frac{e^{-\frac{|t-\mathbb{E}[\sigma_1]l|^2}{\mathrm{Var}\,[\sigma_1]l}}}{\sqrt{l}} \right) = O(1)\frac{e^{-\frac{\boldsymbol{r}^2}{C(1+t)}}}{(1+t)} \text{ for some } C > 0. \qquad (3.52)
\end{aligned}
$$

□

Before proceeding to the proof of Main Theorem A, we denote $\Gamma(\tau)$ a trajectory of the flow $\ddot{\Gamma}(\tau) = -\vec{\boldsymbol{g}}$ with condition $\Gamma(t) = \vec{\boldsymbol{x}}$ and $\dot{\Gamma}(t) = \xi$:

For a given $(\vec{\boldsymbol{x}}, t) \in \mathbb{H}^+ \times \mathbb{R}^+$, the trajectory $\Gamma(\tau)$ is given by

$$\Gamma(\tau) = (\Gamma^1(\tau), \Gamma^2(\tau), \Gamma^3(\tau)),$$
$$\begin{cases} \Gamma^1(\tau) = x + \xi^1(\tau - t), \\ \Gamma^2(\tau) = y + \xi^2(\tau - t), \\ \Gamma^3(\tau) = z + \xi^3(\tau - t) - \frac{g}{2}(t-\tau)^2, \\ \tau_0 = t - \frac{\sqrt{(\xi^3)^2 + 2gz} - \xi^3}{g}. \end{cases} \tag{3.53}$$

Proposition 3.11. *For any fixed $\alpha_0 > 0$ and $\gamma > 0$ there exists $C_* > 0$ such that the trajectory Γ given in (3.53) satisfies*

$$\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}} \mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0} \left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2 + |\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma} + e^{-\frac{3(|\Gamma^1(\tau)| + |\Gamma^2(\tau)| + \tau)}{2C_*}} \right)\Bigg|_{\tau=\tau_0}$$
$$\le O(1) \left(\frac{e^{-\frac{(x^2+y^2)}{C_*(1+t)}}}{2(1+\tau)^\gamma} + e^{-\frac{(|x|+|x|+t)}{C_*}} \right) \tag{3.54}$$

and

$$\int_{\tau_0}^t e^{-\frac{\alpha_0 g \Gamma^3(\tau)}{\theta}} \left(e^{-\frac{g\Gamma^3(\tau)}{\theta}} \mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0}$$
$$\times \left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2 + |\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma} + e^{-\frac{3(|\Gamma^1(\tau)| + |\Gamma^2(\tau)| + \tau)}{2C_*}} \right) d\tau$$
$$\le O(1) \left(\frac{e^{-\frac{(x^2+y^2)}{C_*(1+t)}}}{2(1+\tau)^\gamma} + e^{-\frac{(|x|+|x|+t)}{C_*}} \right) \quad \text{for all } (\vec{\boldsymbol{x}}, \xi, t) \in \mathbb{H}^+ \times \mathbb{R}^3 \times \mathbb{R}^+. \tag{3.55}$$

Proof. We denote

$$\begin{cases} \Gamma_{\|}(\tau) = (\Gamma^1(\tau), \Gamma^2(\tau), 0), \ \ \Gamma_\perp(\tau) = (0, 0, \Gamma^3(\tau)), \\ \vec{\boldsymbol{x}}_{\|} = (x, y, 0), \ \ \vec{\boldsymbol{x}}_\perp = (0, 0, z), \\ d(\tau) = |\Gamma_{\|}(\tau) - \vec{\boldsymbol{x}}_{\|}|. \end{cases}$$

From the conservation law of total energy one has

$$|\dot{\Gamma}(\tau)|^2 + 2g\Gamma^3(\tau) = |\dot{\Gamma}(t)|^2 + 2g\Gamma^3(t) = |\xi|^2 + 2gz. \tag{3.56}$$

We have the estimate for $\tau\in(\tau_0,t)$

$$\max\left\{|\dot{\Gamma}(t)|^2,|\dot{\Gamma}(\tau)|^2\right\}\geq gd(\tau). \tag{3.57}$$

This and (3.56) yield

$$\max\left\{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi),e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot{\Gamma}(\tau))\right\}\leq\frac{1}{(2\pi)^{3/2}}e^{-\frac{gd(\tau)}{2\theta}}. \tag{3.58}$$

Note (3.57) follows directly form the fact that $\frac{v^2}{g}$ is the maximum horizontal displacement of a projectile with fixed speed v under a constant gravitational force.

Case 1. $|\vec{\boldsymbol{x}}_{\|}|\leq\sqrt{1+t}$.

In this case by the property that $\Gamma^3(\tau)=z+\xi^3(t-\tau)-\frac{1}{2}g(t-\tau)^2$ it follows

$$\begin{aligned}&\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0}\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\\ &\leq O(1)\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{1}{2(1+\tau)^\gamma}+e^{-\frac{3\tau}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\leq O(1)\frac{1}{(1+t)^\gamma}\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}&\int_{\tau_0}^t e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0}\\ &\qquad\times\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)d\tau\\ &\leq O(1)\int_{\tau_0}^t e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{1}{2(1+\tau)^\gamma}+e^{-\frac{3\tau}{2C_*}}\right)d\tau\\ &\leq O(1)\frac{1}{(1+t)^\gamma}.\end{aligned} \tag{3.60}$$

Case 2. $|\vec{\boldsymbol{x}}_{\|}|\in(\sqrt{1+t},t)$.

We break this case into two situations:

a. $|\Gamma_{\|}(\tau)|\geq\frac{5}{6}|\vec{\boldsymbol{x}}_{\|}|$ for all $\tau\in[\tau_0,t]$.

In this situation one has

$$\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot\Gamma(\tau))\right)^{\alpha_0}\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)\Bigg|_{\tau=\tau_0}$$

$$\leq O(1)\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{e^{-\frac{|\vec{x}_\parallel|^2}{C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3\tau}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\leq O(1)\frac{e^{-\frac{|\vec{x}_\parallel|^2}{C_*(1+t)}}}{(1+t)^\gamma} \quad (3.61)$$

and

$$\int_{\tau_0}^t e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot\Gamma(\tau))\right)^{\alpha_0}$$
$$\times\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)d\tau$$
$$\leq O(1)\int_{\tau_0}^t e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{e^{-\frac{|\vec{x}_\parallel|^2}{C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3\tau}{2C_*}}\right)d\tau$$
$$\leq O(1)\frac{e^{-\frac{|\vec{x}_\parallel|^2}{C_*(1+t)}}}{(1+t)^\gamma}. \quad (3.62)$$

b. $\tau_0<\tau_*$, where

$$\tau_*\equiv\max_{\substack{\tau<t\\|\Gamma_\parallel(\tau)|=5|\vec{x}_\parallel|/6}}\tau.$$

In this situation by (3.58) one has

$$\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot\Gamma(\tau))\right)^{\alpha_0}\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)\Bigg|_{\tau=\tau_0}$$

$$\leq O(1)\,e^{-\frac{g\alpha_0|\vec{x}_\parallel|}{10\theta}}\left(\frac{1}{2(1+\tau)^\gamma}+e^{-\frac{3\tau}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\leq O(1)\frac{e^{-\frac{|\vec{x}_\parallel|^2}{C_*(1+t)}}}{(1+t)^\gamma} \quad (3.63)$$

and

$$\left(\int_{\tau_*}^t+\int_{\tau_0}^{\tau_*}\right)e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot\Gamma(\tau))\right)^{\alpha_0}$$
$$\times\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^\gamma}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)d\tau$$

$$
\begin{aligned}
&\le O(1)\int_{\tau_*}^{t} e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{e^{-\frac{|\vec{x}_{\|}|^2}{C_*(1+\tau)}}}{2(1+\tau)^{\gamma}}+e^{-\frac{3\tau}{2C_*}}\right)d\tau\\
&\quad +O(1)\int_{\tau_0}^{\tau_*} e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}} e^{-\frac{g\alpha_0|\vec{x}_{\|}|}{10\theta}}\left(\frac{1}{2(1+\tau)^{\gamma}}+e^{-\frac{3\tau}{2C_*}}\right)d\tau\\
&\le O(1)\frac{e^{-\frac{|\vec{x}_{\|}|^2}{C_*(1+t)}}}{(1+t)^{\gamma}}. \qquad (3.64)
\end{aligned}
$$

Case 3. $|\vec{x}_{\|}| \ge (1+t)$. We also break this case into the same situations as (3.62) and (3.64). Here, the estimate (3.62) is also valid for $|\vec{x}_{\|}| > t$. Thus, there is only one situation need to verify:

$$
\begin{aligned}
&\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0}\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^{\gamma}}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\\
&\le O(1)\, e^{-\frac{g\alpha_0|\vec{x}_{\|}|}{10\theta}}\left(\frac{1}{2(1+\tau)^{\gamma}}+e^{-\frac{3\tau}{2C_*}}\right)\Bigg|_{\tau=\tau_0}\\
&\le O(1)\left(\frac{e^{-\frac{|\vec{x}_{\|}|^2}{C_*(1+t)}}}{(1+t)^{\gamma}}+e^{-\frac{|\vec{x}_{\|}|+t}{C_*}}\right) \qquad (3.65)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{\tau_*}^{t}+\int_{\tau_0}^{\tau_*}\right)e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\mathsf{M}(\dot{\Gamma}(\tau))\right)^{\alpha_0}\\
&\qquad\times\left(\frac{e^{-\frac{3(|\Gamma^1(\tau)|^2+|\Gamma^2(\tau)|^2)}{2C_*(1+\tau)}}}{2(1+\tau)^{\gamma}}+e^{-\frac{3(|\Gamma^1(\tau)|+|\Gamma^2(\tau)|+\tau)}{2C_*}}\right)d\tau\\
&\le O(1)\int_{\tau_*}^{t} e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}}\left(e^{-\frac{g\Gamma^3(\tau)}{\theta}}\right)^{\alpha_0}\left(\frac{e^{-\frac{|\vec{x}_{\|}|^2}{C_*(1+\tau)}}}{2(1+\tau)^{\gamma}}+e^{-\frac{3\tau}{2C_*}}\right)d\tau\\
&\quad +O(1)\int_{\tau_0}^{\tau_*} e^{-\frac{\alpha_0 g\Gamma^3(\tau)}{\theta}} e^{-\frac{g\alpha_0|\vec{x}_{\|}|}{10\theta}}\left(\frac{1}{2(1+\tau)^{\gamma}}+e^{-\frac{3\tau}{2C_*}}\right)d\tau\\
&\le O(1)\left(\frac{e^{-\frac{|\vec{x}_{\|}|^2}{C_*(1+t)}}}{(1+t)^{\gamma}}+e^{-\frac{|\vec{x}_{\|}|+t}{C_*}}\right). \qquad (3.66)
\end{aligned}
$$

□

Proof of Main Theorem A. From (3.26), (3.27), and (1.7) one has that

$$|\mathsf{t}^+(x,y,t)| \le O(1)e^{-(\sqrt{x^2+y^2}+t^2)/C} \text{ for some } C>0. \tag{3.67}$$

This, (3.47), and (3.21) yield that

$$\rho(x,y,t) \le O(1)\left(\frac{e^{-\frac{x^2+y^2}{C(1+t)}}}{(1+t)} + e^{-(|x|+|y|+t)/C}\right) \text{ for some } C>0. \tag{3.68}$$

Since $\mathsf{t}(\vec{\boldsymbol{x}},t,\xi)$ is invariant along any characteristic curve $\Gamma(\tau)$ given by (3.53), we have, from (3.54),

$$\begin{aligned}
&\mathsf{t}(x,y,z,t,\xi^1,\xi^2,\xi^3) = \rho\left(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0\right)\mathsf{M}(\dot\Gamma(\tau_0))\\
&= \rho\left(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0\right)\mathsf{M}(\dot\Gamma(\tau_0))^{1-\alpha}\mathsf{M}(\dot\Gamma(\tau_0))^{\alpha}\\
&= \rho\left(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0\right)\mathsf{M}(\dot\Gamma(\tau_0))^{1-\alpha}\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi^1,\xi^2,\xi^3)\right)^{\alpha}\\
&\le O(1)\left(\frac{e^{-\frac{(x^2+y^2)}{C_*(1+t)}}}{(1+t)} + e^{-\frac{|x|+|y|+t}{C_*}}\right)\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi^1,\xi^2,\xi^3)\right)^{\alpha} \text{ for some } C_*>C.
\end{aligned} \tag{3.69}$$

This concludes (1.8) for some $C_*>0$.

To obtain (1.10) we need to obtain a space-time shift estimate for an initial data satisfying (1.7) and (1.9). From (3.25) and (1.10) we have

$$\int_0^\infty\left(\int_{\mathbb{R}^2}\mathsf{t}^+(x,y,t)dxdy\right)dt = 0. \tag{3.70}$$

We decompose $\mathsf{t}^+(x,y,t)$ as follows

$$\begin{aligned}
&\mathsf{t}^+(x,y,t) \equiv \bar{\mathsf{t}}^+(t)\mathsf{G}(x)\mathsf{G}(y) + \mathsf{t}^+_\Delta(x,y,t),\\
&\begin{cases}\bar{\mathsf{t}}^+(t) \equiv \displaystyle\int_{\mathbb{R}^2}\mathsf{t}^+(x,y,t)dxdy,\\ \mathsf{t}^+_\Delta(x,y,t) \equiv \mathsf{t}^+(x,y,t) - \bar{\mathsf{t}}^+(t)\mathsf{G}(x)\mathsf{G}(y).\end{cases}
\end{aligned} \tag{3.71}$$

From (3.70), one has that $\int_0^\infty \bar{\mathsf{t}}^+(t)dt = 0$. This and (3.67) result in that

$$\begin{cases}\left|\displaystyle\int_0^t \bar{\mathsf{t}}^+(\tau)d\tau\right| \le O(1)e^{-t^2/C'} \text{ for some } C'>0,\\ \displaystyle\int_{\mathbb{R}^2}\mathsf{t}^+_\Delta(x,y,t)dxdy \equiv 0,\\ |\mathsf{t}^+_\Delta(x,y,t)| \le O(1)e^{-\frac{t^2+\sqrt{x^2+y^2}}{C'}} \text{ for some } C'>0.\end{cases} \tag{3.72}$$

Substitute the decomposition $\mathsf{t}^+(x,y,t) = \bar{\mathsf{t}}^+(t)\mathsf{G}(x)\mathsf{G}(y) + \mathsf{t}^+_\Delta(x,y,t)$ into (3.21) then it follow from (3.48), (3.49), and (3.72) that

$$\begin{aligned}
\rho(x,y,t) &= \left(1+\sum_{l=1}^{\infty}\mathsf{Q}_l(x,y,t)\right) * \left(\partial_t\int_0^t \bar{\mathsf{t}}^+(\tau)\mathsf{G}(x)\mathsf{G}(y)d\tau + \mathsf{t}^+_\Delta(x,y,t)\right)\\
&= \left(\sum_{l=1}^{\infty}\partial_t\mathsf{Q}_l(x,y,t)\right) * \int_0^t \bar{\mathsf{t}}^+(\tau)\mathsf{G}(x)\mathsf{G}(y)d\tau\\
&\quad + \sum_{l=1}^{\infty}\int_0^t\int_{\mathbb{R}^2}\left(\mathsf{Q}_l(x-\bar{x},y-\bar{y},t-\tau)-\mathsf{Q}_l(x,y,t-\tau)\right)\mathsf{t}^+_\Delta(\bar{x},\bar{y},\tau)d\bar{x}d\bar{y}d\tau\\
&= \left(\sum_{l=1}^{\infty}\partial_t\mathsf{Q}_l(x,y,t)\right) * \int_0^t \bar{\mathsf{t}}^+(\tau)\mathsf{G}(x)\mathsf{G}(y)d\tau\\
&\quad - \int_0^t\int_{\mathbb{R}^2}\int_0^1(\bar{x},\bar{y})\cdot\nabla_{(x,y)}\sum_{l=1}^{\infty}\mathsf{Q}_l(x-s\bar{x},y-s\bar{y},t-\tau)\mathsf{t}^+_\Delta(\bar{x},\bar{y},\tau)dsd\bar{x}d\bar{y}d\tau\\
&\le O(1)\left(\frac{e^{-\frac{x^2+y^2}{C_*(1+t)}}}{(1+t)^{1+\gamma'}}+e^{-(|x|+|y|+t)/C_*}\right)\text{for some } \gamma'\in(0,1/6), C_*>C'. \qquad (3.73)
\end{aligned}$$

This estimates the boundary function $\rho(x,y,t)$. Thus (1.10) follows from the characteristic curve representation in (3.69) with the factor $1/(1+t)$ replaced by $1/(1+t)^{1+\gamma'}$. □

4. Construction of Solutions of the Boltzmann Equation

We will apply the method of continuity to the nonlinear problem (1.1) together with the strong linear estimates yielded by Theorem 1.1. First, we introduce a norm $|||\cdot|||_T$ with $T\ge 0$:

$$|||\mathsf{g}|||_T \equiv \sup_{t\in[0,T]}\left(\sup_{\substack{\vec{\boldsymbol{x}}\in\mathbb{H},\\ \xi\in\mathbb{R}^3}}\frac{|\mathsf{g}(\vec{\boldsymbol{x}},t,\xi)|}{\left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{(1+t)}+e^{-(|x|+|y|+t)/C_0}\right)\sqrt{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)}} + \sup_{\substack{\vec{\boldsymbol{x}}\in\mathbb{H}\\ \xi\in\mathbb{R}^3}}\frac{|\mathsf{g}(\vec{\boldsymbol{x}},t,\xi)|}{\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{\frac{3}{4}}}\right) \qquad (4.1)$$

with $C_0=\frac{3}{2}C_*$, where C_* is the constant given in Theorem 1.1.

Let $\mathsf{f}(\vec{\boldsymbol{x}}, t, \xi)$ be the solution of (1.1) and let $\mathscr{I}$ denote the set

$$\mathscr{I} \equiv \{T : |||\mathsf{f}|||_T \leq \epsilon^{2/3}\}, \tag{4.2}$$

which is not an empty set since $|||\mathsf{f}|||_0 = O(1)\epsilon \ll \epsilon^{2/3}$. Thus

$$0 \in \mathscr{I}.$$

We will show that this set is both open and close.

The close part is a direct consequence of the definition of the norm $||| \cdot |||_T$. Next we continue to prove the open part.

First we decompose the collision operator Q into gain and lost parts $Q = Q_+ - Q_-$. Then we consider the iteration scheme to construct a local solution:

$$\begin{cases} \partial_t \mathsf{f}_0 + \xi \cdot \nabla_x \mathsf{f}_0 - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{f}_0 = -Q_-(\mathsf{f}_0), \\ \partial_t \mathsf{f}_k + \xi \cdot \nabla_x \mathsf{f}_k - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{f}_k = -Q_-(\mathsf{f}_k) + Q_+(\mathsf{f}_{k-1}) \text{ for } k \geq 1. \end{cases} \tag{4.3}$$

Lemma 4.1.(Local Existence) *Suppose that the initial data* $\mathsf{f}(\vec{\boldsymbol{x}}, 0, \xi) \geq 0$ *and that* ϵ *is sufficiently small. Then, there exists* $\alpha > 0$ *and* $\tau_0 > 0$ *such that the following holds for any given* $t_0 \in \mathscr{I}$ *and for any* $t_1 \in (0, \tau_0)$

$$|||\mathsf{f}|||_{t_0+t_1} < \alpha \epsilon^{2/3}. \tag{4.4}$$

Proof. First, let $\tau_0 \equiv 1$. Since the initial data $\mathsf{f}(\vec{\boldsymbol{x}}, t, \xi)$ is a non-negative-valued function, the solution $\mathsf{f}(\vec{\boldsymbol{x}}, t, \xi)$ is non-negative for $t = t_0 \in \mathscr{I}$. Now, we consider the following iteration scheme for $t > t_0$

$$\begin{cases} \partial_t \mathsf{f}_0 + \xi \cdot \nabla_x \mathsf{f}_0 - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{f}_0 = -Q_-(\mathsf{f}_0), \\ \partial_t \mathsf{f}_k + \xi \cdot \nabla_x \mathsf{f}_k - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{f}_k = -Q_-(\mathsf{f}_k) + Q_+(\mathsf{f}_{k-1}) \text{ for } k \geq 1, \\ \mathsf{f}_k(\vec{\boldsymbol{x}}, t_0, \xi) = \mathsf{f}(\vec{\boldsymbol{x}}, t_0, \xi) \text{ for } k \geq 0. \end{cases} \tag{4.5}$$

From Lemma 2.2 the loss operator $Q_-(\mathsf{f}_k)$ can be expressed as a product $Q_-(\mathsf{f}_k)(\xi) = q_-[\mathsf{f}_k](\xi) \cdot \mathsf{f}_k(\xi)$. This, and the positivity of Q_+ as an operator, i.e., $Q_+(\mathsf{g})(\xi) > 0$ for any non-negative valued function g, yields that the functions $\mathsf{f}_k(\vec{\boldsymbol{x}}, t, \xi)$ are all non-negative valued functions

$$\mathsf{f}_k(\vec{\boldsymbol{x}}, t, \xi) \geq 0 \text{ for } t \geq t_0,\ k \geq 0. \tag{4.6}$$

This iteration scheme yields that

$$\begin{cases} \partial_t(\mathsf{f}_1-\mathsf{f}_0)+\xi\cdot\nabla_x(\mathsf{f}_1-\mathsf{f}_0)-\vec{\boldsymbol{g}}\cdot\nabla_\xi(\mathsf{f}_1-\mathsf{f}_0)+q_-[\mathsf{f}_1+\mathsf{f}_0](\mathsf{f}_1-\mathsf{f}_0)=Q_+(\mathsf{f}_0), \\ \partial_t(\mathsf{f}_{k+1}-\mathsf{f}_k)+\xi\cdot\nabla_x(\mathsf{f}_{k+1}-\mathsf{f}_k)-\vec{\boldsymbol{g}}\cdot\nabla_\xi(\mathsf{f}_{k+1}-\mathsf{f}_k)+q_-[\mathsf{f}_{k+1}+\mathsf{f}_k](\mathsf{f}_{k+1}-\mathsf{f}_k) \\ \qquad =Q_+(\mathsf{f}_k)-Q_+(\mathsf{f}_{k-1}) \text{ for } k\geq 1. \end{cases} \tag{4.7}$$

This gives rise to the representations:

$$\begin{aligned} &(\mathsf{f}_1-\mathsf{f}_0)(\vec{\boldsymbol{x}},t,\xi) \\ &=e^{-\int_{\tau_0}^t q_-[\frac{\mathsf{f}_1+\mathsf{f}_0}{2}](\Gamma(s),s,\dot\Gamma(s))ds}\frac{\sqrt{2\pi}(\rho_1-\rho_0)(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)}{\sqrt{\theta}}\mathsf{M}(\dot\Gamma(\tau_0)) \\ &\quad+\int_{\tau_0}^t e^{-\int_\tau^t q_-[\frac{\mathsf{f}_1+\mathsf{f}_0}{2}](\Gamma(s),s,\dot\Gamma(s))ds}Q_+(\mathsf{f}_0)(\Gamma(\tau),\tau,\dot\Gamma(\tau))d\tau, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &(\mathsf{f}_{k+1}-\mathsf{f}_k)(\vec{\boldsymbol{x}},t,\xi) \\ &=e^{-\int_{\tau_0}^t q_-[\frac{\mathsf{f}_{k+1}+\mathsf{f}_k}{2}](\Gamma(s),s,\dot\Gamma(s))ds}\frac{\sqrt{2\pi}(\rho_{k+1}-\rho_k)(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)}{\sqrt{\theta}}\mathsf{M}(\dot\Gamma(\tau_0)) \\ &\quad+\int_{\tau_0}^t e^{-\int_\tau^t q_-[\frac{\mathsf{f}_{k+1}+\mathsf{f}_k}{2}](\Gamma(s),s,\dot\Gamma(s))ds}\left(Q_+(\mathsf{f}_k)-Q_+(\mathsf{f}_{k-1})\right)(\Gamma(\tau),\tau,\dot\Gamma(\tau))d\tau, \end{aligned} \tag{4.9}$$

where $\Gamma(\tau)$ is given in (3.53), and

$$\rho_l(x,y,t)\equiv-\int_{\xi^3<0}\xi^3\mathsf{f}_l(x,y,0,t,\xi)d\xi \text{ for } l\geq 0.$$

Here the function $q_-[\mathsf{f}_{k+1}+\mathsf{f}_k](\vec{\boldsymbol{x}},t,\xi)$ is a non-negative valued function so that the integration factor $e^{-\int_{\tau_0}^t q_-[(\mathsf{f}_{k+1}+\mathsf{f}_k)/2]ds}$ in the representation (4.9) is bounded by 1 from the above. Thus, we can have the comparison

$$\begin{cases} |\mathsf{f}_0(\vec{\boldsymbol{x}},t,\xi)|\leq\mathsf{m}(\vec{\boldsymbol{x}},t,\xi), \\ |(\mathsf{f}_{k+1}-\mathsf{f}_k)(\vec{\boldsymbol{x}},t,\xi)|\leq\mathsf{m}_k(\vec{\boldsymbol{x}},t,\xi) \text{ for } t\geq t_0, \end{cases} \tag{4.10}$$

where $\mathsf{m}(\vec{\boldsymbol{x}},t,\xi)$ and $\mathsf{m}_k(\vec{\boldsymbol{x}},t,\xi)$ are the solutions of

$$\begin{cases} \partial_t\mathsf{m}+\xi\cdot\nabla_x\mathsf{m}-\vec{\boldsymbol{g}}\cdot\nabla_\xi\mathsf{m}=0, \\ \mathsf{m}(\vec{\boldsymbol{x}},t_0,\xi)\equiv|\mathsf{f}(\vec{\boldsymbol{x}},t_0,\xi)|, \end{cases}$$

$$\begin{cases} \partial_t \mathsf{m}_0 + \xi \cdot \nabla_x \mathsf{m}_0 - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{m}_0 = |Q_+(\mathsf{f}_0)|, \\ \mathsf{m}_0(\vec{\boldsymbol{x}}, t_0, \xi) \equiv 0, \end{cases}$$

$$\begin{cases} \partial_t \mathsf{m}_k + \xi \cdot \nabla_x \mathsf{m}_k - \vec{\boldsymbol{g}} \cdot \nabla_\xi \mathsf{m}_k = |Q_+(\mathsf{f}_k) - Q_+(\mathsf{f}_{k-1})|, \\ \mathsf{m}_k(\vec{\boldsymbol{x}}, t_0, \xi) \equiv 0 \text{ for } k \geq 1. \end{cases}$$

Thus, we have the following estimates

$$|\mathsf{f}_0(\vec{\boldsymbol{x}}, t, \xi)| \leq \mathbb{T}^{t-t_0}[\mathsf{f}(\cdot, t_0, \cdot)](\vec{\boldsymbol{x}}, \xi), \tag{4.11}$$

$$|(\mathsf{f}_1 - \mathsf{f}_0)(\vec{\boldsymbol{x}}, t, \xi)| \leq \left| \int_{t_0}^{t} \mathbb{T}^{t-\tau} \left[\left| Q_+\Big(\mathsf{f}_0(\cdot, \tau, \cdot)\Big)\right| \right] d\tau(\vec{\boldsymbol{x}}, \xi) \right|, \tag{4.12}$$

and

$$\begin{aligned} &|(\mathsf{f}_{k+1} - \mathsf{f}_k)(\vec{\boldsymbol{x}}, t, \xi)| \\ &\quad \leq \left| \int_{t_0}^{t} \mathbb{T}^{t-\tau} \left[\left| \mathsf{B}_+\Big((\mathsf{f}_k - \mathsf{f}_{k-1})(\cdot, \tau, \cdot), (\mathsf{f}_k + \mathsf{f}_{k-1})(\cdot, \tau, \cdot)\Big)\right| \right] d\tau(\vec{\boldsymbol{x}}, \xi) \right| \\ &\qquad \text{for } k \geq 1. \end{aligned} \tag{4.13}$$

From (4.11) and Theorem 1.1, there exists $\alpha > 0$ such that

$$|||\mathsf{f}_0|||_{t_0+1} \leq \frac{\alpha}{2} \epsilon^{2/3}. \tag{4.14}$$

This, (4.12), (2.2), and Theorem 1.1 yield that, for $t \in (t_0, t_0 + 1)$,

$$|(\mathsf{f}_1 - \mathsf{f}_0)(\vec{\boldsymbol{x}}, t, \xi)| \leq O(1)\epsilon^{4/3}\alpha^3 \begin{cases} \left(\dfrac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{(1+t)} + e^{-\frac{|x|+|y|+t}{C_0}} \right) \sqrt{e^{-\frac{gz}{\theta}} \mathsf{M}(\xi)}, \\ \left(e^{-\frac{gz}{\theta}} \mathsf{M}(\xi) \right)^{3/4}. \end{cases} \tag{4.15}$$

From (4.15) we can make a priori assumption that, for $k \geq 0$,

$$|||\mathsf{f}_k|||_{t_0+1} \leq 2\epsilon^{2/3}\alpha \tag{4.16}$$

Under this assumption, we have from (4.16), (2.2), (4.13), and Theorem 1.1 that, for $t \in (t_0, t_0 + 1)$,

$$|(\mathsf{f}_{k+1} - \mathsf{f}_k)(\vec{\boldsymbol{x}}, t, \xi)| \leq O(1)\epsilon^{k/3} |(\mathsf{f}_1 - \mathsf{f}_0)(\vec{\boldsymbol{x}}, t, \xi)|$$

$$\leq O(1)\epsilon^{(k+4)/3}\begin{cases}\left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{(1+t)}+e^{-\frac{|x|+|y|+t}{C_0}}\right)\sqrt{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)},\\ \left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{3/4}.\end{cases} \tag{4.17}$$

This, (4.14), and (4.16) yield that

$$|||\mathsf{f}|||_{t_0+1}\leq\frac{1}{2}\alpha\left(1+O(1)\epsilon^{1/3}\right)\epsilon^{2/3}. \tag{4.18}$$

This concludes a priori assumption (4.16), and the lemma follows. □

Lemma 4.2.(Anastz Closure) *The local solution* f *of* (1.1) *constructed in Lemma* 4.1 *with the property* (4.4) *satisfies*

$$|||\mathsf{f}|||_{t_0+1}=O(1)\epsilon. \tag{4.19}$$

Proof. The local solution f constructed in Lemma 4.1 satisfies that

$$|||\mathsf{f}|||_{t_0+1}\leq\alpha\epsilon^{2/3}. \tag{4.20}$$

Next we continue to compute $\int_{t_0}^{t}\mathbb{T}^{t-\tau}[Q(\mathsf{f}(\cdot,\tau,\cdot))](\vec{\boldsymbol{x}},\xi)d\tau$ for $t\in(t_0,t_0+1]$. Consider the partition of unity $\{\chi_{j,l}\}$:

$$\chi_{j,l}(x,y)=\begin{cases}1 \text{ for } (x,y)\in[j-\frac{1}{4},j+\frac{1}{4}]\times[l-\frac{1}{4},l+\frac{1}{4}],\\ 0 \text{ for } (x,y)\notin(j-\frac{3}{4},j+\frac{3}{4}]\times(l-\frac{3}{4}1,l+\frac{3}{4}),\end{cases}$$
$$\sum_{j,l}\chi_{j,l}(x,y)=1.$$

From $(1,\chi_{jl}(x,y)Q(\mathsf{f}))=0$, we can apply (1.10) to $\chi_{jl}(x,y)Q(\mathsf{f})$, to show that there exist $\gamma\in(0,1/6)$ and $\alpha\in(0,1/2)$ such that

$$\left|\mathbb{T}^{t-\tau}[Q(\mathsf{f}(\cdot,\tau,\cdot))](\vec{\boldsymbol{x}},\xi)\right|=\left|\mathbb{T}^{t-\tau}\left[\sum_{j,l}\chi_{jl}Q(\mathsf{f}(\cdot,\tau,\cdot))\right](\vec{\boldsymbol{x}},\xi)\right|$$
$$\leq C_0^2\epsilon^{\frac{4}{3}}\sum_{j,l}\left(\frac{e^{-\frac{(x-j)^2+(y-l)^2}{C_0(t-\tau)}}}{(t-\tau)^{1+\gamma}}+e^{-(|x-j|+|y-l|+t-\tau)/C_0}\right)$$
$$\times\left(\frac{e^{-\frac{2(j^2+l^2)}{C_0\tau}}}{(1+\tau)^2}+e^{-2(|j|+|l|+\tau)/C_0}\right)e^{-\frac{\alpha gz}{\theta}}\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{\alpha}$$

$$= O(1)C_0^2\epsilon^{\frac{4}{3}}\frac{1}{(t-\tau)^\gamma(1+\tau)}\left(\frac{e^{-\frac{x^2+y^2}{C_0 t}}}{(t+1)}+e^{-\frac{|x|+|y|+t}{C_0}}\right)e^{-\frac{\alpha g z}{\theta}}(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi))^\alpha. \quad (4.21)$$

Now, we can use the representation

$$\mathsf{f}(\vec{\boldsymbol{x}},t,\xi)=\mathbb{T}^t[\mathsf{f}(\cdot,0,\cdot)](\vec{\boldsymbol{x}},\xi)+\int_0^t\mathbb{T}^{t-\tau}[Q(\mathsf{f}(\cdot,\tau,\cdot))](\vec{\boldsymbol{x}},\xi)d\tau,$$

for any $t\in(0,t_0+1)$ and (4.21) to estimate $\rho(x,y,t)$

$$\begin{aligned}\rho(x,y,t) \;\le\; & \int_{\xi^3<0}-\xi^3\mathbb{T}^t[\mathsf{f}(\cdot,0,\cdot)](x,y,0,\xi)d\xi \\ & +\left|\int_{\xi^3<0}-\xi^3\left(\int_0^t\mathbb{T}^{t-\tau}\left[Q(\mathsf{f}(\cdot,\tau,\cdot))\right](x,y,0,\xi)d\tau\right)d\xi\right| \\ \le\; & O(1)\epsilon\left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{1+t}+e^{-(|x|+|y|+t)/C_0}\right) \\ & +O(1)\epsilon^{4/3}\left(\frac{e^{-\frac{x^2+y^2}{C_0(1+t)}}}{(1+t)^{1+\gamma}}+e^{-(|x|+|y|+t)/C_0}\right). \qquad (4.22)\end{aligned}$$

Next, we use

$$\partial_t\mathsf{f}+\xi\cdot\nabla_x\mathsf{f}-\vec{\boldsymbol{g}}\cdot\nabla_\xi\mathsf{f}+q_-[\mathsf{f}]\mathsf{f}=Q_+(\mathsf{f})$$

to represent the solution $\mathsf{f}(\vec{\boldsymbol{x}},t,\xi)$ again:

$$\begin{aligned}\mathsf{f}(\vec{\boldsymbol{x}},t,\xi) \;=\; & e^{-\int_{\tau_0}^t q_-[\mathsf{f}](\Gamma(s),s,\dot\Gamma(s))ds}\rho(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)\frac{\sqrt{2\pi}}{\sqrt{\theta}}\mathsf{M}(\dot\Gamma(\tau_0)) \\ & +\int_{\tau_0}^t e^{-\int_\tau^t q_-[\mathsf{f}](\Gamma(s),s,\dot\Gamma(s))ds}Q_+(\mathsf{f})(\Gamma(\tau),\tau,\dot\Gamma(\tau))d\tau, \qquad (4.23)\end{aligned}$$

where the trajectory $\Gamma(\tau)$ is defined in (3.53). From (4.20),

$$\begin{aligned}&|Q_+(\mathsf{f})(\Gamma(\tau),\tau,\dot\Gamma(\tau))| \qquad (4.24)\\ &\le O(1)\epsilon^{\frac{4}{3}}\begin{cases} e^{-\frac{3g\Gamma^3}{4\theta}}\left(e^{-\frac{g\Gamma^3}{\theta}}\mathsf{M}(\dot\Gamma)\right)^{\frac{3}{4}}, \\ \left(\frac{e^{-2\frac{(\Gamma^1)^2+(\Gamma^2)^2}{C_*(1+\tau)}}}{(1+\tau)^2}+e^{-2\frac{\sqrt{(\Gamma^1)^2+(\Gamma^2)^2}+\tau}{C_*}}\right)e^{-\frac{g\Gamma^3}{2\theta}}\left(e^{-\frac{g\Gamma^3}{\theta}}\mathsf{M}(\dot\Gamma)\right)^{\frac{1}{2}}.\end{cases}\end{aligned}$$

From this one has

$$|Q_+(\mathsf{f})(\Gamma(\tau),\tau,\dot\Gamma(\tau))|$$

$$
\begin{aligned}
&\le O(1)\epsilon^{4/3}e^{-\frac{3\alpha_0+2\beta_0}{4\theta}g\Gamma^3}\left(e^{-\frac{g\Gamma^3}{\theta}}\mathsf{M}(\dot\Gamma)\right)^{\frac{3\alpha_0+2\beta_0}{4}}\\
&\quad\times\left(\frac{e^{-2\frac{(\Gamma^1)^2+(\Gamma^2)^2}{C_*(1+\tau)}}}{(1+\tau)^2}+e^{-2\frac{\sqrt{(\Gamma^1)^2+(\Gamma^2)^2}+\tau}{C_*}}\right)^{\beta_0}\\
&= O(1)\epsilon^{4/3}\left(e^{-gz}\mathsf{M}(\xi)\right)^{\frac12}e^{-\frac{3\alpha_0+2\beta_0}{4\theta}g\Gamma^3}\left(e^{-\frac{g\Gamma^3}{\theta}}\mathsf{M}(\dot\Gamma)\right)^{\frac{3\alpha_0+2\beta_0-2}{4}}\\
&\quad\times\left(\frac{e^{-2\frac{(\Gamma^1)^2+(\Gamma^2)^2}{C_*(1+\tau)}}}{(1+\tau)^2}+e^{-2\frac{\sqrt{(\Gamma^1)^2+(\Gamma^2)^2}+\tau}{C_*}}\right)^{\beta_0},
\end{aligned}
\tag{4.25}
$$

where $\alpha_0>0,\ \beta_0\in(1/2,1)$, and $\alpha_0+\beta_0=1$. Let $(\alpha_0,\beta_0)=(\frac14,\frac34)$ and use $q_-[\mathsf{f}]\ge 0$, (3.56), (4.23), (3.53), (4.25), (4.20), and Proposition 3.11 to yield that for all $\xi\in\mathbb{R}^3$

$$
\begin{aligned}
|\mathsf{f}(\vec{\boldsymbol{x}},t,\xi)| &\le \rho(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)\frac{\sqrt{2\pi}}{\sqrt{\theta}}e^{-gz}\mathsf{M}(\xi)+\int_{\tau_0}^t|Q_+(\mathsf{f})(\Gamma(\tau),\tau,\dot\Gamma(\tau))|d\tau\\
&\le O(1)\rho(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)e^{-gz}\mathsf{M}(\xi)\\
&\quad +O(1)\epsilon^{4/3}\left(e^{-gz}\mathsf{M}(\xi)\right)^{\frac12}\int_{\tau_0}^t e^{-\frac{9g\Gamma^3}{16\theta}}\left(e^{-\frac{g\Gamma^3}{\theta}}\mathsf{M}(\dot\Gamma)\right)^{\frac{1}{16}}\\
&\quad\times\left(\frac{e^{-\frac{3[(\Gamma^1)^2+(\Gamma^2)^2]}{2C_*(1+\tau)}}}{(1+\tau)^{3/2}}+e^{-3\frac{\sqrt{(\Gamma^1)^2+(\Gamma^2)^2}+\tau}{2C_*}}\right)d\tau\\
&\le O(1)\epsilon\sqrt{e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)}\left(\frac{e^{-\frac{(x^2+y^2)}{C_*(1+t)}}}{2(1+\tau)}+e^{-\frac{(|x|+|x|+t)}{C_*}}\right).
\end{aligned}
\tag{4.26}
$$

Next, we need to conclude the uniform bound by $(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi))^{3/4}$. We use the first inequality in (4.24) to yield that

$$
\begin{aligned}
|\mathsf{f}(\vec{\boldsymbol{x}},t,\xi)| &\le \rho(\Gamma^1(\tau_0),\Gamma^2(\tau_0),\tau_0)\frac{\sqrt{2\pi}}{\sqrt{\theta}}\mathsf{M}(\dot\Gamma(\tau_0))+\int_{\tau_0}^t|Q_+(\mathsf{f})(\Gamma(\tau),\tau,\dot\Gamma(\tau))|d\tau\\
&\le O(1)\epsilon e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)+O(1)\epsilon^{4/3}\int_{\tau_0}^t e^{-\frac{3g\Gamma^3(\tau)}{4\theta}}d\tau\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{3/4}\\
&= O(1)\epsilon\left(e^{-\frac{gz}{\theta}}\mathsf{M}(\xi)\right)^{3/4}.
\end{aligned}
\tag{4.27}
$$

The estimates (4.26) and (4.27) yield that

$$
|||\mathsf{f}|||_{t_0+1}=O(1)\epsilon. \tag{4.28}
$$

□

Remark 4.3. The extra decaying factor $1/(t-\tau)^\gamma$ in the decaying estimate (4.21) is the key element to close the ansatz assumption. Without this factor the estimate (4.22) would become $\rho(x,y,t) \leq O(1)\epsilon(1+\epsilon^{1/3}\log(1+t))\left(e^{-(x^2+y^2)/(C_0(1+t))}/(1+t)+e^{-(|x|+|y|+t)/C_0}\right)$. It would be difficult to close the global ansatz.

Proof of Main Theorem B. From Lemmas 4.1 and 4.2, we have that the interval (t_0, t_0+1) is contained in $\mathscr{I}$. Thus, $\mathscr{I}$ is an open set. We have mentioned that $\mathscr{I}$ is a closed set due to the definition of $|||\cdot|||_T$. Thus, $\mathscr{I}\equiv\mathbb{R}$, and

$$|||\mathsf{f}|||_t < \epsilon^{2/3} \text{ for all } t\in\mathbb{R}.$$

Again, with this and Lemma 4.2 one has that

$$|||\mathsf{f}|||_t < O(1)\epsilon \text{ for all } t\in\mathbb{R},$$

and the theorem follows. □

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