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DIFFUSIVE PROPERTY OF THE FOKKER-PLANCK-BOLTZMANN EQUATION

BY

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Dedicated to Professor Yoshio Sone for his seventieth birthday

Abstract

We consider the Fokker-Planck-Boltzmann equation consisting of the Boltzmann equation with an additional diffusion term in velocity space to simulate for instance the transport in thermal bath of binary elastic collisional particles, and investigate the (macroscopic) diffusion properties of solutions in large time.

1. Introduction

When concerned with the motion of particles in thermal bath where the bilinear interaction is the one of main characters, we have the Fokker-Planck-Boltzmann type equation. Such type of equations is also used recently in the description of grazing collision [3], in the area of aerosols [15] and driven media [2], and so on. In the present paper, we consider the diffusive property of global classical solutions of the initial value problem (IVP) for Fokker-Planck-Boltzmann equation

$$\widetilde{F}_\tau + \xi \cdot \nabla_y \widetilde{F} = \mathcal{Q}(\widetilde{F}, \widetilde{F}) + \varepsilon \Delta_\xi \widetilde{F}, \tag{1.1}$$

$$\widetilde{F}(y, \xi, 0) = \widetilde{F}_0(y, \xi), \tag{1.2}$$

with $\widetilde{F} = \widetilde{F}(y, \xi, t)$, $(y, \xi, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$ the distribution function and the Knusden number set to be one. The operator Δ_ξ is the Laplacian operator defined on $\mathbb{R}^3$. In general, it is produced by the Brownian motion of random particles, or thermal bath effects on particles, etc, and gives rise to a diffusion

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process of particles. The bilinear collision operator $\mathcal{Q}(\widetilde{F},\widetilde{F})$ is taken as the hard sphere model for elastic collision

$$\mathcal{Q}(\widetilde{F},\widetilde{F}) = \int_{\mathbb{R}^3} d\xi_* \int_{\mathbb{S}^2} |(\xi-\xi_*)\cdot\omega|[\widetilde{F}(\xi')\widetilde{F}(\xi'_*) - \widetilde{F}(\xi)\widetilde{F}(\xi_*)]d\omega \tag{1.3}$$

with

$$\xi' = \xi - [(\xi-\xi_*)\cdot\omega]\omega, \qquad \xi'_* = \xi + [(\xi-\xi_*)\cdot\omega]\omega, \qquad \omega\in\mathbb{S}^2,$$
$$|\xi'|^2 + |\xi'_*|^2 = |\xi_*|^2 + |\xi|^2,$$

where and throughout the paper we often use the abbreviation $g'_* = g(\xi'_*)$, $g_* = g(\xi_*)$, $g' = g(\xi')$. The binary collision term $\mathcal{Q}(\widetilde{F},\widetilde{F})$ can be also represented in (weak) integral form via a test function $\psi = \psi(\xi)$ in phase space

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{Q}(\widetilde{F},\widetilde{F})\psi(\xi)d\xi \\ &= -\frac{1}{4}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} |(\xi-\xi_*)\cdot\omega|[\widetilde{F}'\widetilde{F}'_* - \widetilde{F}\widetilde{F}_*](\psi+\psi_*-\psi'-\psi'_*)d\omega d\xi_* d\xi.\end{aligned} \tag{1.4}$$

Take $\psi(\xi) = \log(\xi)$, we have the so-called entropy production functional

$$D(\widetilde{F}) = \frac{1}{4}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} |(\xi-\xi_*)\cdot\omega|[\widetilde{F}'\widetilde{F}'_* - \widetilde{F}\widetilde{F}_*]\log\frac{\widetilde{F}'\widetilde{F}'_*}{\widetilde{F}\widetilde{F}_*}d\omega d\xi_* d\xi \ge 0, \tag{1.5}$$

due to the inequality $(a-b)\log\frac{a}{b} \ge 0$ for $a,b>0$ and the monotone increase property of logarithmic function. The equality holds only if the distribution function $\widetilde{F}$ is a (local) Maxwellian

$$\widetilde{F}(y,\xi,\tau) = M_{[\rho(y,\tau),u(y,\tau),\theta(y,\tau)]}(\xi) = \frac{\rho(y,\tau)}{(2\pi\theta(y,\tau))^{3/2}}e^{-\frac{|\xi-u(y,\tau)|^2}{2\theta(y,\tau)}}.$$

The simplest consideration is to take the renormalized absolute Maxwellian, i.e., the stationary state uniform with respect to space and time variables

$$\widetilde{F}(y,\xi,\tau) \equiv M(\xi) = (2\pi)^{-3/2}e^{-\frac{|\xi|^2}{2}}.$$

Although it still satisfies the entropy production functional (1.5) and is the equilibrium state of the collision term, the absolute Maxwellian $M(\xi)$, as pointed out in [12], is not any longer a solution of Eq. (1.1). Moreover, affected by the additional diffusion in velocity space, Eq. (1.1) has an important difference from classical Boltzmann equation. In fact, although

Eq. (1.1) still keeps the conservation laws of total mass and total momentum, the total energy (or the temperature), however, grows up with respect to time due to the heating process of the macroscopic fluid-dynamical part caused by the diffusion in the following sense

$$\frac{1}{2}\int_{\mathbb{R}^3_{y,\xi}}|\xi|^2\widetilde{F}(y,\xi,\tau)d\xi dx=\frac{1}{2}\int_{\mathbb{R}^3_{y,\xi}}|\xi|^2\widetilde{F}_0(y,\xi)d\xi dx+3\varepsilon\tau\int_{\mathbb{R}^3_{y,\xi}}\widetilde{F}_0(y,\xi)d\xi dx. \tag{1.6}$$

This finally may influence the qualitative behaviors of solutions of Eq. (1.1). As pointed out in [12], one usually can not expect the convergence of global time-dependent solution of Eq. (1.1) to a global (absolute) Maxwellian even if for small initial perturbation, although from mathematical point of view the Fokker-Planck-Boltzmann equation (1.1) is a velocity regularity approximation to the Boltzmann in phase space [11]. Therefore, it is quite natural to investigate how the diffusion in velocity space influences the qualitative behaviors of solutions of the Fokker-Planck-Boltzmann equation (1.1)

Concerned with the mathematical analysis on the Fokker-Planck-Boltzmann equation (1.1), fewer rigorous analysis was established so far. The zero diffusion limit $\varepsilon \to 0_+$ was investigated by Hadamache when solution is a perturbation of vacuum state [11] and by Li-Matsumura [12] for classical solutions away from vacuum [12]. The global existence theory of IVP (1.1)–(1.2) was proven by DiPerna-Lions [4] in L^1 framework for renormalized solution, and obtained by Hamdache in terms of a direct construction near vacuum state [10]. The global existence of classical solutions of IVP (1.1)–(1.2) was established by Li-Matsumura for initial date near an absolute Maxwellian with the help of the micro-macro decomposition and energy method in [14, 13], where the global classical solution was shown to converge to a time-dependent self-similar Maxwellian in large time with a faster time-decay rate than Boltzmann equation [12].

In the present paper, we shall study the time-decay rates of classical solutions of IVP (1.1)–(1.2) for both homogeneous and inhomogeneous cases and focus on the (macroscopic) diffusive property of solutions caused by the diffusion in velocity space. The main purpose is to understand how the influence of diffusion in velocity space on the time asymptotical behavior of global classical solutions. For simplicity, we first consider the initial value problem for spatial homogeneous case, i.e.,

$$\widetilde{F}_\tau=\mathcal{Q}(\widetilde{F},\widetilde{F})+\varepsilon\Delta_\xi\widetilde{F}, \tag{1.7}$$

$$\widetilde{F}(0,\xi)=\widetilde{F}_1(\xi),\quad \xi\in\mathbb{R}^3, \tag{1.8}$$

with initial data $\widetilde{F}_1$ near the absolute Maxwellian $M_{[1,0,1]}(\xi)$, and we are able to give an exact description on how the diffusion affects the asymptotical behavior of global classical solutions.

Denote the global Maxwellian

$$M(\xi)=M_{[1,0,1]}(\xi)=:(2\pi)^{-3/2}e^{-\frac{|\xi|^2}{2}}$$

and assume that the distribution function is re-normalized at initial time so that

$$\int_{\mathbb{R}^3_\xi}\widetilde{F}_1\,dv=1,\quad \int_{\mathbb{R}^3_\xi}v\widetilde{F}_1\,dv=0,\quad \int_{\mathbb{R}^3_\xi}|v|^2\widetilde{F}_1\,dv=1. \tag{1.9}$$

We prove that the global solution of IVP (1.7)–(1.8) exists and converges to the time-dependent self-similar Maxwellian

$$\widetilde{M}=\widetilde{M}_{[1,0,1]}(\xi,\tau)=:\frac{1}{(2\pi(1+2\varepsilon\tau))^{3/2}}\,e^{-\frac{|\xi|^2}{2(1+2\varepsilon\tau)}} \tag{1.10}$$

in large time. It is easy to verify that the time-dependent self-similar Maxwellian $\widetilde{M}_{[1,0,1]}(\xi,\tau)$ is also a solution of (1.7). We have the following result.

Theorem 1.1. *Assume that* (1.9) *holds and* $g_0=:\widetilde{F}_1-M\in H^k(\mathbb{R}^3, M^{-1}\,d\xi)$, $k\geq 1$, *with* $\eta_0=:\|M^{-1/2}g_0\|_{H^k(\mathbb{R}^3)}$ *small enough. Then, there exists* $\varepsilon_0>0$ *so that for* $\varepsilon\in(0,\varepsilon_0]$, *there exists a unique global solution* $\widetilde{F}$ *of IVP* (1.7)–(1.8) *satisfying*

$$\widetilde{F}-\widetilde{M}\in C([0,\infty);H^k(\mathbb{R}^3,\widetilde{M}^{-1}d\xi)) \tag{1.11}$$

and

$$\|\partial_\xi^\gamma(\widetilde{F}-\widetilde{M})(\tau)\|_{L^1(\mathbb{R}^3_\xi)}\leq C\eta_0(1+2\varepsilon\tau)^{-\gamma/2}e^{-a_1\tau^{3/2}},\quad \gamma=1,2,...,k, \tag{1.12}$$

with $a_1>0$ *a constant.*

Remark 1.2. Theorem 1.1 describes a complete phenomena on how the diffusion affects the asymptotic behaviors of global solution of IVP (1.7)-(1.8). In fact, the exponential time-decay rate on the right hand side term of (1.12) is made of the combined effects of both the binary collision and a pure diffusion process of velocity space, and the algebraic time-decay rate on the right hand side term of (1.12) is produced by the diffusion process. It is different from the case for the homogeneous Boltzmann equation where the optimal time-decay rate in L^1 norm is exponential [1]

$$\|(\hat{F}-M)(\tau)\|_{L^1(\mathbb{R}^3_\xi)} \le Ce^{-a_2\tau}$$

with $a_2 > 0$ a constant.

Similar diffusive phenomena can be also observed for spatial inhomogeneous case, namely, the IVP problem (1.1)–(1.2). Let us recall the definition of the self-similar Maxwellian $\widetilde{M}_{[\rho,u,\theta]}(\xi,\tau)$ and the global Maxwellian $M_{[\rho,u,\theta]}(\xi)$ as follows

$$\begin{aligned} \widetilde{M}_{[\rho,u,\theta]}(\xi,\tau) &=: \frac{\rho}{\sqrt{(2\pi\theta(1+2\varepsilon\tau))^3}}\, e^{-\frac{|\xi-u|^2}{2\theta(1+2\varepsilon\tau)}}, \\ M_{[\rho,u,\theta]}(\xi) &=: \frac{\rho}{\sqrt{(2\pi\theta)^3}}\, e^{-\frac{|\xi-u|^2}{2\theta}}. \end{aligned} \tag{1.13}$$

We have the following result.

Theorem 1.3. *Assume that $\varepsilon > 0$ small. Let the initial data $\widetilde{F}_0(y,\xi)$ to the IVP problem* (1.1)–(1.2) *satisfy $\delta = M^{-1/2}_{[1,0,\theta_*]}(\widetilde{F}_0 - M_{[1,0,1]})\|_{H^5(\mathbb{R}^3_x)\times H^2(\mathbb{R}^3_\xi)}$ small enough with $\theta_* \in (0,1)$. Then, the classical solution $\widetilde{F}$ of the IVP* (1.1)–(1.2) *exists globally in time and satisfies*

$$\|(\widetilde{F}-\widetilde{M}_{[1,0,1]})\|_{L^\infty([0,\infty);H^4(\mathbb{R}^3_x)\times H^2(\mathbb{R}^3_\xi,\widetilde{M}_-^{-1}d\xi)\cap H^5(\mathbb{R}^3_x)\times L^2(\mathbb{R}^3_\xi,\widetilde{M}_-^{-1}d\xi))} \le C\delta, \tag{1.14}$$

where $\widetilde{M}_- = \widetilde{M}_{[1,u_-,\theta_-]}$ with u_- and $\theta_- \in (\theta_*,1)$ two constants satisfying $\mu =: |u_-| + 1 - \theta_-$ small enough. Moreover, it holds for $\gamma = 0,1$ that

$$\|\partial^\gamma_\xi(\widetilde{F}-\widetilde{M}_{[1,0,1]})(\tau)\|_{L^\infty(\mathbb{R}^3_y)\times L^1(\mathbb{R}^3_\xi)} \le C\delta(1+2\varepsilon\tau)^{-\frac{\gamma}{2}-\frac{9}{8}}. \tag{1.15}$$

Remark 1.4.

(1) Theorem 1.1–1.3 is established for Fokker-Planck-Boltzmann equation starting initially away from vacuum (near an absolute Maxwellian) and shows a diffusive phenomena caused by velocity diffusion.

(2) The reason why we only get the time-decay rate (1.15) of macroscopic fluid-dynamical parts of the solutions is mainly because of the transportation in space influenced by streaming operator $\xi\cdot\nabla_y$ (or the transport operator with respect to space) which results in an interaction be-

tween macroscopic fluid-dynamical and microscopic non-fluiddynamical parts of the solutions and makes the convergence slower.

(3) Note that the combination of spatial transport operator and velocity diffusion is a hyper-elliptic operator which can have smoothing effect on solutions. An interesting and important problem is whether this property can be used to study the regularity of DiPerna-Lions' re-normalized solution of Fokker-Planck-Boltzmann equation [4], this is under forthcoming investigation.

2. Proof of Main Results

We shall prove the Theorem 1.1 and Theorem 1.3 about the diffusive property of Fokker-Planck-Boltzmann equation (1.1). For simplicity, we show the property for homogeneous case, i.e., the Theorem 1.1, in the framework of macro-micro decomposition and energy method [14, 13]. The case of inhomogeneous case, i.e., Theorem 1.3, can be proven by a similar argument. Let us introduce the variable transformation to Fokker-Planck-Boltzmann equation (1.7) as

$$t=\frac{(1+2\varepsilon\tau)^{3/2}-1}{3\varepsilon},\quad v=\frac{\xi}{(1+2\varepsilon\tau)^{1/2}},\tag{2.16}$$

and the transformation of distribution function $\widetilde{F}(\xi,\tau)$

$$F(v,t)=(1+3\varepsilon t)\widetilde{F}\left(v(1+3\varepsilon t)^{1/3},\frac{(1+3\varepsilon t)^{2/3}-1}{2\varepsilon}\right).\tag{2.17}$$

Then, by (1.7)–(1.8) the distribution function $F=F(v,t)$ satisfies

$$F_t=\mathcal{Q}(F,F)+\varepsilon(1+3\varepsilon t)^{-1}L_{FP}F,\tag{2.18}$$

$$F(v,0)=\widetilde{F}_1(v),\quad v\in\mathbb{R}^3.\tag{2.19}$$

Here the Fokker-Planck operator L_{FP} is defined on $\mathbb{R}^3$ as

$$L_{FP}F=\nabla_v\cdot[\nabla_vF+vF].$$

Let us introduce a local Maxwellian $M_{[\rho,u,\theta]}(v,t)$ as

$$M_{[\rho,u,\theta]}(v,t)=\frac{\rho(t)}{\sqrt{(2\pi\theta(t))^3}}\,e^{-\frac{|\xi-u(t)|^2}{2\theta(t)}}\tag{2.20}$$

where the macroscopic quantities density ρ, velocity u and temperature θ correspond to the first three moments of F

$$\begin{cases} \rho(t) = \int_{\mathbb{R}^3_v} F(v,t)dv, \quad \rho u(t) = \int_{\mathbb{R}^3_v} vF(v,t)\, dv, \\ \frac{1}{2}\rho(|u|^2 + 3\theta)(t) = \frac{1}{2}\int_{\mathbb{R}^3_v} |v|^2 F(v,t)\, dv. \end{cases} \tag{2.21}$$

We make use of the micro-macro decomposition and energy method established recently in [14, 13] to investigate the diffusive property of Fokker-Planck-Boltzmann equation. Introduce a linear collision operator L_M in terms of the nonlinear collision operator $\mathcal{Q}(f,f)$ by

$$\begin{aligned} L_M f =& \mathcal{Q}(M+f, M+f) - \mathcal{Q}(f,f) \\ =& \mathcal{Q}(M,f) + \mathcal{Q}(f,M). \end{aligned} \tag{2.22}$$

For hard sphere collision model, the linearized collision operator L_M takes the form [7, 3]

$$(L_M f)(v) = -\nu(|v|)f(v) + K(f)(v), \tag{2.23}$$

where K is a L^2-compact symmetric operator and $\nu(v)$ is the collision frequency satisfying

$$\nu_-(1+|v|) \le \nu(|v|) \le \nu_+(1+|v|), \quad v \in \mathbb{R}^3,$$

for two constants $\nu_\pm > 0$.

Denote $\langle .,. \rangle$ the inner product of the Hilbert space $L^2(\mathbb{R}^3)$ with respect to the local Maxwellian $M_{[\rho,u,\theta]}(v,t)$ as

$$\langle f, g \rangle = \int_{\mathbb{R}^3} \frac{1}{M} f(v)g(v)\, dv, \quad \forall f, g \in L^2(\mathbb{R}^3).$$

Following the argument by Grad [7], it is easy to verify that the operator L_M is a self-adjoint, non-positive, Fredhlom operator with respect to the inner product $\langle .,. \rangle$

$$\langle Lf, g \rangle = \langle f, Lg \rangle, \quad \langle Lf, f \rangle \le 0,$$

and its kernel (nullspace) consists of the space of five-fold spanned by the collision invariants parameterized by macroscopic variables. Namely, there exist five normalized orthogonal basis $\chi_i = \chi_i(v,t)$, $i = 0,..,4$, with respect to above inner product

$$\begin{cases} \chi_0 = \dfrac{M_{[\rho,u,\theta]}}{\sqrt{\rho(\tau)}}, \quad \chi_j = \dfrac{\xi - u(\tau)}{\sqrt{\rho(\tau)\theta(\tau)}} M_{[\rho,u,\theta]},\ j = 1,2,3, \\ \chi_4 = \dfrac{1}{\sqrt{6\rho(\tau)}} \left(\dfrac{|\xi - u(t)|^2}{\theta(t)} - 3 \right) M_{[\rho,u,\theta]}. \end{cases}$$

This five functions span a space of fluids component of the solution F, and $Ker(L_M)$ consists of $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ and their linear combination. This is actually one of the important facts related to the macro-micro decomposition, that is, one can take advantage of the dissipation of the collision operator based on the H-Theorem to control the microscopic part [7, 13]

$$-\int_{\mathbb{R}^3_v} M^{-1} h L_M h \, dv \geq \nu_* \int_{\mathbb{R}^3_v} M^{-1}(1+|v|)h^2 \, dv, \quad h \in Ker(L_M)^{\perp} \quad (2.24)$$

where $\nu_* = \nu_*(\rho, u, \theta) > 0$ and we recall M is defined by (2.20).

Thus, we can define the projection operator $\mathbf{P}_0$ onto the space $Ker(L_M)$ and the projection operator $\mathbf{P}_1$ onto the space $Ker(L_M)^{\perp}$ as

$$\mathbf{P}_0(g) = \sum_{j=0}^{4} \langle \chi_j, g \rangle \chi_j, \quad \mathbf{P}_1(g) = (I - \mathbf{P}_0)g.$$

It holds obviously that

$$\mathbf{P}_0^2 = \mathbf{P}_0, \quad \mathbf{P}_1^2 = \mathbf{P}_1, \quad \mathbf{P}_0\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_0 = 0.$$

And we can decompose $F(v,t)$ into two parts as

$$F = M + g, \qquad M = \mathbf{P}_0(F), \quad g = \mathbf{P}_1(F) = F - M. \quad (2.25)$$

By (2.18) we easily conclude that the first three moment of g always is zero

$$\int_{\mathbb{R}^3_v} g dv = \int_{\mathbb{R}^3_v} v g dv = \int_{\mathbb{R}^3_v} |v|^2 g dv = 0.$$

Project the equation (2.18) onto the space of microscopic (non-fluid dynamical) part, the IVP problem for $g(v,t) = F(v,t) - M(v,t)$ is

$$g_t = L_M g + Q(g,g) + \varepsilon(1+3\varepsilon t)^{-1} \nabla \cdot [\nabla g + v g], \quad (2.26)$$

$$g(v,t) = g_0 =: \widetilde{F}_1(v) - M(v), \quad v \in \mathbb{R}^3, \tag{2.27}$$

where we recall that the local Maxwellian M is defined by (2.20).

We need to understand more about the local Maxwellian defined by (2.20). The evolutional equations for macroscopic quantities $\rho(t), u(t), \theta(t)$ are obtained from equation (2.18) by taking the first three moments as

$$\begin{cases} \rho_t = 0, \quad (\rho u)_t = \varepsilon(1+3\varepsilon t)^{-1} \displaystyle\int_{\mathbb{R}^3_v} v L_{FP} F \, dv, \\ \dfrac{1}{2}(\rho|u|^2 + 3\rho\theta)_t = \dfrac{1}{2}\varepsilon(1+3\varepsilon t)^{-1} \displaystyle\int_{\mathbb{R}^3_v} |v|^2 L_{FP} F \, dv, \end{cases} \tag{2.28}$$

which indeed gives

$$\begin{cases} \rho_t = 0, \quad (\rho u)_t = -2\varepsilon(1+3\varepsilon t)^{-1} \rho u, \\ \dfrac{1}{2}(\rho|u|^2 + 3\rho\theta)_t = -\varepsilon(1+3\varepsilon t)^{-1} \rho(|u|^2 + 3\theta - 3). \end{cases} \tag{2.29}$$

Since we can always re-normalize the distribution function so that

$$\int_{\mathbb{R}^3} \widetilde{F}_1 \, dv = 1, \quad \int_{\mathbb{R}^3} v \widetilde{F}_1 \, dv = 0, \quad \int_{\mathbb{R}^3_v} |v|^2 \widetilde{F}_1 \, dv = 3,$$

it follows from (2.29) that

$$\rho(t) = 1, \quad u(t) = 0, \quad \theta(t) = 1, \quad t \geq 0. \tag{2.30}$$

This implies that the local Maxwellian $M_{[\rho,u,\theta]}(v)$ is in fact an absolute one, namely

$$M = M_{[1,0,1]}(v) = (2\pi)^{-3/2} \, e^{-\frac{|\xi|^2}{2}}. \tag{2.31}$$

What we shall do is to establish the following a-priori estimates.

Proposition 2.1. *Let $T > 0$. Under the assumptions of Theorem 1.1, the solutions F of IVP (2.26)–(2.27) satisfies for $t \in [0,T]$ that*

$$\sum_{|\alpha|=0}^{k} \int_0^t \int_{\mathbb{R}^3} \frac{(1+|v|)|\partial_v^\alpha (F-M)(v,t)|^2 + \varepsilon(1+3\varepsilon s)^{-1} |\nabla \partial_v^\alpha (F-M)(v,t)|^2}{M} \cdot dv ds \leq C\eta_0, \tag{2.32}$$

$$\sum_{|\alpha|=0}^{k}\int_{\mathbb{R}^3}\frac{|\partial_v^\alpha(F-M)(v,t)|^2}{M}\,dv\le C\eta_0 e^{-\nu t},\tag{2.33}$$

with $C>0$ and $\nu>0$ two constants independent of $\varepsilon>0$.

Proof. Assume that the solution F of Eq. (2.26)–(2.27) satisfies

$$\delta_T=\sup_{t\in[0,T]}\sqrt{\sum_{0\le\gamma\le k}\int_{\mathbb{R}^3}\frac{|\partial_v^\gamma g(v,t)|^2}{M}\,dv}\ll 1.\tag{2.34}$$

Take inner product between (2.26) and gM^{-1} with respect to v. We have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\frac{g^2}{M}\,dv-\int_{\mathbb{R}^3}\frac{gL_Mg}{M}\,dv-\varepsilon(1+3\varepsilon t)^{-1}\int_{\mathbb{R}^3}\frac{g\nabla\cdot[\nabla g+vg]}{M}\,dv\\&=\int_{\mathbb{R}^3}\frac{g\,Q(g,g)}{M}\,dv.\end{aligned}\tag{2.35}$$

Recall the definition of the global Maxwellian $M(v)=(2\pi)^{-3/2}e^{-\frac{|v|^2}{2}}$, note that it still holds

$$\int_{\mathbb{R}^3}M^{-1}g\nabla\cdot[\nabla g+vg]\,dv=-\int_{\mathbb{R}^3}M^{-1}|\nabla g+vg|^2\,dv,\tag{2.36}$$

$$\int_{\mathbb{R}^3}M^{-1}g\nabla\cdot[\nabla g+vg]\,dv=-\int_{\mathbb{R}^3}M^{-1}(|\nabla g|^2-3g^2)\,dv,\tag{2.37}$$

$$\nu_0\int_{\mathbb{R}^3}(1+|v|)g^2M^{-1}\,dv\le-\int_{\mathbb{R}^3}gL_MgM^{-1}\,dv,\tag{2.38}$$

$$\begin{aligned}\int_{\mathbb{R}^3}M^{-1}g\mathcal{Q}(g,g)\,dv\le c_0\sup_{s\ge0}&\sqrt{\int_{\mathbb{R}^3}M^{-1}|g(s,v)|^2dv}\\&\cdot\int_{\mathbb{R}^3}M^{-1}(1+|v|)g^2dv,\end{aligned}\tag{2.39}$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\frac{g^2}{M}\,dv+a_0\int_{\mathbb{R}^3}\frac{(1+|v|)g^2}{M}\,dv\\&\quad+\frac{1}{2}\varepsilon(1+3\varepsilon t)^{-1}\int_{\mathbb{R}^3}\frac{|\nabla g|^2+|\nabla g+vg|^2}{M}\,dv\le0\end{aligned}\tag{2.40}$$

with $a_0=:\nu_0 3\varepsilon_0-c_0\delta_T>0$. Thus, as $\varepsilon>0$ and initial perturbation

$\|M^{-1/2}g_0^2\|_{L^2(R_v^3)}$ small enough, we can obtain

$$\int_0^t \int_{\mathbb{R}^3} \frac{(1+|v|)g^2}{M}\,dv + \varepsilon \int_0^t (1+3\varepsilon s)^{-1} \int_{\mathbb{R}^3} \frac{|\nabla g|^2 + |\nabla g + vg|^2}{M}\,dv$$
$$\leq \int_{\mathbb{R}^3} \frac{g_0^2}{M}\,dv, \tag{2.41}$$

and

$$\int_{\mathbb{R}^3} \frac{|g(v,t)|^2}{M}\,dv \leq e^{-\nu_0 t} \int_{\mathbb{R}^3} \frac{g_0^2}{M}\,dv. \tag{2.42}$$

To consider the derivative with respect to v, let us differentiate the equation (2.26) on v to have

$$\begin{aligned}\partial_v g_t \;=\; & L_M \partial_v g + (\partial_v L_M)g + Q(\partial_v g, g) + Q(g, \partial_v g) \\ & + \varepsilon(1+3\varepsilon t)^{-1}\nabla\cdot[\nabla \partial_v g + \partial_v(vg)]. \end{aligned} \tag{2.43}$$

Take inner product between the equation (2.43) and g_v. After integration by parts and the application of the similar properties to (2.36)–(2.39), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \frac{|\partial_v g|^2}{M}\,dv &+ (a_0 - \delta_T)\int_{\mathbb{R}^3} \frac{(1+|v|)|\partial_v g|^2}{M}\,dv \\ &+ \frac{1}{2}\varepsilon(1+3\varepsilon t)^{-1} \int_{\mathbb{R}^3} \frac{|\nabla \partial_v g|^2 + |\nabla \partial_v g + v\partial_v g|^2}{M}\,dv \\ \leq\; & \int_{\mathbb{R}^3} \frac{|\partial_v g(\partial_v L_M)g|}{M}\,dv + C\varepsilon(1+3\varepsilon t)^{-1}\int_{\mathbb{R}^3} \frac{|\partial_v g|^2}{M}\,dv \\ \leq\; & C_\beta \int_{\mathbb{R}^3} \frac{g^2}{M}\,dv + \beta \int_{\mathbb{R}^3} \frac{|\partial_v g|^2}{M}\,dv + C\varepsilon(1+3\varepsilon t)^{-1}\int_{\mathbb{R}^3} \frac{|\partial_v g|^2}{M}\,dv \end{aligned} \tag{2.44}$$

with $\beta > 0$ a constant small enough, where we have used the fact that $\partial_v \nu(|v|)$ and $\partial_v K$ are bounded operators from $H^k(\mathbb{R}_v^3, M^{-1}\,dv)$ to $H^k(\mathbb{R}_v^3, M^{-1}\,dv)$ (see [7, 8] for instance) and that it holds for collision operator [5]

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\mathcal{Q}(f,g)^2}{(1+|v|)M}\,dv \;\leq\; & C\int_{\mathbb{R}^3} \frac{(1+|v|)f^2}{M}\,dv \cdot \int_{\mathbb{R}^3} \frac{g^2}{M}\,dv \\ & + C\int_{\mathbb{R}^3} \frac{f^2}{M}\,dv \cdot \int_{\mathbb{R}^3} \frac{(1+|v|)g^2}{M}\,dv \end{aligned} \tag{2.45}$$

with $C > 0$ a constant.

Making a summation between $c_0 \times (2.40)$ and (2.44) with $c_0 = 2\max\{C_\beta,$

$C\}$ and applying the Gronwall's inequality, we are able to obtain

$$\int_0^t\int_{\mathbb{R}^3}\frac{(1+|v|)|\partial_v g|^2}{M}dvds+\varepsilon\int_0^t(1+3\varepsilon s)^{-1}\int_{\mathbb{R}^3}\frac{|\partial_v^2 g|^2+|\nabla\partial_v g+v\partial_v g|^2}{M}dvds$$
$$\leq \int_{\mathbb{R}^3} M^{-1}(|g_0(v)|^2+|\partial_v g_0(v)|^2)\,dv, \tag{2.46}$$

and

$$\int_{\mathbb{R}^3}\frac{|g(v,t)|^2}{M}\,dv+\int_{\mathbb{R}^3}\frac{|\partial_v g(v,t)|^2}{M}\,dv\leq e^{-\nu_1 t}\int_{\mathbb{R}^3}\frac{|g_0(v)|^2+|\partial_v g_0(v)|^2}{M}\,dv \tag{2.47}$$

with $\nu_1>0$ a constant.

The higher order estimates can be obtained by taking inner product between ∂_v^α(2.26) and $M^{-1}\partial_v^\alpha g$, $1\leq|\alpha|\leq k$, and a complicated but straightforward computation (we omit the details) and using (2.41)–(2.42), (2.46)–(2.47). Finally, we have (2.32)–(2.33) for $F=M+g$ based on (2.25).

Proof of Theorems 1.1-1.3. With the help of local existence [12] and the application of standard continuity argument, we can obtain the global existence of solutions $F=M+g$ of the IVP (2.18)–(2.19), and hence by (2.17), we can obtain the global existence of solutions $\widetilde{F}$ of IVP (1.7)–(1.8) and its time-convergence rate to the self-similar Maxwellian for general initial data satisfying (1.9) and small perturbation as

$$\|\partial_\xi^\gamma(\widetilde{F}-\widetilde{M})(\tau)\|_{L^1(\mathbb{R}^3_\xi)}\leq C\eta_0(1+2\varepsilon\tau)^{-\gamma/2}e^{-a_1\tau^{3/2}}. \tag{2.48}$$

The proof of Theorem 1.1 is completed.

The proof of Theorem 1.3 can be established in a similar argument to Theorem 1.1, besides a complicated resolution of fluid-dynamical part and its coupling to Fokker-Planck operator, the reader can refer to [12], we omit the details.

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