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# THE CLASSICAL LIMIT FOR THE UEHLING–UHLENBECK OPERATOR

BY

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**Abstract**

We show that the Uehling–Uhlenbeck operator, the one arising in the Boltzmann equations for Fermions and Bosons, converges, when the Planck constant goes to zero, to the diffusion operator appearing in the Fokker–Planck–Landau equation.

## 1. Introduction

Consider a classical system of $N$ identical particles. We are interested in a situation where the number of particles $N$ is very large and the interaction strength quite moderate. In addition we look for a reduced or macroscopic description of the system. According to the general prescription of the kinetic theory, we introduce $r > 0$, a small parameter expressing the ratio between the macro and the micro scales. The weakness of the interaction is expressed by assuming that the potential is $O(\sqrt{r})$. Since many of the physical quantities of interest are varying on a macroscopic scale and are almost constant on the microscopic scale, we rescale the equation of motion. Then the behavior of the one-particle distribution function $f(x, v)$ (being $x$ the position and $v$ the velocity of a test particle) in the limit $r \to 0$, $N = O(r^{-3})$, is expected to solve the Fokker–Plank–Landau nonlinear diffusion equation:

$$(\partial_t + v \cdot \nabla_x) f = Q_{FP}(f), \tag{1.1}$$

where the operator $Q_{FP}(f)$ is defined by

$$Q_{FP}(f)(v) = B \int dv_1 \, \mathrm{div}_v \left[ A(v - v_1) \, (\nabla_v - \nabla_{v_1}) f(v) f(v_1) \right]. \tag{1.2}$$

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Here $A$ is the matrix

$$A(w) = \frac{|w|^2 Id - w \otimes w}{|w|^3} \tag{1.3}$$

and

$$B = \frac{1}{8\pi} \int_0^{+\infty} d\mu \, \mu^3 \hat{\phi}(\mu)^2 \tag{1.4}$$

being

$$\hat{\phi}(k) = \int e^{-ik\cdot x} \phi(x) \, dx$$

the Fourier transform of the interaction potential $\phi$. Note that the details of the interaction enter in the definition of $B$ only.

As regards the mathematical side, a derivation of the Fokker–Planck–Landau equation (in the previous scaling usually called weak–coupling limit) is a challenging and interesting still open problem. However there are results for the linear case, namely the convergence to a diffusion of a test particle in a random distribution of scatterers (see [7], [8] and [10]). We address the reader to Ref.s [2] and [15] for a formal derivation in the present nonlinear case.

A kinetic equation of the same type has been introduced by Landau in 1936 (see for instance [13] [18] [9]) for describing a plasma on the basis of the Boltzmann equation, whenever the grazing collisions become dominant. It comes out from the asymptotics (for a large ratio between the Debye and Landau lengths) of the Boltzmann collision operator in case of screened Coulomb potential, for the study of a dilute plasma. More generally, for a reasonable power law potential, the matrix $A$ takes the form

$$A_L(w) = a(|w|)(|w|^2 Id - w \otimes w). \tag{1.5}$$

where the function $a \approx \frac{1}{|w|^\nu}$ for small $|w|$, with $\nu < 1$, depends on the specific form of the cross-section appearing in the Boltzmann collision operator. For these reasons the behavior $a \approx \frac{1}{|w|}$ is usually associated to the Coulomb potential, although it arises also in the weak-coupling limit context, even for a smooth and short-range potential!

Let us now analyze the case of a quantum system under the previous scaling. This time, due to a macroscopic tunnel effect, we expect a kinetic equation of Boltzmann type. Formal arguments (see [2], [15] and [3]) yields:

$$(\partial_t + v \cdot \nabla_x) f = Q_{UU}(f) \tag{1.6}$$

where

$$Q_{UU}(f)(v) = \frac{1}{8\pi^2\hbar^2} \int dv_1 \int dk \big[\hat{\phi}(k) \pm \hat{\phi}\left(k + \tfrac{w}{\hbar}\right)\big]^2 \delta(\hbar k^2 + w \cdot k)$$
$$\big\{(1 \pm (2\pi\hbar)^3 f)(1 \pm (2\pi\hbar)^3 f_1) f' f_1' - (1 \pm (2\pi\hbar)^3 f')(1 \pm (2\pi\hbar)^3 f_1') f f_1\big\}. \tag{1.7}$$

Here $\hbar = h/(2\pi) = 1.0546 \cdot 10^{-34}$ Js, $h$ is the Planck constant, the sign $\pm$ stands for Bosons and Fermions respectively, $w = v - v_1$ denotes the relative velocity,

$$v' = v + \hbar k, \qquad v_1' = v_1 - \hbar k$$

denote the postcollisional velocities and, finally, we used the usual notation

$$f = f(v),\ f_1 = f(v_1),\ f' = f(v'),\ f_1' = f(v_1').$$

Note that the momentum is automatically conserved in the collision, while the energy is also conserved due to the presence of the $\delta$ function since:

$$\hbar k^2 + w \cdot k = \frac{1}{2}(v + \hbar k)^2 + \frac{1}{2}(v_1 - \hbar k)^2 - \frac{1}{2}v^2 - \frac{1}{2}v_1^2.$$

Equation (1.6) has been derived on the basis of purely phenomenological arguments by Nordheim in 1928 [14] and by Uehling–Uhlenbeck in 1933 [17]. We call the collision operator $Q_{UU}$ because in [17] we find the precise form we make use in the present paper. Again as regards the mathematical analysis of the derivation of this equation starting from the Schrödinger evolution, very little is known. In [3] we try a perturbative approach and find an agreement up to the second order (in the potential) of the expansion.

In [4] and [5] we consider a quantum particle system with the classical statistics and show that the agreement holds at any order of the expansion (under suitable assumptions on the interaction potential) without being able to sum the series. The limiting kinetic equation in this case is:

$$(\partial_t + v \cdot \nabla_x) f = Q_{MB}(f) \tag{1.8}$$

where

$$Q_{MB}(f)(v) = \frac{1}{4\pi^2\hbar^2}\int dv_1 \int dk\, \hat{\phi}(k)^2 \delta(\hbar k^2 + w\cdot k)\big\{f'f_1' - ff_1\big\}. \quad (1.9)$$

Here the index $MB$ stands to indicate the Maxwell–Boltzmann statistics. The situations is much better for the linear case, namely a quantum particle under the action of a random potential. Now a linear Boltzmann equation can indeed be rigorously derived ([16], [11], [12]).

Since $\hbar$ is small it is natural to look at the classical limit $\hbar \to 0$ for the solutions to equations (1.6) and (1.8), at least in the homogeneous case. Of course we expect convergence to the solutions of the Fokker–Planck–Landau equation (1.1). Indeed looking at the structure of $Q_{MB}$ we realize that the collisions become grazing when $\hbar \to 0$, and the grazing collision limit of the homogeneous Boltzmann equation has been investigated in [1], [6], [19]. Unfortunately in such papers the hypotheses are not suitable for the case we are dealing with here: indeed the case $a \approx \frac{1}{|w|}$ is excluded by the mentioned literature, so that we cannot hope to extend those results to the operator $Q_{UU}$. After the proof of Proposition 3.2 below we shall discuss further the problem.

In the present paper we approach the more modest problem of recovering the asymptotic of the operators $Q_{MB}$ and $Q_{UU}$ as preliminary step.

## 2. Organizing the Terms

In this section we set $\hbar = \varepsilon$. The collision operator is

$$\begin{aligned} Q^{\varepsilon}_{UU}(f)(v) = \frac{1}{8\pi^2\varepsilon^2}\int dv_1 \int dk\, \big[\hat{\phi}(k) \pm \hat{\phi}\left(k + \tfrac{w}{\varepsilon}\right)\big]^2 \delta(\varepsilon k^2 + w\cdot k) \\ \big\{(1\pm(2\pi\varepsilon)^3 f)(1\pm(2\pi\varepsilon)^3 f_1)f'f_1' - (1\pm(2\pi\varepsilon)^3 f')(1\pm(2\pi\varepsilon)^3 f_1')ff_1\big\}. \end{aligned} \quad (2.1)$$

The interaction potential is assumed real, spherically symmetric and suitably smooth. As a consequence $\hat{\phi}$ is real and spherically symmetric as well. Expanding $\big[\hat{\phi}(k) \pm \hat{\phi}\left(k+\frac{w}{\varepsilon}\right)\big]^2$, we consider the term

$$\frac{1}{2\varepsilon^2}\int dv_1 \int dk\, \hat{\phi}\left(k + \tfrac{w}{\varepsilon}\right)^2\, \delta(\dots)\,\{\dots\}. \quad (2.2)$$

Setting

$$k' = -\left(k + \tfrac{w}{\varepsilon}\right), \quad (2.3)$$

then

$$v + \varepsilon k = v_1 - \varepsilon k', \qquad v_1 - \varepsilon k = v + \varepsilon k' \tag{2.4}$$

and $\delta(\dots)\{\dots\}$ remains invariant. Using that $\hat{\phi}(k) = \hat{\phi}(-k)$ we can rewrite

$$Q^\varepsilon_{UU} = Q^\varepsilon_1 + Q^\varepsilon_2 \tag{2.5}$$

with

$$Q^\varepsilon_1 = \frac{1}{4\pi^2\varepsilon^2}\int dv_1 \int dk\, \hat{\phi}(k)^2\, \delta(\dots)\{\dots\} \tag{2.6}$$

and

$$Q^\varepsilon_2 = \pm\frac{1}{4\pi^2\varepsilon^2}\int dv_1 \int dk\, \hat{\phi}(k)\hat{\phi}\left(k + \tfrac{w}{\varepsilon}\right)\, \delta(\dots)\{\dots\}. \tag{2.7}$$

Obviously the $\delta$ function appearing in the collision operator can be solved so that we arrive to the more conventional form:

$$Q^\varepsilon_1 = \frac{1}{4\pi^2\varepsilon^4}\int dv_1 \int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w| \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\hat{k}\right)^2$$
$$\left\{(1 \pm (2\pi\varepsilon)^3 f)(1 \pm (2\pi\varepsilon)^3 f_1) f' f'_1 - (1 \pm (2\pi\varepsilon)^3 f')(1 \pm (2\pi\varepsilon)^3 f'_1) f f_1\right\}, \tag{2.8}$$

$$Q^\varepsilon_2 = \pm\frac{1}{4\pi^2\varepsilon^4}\int dv_1 \int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w| \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\hat{k}\right)\hat{\phi}\left(\tfrac{w-(\hat{k}\cdot w)\,\hat{k}}{\varepsilon}\right)$$
$$\left\{(1 \pm (2\pi\varepsilon)^3 f)(1 \pm (2\pi\varepsilon)^3 f_1) f' f'_1 - (1 \pm (2\pi\varepsilon)^3 f')(1 \pm (2\pi\varepsilon)^3 f'_1) f f_1\right\}, \tag{2.9}$$

where

$$S^2_- = \{\hat{k}|\, |\hat{k}| = 1,\, w\cdot\hat{k} \le 0\}$$

and

$$v' = v - (\hat{k}\cdot w)\,\hat{k}, \qquad v'_1 = v_1 + (\hat{k}\cdot w)\,\hat{k}. \tag{2.10}$$

Equations (2.8) and (2.9) follow from the formula

$$\int dk \delta(\varepsilon k^2 + w\cdot k)\, g(k) = \frac{1}{\varepsilon^2}\int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w|\, g\left(-\tfrac{\hat{k}\cdot w}{\varepsilon}\hat{k}\right) \tag{2.11}$$

which is valid for any test function $g$ as follows by using polar coordinates.

The quadratic part in $f$ of $Q^\varepsilon_1$ is exactly the collision operator (1.9) for the case of Maxwell–Boltzmann statistics:

$$Q^\varepsilon_{MB} = \frac{1}{4\pi^2\varepsilon^4}\int dv_1 \int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w| \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)^2 (f' f'_1 - f f_1). \tag{2.12}$$

Here we used the standard notational abuse $\hat{\phi}(|x|) = \hat{\phi}(x)$. Separating the contributions of order two and three in $f$ in $Q_1^\varepsilon$ and $Q_2^\varepsilon$ we can write

$$Q_{UU}^\varepsilon = Q_1^\varepsilon + Q_2^\varepsilon = Q_{MB}^\varepsilon + R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon.$$

where

$$R_1^\varepsilon = \pm\frac{1}{4\pi^2\varepsilon^4}\int dv_1 \int_{S_-^2} d\hat{k}\,|\hat{k}\cdot w|\hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)\hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-(\hat{k}\cdot\hat{w})^2}\right)(f'f_1'-ff_1), \tag{2.13}$$

$$R_2^\varepsilon = \pm\frac{2\pi}{\varepsilon}\int dv_1 \int_{S_-^2} d\hat{k}\,|\hat{k}\cdot w|\hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)^2 \left(f'f_1'(f+f_1) - ff_1(f'+f_1')\right), \tag{2.14}$$

$$R_3^\varepsilon = \frac{2\pi}{\varepsilon}\int dv_1 \int_{S_-^2} d\hat{k}\,|\hat{k}\cdot w|\hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)\hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-(\hat{k}\cdot\hat{w})^2}\right) \times \left(f'f_1'(f+f_1) - ff_1(f'+f_1')\right), \tag{2.15}$$

$Q_{MB}^\varepsilon$ is the leading part in the asymptotics of $Q_{UU}^\varepsilon$. We analyze it in the next section, then we will prove that $R_i^\varepsilon$, $i = 1, 2$, vanish as $\varepsilon \to 0$.

## 3. The Main Term

In this section we prove that

$$\lim_{\varepsilon\to 0} Q_{MB}^\varepsilon(f) = Q_{FP}(f), \quad \text{in } \mathcal{S}'.$$

A standard computation show that, for any $u \in \mathcal{S}$:

$$\int dv\, u(v) Q_{FP}(f)(v) = \int dv\, dv_1\, \mathcal{L}u(v, v_1)\, ff_1 \tag{3.1}$$

where

$$\mathcal{L}u = -2\frac{B}{|w|^2}\hat{w}\cdot(\nabla_v u(v) - \nabla_{v_1}u(v_1)) + B\,\mathrm{Tr}\,(A\,D^2u(v)) \tag{3.2}$$

and where $\hat{w} = \frac{w}{|w|}$ and $\mathrm{Tr}\,(A\,D^2 u(v)) = \sum_{i,j} A_{ij}\partial^2_{v_i v_j} u(v)$. Note that the right hand side of (3.2) is $O(1/|w|)$, for $|w|$ small, then (3.1) makes sense if

$$\int \frac{dv\, dv_1\, ff_1}{|v - v_1|} < +\infty.$$

This is assured for a probability distribution $f \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, with $p > 3/2$. Moreover this condition give the extra summability

$$\int \frac{dv\, dv_1\, f f_1}{|v - v_1|^{1+\eta}} < +\infty \tag{3.3}$$

for $\eta < 2 - 3/p$, which will be needed in performing the limit, as we shall see in the sequel.

We start by analyzing the properties of the cross–section. We introduce the notation for $a \geq 0$

$$b_a(\mu) = |\mu|^a \hat{\phi}(|\mu|)^2, \quad n_a = \int_0^{+\infty} (1 + \mu^a)\hat{\phi}(\mu)^2. \tag{3.4}$$

**Lemma 3.1.**

$$\int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) \hat{k} = -\frac{2\pi\varepsilon^2}{|w|^2}\hat{w} \int_0^{|w|/\varepsilon} d\mu\, \mu^3\, \hat{\phi}(\mu)^2 \tag{3.5}$$

$$\int_{S^2_-} d\hat{k}\; b_3\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) \hat{k}\otimes\hat{k} = \frac{\pi\varepsilon}{|w|}\left(Id - \hat{w}\otimes\hat{w}\right) \int_0^{|w|/\varepsilon} d\mu\, \mu^3\, \hat{\phi}(\mu)^2 + r \tag{3.6}$$

$$\text{with} \qquad |r| \leq c\, n_{3+\eta} \left(\frac{\varepsilon}{|w|}\right)^{1+\eta}, \qquad \eta \in [0, 2], \tag{3.7}$$

$$\int_{S^2_-} d\hat{k}\; b_a\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) \leq c\, n_{a-(1-\eta)} \left(\frac{\varepsilon}{|w|}\right)^{\eta}, \quad a \geq 1,\ \eta \in [0, 1]. \tag{3.8}$$

*Proof.* We set:

$$\hat{k} = -\lambda\hat{w} + \sqrt{1-\lambda^2}\xi, \tag{3.9}$$

where $\lambda \in [0, 1]$ and the unitary vector $\xi$ varies in the circle $S^1(w)$ lying in the plane orthogonal to $\hat{w}$. Then:

$$\int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) \hat{k} = \int_{S^1(w)} d\xi \int_0^1 d\lambda\; b_2\left(\tfrac{\lambda|w|}{\varepsilon}\right) (\sqrt{1-\lambda^2}\xi - \lambda\hat{w}).$$

Using that $\int_{S^1(w)} \xi d\xi = 0$, and setting $\mu = \frac{\lambda|w|}{\varepsilon}$, we obtain (3.5). With the same change of variable:

$$\int_{S^2_-} d\hat{k}\; b_3\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) \hat{k}\otimes\hat{k}$$

$$= \int_{S^1(w)} d\xi \int_0^1 d\lambda\, b_3\left(\tfrac{\lambda|w|}{\varepsilon}\right) (\sqrt{1-\lambda^2}\xi - \lambda\hat{w}) \otimes (\sqrt{1-\lambda^2}\xi - \lambda\hat{w}).$$

Now we can use that $\int_{S^1(w)} d\xi\, \xi \otimes \xi = \pi(Id - \hat{w} \otimes \hat{w})$, i.e. $2\pi \cdot 1/2$ times the identity matrix on the plane orthogonal to $\hat{w}$. We obtain (3.6) with

$$r = -\pi(Id - 3\hat{w} \otimes \hat{w}) \int_0^1 d\lambda\, \lambda^2\, b_3\left(\tfrac{\lambda|w|}{\varepsilon}\right).$$

Moreover:

$$|r| \le c\frac{\varepsilon}{|w|} \int_0^{|w|/\varepsilon} d\mu \left(\frac{\mu\varepsilon}{|w|}\right)^2 \mu^3 \hat{\phi}(\mu)^2 \le c\left(\frac{\varepsilon}{|w|}\right)^{1+\eta} \int_0^{+\infty} d\mu\, \mu^{3+\eta}\hat{\phi}(\mu)^2,$$

for any $\eta \in [0,2]$. Finally:

$$\int_{S^2_-} d\hat{k}\, b_a\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) = \int_{S^1(w)} d\xi \int_0^1 d\lambda\, b_a\left(\tfrac{\lambda|w|}{\varepsilon}\right) = \frac{2\pi\varepsilon}{|w|} \int_0^{|w|/\varepsilon} d\mu\, \mu^a\, \hat{\phi}(\mu)^2$$
$$\le c\left(\frac{\varepsilon}{|w|}\right)^{1-q} \int_0^{+\infty} d\mu\, \mu^{a-q}\, \hat{\phi}(\mu)^2,$$

for any $q \in [0,a]$. Choosing $q = 1-\eta$ we obtain (3.8). □

**Proposition 3.2.** *Let $f$ be a probability distribution such that $f \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $p > 3/2$. Suppose also that:*

$$\int_0^\infty d\mu\, (1+\mu^\alpha)\hat{\phi}(\mu)^2 < +\infty \tag{3.10}$$

*for some $\alpha > 3$. Then*

$$\lim_{\varepsilon\to 0} Q^\varepsilon_{MB}(f) = Q_{FP}(f) \quad in \quad \mathcal{S}' \tag{3.11}$$

*where $Q_{FP}(f)$ is given by equations* (1.2), (1.3), (1.4).

*Proof.* For $u \in \mathcal{S}$, we have:

$$\int dv\, u(v)\, Q^\varepsilon_{MB}(f)(v) = \frac{1}{4\pi^2\varepsilon^4} \int dv\, dv_1\, f f_1 \int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w|\, \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)^2 [u(v') - u(v)], \tag{3.12}$$

where $v' = v - (\hat{k}\cdot w)\,\hat{k}$. Expanding

$$u(v') - u(v) = |\hat{k}\cdot w|\, \hat{k}\cdot \nabla_v u + \frac{1}{2}|\hat{k}\cdot w|^2 \mathrm{Tr}(\hat{k}\otimes\hat{k}\; D^2u(v)) + r_3\, |\hat{k}\cdot w|^3,$$

we can write

$$\int dv\, u(v)\, Q^{\varepsilon}_{_{MB}}(f)(v) = \int dv\, dv_1\, f f_1\, (T_1 + T_2 + T_3)\,,$$

where

$$T_1 = \frac{1}{8\pi^2\varepsilon^2}\int_{S^2_-} d\hat{k}\ b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\ \hat{k}\cdot(\nabla_v u(v) - \nabla_{v_1} u(v_1)), \tag{3.13}$$

$$T_2 = \frac{1}{8\pi^2\varepsilon}\int_{S^2_-} d\hat{k}\ b_3\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\ \mathrm{Tr}(\hat{k}\otimes\hat{k}\ D^2u(v)), \tag{3.14}$$

$$T_3 = \frac{1}{4\pi}\int_{S^2_-} d\hat{k}\ b_4\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\ r_3, \tag{3.15}$$

For (3.13) we have symmetrized the term using the changes of variables $v \leftrightarrow v_1$, $k \rightarrow -k$. Now we can apply Lemma 3.1 to obtain (3.11). □

We note that $Q^{\varepsilon}_{_{BQ}}(f)$ is a collision operator for a classical Boltzmann equation, with a quantum cross–section $\hat{\phi}(\hat{k}\cdot w/\varepsilon)^2$. This expression implies that, when $\varepsilon \rightarrow 0$, $w\cdot\hat{k} = O(\varepsilon)$ so that the collision operator concentrates on grazing collisions. The behavior of the solutions of the homogeneous Boltzmann equation in the grazing collision limit is well known. In [1] the authors show that, under suitable assumptions on the cross-section, a diffusion Fokker–Planck–Landau equation, with a matrix $A_L$ given by equation (1.5) and a smooth function $a$, can indeed be derived. Next in [6] and [19] steps forward were performed to arrive to cover the case $a \approx \frac{1}{|w|^\nu}$ , with $\nu < 1$, so that this analysis cannot be implemented in our context. Actually we would need a control on the quantity (3.3) uniform in $\varepsilon$ and this does not follow by the usual estimates available up to now.

## 4. The Remainders

For $R^{\varepsilon}_1$, we have:

$$\int dv\, u(v)\, R^{\varepsilon}_1(f)(v)$$
$$= \frac{\pm 1}{4\pi^2\varepsilon^4}\int dv dv_1 f f_1 \int_{S^2_-} d\hat{k}|\hat{k}\cdot w|\hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)\hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-(\hat{k}\cdot\hat{w})^2}\right)$$
$$\times[u(v')-u(v)]. \tag{4.1}$$

The term $\hat{\phi}\left(\frac{|\hat{k}\cdot w|}{\varepsilon}\right)$ concentrates the collisions on $\hat{k}\cdot w = O(\varepsilon)$, while $\hat{\phi}\left(\frac{|w|}{\varepsilon}\sqrt{1-(\hat{k}\cdot\hat{w})^2}\right)$ concentrates on $(Id-\hat{k}\otimes\hat{k})w = O(\varepsilon)$. We will take advantage from this two simultaneous concentrations, as stated in the following lemma.

**Lemma 4.1.** *For any* $\eta \geq 0$

$$\left|\int_{S^2_-} d\hat{k}\, \frac{|\hat{k}\cdot w|^2}{\varepsilon^2}\, \hat{\phi}\left(\frac{|\hat{k}\cdot w|}{\varepsilon}\right)\, \hat{\phi}\left(\frac{|w|}{\varepsilon}\sqrt{1-\lambda^2}\right) \hat{k}\right| \leq c\, n_{5+2\eta}\left(\frac{\varepsilon}{|w|}\right)^{2+\eta}, \tag{4.2}$$

$$\left|\int_{S^2_-} d\hat{k}\, \frac{|\hat{k}\cdot w|^3}{\varepsilon^3}\, \hat{\phi}\left(\frac{|\hat{k}\cdot w|}{\varepsilon}\right)\, \hat{\phi}\left(\frac{|w|}{\varepsilon}\sqrt{1-\lambda^2}\right)\right| \leq n_{5+2\eta}\left(\frac{\varepsilon}{|w|}\right)^{1+\eta}, \tag{4.3}$$

*Proof.* We proceed as in Lemma 3.1. Integrating in $d\xi$ as for (3.5), we easily bound the first term in terms of

$$c\frac{\varepsilon^2}{|w|^2}\int_0^\gamma d\mu\, \mu^3 \left|\hat{\phi}\,(\mu)\ \hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right| \tag{4.4}$$

where $\gamma = |w|/\varepsilon$. Using the change of variable $\mu \leftrightarrow \sqrt{\gamma^2-\mu^2}$, for which $\mu\, d\mu$ is invariant, we obtain

$$\begin{aligned}
&\int_0^{\gamma/\sqrt{2}} d\mu\, \mu^3 \left|\hat{\phi}\,(\mu)\,\hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right| + \int_{\gamma/\sqrt{2}}^{\gamma} d\mu\, \mu^3 \left|\hat{\phi}\,(\mu)\ \hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right| \\
&\quad = \int_{\gamma/\sqrt{2}}^{\gamma} d\mu\, (\mu^3+\mu\,(\gamma^2-\mu^2)) \left|\hat{\phi}\,(\mu)\,\hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right| \\
&\quad = \gamma^2 \int_{\gamma/\sqrt{2}}^{\gamma} d\mu\, \mu \left|\hat{\phi}\,(\mu)\,\hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right|
\end{aligned}$$

Finally

$$\begin{aligned}
&\gamma^2 \int_{\gamma/\sqrt{2}}^{\gamma} d\mu\, \mu \left|\hat{\phi}\,(\mu)\,\hat{\phi}\left(\sqrt{\gamma^2-\mu^2}\right)\right| \\
&\quad \leq \gamma^2 \left(\int_0^{+\infty} d\mu\, \mu\, \hat{\phi}(\mu)^2\right)^{1/2} \left(\int_{\gamma/\sqrt{2}}^{+\infty} d\mu\, \mu\, \hat{\phi}(\mu)^2\right)^{1/2} \\
&\quad \leq c\frac{\varepsilon^\eta}{|w|^\eta} \left(\int_0^{+\infty} d\mu\, \mu\, \hat{\phi}(\mu)^2\right)^{1/2} \left(\int_0^{+\infty} d\mu\, \mu^{5+2\eta}\, \hat{\phi}(\mu)^2\right)^{1/2},
\end{aligned}$$

which inserted in (4.4) gives (4.2).

For (4.3), with the same change of variables, we obtain the same estimate (4.4) replacing the exponent 2 for $\varepsilon/|w|$ by 1. □

**Proposition 4.2.** *Under the hypotheses of Proposition* 3.2 *with* $\alpha > 5$,

$$\lim_{\varepsilon\to 0} R_1^\varepsilon = 0 \quad \text{in} \quad \mathcal{S}'. \tag{4.5}$$

*Proof.* We expand $u$ in (4.1) up to the second order:

$$u(v') - u(v) = |\hat{k}\cdot w|\,\hat{k}\cdot\nabla_v u(u) + |\hat{k}\cdot w|^2 r_2,$$

obtaining

$$\int dv\, u(v)\, R_1^\varepsilon(f)(v) = \int dv\, dv_1\, f f_1(\tilde{T}_1 + \tilde{T}_2)$$

where

$$\tilde{T}_1 = \frac{1}{8\pi^2\varepsilon^2}\int_{S^2_-} d\hat{k}\, \tfrac{|\hat{k}\cdot w|^2}{\varepsilon^2}\, \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)\, \hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-\lambda^2}\right)\hat{k}\cdot(\nabla_v u(v) - \nabla_{v_1}u(v_1)), \tag{4.6}$$

$$\tilde{T}_2 = \frac{1}{4\pi^2\varepsilon}\int_{S^2_-} d\hat{k}\, \tfrac{|\hat{k}\cdot w|^3}{\varepsilon^3}\, \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)\, \hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-\lambda^2}\right) r_2. \tag{4.7}$$

Now we can apply Lemma 4.1 to conclude. Let us note that the symmetrized expression for $\tilde{T}_1$ is needed in order to compensate $1/|w|^{2+\eta}$ with $\nabla_v u(v) - \nabla_{v_1}u(v_1)$. □

It remains to prove that also the terms $R_2^\varepsilon$ and $R_3^\varepsilon$ are vanishing. For them we have an extra $\varepsilon^3$ which, in principle, give makes easier the convergence to 0. Hoverer such terms are cubic in $f$, so that we need more summability. For $u \in \mathcal{S}$:

$$\int dv\, u(v)\, R_2^\varepsilon = \frac{2\pi}{\varepsilon}\int dv\, dv_1 \int_{S^2_-} d\hat{k}\, |\hat{k}\cdot w|\, \hat{\phi}\left(\tfrac{|\hat{k}\cdot w|}{\varepsilon}\right)^2 f f_1(f'+f_1')[u(v')-u(v)].$$

Using $|u(v') - u(v)| \le \|\nabla u\|_\infty |\hat{k}\cdot w|$, we obtain

$$\left|\int dv\, u(v)\, R_2^\varepsilon\right| \le c\varepsilon \int dv\, dv_1 \int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right) f f_1(f' + f_1') \le c\varepsilon(I_1\, I_2)^{1/2},$$

where

$$I_1 = \int dv\, dv_1 \int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)(1+|w|)^\beta f^2 f_1^2$$

$$I_2 = \int dv\, dv_1 \int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\frac{(f'+f_1')^2}{(1+|w|)^\beta}$$

We estimate $I_1$ by $c\int dv\,(1+|v|^\beta)\,f^2$ using (3.8) with $a=2$, $\eta=0$. We can estimate $I_2$ after the change of variable $(v,\,v_1)\leftrightarrow(v',\,v_1')$, using that $\int dv/(1+|v|^\beta)<+\infty$ if $\beta>3$:

$$\int dv\, dv_1 \int_{S^2_-} d\hat{k}\; b_2\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\tfrac{(f+f_1)^2}{1+|w|^\beta} \le c\int dv\, f^2.$$

Collecting these two estimate we obtain

$$\left|\int dv\, u(v)\, R_2^\varepsilon\, u(v)\right| \le c\varepsilon\,\|(1+|v|^\beta)f(v)\|^3_{L^2(\mathbb{R}^3)}.$$

For $R_3^\varepsilon$ we use the Cauchy–Schwartz inequality:
$\left|\int dv\, u(v)\, R_3^\varepsilon\right| \le c\varepsilon(I_1\, I_2)^{1/2}$, where

$$I_1 = \int dv\, dv_1 \int_{S^2_-} d\hat{k}\; \tfrac{|\hat{k}\cdot w|}{\varepsilon}\left|\hat{\phi}\left(\tfrac{|w|}{\varepsilon}\sqrt{1-\lambda^2}\right)\right|(1+|w|)^\beta f^2 f_1^2$$

$$I_2 = \int dv\, dv_1 \int_{S^2_-} d\hat{k}\; b_3\left(\tfrac{\hat{k}\cdot w}{\varepsilon}\right)\frac{(f'+f_1')^2}{(1+|w|)^\beta}$$

Using that $\int_0^\gamma d\mu\,\mu\,\hat{\phi}\left(\sqrt{\gamma^2-\lambda^2}\right)=\int_0^\gamma d\mu\,\mu\,\hat{\phi}(\mu)$ we obtain the same estimate as for $R_2^\varepsilon$.

The estimates of this section show that the effects of the quantum statistics become negligible in the limit $\varepsilon\to 0$. We summarize all the propositions in the following theorem.

**Theorem 4.3.** *If $f\in L^1(\mathbb{R}^3)$, $\int(1+|v|^\beta)f^2<+\infty$ with $\beta>3$, and $\int_0^{+\infty}(1+\mu^\alpha)\hat{\phi}(\mu)^2$ with $\alpha>5$, then*

$$\lim_{\varepsilon\to 0} Q^\varepsilon_{UU} = Q_{FP}, \quad \text{in } \mathcal{S}'$$

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