Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **2** (2007), No. 3, pp. 785-795

EXISTENCE OF NONOSCILLATORY SOLUTION OF SECOND ORDER NEUTRAL DIFFERENTIAL EQUATION

BY

GONG WEIMING, CHENG JINFA AND CHU YUMING

Abstract

Consider the neutral delay differential equation with positive and negative coefficients:

$$(r(t)(x(t)+px(t-\tau)')'+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0,$$

where $p\in R$ and $\tau\in(0,\infty)$, σ_1, $\sigma_2\in[0,\infty)$ and $Q_1(t)$, $Q_2(t)$, $r(t)\in C([t_0,\infty),R^+)$. Some sufficient conditions for the existence of a nonoscillatory solution of the above equation in terms of $\int^\infty R(s)Q_i ds<\infty, i=1,2$ are obtained.

1. Introduction

Consider the neutral delay differential equation of second order with positive and negative coefficients:

$$(r(t)(x(t)+px(t-\tau)')'+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)=0, \tag{1}$$

where $p\in R$ and

$$\tau\in(0,\infty),\sigma_{1,}\sigma_2\in[0,\infty) \quad \text{and} \quad Q_1(t),Q_2(t),r(t)\in C([t_0,\infty),R^+). \tag{2}$$

Received June 12, 2006 and in revised form February 12, 2007.

AMS Subject Classification: 39A10, 34K11, 34K40.

Key words and phrases: Nonoscillatory solution, delay, Neutral differential equation.

Research supported by the 973 Project(2006CB708304), the NSF of China(10471039) and Foundation of the Educational Committee of Zhejiang Province(20060306).

$$\int^{\infty} R(s)Q_i ds < \infty, i = 1, 2, \tag{3}$$

where $R(t) = \int^t r(s)ds$.

When $r(t) = 1$, the equation (1) has been reduced to the following equation:

$$\frac{d^2}{dt^2}[x(t) + px(t-\tau)] + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0. \tag{4}$$

When $q(t) = 0$, the Eq.(4) has been investigated by many authors, see [1-7] and the reference therein.

For Eq.(4), Recently M. R. S. Kulenović and S.Hadžiomerspahić in [1] obtained the following result:

Theorem A. *Consider the Eq.*(1)*, if condition* (2) *holds, and*

$$\int^{\infty} sQ_i ds < \infty, i = 1, 2, \tag{5}$$

$$aQ_1(t) - Q_2(t) \geq 0, \quad for \quad every \quad t \geq T_1 \quad and \quad a > 0, \tag{6}$$

where $p \neq \pm 1$*, and* T_1 *is large enough, then Eq.*(1) *has a nonoscillatory solution.*

So far, this is the first global result (with respect to p) in the nonconstant coefficient case, which is a sufficient condition for the existence of a nonoscillatory solution for all values of p.

But, condition (6) is too restrictive. In [6] the first author of this paper deleted the strong condition (6), and permitting p=1, obtained the following global sufficient condition(with respect to p) for the existence of a nonoscillatory solution for equation (1):

Theorem B. *Consider equation* (4)*, if condition* (2)*,* (5) *hold, where* $p \neq -1$*, then equation* (1) *has a nonoscillatory solution.*

For Eq(1), only in special case, for example when $p = 0$, $Q_2(t) = 0$, Hooker and Patula in [7] had investigated the existence of positive solution.

However results for the existence of nonoscillatory solution of Eq.(1) are relatives scarce. Motived by the paper [6] and [7], the purpose of this paper will investigate the existence of nonoscillatory solution of Eq.(1).

Our main result is the following.

Theorem. *Consider equation* (1)*, if condition* (2)*,* (3) *hold, where* $p \neq \pm 1$*, then equation* (1) *has a nonoscillatory solution.*

This result extends the relevant result in [6] for $p \neq -1$.

2. The Proof of Theorem

The proof of theorem will be divided into four claims, depending on the four different ranges of the parameter p.

Claim 1. $p \in (0,1)$. Choose $t_1 > t_0$ large enough such that

$$t_1 \geq t_0 + \sigma, \sigma = \max\{\tau, \sigma_1, \sigma_2\},$$
$$\int_{t_1}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds < 1 - p,$$
$$\int_{t_1}^{\infty} R(s)Q_1(s)ds \leq \frac{p - (1 - M_1)}{M_1}$$

and

$$\int_{t_1}^{\infty} R(s)Q_2(s)ds \leq \frac{1 - p - pM_2 - M_1}{M_2}$$

hold, where M_1 and M_2 are positive constants which satisfy

$$1 - M_2 < p < \frac{1 - M_1}{1 - M_2}.$$

Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the sup norm. Set

$$A = \{x \in X : M_1 \leq x(t) \leq M_2, t \geq t_0\}.$$

Define mapping $T : A \to X$ as follows:

$$(Tx)(t)=\begin{cases} 1-p-px(t-\tau) \\ \quad +R(t)\int_t^\infty [Q_1(s)x\,(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \\ \quad +\int_{t_1}^t R(s)[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds, & t\geq t_1; \\ (Tx)(t_1), & t_0\leq t\leq t_1. \end{cases}$$

We have

$$\begin{aligned} (Tx)(t) &\leq 1-p+R(t)\int_t^\infty M_2Q_1(s)ds+\int_{t_1}^t R(s)M_2Q_1(s)ds \\ &\leq 1-p+M_2\int_t^\infty R(s)Q_1(s)ds\leq M_2 \end{aligned}$$

and

$$\begin{aligned} (Tx)(t) &\geq 1-p-pM_2-R(t)\int_t^\infty Q_2(s)x\,(s-\sigma_2)\,ds \\ &\qquad -\int_{t_1}^t R(s)[Q_2(s)x(s-\sigma_2)]ds \\ &\geq 1-p-pM_2-M_2\int_t^\infty R(s)Q_2(s)ds\geq M_1, \end{aligned}$$

so TA $\subseteq A$.

Now for $x_1,x_2\in A$ and $t\geq t_1$,we have

$$\begin{aligned} |(Tx_1)(t)-(Tx_2)(t)| &\leq p|x_1(t-\tau)-x_2(t-\tau)| \\ &\quad +R(t)\int_t^\infty Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_1)|ds \\ &\quad +R(t)\int_t^\infty Q_2(s)|x_1(s-\sigma_2)-x_2(s-\sigma_2)|ds \\ &\quad +\int_{t_1}^t R(s)Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_1)|ds \\ &\quad +\int_{t_1}^t R(s)Q_2(s)|x_1(s-\sigma_2)-x_2(s-\sigma_2)|ds \\ &\leq p||x_1-x_2||+||x_1-x_2||\{\int_t^\infty R(s)[Q_1(s)+Q_2(s)]ds \\ &\quad +\int_{t_1}^t R(s)[Q_1(s)+Q_2(s)]ds\} \\ &= ||x_1-x_2||\{p+\int_{t_1}^\infty R(s)[Q_1(s)+Q_2(s)]ds\} \end{aligned}$$

$$= q_1||x_1 - x_2||, \qquad q_1 < 1.$$

Thus we know that T is a contraction mapping . Consequently T has the unique fixed point x, which is obviously a positive solution of Eq.(1). This completes the proof of Claim 1.

Claim 2. $p \in (1, \infty)$. Choose $t_1 \geq t_0$ large enough such that

$$\int_{t_1}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds < p - 1,$$
$$\int_{t_1}^{\infty} R(s)Q_1(s)ds \leq \frac{1 - p(1 - N_1)}{N_1}$$

and

$$\int_{t_1}^{\infty} R(s)Q_2(s)ds \leq \frac{(1 - N_1)p - (1 + N_2)}{N_2}, \tag{7}$$

where N_1, N_2 are positive constants which satisfy

$$(1 - N_1)p \geq 1 + N_2 \quad \text{and} \quad p(1 - N_2) < 1.$$

Let X be the same set as in Claim 1.Set

$$A = \{x \in X : N_1 \leq x(t) \leq N_2, t \geq t_0\}.$$

Define mapping $T : A \to X$ as follows:

$$(Tx)(t) = \begin{cases} 1 - \frac{1}{p} - \frac{1}{p}x(t + \tau) \\ \quad + \frac{R(t+\tau)}{p} \int_{t+\tau}^{\infty} [Q_1(s)x(s - \sigma_1) - Q_2(s)x(s - \sigma_2)]ds \\ \quad + \frac{1}{p} \int_{t_1}^{t+\tau} R(s)[Q_1(s)x(s - \sigma_1) - Q_2(s)x(s - \sigma_2)]ds, & t \geq t_1; \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, we get

$$\begin{aligned} (Tx)(t) &\leq 1 - \frac{1}{p} + \frac{R(t + \tau)}{p} \int_{t+\tau}^{\infty} N_2 Q_1(s)ds + \frac{1}{p} \int_{t_1}^{\infty} N_2 R(s)Q_1(s)ds \\ &\leq 1 - \frac{1}{p} + \frac{N_2}{p} \int_{t_1}^{\infty} R(s)Q_1(s)ds \leq N_2 \end{aligned}$$

and

$$\begin{aligned}(Tx)(t) &\geq 1-\frac{1}{p}-\frac{N_2}{p}+\frac{R(t+\tau)}{p}\int_{t+\tau}^{\infty}(-N_2Q_2(s))ds \\ &\quad +\frac{1}{p}\int_{t_1}^{t+\tau}(-N_2R(s)Q_2(s))ds \\ &\geq 1-\frac{1}{p}-\frac{N_2}{p}-\frac{N_2}{p}\int_{t_1}^{\infty}R(s)Q_2(s)ds \geq N_1.\end{aligned}$$

Thus we know that $TA \subset A$. Since A is a bounded, closed, and convex subset of X, hence we can prove that T is a contraction mapping on A by the contraction principle.

In fact, for $x_1, x_2 \in A$, we have

$$\begin{aligned}&|(Tx_1)(t)-(Tx_2)(t)| \\ &\leq \frac{1}{p}|x_1(t+\tau)-x_2(t+\tau)| \\ &\quad +\frac{R(t+\tau)}{p}\left[\int_{t+\tau}^{\infty}Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_2)|ds\right. \\ &\quad \left.+\int_{t+\tau}^{\infty}Q_2(s)|x_1(s-\sigma_2)-x_2(s-\sigma_2)|ds\right] \\ &\quad +\frac{1}{p}\left[\int_{t_1}^{t+\tau}R(s)Q_1(s)|x_1(s-\sigma_1)-x_2(s-\sigma_1)|ds\right. \\ &\quad \left.+\int_{t_1}^{t+\tau}R(s)Q_1(s)|x_1(s-\sigma_2)-x_2(s-\sigma_2)|ds\right] \\ &\leq \frac{1}{p}||x_1-x_2||+\frac{1}{p}||x_1-x_2||\left\{\int_{t+\tau}^{\infty}R(s)[Q_1(s)+Q_2(s)]ds\right. \\ &\quad \left.+\int_{t_1}^{t+\tau}R(s)[Q_1(s)+Q_2(s)]ds\right\} \\ &= \frac{1}{p}||x_1-x_2||\{1+\int_{t_1}^{\infty}R(s)[Q_1(s)+Q_2(s)]ds\} \\ &= q_2||x_1-x_2||, \qquad\qquad q_2<1.\end{aligned}$$

This implies that

$$||Tx_1-Tx_2|| \leq q_2||x_1-x_2||.$$

Thus we know that T is a contraction mapping. Consequently T has the unique fixed point x, which is obviously a positive solution of Eq.(1). This completes the proof of Claim 2.

Claim 3. $p \in (-1,0)$. Choose $t_1 > t_0$ large enough such that the inequalities

$$\int_{t_1}^{\infty} R(s)[Q_1(s)+Q_2(s)]ds < p+1,$$

$$0 \leq \int_{t_1}^{\infty} R(s)Q_1(s)ds \leq \frac{M_3(1+p)-(1+p)}{M_3}$$

and

$$\int_{t_1}^{\infty} R(s)Q_2(s)ds \leq \frac{(1+p)-M_3(1+p)}{M_4}. \tag{8}$$

hold, where the positive constants M_3 and M_4 satisfy

$$0 < M_3 < 1 < M_4.$$

Let X be the same set as in Claim 1. Set

$$A = \{x \in X : M_3 \leq x(t) \leq M_4, t \geq t_0\}. \tag{9}$$

Define mapping $T : A \to X$ as follows:

$$(Tx)(t) = \begin{cases} 1+p-px(t-\tau) \\ \quad +R(t)\int_t^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \\ \quad +\int_{t_1}^{t} R(s)[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds, & t \geq t_1; \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \geq t_1$, by (8) we get

$$\begin{aligned} (Tx)(t) &\leq 1+p-pM_4+R(t)\int_t^{\infty} M_4 Q_1(s)ds+\int_{t_1}^{t} R(s)M_4Q_1(s)ds \\ &\leq 1+p-pM_4+M_4\int_{t_1}^{\infty} R(s)Q_1(s)ds \leq M_4 \end{aligned}$$

and

$$\begin{aligned}(Tx)(t) &\geq 1+p-pM_3-R(t)\int_t^{\infty} M_4Q_2(s)ds-\int_{t_1}^{t} R(s)M_4Q_2(s)ds\\ &= 1+p-pM_3-M_4\int_t^{\infty} R(s)Q_2(s)ds \geq M_3.\end{aligned}$$

Thus we know that $TA \subset A$. Since A is a bounded, closed, and convex subset of X, hence we can prove that T is a contraction mapping on A by the contraction principle.

In fact, for $x_1, x_2 \in A$, we have

$$\begin{aligned}&|(Tx_1)(t)-(Tx_2)(t)|\\ &\leq -p|x_1(t-\tau)-x_2(t-\tau)|\\ &\quad +R(t)\int_t^{\infty} Q_1(s)\,|x_1(s-\sigma_2)-x_2(s-\sigma_2)\,|ds\\ &\quad +R(t)\int_t^{\infty} Q_2(s)\,|x_1(s-\sigma_2)-x_2(s-\sigma_2)\,|ds\\ &\quad +\int_{t_1}^{t} R(s)Q_1(s)\,|x_1(s-\sigma_1)-x_2(s-\sigma_1)\,|ds\\ &\quad +\int_{t_1}^{t} R(s)Q_2(s)\,|x_1(s-\sigma_2)-x_2(s-\sigma_2)\,|ds\\ &\leq -p||x_1-x_2||+||x_1-x_2||\left(\int_t^{\infty} R(s)\left[Q_1(s)+Q_2(s)\right]ds\right)\\ &= ||x_1-x_2||\{-p+\int_{t_1}^{\infty} R(s)\left[Q_1(s)+Q_2(s)\right]ds\}\\ &= q_3||x_1-x_2||, \qquad\qquad q_3<1.\end{aligned}$$

This implies that

$$||Tx_1-Tx_2|| \leq q_3||x_1-x_2||.$$

Thus we know that T is a contraction mapping. Consequently T has the unique fixed point x, which is obviously a positive solution of Eq.(1). This completes the proof of Claim 3.

Claim 4. $p \in (-\infty,-1)$. Choose $t_1 > t_0$ large enough such that the inequalities

$$\int_{t_1}^{\infty} R(s)\,[Q_1\,(s)+Q_2\,(s)]\,ds < -(p+1), \tag{10}$$

$$\int_{t_1}^{\infty} R(s)Q_2(s)ds < \frac{-(p+1)(N_3-1)}{N_3} \tag{11}$$

and

$$\int_{t_1}^{\infty} R(s)Q_1(s) < \frac{-(1+p)(1-N_3)}{N_4} \tag{12}$$

hold, where the positive constants N_3 and N_4 satisfy

$$0 < N_3 < 1 < N_4.$$

Let X be the same set as in Claim 1. Set

$$A = \{x \in X : N_3 \le x(t) \le N_4, t \ge t_0\}. \tag{13}$$

Define mapping $T : A \to X$ as follows:

$$(Tx)(t) = \begin{cases} 1+\frac{1}{p}-\frac{1}{p}x(t+\tau) \\ \quad +\frac{R(t+\tau)}{p}\int_{t+\tau}^{\infty}[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds \\ \quad +\frac{1}{p}\int_{t_1}^{t+\tau} R(s)[Q_1(s)x(s-\sigma_1)-Q_2(s)x(s-\sigma_2)]ds, & t \ge t_1; \\ (Tx)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Clearly, Tx is continuous. For every $x \in A$ and $t \ge t_1$, using (10) and (12) we get

$$(Tx)(t) \le 1+\frac{1}{p}-\frac{N_4}{p}-\frac{N_4}{p}\int_t^{\infty} R(s)Q_2(s)ds \le N_4.$$

Furthermore, in view of (11) and (12) we have

$$(Tx)(t) \ge 1+\frac{1}{p}-\frac{N_3}{p}+\frac{N_4}{p}\int_t^{\infty} R(s)Q_1(s)ds \ge N_3.$$

Thus we know that $TA \subset A$. Since A is a bounded, closed, and convex subset of X, hence we can prove that T is a contraction mapping on A by the contraction principle.

In fact, for $x_1, x_2 \in A$, we have

$$\begin{aligned}
&|(Tx_1)(t) - (Tx_2)(t)| \\
&\leq -\frac{1}{p}|x_1(t+\tau) - x_2(t+\tau)| \\
&\quad -\frac{R(t+\tau)}{p}\left[\int_{t+\tau}^{\infty} Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)|ds\right. \\
&\quad \left.+\int_{t+\tau}^{\infty} Q_2(s)|x_1(s-\sigma_2) - x_2(s-\sigma_2)|ds\right] \\
&\quad -\frac{1}{p}\left[\int_{t_1}^{t+\tau} R(s)Q_1(s)|x_1(s-\sigma_1) - x_2(s-\sigma_1)|ds\right. \\
&\quad \left.+\int_{t_1}^{t+\tau} R(s)Q_2(s)|x_1(s-\sigma_2) - x_2(s-\sigma_2)|ds\right] \\
&\leq -\frac{1}{p}||x_1 - x_2|| - \frac{1}{p}||x_1 - x_2||\int_{t+\tau}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds \\
&\quad +\int_{t_1}^{t+\tau} R(s)[Q_1(s) + Q_2(s)]ds \\
&= -\frac{1}{p}||x_1 - x_2||\{1 + \int_{t_1}^{\infty} R(s)[Q_1(s) + Q_2(s)]ds\} \\
&= q_4||x_1 - x_2||, \qquad\qquad q_4 < 1.
\end{aligned}$$

This implies that

$$||Tx_1 - Tx_2|| \leq q_4||x_1 - x_2||.$$

Thus we know that T is a contraction mapping. Consequently T has the unique fixed point x, which is obviously a positive solution of Eq.(1). This completes the proof of Claim 4.

References

1. M. R. S. Kulenović and S.Hadžiomerspahić, Existence of nonoscillatory solution of second order linear neutral delay equation, *J. Math. Anal. Appl.*, **228**(1998), no.2, 436-448.

2. I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Univ. Press, New York, 1991.

3. D. D. Baǐnov and D. P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, Adam Hilger, Bristol, 1991.

4. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, 1992.

5. B. G. Zhang and J. S. Yu, Existence of positive solutions for neutral differential equations, *Sci. China Ser. A*, **35**(1992), no.11, 1306-1313.

6. J. F. Cheng, Existence of nonoscillatory solution of second order linear neutral difference equation, *Appl. Math. Lett.*, **20**(2007), no.8, 892-899.

7. J. W. Hooker and W. T. Patula, A second-order nonlinear difference equation: oscillation and asymptotic behavior, *J. Math. Anal. Appl.*, **91**(1983), no.1, 9-29.

Department of mathematics, Hunan City University, Yiyang 413000, P.R. China.

E-mail: gwm@mail.hncul.net

Department of mathematics, Xiamen University, Xiamen 361005, P.R. China.

E-mail: jfcheng@xmu.edu.cn

Department of mathematics, Huzhou Teachers College, Huzhou 313000, P.R. China.

E-mail: chuyuming@hutc.zj.cn