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STABLE SPLITTINGS OF THE COMPLEX CONNECTIVE *K*-THEORY OF *BG* FOR SOME INFINITE GROUPS *G*

BY

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Abstract

We show that $bu \wedge BO(n)$ splits as a wedge product of suspended copies of $HZ/2$, bu, and $bu \wedge BO(1)$ at prime 2 for $n = 2$, 3, and 4. Similarly, we show that $bu \wedge BSO(2n+1)$ splits as a wedge product of suspended copies of $HZ/2$ and bu at prime 2 for $n = 1$ and 2.

1. Introduction

Let bu be the complex connective K-theory, $RP^\infty = BO(1)$ be the infinite real projective space, $HZ/2$ be the $Z/2$ Eilenberg-Mac Lane spectrum, $BO(n)$ be the classifying space of the n-th orthogonal group, $BSO(n)$ be the classifying space of the n-th special orthogonal group and $\widetilde{H}^*(X)$ be the reduced mod 2 cohomology of X. For simplicity of notation, we write $\otimes$ instead of $\otimes_{Z/2}$.

Eric Ossa [1] has showed that

$$bu \wedge RP^\infty \wedge RP^\infty \simeq [\bigvee_{0<i,j} \sum^{2i+2j-2} HZ/2] \vee [\sum^2 bu \wedge RP^\infty].$$

D. C. Johnson and W. S. Wilson [2] gave a brief proof of this theorem and

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split $bu\wedge RP^\infty\wedge\cdots\wedge RP^\infty$ into suspended copies of $HZ/2$ and one suspended copy of $bu\wedge RP^\infty$ inductively. Also, R. R. Bruner [3] provided the analogous results in the real case ko. So far, we only know the stable splittings of $bu\wedge BG$ for particular finite groups G at prime 2 (see [8]). For infinite groups G, we never know the stable splittings of $bu\wedge BG$ at prime 2. The purpose of this paper is to give the stable splittings of $bu\wedge BO(n)$ ($n=2$, 3, and 4) and $bu\wedge BSO(2n+1)$ ($n=1$ and 2). We consider $bu\wedge BO(n)$ for $n\geq 2$ first.

Thoerem 1. *There is a stable homotopy equivalence*

$$bu\wedge BO(2)\simeq [\bigvee_{0\leq i,j}\sum^{2i+4j+2}HZ/2]\vee[\bigvee_{0<j}\sum^{4j}bu]\vee[bu\wedge RP^\infty]$$

at prime 2.

Thoerem 2. *There is a stable homotopy equivalence*

$$\begin{aligned}bu\wedge BO(3)\ \simeq\ &[\bigvee_{0\leq i,j,k}\sum^{2i+4j+2+6k}HZ/2]\vee[\bigvee_{0\leq i,j,k}\sum^{2i+4j+6k+3}HZ/2]\\ &\vee[\bigvee_{0<j}\sum^{4j}bu]\vee[bu\wedge RP^\infty]\vee[\bigvee_{0<j}\sum^{4j}bu\wedge RP^\infty]\end{aligned}$$

at prime 2.

Thoerem 3. *There is a stable homotopy equivalence*

$$\begin{aligned}bu\wedge BO(4)\ \simeq\ &[\bigvee_{\alpha}\sum^{\alpha}HZ/2]\vee[\bigvee_{0<j+l}\sum^{4j+8l}bu]\\ &\vee[bu\wedge RP^\infty]\vee[\bigvee_{0<j}\sum^{4j}bu\wedge RP^\infty]\end{aligned}$$

at prime 2*, where* $\alpha=2i+4j+6k+8l+4$, $2i+4j+6k+3+8l$, $2i+4j+2+6k+8l$, $2i+4j+2+6k+3+8l+4$, *and* $2i+4j+2+8l+4$ *for all* i, j, k, $l\geq 0$.

Also we can consider the classifying space of the n-th special orthogonal group $BSO(n)$. The splitting of $bu\wedge BSO(2)=bu\wedge CP^\infty$ was given by D. C. Johnson and W. S. Wilson [2]. Unfortunately, $bu\wedge BSO(4)$ can not split into the similar parts as above. It seems that only $bu\wedge BSO(2n+1)$ can split as a wedge of suspended copies of $HZ/2$ and bu. Here we provide the splittings of $bu\wedge BSO(3)$ and $bu\wedge BSO(5)$.

Theorem 4. *There is a stable homotopy equivalence*

$$bu \wedge BSO(3) \simeq [\bigvee_{0\leq j,k} \sum^{4j+2+6k} HZ/2] \vee [\bigvee_{0<j} \sum^{4j} bu]$$

at prime 2.

Theorem 5. *There is a stable homotopy equivalence*

$$bu \wedge BSO(5) \simeq [\bigvee_{\alpha} \sum^{\alpha} HZ/2] \vee [\bigvee_{0<j+l} \sum^{4j+8l} bu]$$

at prime 2, *where* $\alpha = 4j+6k+8l+4+10m$, $4j+2+6k+8l+10m$, $4j+2+6k+8l+4+10m+5$, $4j+2+6k+3+8l+4+10m$, *and* $4j+2+8l+4+10m$ *for all* j, k, l, $m \geq 0$.

Our main idea comes from [2]. The first step is to show that the E-module $\widetilde{H}^*(BO(n))$ is isomorphic to the direct sum of an E-module D^* and a free E-module M where $E = E[Q_0, Q_1]$ ($Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$) is an exterior algebra which is a subalgebra of the mod 2 Steenrod algebra A. The second step is to construct the space X and to determine α such that $\widetilde{H}^*(X) \cong D^*$ and $\widetilde{H}^*(\bigvee_{\alpha} \sum^{\alpha} HZ/2) \cong A \otimes_E M$. Finally, we construct a map from $bu \wedge BO(n)$ to $[\bigvee_{\alpha} \sum^{\alpha} HZ/2] \vee [bu \wedge X]$ and prove that this map is a homotopy equivalence at prime 2. The difficulty is to construct the space X and the homotopy equivalence map. We show that X is a wedge of suspended copies of bu and $bu \wedge RP^{\infty}$ and construct the homotopy equivalence map for $n = 2$, 3, and 4. We shall describe the construction of the difficult part of this map.

It is well-known that $bu_* = Z[v_1]$ where $deg v_1 = 2$ and $\widetilde{H}^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E Z/2$ where $A(Q_0, Q_1)$ is the ideal generated by Q_0 and Q_1. D. C. Johnson and W. S. Wilson [4] showed that $bu_{(2)}$, the spectrum for complex connective K-theory localized at prime 2, is homotopic to the Johnson-Wilson Spectrum $BP\langle 1\rangle$. Also, W. S. Wilson [5] showed that a tower of BP module spectra was constructed using Sullivan's theory of manifolds with singularities:

$$\begin{aligned} BP \ \rightarrow\ \cdots \rightarrow BP\langle n+1\rangle \rightarrow BP\langle n\rangle \rightarrow \\ \cdots \ \rightarrow\ BP\langle 1\rangle \rightarrow BP\langle 0\rangle \rightarrow BP\langle -1\rangle, \end{aligned}$$

where $BP\langle 0\rangle$ is the $Z_{(2)}$ Eilenberg-Mac Lane spectrum and $BP\langle -1\rangle$ is the $Z/2$ Eilenberg-Mac Lane spectrum. We construct the 2-local stable map from $BO(n)$ to $\sum^{4j} bu$ for each $j>0$ by the Adams spectral sequence.

Lemma 1.1. *There is a 2-local stable map* $W_2^{2j} : BO(n) \to \sum^{4j} bu$ *which is detected by* $w_2^{2j} \in \widetilde{H}^* BO(n)$ *for each* $j > 0$.

Proof. Let A be the mod 2 Steenrod algebra and $Z_{(2)}$ be the integers localized at prime 2. The Adams spectral sequence (for the appropriate spaces or spectra X and Y)

$$E_2^{*,*} \cong Ext_A^{*,*}(\widetilde{H}^*(X),\ \widetilde{H}^*(Y)) \Longrightarrow \{Y,\ X\}_* \otimes Z_{(2)}$$

can be used to compute $\widetilde{BP}^* Y = \{Y, BP\}_{-*}$ and $\widetilde{bu}^*_{(2)} Y = \{Y, bu_{(2)}\}_{-*}$. By a well-known change-of-rings isomorphism [7] we can replace

$$Ext_A^{*,*}(\widetilde{H}^*(BP \wedge X),\ \widetilde{H}^*(Y)) \text{ with } Ext^{*,*}_{E[Q_0,Q_1,\cdots]}(\widetilde{H}^*(X),\ \widetilde{H}^*(Y))$$

and replace

$$Ext_A^{*,*}(\widetilde{H}^*(bu_{(2)} \wedge X),\ \widetilde{H}^*(Y)) \text{ with } Ext_E^{*,*}(\widetilde{H}^*(X),\ \widetilde{H}^*(Y)),$$

where $E[Q_0,Q_1,\cdots]$ is the exterior algebra on the Milnor primitives [6] and $E = E[Q_0, Q_1]$ (see [7] and [8]). The forms of Adams spectral sequence we use are

$$Ext^{*,*}_{E[Q_0,Q_1,\cdots]}(Z/2,\ \widetilde{H}^*(Y)) \Longrightarrow \widetilde{BP}^{-*} Y$$

and

$$Ext_E^{*,*}(Z/2,\ \widetilde{H}^*(Y)) \Longrightarrow \widetilde{bu}^{-*}_{(2)} Y.$$

After changing grading, we compare $\widetilde{BP}^* BO(n)$ with $\widetilde{bu}^*_{(2)} BO(n)$ under the map $BP \to bu_{(2)}$ described above. For each $j > 0$, W. S. Wilson [9] has showed that $w_2^{2j} \in \widetilde{H}^*(BO(n))$ is a permanent cycle for $\widetilde{BP}^* BO(n)$. Hence $w_2^{2j} \in \widetilde{H}^*(BO(n))$ is a permanent cycle for $\widetilde{bu}^*_{(2)} BO(n)$ for each $j > 0$. Thus we have a 2-local stable map $W_2^{2j} : BO(n) \to \sum^{4j} bu$ for each $j > 0$. This completes the proof. □

Remark. If $n \geq 4$, Lemma 1.1 is also true for $w_2^{2j} w_4^{2l} \in \widetilde{H}^* BO(n)$ by the same argument as above. It means there is a 2-local stable map

$W_2^{2j}W_4^{2l} : BO(n) \to \sum^{4j+8l} bu$ which is detected by $w_2^{2j}w_4^{2l} \in \widetilde{H}^*BO(n)$ for each $j+l>0$.

Let A be the mod 2 Steenrod algebra and $E = E[Q_0, Q_1]$ ($Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$) be an exterior algebra which is a subalgebra of A. Since E is a subalgebra of A, $\widetilde{H}^*X$ is an E-module for any space or spectrum X. If we know $Sq^k(w_m)$ for each k and m, then we can describe the E-module structure of $\widetilde{H}^*(BO(n))$ by Cartan formula $Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$ and the fact that Q_0 and Q_1 act as derivations (that is, $Q_k(xy) = Q_k(x)y + xQ_k(y)$). We provide Wu formula here.

Proposition 1.2.(Wu formula) $Sq^k(w_m) = \sum_{t=0}^{k} \binom{m-k+t-1}{t} w_{k-t}w_{m+t}$ *where the binomial coefficient* $\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{1\cdot 2\cdots b}$ *is taken mod* 2.

Proof. See [10]. □

To show that $\widetilde{H}^*(\bigvee_\alpha \sum^\alpha HZ/2) \cong A \otimes_E M$ for an appropriate α and a free E-module M, we need more information. We state the notation first. Suppose M and N are left A-modules with the actions μ_M and μ_N, then $M \otimes N$ is also a left A-module with the action defined by the composite

$$A \otimes M \otimes N \overset{\psi\otimes M\otimes N}{\to} A \otimes A \otimes M \otimes N \overset{A\otimes T\otimes N}{\to} A \otimes M \otimes A \otimes N \overset{\mu_M\otimes\mu_N}{\to} M \otimes N,$$

where ψ is the diagonal map of A and $T(a \otimes b) = (-1)^{\dim a \dim b}(b \otimes a)$ is the twist map. We write ${}_D(M \otimes N)$ to indicate $M \otimes N$ with this left action. Similarly, ${}_L(M \otimes N)$ indicates the extended A action over M.

Proposition 1.3.(Proposition 1.7 of [11]) *If B is a Hopf subalgebra of A, M a left A-module, N a left B-module, then*

$${}_D[M \otimes (A \otimes_B N)] \cong_L [A \otimes_B {}_D(M \otimes N)]$$

as left A-modules.

Since B is a subalgebra of A, we know that M is a left B-module. Hence ${}_D(M \otimes N)$ is a left B-module with the action:

$$B\otimes M\otimes N \overset{\psi|_B\otimes M\otimes N}{\to} B\otimes B\otimes M\otimes N \overset{B\otimes T\otimes N}{\to} B\otimes M\otimes B\otimes N \overset{\mu_M|_B\otimes\mu_N}{\to} M\otimes N,$$

where $\psi|_B$ is the diagonal map of A restricted on B and $\mu_M|_B$ is the action of M restricted on B. Also we know that A is both a right B-module and a left A-module, hence $A \otimes_B N$ is a left A-module with the extended action over A. For the detail proof we refer the reader to [11].

Note: Let N be $Z/2$ and B be E in Proposition 1.3. Since

$$_D[M \otimes (A \otimes_E Z/2)] \cong_D [(A \otimes_E Z/2) \otimes M] \text{ and } _D(M \otimes Z/2) \cong M,$$

this isomorphism (see [12] and Proposition 1.1 of [11])

$$\theta :_L [A \otimes_E M] \overset{\cong}{\to}_D [(A \otimes_E Z/2) \otimes M]$$

is given by $\theta(a \otimes x) = \sum a' \otimes 1 \otimes a''x$, with inverse $\theta^{-1}(a \otimes 1 \otimes x) = \sum a' \otimes \chi(a'')x$, where $\psi(a) = \sum a' \otimes a''$ and χ is the conjugation map.

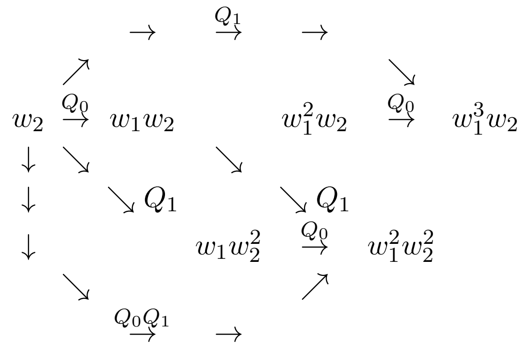
Now we are ready to prove our theorems, that is, the stable splittings of $bu \wedge BO(n)$ ($n = 2$, 3, and 4) and $bu \wedge BSO(2n+1)$ ($n = 1$ and 2). Although the proof of the general case still escapes us, it seems that the general case can be solved by the same argument which we provide later. We believe that $bu \wedge BO(n)$ splits as a wedge product of suspended copies of $HZ/2$, bu, and $bu \wedge RP^\infty$ at prime 2 for each $n \geq 2$. Also, we believe that $bu \wedge BSO(2n+1)$ splits as a wedge product of suspended copies of $HZ/2$ and bu at prime 2 for each $n \geq 1$.

2. The proof of theorem 1

Lemma 2.1. *By Cartan formula, Wu formula, and the fact that Q_0 and Q_1 act as derivations, the E-module structure of $\widetilde{H}^*(BO(2))$ is as follows:*

$Q_0(w_1^{2i}w_2^{2j}) = Q_1(w_1^{2i}w_2^{2j}) = Q_0Q_1(w_1^{2i}w_2^{2j}) = 0.$
$Q_0(w_1^{2i}w_2^{2j+1}) = w_1^{2i+1}w_2^{2j+1},\ Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+3}w_2^{2j+1} + w_1^{2i+1}w_2^{2j+2},$
$Q_0Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+2}w_2^{2j+2}.$
$Q_0(w_1^{2i+1}w_2^{2j}) = w_1^{2i+2}w_2^{2j},\ Q_1(w_1^{2i+1}w_2^{2j}) = w_1^{2i+4}w_2^{2j},$
$Q_0Q_1(w_1^{2i+1}w_2^{2j}) = 0.$
$Q_0(w_1^{2i+1}w_2^{2j+1}) = 0,\ Q_1(w_1^{2i+1}w_2^{2j+1}) = w_1^{2i+2}w_2^{2j+2},$
$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}) = 0.$

We omit the action 1 in E since $1(x) = x$ for any $x \in \widetilde{H}^*(BO(2))$. We illustrate the E action on $w_2 \in \widetilde{H}^*(BO(2))$ as follows:



One should notice that $w_1^2w_2$ and $w_1w_2^2$ also have the nontrivial E-action, although we do not point out.

Lemma 2.2. *As an E-module, $\widetilde{H}^*(BO(2))$ is isomorphic to $D^* \oplus M$, where D^* is an E-module with the $Z/2$-basis $\{w_1^i,\ w_2^{2j} \mid i,\ j > 0\}$ and M is isomorphic to a free E-module $\widetilde{H}^*(BO(2))/D^*$ with E-basis $\{w_1^{2i}w_2^{2j+1} \mid i,\ j \geq 0\}$.*

Proof. By Lemma 2.1, we know

$(*a)$ $Q_0(w_2) = w_1w_2,\ Q_1(w_2) = w_1^3w_2 + w_1w_2^2,\ Q_0Q_1(w_2) = w_1^2w_2^2.$

$(*b)$ $Q_0(w_1) = w_1^2,\ Q_1(w_1) = w_1^4,\ Q_0Q_1(w_1) = 0.$

Using that Q_0 and Q_1 act as derivations, that is $Q_k(xy) = Q_k(x)y + xQ_k(y)$, it is easy to see that D^* is an E-module by $(*b)$ since the E-action is closed on D^*. Hence it remains to prove $\{w_1^{2i}w_2^{2j+1} \mid i,\ j \geq 0\}$ is a basis of the free E-module $\widetilde{H}^*(BO(2))/D^*$. Since $1(w_1^{2i}w_2^{2j+1}) = w_1^{2i}w_2^{2j+1}$ and $Q_0(w_1^{2i}w_2^{2j+1}) = w_1^{2i+1}w_2^{2j+1}$ by $(*a)$, we know $w_1^{2i}w_2^{2j+1}$ and $w_1^{2i+1}w_2^{2j+1}$ can be generated uniquely. By considering $(*a)$ $Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+3}w_2^{2j+1} + w_1^{2i+1}w_2^{2j+2}$, we know $w_1^{2i+1}w_2^{2j}$ $(j \geq 1)$ can be generated uniquely since we have shown that $w_1^{2i+3}w_2^{2j+1}$ can be generated uniquely. Finally, we know

$(*a)$ $Q_0Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+2}w_2^{2j+2},$

hence $w_1^{2i}w_2^{2j}$ $(i,\ j \geq 1)$ can be generated uniquely.

This completes the proof. □

Proof of Theorem 1. The proof of Theorem 1 is similar to that proof in [2]. Define g_1 to be the composition

$$g_1 : bu \wedge BO(2) \overset{bu\wedge \underset{0<j}{\vee} W_2^{2j}}{\to} bu \wedge [\underset{0<j}{\vee}(bu \wedge S^{4j})] \overset{\underset{j}{\vee}\mu}{\to} \underset{0<j}{\vee}(bu \wedge S^{4j}),$$

where W_2^{2j} is the 2-local stable map in Lemma 1.1 and μ is the multiplication of the bu spectrum. Define g_2 to be the map

$$g_2 : bu\wedge BO(2) \overset{bu\wedge \det}{\to} bu\wedge RP^\infty,$$

where det denotes the map which classifies the determinant bundle. For $b = w_1^{2i}w_2^{2j+1} \in \widetilde{H}^{2i+4j+2}(BO(2))$, let $g_b : BO(2) \to \sum^{2i+4j+2} HZ/2$ represent b. Let $i : bu \to HZ/2$ be the multiplicative map and μ' be the ring structure map of $HZ/2$. Now we construct the map g_0 by the following composition:

$$g_0 : bu \wedge BO(2) \overset{bu\wedge \underset{b}{\vee} g_b}{\to} bu \wedge [\underset{0\le i,j}{\vee} \sum^{2i+4j+2} HZ/2] \overset{\underset{b}{\vee}\nu}{\to} \underset{0\le i,j}{\vee} \sum^{2i+4j+2} HZ/2,$$

where $\nu : bu \wedge HZ/2 \overset{i\wedge HZ/2}{\to} HZ/2 \wedge HZ/2 \overset{\mu'}{\to} HZ/2$. Hence we have the map

$$\begin{aligned} g \;=\; & g_0 \vee g_1 \vee g_2 : bu \wedge BO(2) \to [\underset{0\le i,j}{\vee} \sum^{2i+4j+2} HZ/2] \\ & \vee[\underset{0<j}{\vee} \sum^{4j} bu] \vee [bu \wedge RP^\infty]. \end{aligned}$$

Now we show that g induces an isomorphism in mod 2 cohomology. Recall that $\widetilde{H}^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E Z/2$ and the Künneth theorem gives

$$\widetilde{H}^*(bu \wedge X) \cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*(X) \cong (A \otimes_E Z/2) \otimes \widetilde{H}^*(X) \overset{\theta^{-1}}{\cong} A \otimes_E \widetilde{H}^*(X)$$

for any space or spectrum X where θ^{-1} is an isomorphism described in the note after Proposition 1.3. In Lemma 2.2, we show that $\widetilde{H}^*(BO(2))$ is isomorphic to $D^* \oplus M$ as an E-module, hence

$$\begin{aligned}
&\widetilde{H}^*(bu \wedge BO(2)) \\
&\cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*(BO(2)) \cong (A \otimes_E Z/2) \otimes \widetilde{H}^*(BO(2)) \\
&\overset{\theta^{-1}}{\cong} A \otimes_E \widetilde{H}^*(BO(2)) \cong A \otimes_E (D^* \oplus M) \cong A \otimes_E D^* \oplus A \otimes_E M.
\end{aligned}$$

The class $\{w_1^i,\, w_2^{2j} \mid i,\, j > 0\}$ give a $Z/2$-basis for the E-module D^* which is isomorphic to $\widetilde{H}^*((\underset{0<j}{\vee} S^{4j}) \vee RP^\infty)$. Consider the composite maps

$$\begin{aligned}
&\alpha_1 : A \otimes_E [\widetilde{H}^*(\underset{0<j}{\vee} S^{4j}) \oplus \widetilde{H}^*(RP^\infty)] \\
&\overset{\theta}{\cong} (A \otimes_E Z/2) \otimes \widetilde{H}^*((\underset{0<j}{\vee} S^{4j}) \vee RP^\infty) \cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*((\underset{0<j}{\vee} S^{4j}) \vee RP^\infty) \\
&\cong \widetilde{H}^*([\underset{0<j}{\vee} bu \wedge S^{4j}] \vee [bu \wedge RP^\infty]) \overset{(g_1 \vee g_2)^*}{\rightarrow} \widetilde{H}^*(bu \wedge BO(2)) \\
&\cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*(BO(2)) \cong (A \otimes_E Z/2) \otimes \widetilde{H}^*(BO(2)) \\
&\overset{\theta^{-1}}{\cong} A \otimes_E \widetilde{H}^*(BO(2)) \cong A \otimes_E (D^* \oplus M) \\
&\cong A \otimes_E D^* \oplus A \otimes_E M \overset{p_1}{\rightarrow} A \otimes_E D^*,
\end{aligned}$$

where p_1 is the projection map. For $1 \in A$ and $\sum^{4j} 1 \in \widetilde{H}^{4j}(\underset{0<j}{\vee} S^{4j})$, since

$$\psi(1) = 1 \otimes 1 \text{ and } \chi(1) = 1,$$

we follow the above α_1 diagram then we have the following diagram

$$\begin{aligned}
&\alpha_1 : 1 \otimes (\sum\nolimits^{4j} 1 \oplus 0) \overset{\theta}{\mapsto} 1 \otimes 1 \otimes (\sum\nolimits^{4j} 1 \oplus 0) \\
&\mapsto 1 \otimes (\sum\nolimits^{4j} 1 \oplus 0) \mapsto 1 \otimes (\sum\nolimits^{4j} 1 \oplus 0) \\
&\overset{(g_1 \vee g_2)^*}{\mapsto} 1 \otimes w_2^{2j} \mapsto 1 \otimes w_2^{2j} \mapsto 1 \otimes 1 \otimes w_2^{2j} \\
&\overset{\theta^{-1}}{\mapsto} 1 \otimes w_2^{2j} \mapsto 1 \otimes (w_2^{2j} \oplus 0) \mapsto 1 \otimes w_2^{2j} \oplus 0 \overset{p_1}{\mapsto} 1 \otimes w_2^{2j}.
\end{aligned}$$

The A-action on $A \otimes_E [\widetilde{H}^*(\underset{0<j}{\vee} S^{4j}) \oplus \widetilde{H}^*(RP^\infty)]$ is just on A and so is $A \otimes_E D^*$, thus

$$\alpha_1(a \otimes (\sum\nolimits^{4j} 1 \oplus 0)) = a \otimes w_2^{2j},$$

for each $a \in A$ and $\sum^{4j} 1 \in \widetilde{H}^{4j}(\bigvee_{0<j} S^{4j})$. Similarly,

$$\alpha_1(a \otimes (0 \oplus w_1^i)) = a \otimes w_1^i,$$

for each $a \in A$ and $w_1^i \in \widetilde{H}^i(RP^\infty)$. Hence α_1 is an isomorphism and this implies $(g_1 \vee g_2)^*$ takes $\widetilde{H}^*([\bigvee_{0<j} bu \wedge S^{4j}] \vee [bu \wedge RP^\infty])$ isomorphically onto $A \otimes_E D^* \cong \widetilde{H}^*(bu) \otimes D^*$. Now we consider the map

$$\begin{aligned}\alpha_2 : \widetilde{H}^*(\bigvee_{0\le i,j} \sum^{2i+4j+2} HZ/2) &\overset{g_0^*}{\to} \widetilde{H}^*(bu \wedge BO(2)) \cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*(BO(2))\\ \cong\ & (A \otimes_E Z/2) \otimes \widetilde{H}^*(BO(2)) \overset{\theta^{-1}}{\cong} A \otimes_E \widetilde{H}^*(BO(2)) \cong A \otimes_E (D^* \oplus M)\\ \cong\ & A \otimes_E D^* \oplus A \otimes_E M \overset{p_2}{\to} A \otimes_E M,\end{aligned}$$

where p_2 is the projection map. By the construction of the map g_0, we see that g_0^* sends the generator $\sum^{2i+4j+2} 1 \in \widetilde{H}^*(\bigvee_{0\le i,j} \sum^{2i+4j+2} HZ/2)$ to $1 \otimes w_1^{2i} w_2^{2j+1} \in \widetilde{H}^*(bu \wedge BO(2))$ for each $i \geq 0$ and $j \geq 0$. Let N be $Z/2$, A be A, B be E and M be M in Proposition 1.3. We have

$$_D[M \otimes (A \otimes_E Z/2)] \cong_L [A \otimes_{E\ D} (M \otimes Z/2)] \cong_L [A \otimes_E M].$$

The A-action on the left is by the diagonal and this is isomorphic to $(A \otimes_E Z/2) \otimes M$. The A-action on the right-hand side is just on A. Since $\chi(1) = 1$, we follow the above α_2 diagram then we have the following diagram

$$\begin{aligned}\alpha_2 : \sum^{2i+4j+2} 1 &\overset{g_0^*}{\mapsto} 1 \otimes w_1^{2i} w_2^{2j+1} \mapsto 1 \otimes w_1^{2i} w_2^{2j+1}\\ \mapsto\ & 1 \otimes 1 \otimes w_1^{2i} w_2^{2j+1} \overset{\theta^{-1}}{\mapsto} 1 \otimes w_1^{2i} w_2^{2j+1} \mapsto 1 \otimes (0 \oplus w_1^{2i} w_2^{2j+1})\\ \mapsto\ & 0 \oplus 1 \otimes w_1^{2i} w_2^{2j+1} \overset{p_2}{\mapsto} 1 \otimes w_1^{2i} w_2^{2j+1}.\end{aligned}$$

Hence

$$\alpha_2(\sum^{2i+4j+2} 1) = 1 \otimes w_1^{2i} w_2^{2j+1}$$

for each generator $\sum^{2i+4j+2} 1$ of the free A-module $\widetilde{H}^*(\bigvee_{0\le i,j} \sum^{2i+4j+2} HZ/2)$. Since M is a free E-module with basis $\{w_1^{2i} w_2^{2j+1} \mid i,\ j \geq 0\}$, this means $A \otimes_E M$ is a free A-module with basis $\{1 \otimes w_1^{2i} w_2^{2j+1} \mid i,\ j \geq 0\}$. Hence $\widetilde{H}^*(\bigvee_{0\le i,j} \sum^{2i+4j+2} HZ/2)$ and $A \otimes_E M$ are both free A-modules and have the same rank. It follows that α_2 is an isomorphism and this implies g_0^* take $\widetilde{H}^*(\bigvee_{0\le i,j}\sum^{2i+4j+2}HZ/2)$ isomorphically onto $A\otimes_E M$. Now we have shown that

$$\begin{aligned}
&\widetilde{H}^*([\bigvee_{0\le i,j}\sum^{2i+4j+2}HZ/2]\vee[\bigvee_{0<j}\sum^{4j}bu]\vee[bu\wedge\widetilde{H}^*P^\infty])\\
&\quad\overset{g^*=(g_0\vee g_1\vee g_2)^*}{\to}\ \widetilde{H}^*(bu\wedge BO(2))\cong\widetilde{H}^*(bu)\otimes\widetilde{H}^*(BO(2))\\
&\quad\cong(A\otimes_E Z/2)\otimes\widetilde{H}^*(BO(2))\overset{\theta^{-1}}{\cong}A\otimes_E\widetilde{H}^*(BO(2))\cong A\otimes_E(D^*\oplus M)\\
&\quad\cong A\otimes_E D^*\oplus A\otimes_E M
\end{aligned}$$

is an isomorphism, hence g induces an isomorphism in mod 2 cohomology and this is an equivalence at prime 2. □

3. The Proof of Theorem 2

Lemma 3.1. *By Cartan formula, Wu formula, and the fact that Q_0 and Q_1 act as derivations, the E-module structure of $\widetilde{H}^*(BO(3))$ is as follows:*

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k})=Q_1(w_1^{2i}w_2^{2j}w_3^{2k})=Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k})=0.$$

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k+1})=w_1^{2i+1}w_2^{2j}w_3^{2k+1},$$

$$Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1})=w_1^{2i+3}w_2^{2j}w_3^{2k+1}+w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}+w_1^{2i}w_2^{2j}w_3^{2k+2},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1})=w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}+w_1^{2i+1}w_2^{2j}w_3^{2k+2}.$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k})=w_1^{2i+1}w_2^{2j+1}w_3^{2k}+w_1^{2i}w_2^{2j}w_3^{2k+1},$$

$$\begin{aligned}Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k})=&\,w_1^{2i+3}w_2^{2j+1}w_3^{2k}+w_1^{2i+2}w_2^{2j}w_3^{2k+1}+w_1^{2i+1}w_2^{2j+2}w_3^{2k}\\&+w_1^{2i}w_2^{2j+1}w_3^{2k+1},\end{aligned}$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k})=w_1^{2i+2}w_2^{2j+2}w_3^{2k}+w_1^{2i}w_2^{2j}w_3^{2k+2}.$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k+1})=w_1^{2i}w_2^{2j}w_3^{2k+2},$$

$$Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1})=w_1^{2i+2}w_2^{2j}w_3^{2k+2},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1})=0.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k})=w_1^{2i+2}w_2^{2j}w_3^{2k},\ Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k})=w_1^{2i+4}w_2^{2j}w_3^{2k},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k})=0.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k+1})=0,$$

$Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+2},$

$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}) = 0.$

$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k}) = w_1^{2i+1}w_2^{2j}w_3^{2k+1},$

$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}) = w_1^{2i+3}w_2^{2j}w_3^{2k+1}+w_1^{2i+2}w_2^{2j+2}w_3^{2k}+w_1^{2i+1}w_2^{2j+1}w_3^{2k+1},$

$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}) = w_1^{2i+1}w_2^{2j}w_3^{2k+2} + w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}.$

$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}) = w_1^{2i+1}w_2^{2j}w_3^{2k+2} + w_1^{2i+2}w_2^{2j+1}w_3^{2k+1},$

$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}) = w_1^{2i+3}w_2^{2j}w_3^{2k+2} + w_1^{2i+4}w_2^{2j+1}w_3^{2k+1},$

$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}) = 0.$

Lemma 3.2. *As an E-module, $\widetilde{H}^*(BO(3))$ is isomorphic to $D^* \oplus M$, where D^* is an E-module with the $Z/2$-basis $\{w_1^i w_2^{2j} \mid i+j > 0\}$ and M is isomorphic to a free E-module $\widetilde{H}^*(BO(3))/D^*$ with E-basis $\{w_1^{2i}w_2^{2j+1}w_3^{2k}$, $w_1^{2i}w_2^{2j}w_3^{2k+1} \mid i,\ j,\ k \geq 0\}$.*

Proof. By Lemma 3.1, we know

$(*a)$ $Q_0(w_3) = w_1w_3,\ Q_1(w_3) = w_1^3w_3 + w_1w_2w_3 + w_3^2,$

$Q_0Q_1(w_3) = w_1^2w_2w_3 + w_1w_3^2.$

$(*b)$ $Q_0(w_2) = w_1w_2 + w_3,\ Q_1(w_2) = w_1^3w_2 + w_1^2w_3 + w_1w_2^2 + w_2w_3,$

$Q_0Q_1(w_2) = w_1^2w_2^2 + w_3^2.$

$(*c)$ $Q_0(w_1) = w_1^2,\ Q_1(w_1) = w_1^4,\ Q_0Q_1(w_1) = 0.$

Using that Q_0 and Q_1 act as derivations, it is easy to see that D^* is an E-module by $(*c)$. Hence it remains to prove $\{w_1^{2i}w_2^{2j+1}w_3^{2k}, w_1^{2i}w_2^{2j}w_3^{2k+1} \mid i,\ j,\ k \geq 0\}$ is a basis of the free E-module $\widetilde{H}^*(BO(3))/D^*$. Since $1(w_1^{2i}w_2^{2j+1}w_3^{2k}) = w_1^{2i}w_2^{2j+1}w_3^{2k}$ and $1(w_1^{2i}w_2^{2j}w_3^{2k+1}) = w_1^{2i}w_2^{2j}w_3^{2k+1}$ by $(*b)$ and $(*a)$ respectively, we know that $w_1^{2i}w_2^{2j+1}w_3^{2k}$ and $w_1^{2i}w_2^{2j}w_3^{2k+1}$ can be generated uniquely. By

$(*a)$ $Q_0(w_1^{2i}w_2^{2j}w_3^{2k+1}) = w_1^{2i+1}w_2^{2j}w_3^{2k+1}$

and

$(*b)$ $Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k}) = w_1^{2i}w_2^{2j}w_3^{2k+1} + w_1^{2i+1}w_2^{2j+1}w_3^{2k},$

we know $w_1^{2i+1}w_2^{2j}w_3^{2k+1}$ and $w_1^{2i+1}w_2^{2j+1}w_3^{2k}$ can be generated uniquely since $w_1^{2i}w_2^{2j}w_3^{2k+1}$ can be generated uniquely. Since $(*b)$ $Q_0Q_1(w_1^{2i}w_2^{2j+1})=w_1^{2i+2}$ $w_2^{2j+2}+w_1^{2i}w_2^{2j}w_3^2$ and $w_1^{2i+2}w_2^{2j+2}\in D^*$, it means $w_1^{2i}w_2^{2j}w_3^2$ can be generated uniquely. Therefore $w_1^{2i}w_2^{2j}w_3^4$ can be generated uniguely by considering

$$(*b)\ Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^2) = w_1^{2i+2}w_2^{2j+2}w_3^2 + w_1^{2i}w_2^{2j}w_3^4.$$

Repeat this argument, hence $w_1^{2i}w_2^{2j}w_3^{2k}$ $(k\geq 1)$ can be generated uniquely. Also we see that

$$(*a)\ Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}) = w_1^{2i+3}w_2^{2j}w_3^{2k+1}+w_1^{2i}w_2^{2j}w_3^{2k+2}+w_1^{2i+1}w_2^{2j+1}w_3^{2k+1},$$

hence $w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}$ can be generated uniquely since we have shown $w_1^{2i+3}w_2^{2j}w_3^{2k+1}$ and $w_1^{2i}w_2^{2j}w_3^{2k+2}$ can be generated uniquely. Now

$$(*a)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+2}$$

and

$$(*b)\ Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}) = w_1^{2i+1}w_2^{2j+2}w_3^{2k} + w_1^{2i}w_2^{2j+1}w_3^{2k+1} + \alpha$$

where $\alpha = w_1^{2i+3}w_2^{2j+1}w_3^{2k} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}$. We only concentrate on w_1^{2i+1} $w_2^{2j+2}w_3^{2k} + w_1^{2i}w_2^{2j+1}w_3^{2k+1}$ since we have shown that $w_1^{2i+3}w_2^{2j+1}w_3^{2k}$ and $w_1^{2i+2}w_2^{2j}w_3^{2k+1}$ can be generated uniquely. For $k=0$, $(*b)$ $Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+1}w_2^{2j+2}+w_1^{2i}w_2^{2j+1}w_3+\alpha'$, where $w_1^{2i+1}w_2^{2j+2}\in D^*$ and α' can be generated uniquely, hence $w_1^{2i}w_2^{2j+1}w_3$ can be generated uniquely. Also we know

$$(*a)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3) = w_1^{2i+2}w_2^{2j+1}w_3 + w_1^{2i+1}w_2^{2j}w_3^2,$$

hence $w_1^{2i+1}w_2^{2j}w_3^2$ can be generated uniquely. Repeat this argument,

$$(*b)\ Q_1(w_1^{2i}w_2^{2j+1}w_3^2) = w_1^{2i+1}w_2^{2j+2}w_3^2 + w_1^{2i}w_2^{2j+1}w_3^3 + \alpha'',$$

where $w_1^{2i+1}w_2^{2j+2}w_3^2$ and α'' can be generated uniquely, hence $w_1^{2i}w_2^{2j+1}w_3^3$ can be generated uniquely. By induction, we know $w_1^{2i}w_2^{2j+1}w_3^{2k+1}$ and $w_1^{2i+1}w_2^{2j}w_3^{2k}$ $(k\geq 1)$ can be generated uniquely.

This completes the proof. □

Proof of Theorem 2. It suffices to define the homotopy equivalence map. Define

$$g_1 : bu \wedge BO(3) \overset{bu\wedge \bigvee_{0<j} W_2^{2j}}{\rightarrow} bu \wedge [\bigvee_{0<j}(bu \wedge S^{4j})] \overset{\bigvee_j \mu}{\rightarrow} \bigvee_{0<j}(bu \wedge S^{4j}),$$

$$g_2 : bu \wedge BO(3) \overset{bu\wedge \det}{\rightarrow} bu \wedge RP^{\infty},$$

$$g_3 : bu \wedge BO(3) \overset{bu\wedge\Delta}{\rightarrow} bu \wedge [BO(3) \times BO(3)] \overset{bu\wedge q}{\rightarrow} bu \wedge [BO(3) \wedge BO(3)]$$
$$\overset{bu\wedge \bigvee_{0<j} W_2^{2j}\wedge\det}{\rightarrow} bu \wedge [[\bigvee_{0<j}(bu \wedge S^{4j})] \wedge RP^{\infty}] \overset{\bigvee_j \mu\wedge RP^{\infty}}{\rightarrow} \bigvee_{0<j}(bu \wedge S^{4j} \wedge RP^{\infty}),$$

and

$$g_0 : bu \wedge BO(3)$$
$$\overset{bu\wedge\bigvee_b g_b}{\rightarrow} bu \wedge [[\bigvee_{0\le i,j,k} \sum^{2i+4j+2+6k} HZ/2] \vee [\bigvee_{0\le i,j,k} \sum^{2i+4j+6k+3} HZ/2]]$$
$$\overset{\bigvee_b \nu}{\rightarrow} [\bigvee_{0\le i,j,k} \sum^{2i+4j+2+6k} HZ/2] \vee [\bigvee_{0\le i,j,k} \sum^{2i+4j+6k+3} HZ/2],$$

where Δ is the diagonal map, q is the quotient map and the other maps are defined as the proof of Theorem 1. The map

$$g_3^* : \widetilde{H}^*(\bigvee_{0<j}(bu \wedge S^{4j} \wedge RP^{\infty})) \overset{(\bigvee_j \mu\wedge RP^{\infty})^*}{\rightarrow} \widetilde{H}^*(bu \wedge [[\bigvee_{0<j}(bu \wedge S^{4j})] \wedge RP^{\infty}])$$
$$\overset{(bu\wedge \bigvee_{0<j} W_2^{2j}\wedge\det)^*}{\rightarrow} \widetilde{H}^*(bu \wedge [BO(3) \wedge BO(3)])$$
$$\overset{(bu\wedge q)^*}{\rightarrow} \widetilde{H}^*(bu \wedge [BO(3) \times BO(3)]) \overset{(bu\wedge\Delta)^*}{\rightarrow} \widetilde{H}^*(bu \wedge BO(3))$$

shows that

$$g_3^* : 1 \otimes \sum^{4j} 1 \otimes w_1^i \overset{(\bigvee_j \mu\wedge RP^{\infty})^*}{\mapsto} 1 \otimes 1 \otimes \sum^{4j} 1 \otimes w_1^i$$
$$\overset{(bu\wedge \bigvee_{0<j} W_2^{2j}\wedge\det)^*}{\mapsto} 1 \otimes w_2^{2j} \otimes w_1^i \overset{(bu\wedge q)^*}{\mapsto} 1 \otimes w_2^{2j} \otimes w_1^i \overset{(bu\wedge\Delta)^*}{\mapsto} 1 \otimes w_1^i w_2^{2j}.$$

By the same argument as the proof of Theorem 1, this implies $(g_1 \vee g_2 \vee g_3)^*$ takes

$$\widetilde{H}^*([\bigvee_{0<j} bu \wedge S^{4j}] \vee [bu \wedge RP^{\infty}] \vee [\bigvee_{0<j}(bu \wedge S^{4j} \wedge RP^{\infty})])$$

isomorphically onto $A \otimes_E D^* \cong \widetilde{H}^*(bu) \otimes D^*$. Repeat the argument in the proof of Theorem 1, we know $g = g_0 \vee g_1 \vee g_2 \vee g_3$ is an equivalence at prime 2. □

4. The proof of Theorem 3

Lemma 4.1. By Cartan formula, Wu formula, and the fact that Q_0 and Q_1 act as derivations, the E-module structure of $\widetilde{H}^*(BO(4))$ is as follows:

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}) = Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}) = Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}) = 0.$$

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1},$$

$$Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+3}w_2^{2j}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}.$$

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l},$$

$$Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+3}w_2^{2j}w_3^{2k+1}w_4^{2l} + w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+3}w_2^{2j}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l}.$$

$$Q_0(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = 0,\ Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = 0$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l},$$

$$Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+3}w_2^{2j+1}w_3^{2k}w_4^{2l} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l} + w_1^{2i+1}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l}.$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1},$$

$$Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l+2}.$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l},$$

$$Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = w_1^{2i+2}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = 0.$$

$$Q_0(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1} + w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l+1},$$

$$Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+3}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+2}w_2^{2j}w_3^{2k+2}w_4^{2l+1} + w_1^{2i+1}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1} + w_1^{2i}w_2^{2j+1}w_3^{2k+2}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+2} + w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1} + w_1^{2i}w_2^{2j}w_3^{2k+3}w_4^{2l+1} + w_1^{2i+3}w_2^{2j+1}w_3^{2k}w_4^{2l+2} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+2}.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l}) = w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l},$$

$$Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l}) = w_1^{2i+4}w_2^{2j}w_3^{2k}w_4^{2l},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l}) = 0.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1}) = 0,$$

$$Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1}) = 0.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l}) = 0,$$

$$Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+3}w_2^{2j}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l}) = 0.$$

$$Q_0(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+1},$$

$$Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+4}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+3}w_2^{2j}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+4}w_2^{2j}w_3^{2k}w_4^{2l+2}.$$

$$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l},$$

$$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+3}w_2^{2j}w_3^{2k+1}w_4^{2l} + w_1^{2i+2}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+3}w_2^{2j}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}w_4^{2l}.$$

$$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1},$$

$$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = w_1^{2i+4}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + w_1^{2i+3}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+2}w_2^{2j}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1}) = 0.$$

$$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l},$$

$$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = w_1^{2i+4}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+3}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+3}w_2^{2j+1}w_3^{2k}w_4^{2l+1},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l}) = w_1^{2i+4}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + w_1^{2i+3}w_2^{2j}w_3^{2k+1}w_4^{2l+1}.$$

$$Q_0(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l+1},$$

$$Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+3}w_2^{2j}w_3^{2k+2}w_4^{2l+1} + w_1^{2i+2}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+2} + w_1^{2i+1}w_2^{2j+1}w_3^{2k+2}w_4^{2l+1} + w_1^{2i+3}w_2^{2j+1}w_3^{2k}w_4^{2l+2},$$

$$Q_0Q_1(w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+2}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+3}w_4^{2l+1}.$$

Lemma 4.2. *As an E-module, $\widetilde{H}^*(BO(4))$ is isomorphic to $D^* \oplus M$, where D^* is an E-module with the $Z/2$-basis $\{w_1^iw_2^{2j},\ w_2^{2j}w_4^{2l} \mid i > 0,\ j+l > 0\}$ and M is isomorphic to a free E-module $\widetilde{H}^*(BO(4))/D^*$ with E-basis $\{w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1},\ w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l},\ w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l},\ w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1},\ w_1^{2i}w_2^{2j+1}w_4^{2l+1} \mid i,\ j,\ k,\ l \geq 0\}$.*

Proof. By Lemma 4.1, we know

$(*a)$ $Q_0(w_4) = w_1w_4,\ Q_1(w_4) = w_3w_4 + w_1^3w_4 + w_1w_2w_4,$
$Q_0Q_1(w_4) = w_1^2w_2w_4 + w_1w_3w_4.$

$(*b)$ $Q_0(w_3) = w_1w_3,\ Q_1(w_3) = w_3^2 + w_1^3w_3 + w_1w_2w_3 + w_1^2w_4,$
$Q_0Q_1(w_3) = w_1^2w_2w_3 + w_1^3w_4 + w_1w_3^2.$

$(*c)$ $Q_0(w_2) = w_1w_2 + w_3,\ Q_1(w_2) = w_1^3w_2 + w_1^2w_3 + w_1w_2^2 + w_2w_3 + w_1w_4,$
$Q_0Q_1(w_2) = w_1^2w_2^2 + w_3^2.$

$(*d)$ $Q_0(w_2w_3w_4) = w_1w_2w_3w_4 + w_3^2w_4,$
$Q_1(w_2w_3w_4) = w_1^3w_2w_3w_4 + w_1^2w_3^2w_4 + w_1w_2^2w_3w_4 + w_2w_3^2w_4 + w_1w_3w_4^2 + w_1^2w_2w_4^2,$

$$Q_0Q_1(w_2w_3w_4) = w_1^2w_2^2w_3w_4 + w_3^3w_4 + w_1^3w_2w_4^2 + w_1^2w_3w_4^2.$$

$(*e)$ $Q_0(w_2w_4) = w_3w_4,\ Q_1(w_2w_4) = w_1^2w_3w_4 + w_1w_4^2,\ Q_0Q_1(w_2w_4) = w_1^2w_4^2.$

$(*f)$ $Q_0(w_1) = w_1^2,\ Q_1(w_1) = w_1^4,\ Q_0Q_1(w_1) = 0.$

Using that Q_0 and Q_1 act as derivations, it is easy to see that D^* is an E-module by $(*f)$. Hence it remains to prove that $\{w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}$, $w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}$, $w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}$, $w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}$, $w_1^{2i}w_2^{2j+1}w_4^{2l+1} \mid i$, j, k ,$l \geq 0\}$ is a basis of the free E-module $\widetilde{H}^*(BO(4))/D^*$. Since $1(x) = x$ for all $x \in \widetilde{H}^*(BO(4))/D^*$, the basis can be generated uniquely. Consider the Q_0 action on the basis, hence $w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l+1}$, $w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l}$, $w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l}$, $w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}$, and $w_1^{2i}w_2^{2j}w_3w_4^{2l+1}$ can be generated uniquely. We consider the complicated cases below.

First, we consider

$$(*d)\ \ Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1}$$
$$+w_1^{2i}w_2^{2j}w_3^{2k+3}w_4^{2l+1} + w_1^{2i+3}w_2^{2j+1}w_3^{2k}w_4^{2l+2} + w_1^{2i+2}w_2^{2j}w_3^{2k+1}w_4^{2l+2}.$$

We concentrate on $w_1^{2i+2}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1}+w_1^{2i}w_2^{2j}w_3^{2k+3}w_4^{2l+1}$ since we have shown that other elements can be generated uniquely. Since $w_1^{2i}w_2^{2j}w_3w_4^{2l+1}$ can be generated uniquely, so can $w_1^{2i+2}w_2^{2j+2}w_3w_4^{2l+1}$. Hence $w_1^{2i}w_2^{2j}w_3^3w_4^{2l+1}$ can be generated uniquely. This means $w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1}$ can be generated uniquely by induction. Also we know $(*a)$ $Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l+1} + w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + \alpha$, where α can be generated uniquely. Hence $w_1^{2i+1}w_2^{2j+1}w_3^{2k}w_4^{2l+1}$ can be generated uniquely.

Secondly, we consider

$$(*c)\ Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l}$$

and $(*e)$ $Q_0Q_1(w_1^{2i}w_2^{2j+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_4^{2l+2}$. Since $w_1^{2i}w_2^{2j} \in D^*$, this implies $w_1^{2i}w_2^{2j}w_3^2$ can be generated uniquely by considering

$$(*c)\ Q_0Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+2}w_2^{2j+2} + w_1^{2i}w_2^{2j}w_3^2.$$

Repeat this argument on $(*c)$ $Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k} + w_1^{2i}$ $w_2^{2j}w_3^{2k+2}$, it follows that $w_1^{2i}w_2^{2j}w_3^{2k}$ $(k \geq 1)$ can be generated uniquely. Also we know $w_1^{2i}w_2^{2j}w_4^{2l}$ $(i,\ l \geq 1)$ can be generated uniquely by $(*e)$ Q_0Q_1

$(w_1^{2i}w_2^{2j+1}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j}w_4^{2l+2}$, hence $w_1^{2i}w_2^{2j}w_3^2w_4^{2l}$ $(l \geq 1)$ can be generated uniquely by considering

$$(*c)\ Q_0Q_1(w_1^{2i}w_2^{2j+1}w_4^{2l}) = w_1^{2i+2}w_2^{2j+2}w_4^{2l} + w_1^{2i}w_2^{2j}w_3^2w_4^{2l}.$$

Repeat this argument on

$$(*c)\ Q_0Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+2}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l},$$

we see that $w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}$ $(k, l \geq 1)$ can be generated uniquely by induction. Recall that D^* is an E-module with the $Z/2$-basis $\{w_1^iw_2^{2j},\ w_2^{2j}w_4^{2l} \mid i > 0, j+l > 0\}$. If $k = 0$ and $l = 0$, then $w_1^{2i}w_2^{2j} \in D^*$ $(i+j > 0)$. If $k = 0$ and $l \geq 1$, then $w_2^{2j}w_4^{2l} \in D^*$ $(i = 0,\ l \geq 1)$ and we have shown that $w_1^{2i}w_2^{2j}w_4^{2l}$ $(i \geq 1, l \geq 1)$ can be generated uniquely. If $k \geq 1$ and $l = 0$, then we have shown that $w_1^{2i}w_2^{2j}w_3^{2k}$ $(k \geq 1)$ can be generated uniquely. If If $k \geq 1$ and $l \geq 1$, then we have shown that $w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}$ $(k \geq 1,\ l \geq 1)$ can be generated uniquely. Hence the case $w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l}$ which belongs to $\widetilde{H}^*(BO(4))/D^*$ can be generated uniquely. By considering $(*b)$ $Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i}w_2^{2j}w_3^{2k+2}w_4^{2l} + w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + \alpha$, where α can be generated uniquely, it follows that $w_1^{2i+1}w_2^{2j+1}w_3^{2k+1}w_4^{2l}$ can be generated uniquely.

Thirdly, we consider

$$(*a)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}$$

and

$$(*d)\ Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j+2}w_3^{2k+1}w_4^{2l+1} + w_1^{2i}w_2^{2j+1}w_3^{2k+2}w_4^{2l+1} + \alpha,$$

where α can be generated uniquely. Since $w_1^{2i}w_2^{2j+1}w_4^{2l+1}$ is contained in the basis, $w_1^{2i+1}w_2^{2j}w_3w_4^{2l+1}$ can be generated uniquely by considering

$$(*a)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_4^{2l+1}) = w_1^{2i+2}w_2^{2j+1}w_4^{2l+1} + w_1^{2i+1}w_2^{2j}w_3w_4^{2l+1}.$$

Therefore $w_1^{2i}w_2^{2j+1}w_3^2w_4^{2l+1}$ can be generated uniquely by

$$(*d)\ Q_1(w_1^{2i}w_2^{2j+1}w_3w_4^{2l+1}) = w_1^{2i+1}w_2^{2j+2}w_3w_4^{2l+1} + w_1^{2i}w_2^{2j+1}w_3^2w_4^{2l+1} + \alpha,$$

where α can be generated uniquely. Repeat this argument on $(*a)$ $Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k}w_4^{2l+1})$ and $(*d)$ $Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l+1})$, it follows that $w_1^{2i}w_2^{2j+1}$

$w_3^{2k}w_4^{2l+1}$ $(k \geq 1)$ and $w_1^{2i+1}w_2^{2j}w_3^{2k+1}w_4^{2l+1}$ can be generated uniquely by induction.

Finally, we consider

$$(*b)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}) = w_1^{2i+2}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + w_1^{2i+1}w_2^{2j}w_3^{2k+2}w_4^{2l} + \alpha$$

and

$$(*c)\ Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l}) = w_1^{2i+1}w_2^{2j+2}w_3^{2k}w_4^{2l} + w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l} + \alpha',$$

where α and α' can be generated uniquely. Notice that $w_1^{2i+1}w_2^{2j} \in D^*$. If $k = l = 0$, then $w_1^{2i}w_2^{2j+1}w_3$ can be generated uniquely by considering

$$(*c)\ Q_1(w_1^{2i}w_2^{2j+1}) = w_1^{2i+1}w_2^{2j+2} + w_1^{2i}w_2^{2j+1}w_3 + \alpha',$$

where α' can be generated uniquely. By considering

$$(*b)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3) = w_1^{2i+2}w_2^{2j+1}w_3 + w_1^{2i+1}w_2^{2j}w_3^2 + \alpha,$$

where α can be generated uniquely, we know that $w_1^{2i+1}w_2^{2j}w_3^2$ can be generated uniquely. Repeat this argument on

$$(*c)\ Q_1(w_1^{2i}w_2^{2j+1}w_3^{2k}w_4^{2l})$$

and

$$(*b)\ Q_0Q_1(w_1^{2i}w_2^{2j}w_3^{2k+1}w_4^{2l}),$$

this implies that $w_1^{2i}w_2^{2j+1}w_3^{2k+1}$ and $w_1^{2i+1}w_2^{2j}w_3^{2k}$ $(k \geq 1)$ can be generated uniquely by induction. Notice that

$$(*e)\ Q_1(w_1^{2i}w_2^{2j+1}w_4^{2l+1}) = w_1^{2i+1}w_2^{2j}w_4^{2l+2} + \alpha,$$

where α can be generated uniquely, hence $w_1^{2i+1}w_2^{2j}w_4^{2l}$ $(l \geq 1)$ can be generated uniquely. By the same argument as above, we see that $w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l}$ $(l \geq 1)$ and $w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l}$ $(k,\ l \geq 1)$ can be generated uniquely. From the above argument, we know that $w_1^{2i}w_2^{2j+1}w_3^{2k+1}w_4^{2l}$ and $w_1^{2i+1}w_2^{2j}w_3^{2k}w_4^{2l}$ $(k + l > 0)$ can be generated uniquely. This completes the proof. □

Proof of Theorem 3. It suffices to define the homotopy equivalence map. Since $n = 4$, there is a 2-local stable map $W_2^{2j}W_4^{2l} : BO(4) \to \sum^{4j+8l} bu$

which is detected by $w_2^{2j}w_4^{2l} \in \widetilde{H}^*BO(4)$ for each $j+l>0$ by the remark after Lemma 1.1.

Define

$$g_1 : bu \wedge BO(4) \overset{bu\wedge \underset{0<j+l}{\vee} W_2^{2j}W_4^{2l}}{\rightarrow} bu \wedge [\underset{0<j+l}{\vee}(bu \wedge S^{4j+8l})] \overset{\underset{j+l}{\vee}\mu}{\rightarrow} \underset{0<j+l}{\vee}(bu \wedge S^{4j+8l}),$$

$$g_2 : bu \wedge BO(4) \overset{bu\wedge \det}{\rightarrow} bu \wedge RP^\infty,$$

$$g_3 : bu \wedge BO(4) \overset{bu\wedge\Delta}{\rightarrow} bu \wedge [BO(4)\times BO(4)] \overset{bu\wedge q}{\rightarrow} bu \wedge [BO(4)\wedge BO(4)]$$
$$\overset{bu\wedge \underset{0<j}{\vee} W_2^{2j}\wedge \det}{\rightarrow} bu \wedge [[\underset{0<j}{\vee}(bu\wedge S^{4j})] \wedge RP^\infty] \overset{\underset{j}{\vee}\mu\wedge RP^\infty}{\rightarrow} \underset{0<j}{\vee}(bu \wedge S^{4j}\wedge RP^\infty),$$

and

$$g_0 : bu \wedge BO(4) \overset{bu\wedge \underset{b}{\vee} g_b}{\rightarrow} bu \wedge [\underset{\alpha}{\vee}\sum{}^{\alpha} HZ/2] \overset{\underset{b}{\vee}\nu}{\rightarrow} [\underset{\alpha}{\vee}\sum{}^{\alpha} HZ/2],$$

where Δ is the diagonal map, q is the quotient map, $\alpha = 2i+4j+6k+8l+4$, $2i+4j+6k+3+8l$, $2i+4j+2+6k+8l$, $2i+4j+2+6k+3+8l+4$, and $2i+4j+2+8l+4$ for all i, j, k, $l \geq 0$, and the other maps are defined as the proof of Theorem 1. By the same argument as the proof of Theorem 1, we know $g = g_0 \vee g_1 \vee g_2 \vee g_3$ is an equivalence at prime 2. $\square$

5. The proof of Theorem 4

Lemma 5.1. *By Cartan formula, Wu formula, and the fact that Q_0 and Q_1 act as derivations, the E-module structure of $\widetilde{H}^*(BSO(3))$ is as follows:*

$$Q_0(w_2^{2j}w_3^{2k}) = Q_1(w_2^{2j}w_3^{2k}) = Q_0Q_1(w_2^{2j}w_3^{2k}) = 0.$$
$$Q_0(w_2^{2j}w_3^{2k+1}) = 0,\ Q_1(w_2^{2j}w_3^{2k+1}) = w_2^{2j}w_3^{2k+2},\ Q_0Q_1(w_2^{2j}w_3^{2k+1}) = 0.$$
$$Q_0(w_2^{2j+1}w_3^{2k}) = w_2^{2j}w_3^{2k+1},\ Q_1(w_2^{2j+1}w_3^{2k}) = w_2^{2j+1}w_3^{2k+1},$$
$$Q_0Q_1(w_2^{2j+1}w_3^{2k}) = w_2^{2j}w_3^{2k+2}.$$
$$Q_0(w_2^{2j+1}w_3^{2k+1}) = w_2^{2j}w_3^{2k+2},\ Q_1(w_2^{2j+1}w_3^{2k+1}) = Q_0Q_1(w_2^{2j+1}w_3^{2k+1}) = 0.$$

Proof of Theorem 4. By Lemma 5.1, we know

$$Q_0(w_2) = w_3,\ Q_1(w_2) = w_2w_3,\ Q_0Q_1(w_2) = w_3^2.$$

Using that Q_0 and Q_1 act as derivations, it is easy to see that $\widetilde{H}^*(BSO(3))$ is isomorphic to $D^* \oplus M$ as an E-module, where D^* is an E-module with the $Z/2$-basis $\{w_2^{2j} \mid j > 0\}$ and M is isomorphic to a free E-module $\widetilde{H}^*(BSO(3))/D^*$ with E-basis $\{w_2^{2j+1}w_3^{2k} \mid j,k \geq 0\}$. Let $h_3 : BSO(3) \to BO(3)$ be the usual 2-folds map, then we have a 2-local stable map $BSO(3) \stackrel{h_3}{\to} BO(3) \stackrel{W_2^{2j}}{\to} \sum^{4j} bu$ for each $j > 0$ by Lemma 1.1. Define the homotopy equivalence map as the proof of Theorem 1. This completes the proof by the same argument as the proof of Theorem 1. □

6. The proof of Theorem 5

Lemma 6.1. *By Cartan formula, Wu formula, and the fact that Q_0 and Q_1 act as derivations, the E-module structure of $\widetilde{H}^*(BSO(5))$ is as follows:*

$$Q_0(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}) = Q_1(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}) = Q_0Q_1(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}) = 0.$$

$$Q_0(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}) = 0,\ Q_1(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1},$$

$$Q_0Q_1(w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}) = 0.$$

$$Q_0(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m},$$

$$Q_0Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1}.$$

$$Q_0(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+2},$$

$$Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = Q_0Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = 0.$$

$$Q_0(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m}) = 0,\ Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m},$$

$$Q_0Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m}) = 0.$$

$$\begin{aligned}Q_0(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1}) &= Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1})\\ &= Q_0Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1}) = 0.\end{aligned}$$

$$Q_0(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = Q_0Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = 0.$$

$$Q_0(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+2},$$

$$Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+2}w_4^{2l+1}w_5^{2m+1},$$

$$Q_0Q_1(w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m+2}.$$

$$Q_0(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m},$$

$$Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m}) = w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m} + w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1},$$

$$Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m}.$$

$$Q_0(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1}) = w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+2},$$

$$Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1}) = 0.$$

$$Q_0(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m} + w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1},$$

$$Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+2}.$$

$$Q_0(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1} + w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+2},$$

$$Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1} + w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+2},$$

$$\begin{aligned}Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = {} & w_2^{2j}w_3^{2k+2}w_4^{2l+1}w_5^{2m+1} \\ & + w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+2} + w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+3}.\end{aligned}$$

$$Q_0(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m},$$

$$Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1},$$

$$Q_0Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m}) = 0.$$

$$Q_0(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+1}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+1}) = w_2^{2j+1}w_3^{2k+2}w_4^{2l}w_5^{2m+1} + w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+2},$$

$$Q_0Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+1}) = w_2^{2j}w_3^{2k+3}w_4^{2l}w_5^{2m+1}.$$

$$Q_0(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+2}w_4^{2l+1}w_5^{2m} + w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+1},$$

$$Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j+1}w_3^{2k+2}w_4^{2l+1}w_5^{2m} + w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1},$$

$$\begin{aligned}Q_0Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = {} & w_2^{2j}w_3^{2k+3}w_4^{2l+1}w_5^{2m} + w_2^{2j+1}w_3^{2k+2}w_4^{2l}w_5^{2m+1} \\ & + w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+2}.\end{aligned}$$

$$\begin{aligned}Q_0(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = {} & w_2^{2j}w_3^{2k+2}w_4^{2l+1}w_5^{2m+1} \\ & + w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+2},\end{aligned}$$

$$Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+2},$$

$Q_0Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+3}$.

Lemma 6.2. As an E-module, $\widetilde{H}^*(BSO(5))$ is isomorphic to $D^* \oplus M$, where D^* is an E-module with the $Z/2$-basis $\{w_2^{2j}w_4^{2l} \mid j+l>0\}$ and M is isomorphic to a free E-module $\widetilde{H}^*(BSO(5))/D^*$ with E-basis $\{w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m},\ w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m},\ w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1},\ w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m},\ w_2^{2j+1}w_4^{2l+1}w_5^{2m} \mid j, k, l, m \geq 0\}$.

Proof. By Lemma 6.1, we know

$(*a)$ $Q_0(w_4) = w_5,\ Q_1(w_4) = w_3w_4,\ Q_0Q_1(w_4) = w_3w_5$.

$(*b)$ $Q_0(w_2) = w_3,\ Q_1(w_2) = w_2w_3 + w_5,\ Q_0Q_1(w_2) = w_3^2$.

$(*c)$ $Q_0(w_2w_4w_5) = w_3w_4w_5 + w_2w_5^2,\ Q_1(w_2w_4w_5) = w_2w_3w_4w_5 + w_4w_5^2,$
$Q_0Q_1(w_2w_4w_5) = w_3^2w_4w_5 + w_2w_3w_5^2 + w_5^3$.

$(*d)$ $Q_0(w_2w_3w_4) = w_3^2w_4 + w_2w_3w_5,\ Q_1(w_2w_3w_4) = w_2w_3^2w_4 + w_3w_4w_5,$
$Q_0Q_1(w_2w_3w_4) = w_3^3w_4 + w_2w_3^2w_5 + w_3w_5^2$.

$(*e)$ $Q_0(w_2w_4) = w_3w_4 + w_2w_5,\ Q_1(w_2w_4) = w_4w_5,\ Q_0Q_1(w_2w_4) = w_5^2$.

It easy to see that D^* is an E-module. Hence it remains to prove $\{w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m},\ w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m},\ w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1},\ w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m},\ w_2^{2j+1}w_4^{2l+1}w_5^{2m} \mid j, k, l, m \geq 0\}$ is a basis of the free E-module $\widetilde{H}^*(BSO(5))/D^*$. Since $1(x) = x$ for all $x \in \widetilde{H}^*(BSO(5))/D^*$, the basis can be generated uniquely. Consider the Q_0 action on the basis, hence $w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}$, $w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m}$, $w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}$, $w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m+1}$, and $w_2^{2j+1}w_4^{2l}w_5^{2m+1}$ can be generated uniquely. Since

$(*a)$ $Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m}$

and $(*a)$ $Q_0Q_1(w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1}$, it means $w_2^{2j}w_3^{2k+1}w_4^{2l+1}w_5^{2m}$ and $w_2^{2j}w_3^{2k+1}w_4^{2l}w_5^{2m+1}$ can be generated uniquely. Since

$(*b)$ $Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m}) = w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m} + w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}$

and we have shown that $w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m+1}$ can be generated uniquely, this implies $w_2^{2j+1}w_3^{2k+1}w_4^{2l}w_5^{2m}$ can be generated uniquely. Now we want to show $w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}$ which belongs to $\widetilde{H}^*(BSO(5))/D^*$ can be generated uniquely.

If $k=0$ and $m=0$, then $w_2^{2j}w_4^{2l}\in D^*$. If $k=0$ and $m\geq 1$, then $w_2^{2j}w_4^{2l}w_5^{2m}$ can be generated uniquely by considering $(*e)$ $Q_0Q_1(w_2^{2j+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_4^{2l}w_5^{2m+2}$. If $k\geq 1$, then $w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}$ can be generated uniquely by conisdering

$$(*b)Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m}) = w_2^{2j}w_3^{2k+2}w_4^{2l}w_5^{2m}.$$

Hence the case $w_2^{2j}w_3^{2k}w_4^{2l}w_5^{2m}$ which belongs to $\widetilde{H}^*(BSO(5))/D^*$ can be generated uniquely. Next we want to show the case $w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1}$ can be generated uniquely. If $k=0$, then $w_2^{2j}w_4^{2l+1}w_5^{2m+1}$ can be generated uniquely by considering

$$(*e)\ Q_1(w_2^{2j+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j}w_4^{2l+1}w_5^{2m+1}.$$

If $k\geq 1$, then $w_2^{2j}w_3^{2k}w_4^{2l+1}w_5^{2m+1}$ can be generated uniquely by considering

$$(*c)\ Q_0Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j}w_3^{2k+2}w_4^{2l+1}w_5^{2m+1}+\alpha,$$

where α can be can be generated uniquely. For the case $w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1}$, we can consider $(*e)$ $Q_0(w_2^{2j+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j+1}w_4^{2l}w_5^{2m+1}$ and

$$(*d)\ Q_0Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j+1}w_3^{2k+2}w_4^{2l}w_5^{2m+1}+\alpha,$$

where α can be generated uniquely. Hence the case $w_2^{2j+1}w_3^{2k}w_4^{2l}w_5^{2m+1}$ can be generated uniquely. Finally we consider

$$(*c)\ Q_1(w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m+1}) = w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}+\alpha$$

and $(*d)$ $Q_1(w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m}) = w_2^{2j+1}w_3^{2k+2}w_4^{2l+1}w_5^{2m}+\alpha'$, where α and α' can be generated uniquely, hence $w_2^{2j+1}w_3^{2k+1}w_4^{2l+1}w_5^{2m+1}$ and $w_2^{2j+1}w_3^{2k}w_4^{2l+1}w_5^{2m}$ $(k\geq 1)$ can be generated uniquely. This completes the proof. $\square$

Proof of Theorem 5. Let $h_5 : BSO(5)\to BO(5)$ be the usual 2-folds map, then we have a 2-local stable map $BSO(5)\overset{h_3}{\to} BO(5)\overset{W_2^{2j}W_4^{2l}}{\to}\sum^{4j+8l}bu$ for each $j+l>0$ by the remark after Lemma 1.1. By the same argument, we can define the homotopy equivalence map as the proof of Theorem 3. This completes the proof. $\square$

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