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# INSTABILITY AND FORMATION OF CLUSTERING SOLUTIONS OF TRAFFIC FLOW

BY

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**Abstract**

We review recent results on traffic instabilities. An innovative approach to traffic dynamics is proposed. The self-organized oscillatory and chaotic behavior of traffic flow are identified and formulated. The results agree with the empirical findings for freeway traffic and with the previous numerical simulations. Thus the work gives a justification for observed and simulated traffic instabilities and some insight into their meanings.

## 1. Introduction

We are interested in modeling rich nonlinear phenomena in traffic: the formation of traffic jams, stop-and-go waves, chaos, hysteresis and phase transitions, see Daganzo et al. [6], Helbing [13], Kerner [18], Kerner and Rehborn [20], Knospe, Santen, Schadschneider and Schreckenberg [22], Mauch and Cassidy [40], Treiterer and Myers [50], Safonov and co-authors [49], Intrigued by the phenomenology of real-world traffic and the simulations of others, we propose an innovative approach to the nonlinear dynamics of traffic flow. Our goal is to identify and precisely formulate the self-organized oscillatory behavior and chaotic behavior in traffic.

There are many important approaches to the modeling of traffic phenomena: microscopic models which explain traffic phenomena on the basis

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of the behavior of single vehicles, see Gazis, Herman and Rothery [9], mesoscopic models such as kinetic or Boltzmann-like models which deal with the statistical number-of-car distribution function and its time, space and velocity dependence, see Prigogine and Herman [45], and macroscopic models which describe traffic phenomena through parameters which characterize collective traffic properties, see Daganzo [5], Li [31, 32, 33], Lighthill and Whitham [38], Payne [44], Richards [46], Whitham [52] and Zhang [55]. The relationship between microscopic models and macroscopic models and the relationship between kinetic models and macroscopic models are addressed by Helbing [13], Klar and Wegener [21] and Lee et al. [28]. There are models using either discrete or continuous state space, with discrete or continuous time and/or space. For example, traffic cellular automaton (CA) model of Gray and Griffeath [10] and Nagel [43] is discrete in both time and space.

We review instability results in Section 2. An innovative approach to traffic dynamics is proposed in Section 3. Our results will be compared with empirical and numerical findings in Section 4.

## 2. Traffic Instability

Traffic instability has been empirically studied by many authors, including Helbing [13], Kerner [18], Kerner and Rehborn [20], Knospe, Santen, Schadschneider and Schreckenberg [22], Mauch and Cassidy [40], Treiterer and Myers [50], where stop-and-go waves and spontaneous formation of traffic jams were observed.

The instability phenomena have been modeled and simulated by many. Gray and Griffeath [10] studied the ergodic theory of traffic jams. They analyzed a probabilistic cellular automaton (CA) as a prototype for the emergence of traffic jams at some intermediate density range. Different clusters of vehicles, including the cluster of large amplitude moving against the flow, have been observed in the numerical investigations of the cellular automation model by Nagel [43].

Assuming that there exists a function relation between the velocity and the density $v = v_e(\rho)$, Lighthill and Whitham [38] and Richards [46] developed the first macroscopic model of traffic flow, LWR (Lighthill, Whitham and Richards) theory,

$$\rho_t + (\rho v_e(\rho))_x = 0. \tag{1}$$

The function $v_e(\rho)$ is the equilibrium velocity and is non-increasing

$$v_e'(\rho) \leq 0. \tag{2}$$

$v_e(0) = v_f$ and $v_e(\rho_j) = 0$, where $v_f$ is the free flow speed and $\rho_j$ is the jam concentration.

$$q(\rho) = \rho v_e(\rho) \tag{3}$$

is the so-called fundamental diagram in traffic flow.

A fundamental diagram gives a correspondence of vehicle density to the flow rate in traffic which can in general be determined by measurements. It is well-known that experimental flux-density plots show wide scatter in certain density regime. It can be speculated that this scatter hides a more complicated behavior of the speed-density relation. There are a huge number of different suggestions about the speed-density relation. The fundamental diagrams may be concave, nonconcave, smooth, discontinuous or have multiple branches. According to Nagel [43] and others, a typical fundamental diagram looks like a 'reverse $\lambda$' in the $(\rho, q)$ plane. Multivalued fundamental diagrams are suggested by Illner, Klar and Materne [15] and Günther, Klar, Materne and Wegener [12]. In this paper, we adopt nonconcave fundamental diagrams, see (10) and Figure 1. Thus the characteristic family of equation (1) is not genuinely nonlinear. The characteristic speed of (1) is

$$\lambda_*(\rho) = q'(\rho) = \rho v_e'(\rho) + v_e(\rho). \tag{4}$$

which is not faster than the traffic speed $v_e(\rho)$ under the assumption that $v_e'(\rho) \leq 0$. This is the so-called anisotropic property.

LWR model can explain the formation of shock waves which correspond to congestion formation in traffic flow. The LWR theory fails in describing more complicated traffic flow patterns including hysteresis phenomena and stop-and-go traffic. This is due to the unrealistic assumption that the equilibrium speed is adapted instantaneously. The LWR model (1) has been extended to nonequilibrium macroscopic models that include the dynamics of the velocity [19, 23, 33, 34, 44, 52, 55].

Payne [44] and Whitham [52]. proposed the nonequilibrium PW model,

$$\rho_t + (\rho v)_x = 0 \tag{5}$$

$$v_t + v v_x + \frac{c_0^2}{\rho}\rho_x = \frac{v_e(\rho) - v}{\tau} \tag{6}$$

where $\tau > 0$ is the relaxation time, $c_0$ is the traffic sound speed coefficient and $v_e(\rho)$ is the desired speed.

Equation (6) describes drivers' acceleration behavior. The acceleration consists of a relaxation to the static equilibrium speed-density relation and an anticipation which expresses the effect of drivers reacting to conditions downstream. The third term on the left hand side of (6) accounts for the anticipation effect. Its physical meaning is that one tends to reduce speed when the density increases. The right hand side of (6) is the relaxation term. Schochet [47] established the well-posedness for certain Payne type higher order traffic flow models. Lattanzio and Marcati [26] showed the convergence to the equilibrium solution as the relaxation parameter $\tau$ tends to zero, away from the vacuum for certain PW type higher order traffic flow models.

The model (5) (6) is a system of nonlinear hyperbolic equations. Its characteristic speeds are

$$\lambda_1 = v - c_0 < \lambda_2 = v + c_0.$$

The model (5) (6) is stable if

$$\lambda_1 \ < \ \lambda_* < \lambda_2 \tag{7}$$

or

$$-c_0 \ < \ \rho v_e'(\rho) < c_0 \tag{8}$$

on the equilibrium curve $v = v_e(\rho)$, see Whitham [52]. Condition (7) is the so-called strict subcharacteristic condition. Under the strict subcharacteristic condition (7), Li and Liu [36] and Liu [39] derived the nonlinear stability of elementary waves. Indeed, under the assumption (7), it can be derived, in the same spirit as the Chapman-Enskog expansion, that the relaxation process is approximated by a viscous conservation law

$$\rho_t + (\rho v_e(\rho))_x = (\beta(\rho)\rho_x)_x \tag{9}$$

where $\beta(\rho) = -\frac{\tau}{\rho}(\lambda_* - \lambda_1)(\lambda_* - \lambda_2) > 0$.

The following is a fundamental diagrams that changes concavity

$$q(\rho) = 5.0461\rho((1 + e^{\frac{\rho-0.25}{0.06}})^{-1} - 3.72 \cdot 10^{-6}), \tag{10}$$

see Figure 1.

If $q(\rho)$ is nonconcave, such as in (10), then there exists an unstable region of Payne-Whitham model (5) (6)

$$[\rho_{c1}, \rho_{c2}], \tag{11}$$

where $\rho_{c1} < \rho_i < \rho_{c2}$ are solutions of

$$-c_0 = \rho v_e'(\rho), \tag{12}$$

where $\rho_i$ is the inflection point of the nonconcave fundamental diagram (10). Kühne [23], Kerner and Konhäuser [19] and Jin and Zhang [16] observed the formation of vehicle clusters in the numerical solutions of the Payne-Whitham model in *the unstable regions*. In particular, numerically solving the viscous PW model by the centered Euler scheme, Kerner and Konhäuser [19] found that if the density of vehicles exceeds some critical value, a small perturbation in a homogeneous traffic flow on a circular ring road can grow to a stationary moving cluster. If the density is increased further, the avalanche-like process of cluster formation can start. Therefore, a sequence of clusters, which appear subsequently in space and time, is created. The clusters in this sequence have different amplitudes, different widths, different velocities, and are not situated periodically in space. Similar results were found by Jin and Zhang [16] when they numerically solved PW model (5) (6) by the Godunov method. They found that the number of clusters, the position, height and the width of each cluster were not predictable. Kerner, Klenov and Konhäuser [17] developed an asymptotic theory of traffic jams of large amplitude based on singular perturbation methods.

Kühne and co-workers found that there are stable and unstable fixed points and limit cycles of an approximation of the viscous PW model (5) (6) with the following nonconcave fundamental diagram

$$q(\rho) = v_f \rho (1 - \rho^{1.4})^4, \tag{13}$$

where $v_f = 120\frac{km}{h}$. The approximation is based on a truncated expansion using the eigenmode expansion from linear stability as a starting point, which suggest that traffic near maximum flow operates on a strange attractor [24]. Under certain circumstances, chaotic motion is observed. The unstable traffic patterns are connected with nonlinear stochastics. Free traffic would correspond to a point attractor and the oscillating traffic state to a stable limit cycle. This corresponds to a subcritical Hopf bifurcation.

In their numerical simulations of Payne-Whitham equations (5) (6) with a nonconcave fundamental diagram

$$q(\rho) = V_0 \frac{\rho(1-\frac{\rho}{\rho_0})}{1+E(\frac{\rho}{\rho_0})^\theta} \tag{14}$$

where $V_0 = 120\frac{km}{h}$, $\rho_0 = 140\frac{veh}{km}$, $E = 100$ and $\theta = 4$, Lee et al. [27] managed to trigger a form of stop-and-go traffic that was propagating upstream, but its downstream front was pinned at the location of the ramp.

Bando et al. [1] proposed an optimal velocity model in which the acceleration of every car is determined by its velocity $v_n$ and a desired speed $V_0(b_n)$ depending on the headway $b_n$ to the car in front

$$\frac{dv_n}{dt} = a(V_0(b_n) - v_n)$$

where the optimal velocity $V_0(h) = \tanh(h-2) + \tanh(2)$ is a monotone increasing function, $a$ is a constant called sensitivity which equals the inverse of the reaction time. The evolution of traffic congestion was observed with the development of time. Berg and Woods [3] gave an analogous continuum counterpart of the optimal velocity model which is in good agreement with its discrete version. The fundamental diagram is

$$q(\rho) = \rho V(\rho) = \rho V_0(\frac{1}{\rho}) \tag{15}$$

which is nonconcave, $x_{n+1} - x_n \in (\Delta x_{min}, \Delta x_{max})$ is unstable, where $\Delta x_{min}$ $= 1$ and $\Delta x_{max} = 3$. Berg and Woods [2] defined a recursive map which specify the plateau headway as a function of the upstream headway. In all cases, nonconcave fundamental diagrams were needed to obtain the cluster solutions.

In [8], Gasser and co-authors presented the bifurcation analysis on follow-the-car traffic models describing dynamics of $N$ cars on a circular road. It is proved that the loss of stability generally is due to Hopf bifurcation. It is also shown that variable reaction time changes the periodic dynamics and that aggressive driving behavior increases the stability. Whitham [51] obtained exact solutions representing periodic waves and solitary waves for a car-following model including time lag. Nagatani [41] studied the physics of traffic jams through microscopic car-following models. Safonov and co-authors [49] studied chaos and multifractality in a time-delay car-following traffic model. Numerical simulations with fundamental diagram (15) by Greenberg [11] yielded roll-wave solutions.

## 3. A Class of Discrete Models

In [35], the author derived a class of dynamic traffic flow models that capture the essential features of traffic jams. One key result is the identification of the physical condition needed to obtain clustering solutions, namely that, the fundamental diagrams change concavity.

Consider the traveling wave solutions of (5) (6), namely, solutions of form

$$(P, V)(x - ct) = (P, V)(\xi)$$

where $\xi = x - ct$ is the traveling wave variable. $c$ is the traveling wave speed.

We are looking for clustering solutions on a ring road propagating with a speed which is less than the vehicle speed

$$c < V. \tag{16}$$

This reflects the fact that the vehicle clustering travels against the traffic flow.

$$\frac{1}{P^3}(-Q^2 + c_0^2 P^2)P' = \frac{v_e(P) - c - \frac{Q}{P}}{\tau}. \tag{17}$$

We denote the coefficient of $P'$ in (17) as

$$B(P) = \frac{1}{P^3}(-Q^2 + c_0^2 P^2). \tag{18}$$

and the numerator of the right hand side of (17) as

$$D(P) = v_e(P) - c - \frac{Q}{P}. \tag{19}$$

Thus the right hand side of (17) is zero if and only if

$$PD(P) = q(P) - (cP + Q) = 0, \tag{20}$$

see (3).

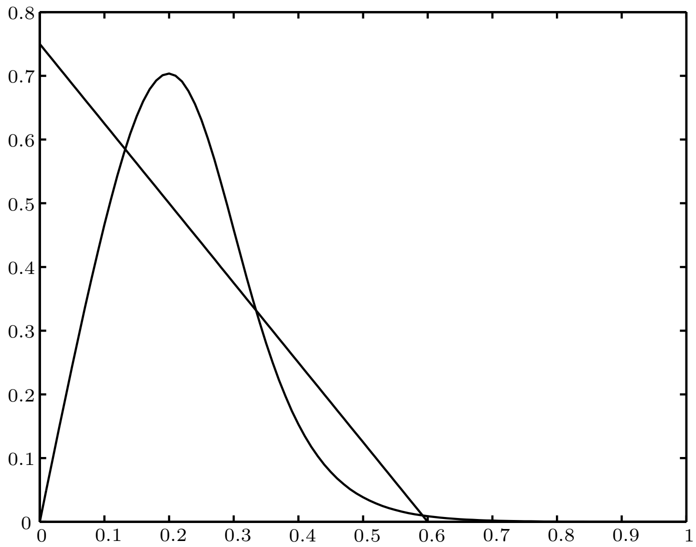


**Figure 1.** A Nonconcave Fundamental Diagram $q(P)$ intersects $cP+Q$ at three points.

$B(P)$ vanishes at a single point $0<P=P_0(Q)<1$.

For nonconcave fundamental diagram $q(\rho)$ defined in (10), $Q>0$, and thus $c=c(Q)<V$ can then be chosen so that

$$D(P)=0$$

holds at

$$P=P_0(Q)$$

and at two other points,

$$P=P_-(Q),\ \ P=P_+(Q),$$

where $0<P_-(Q)<P_+(Q)<1$ satisfying

$$0<\min\{P_-(Q),P_0(Q)\}<\rho_i<\max\{P_+(Q),P_0(Q)\}<1 \tag{21}$$

where $0<\rho_i<1$ is the only inflection point of $q(\rho)$. (21) is consistent with previous instability result (11) for PW model.

Define

$$f(P) = P + \frac{\gamma}{\tau}\frac{D(P)}{B(P)} \tag{22}$$

for some $\gamma > 0$. Thus

$$f(P_\pm) = P_\pm \tag{23}$$

and

$$f(P) - P < 0,\ P_- < P < P_+. \tag{24}$$

Moreover, $f(P)$ is smooth and bounded on $0 \leq P \leq 1$. Equation (17) becomes

$$P' = \frac{1}{\gamma}(f(P) - P). \tag{25}$$

(23) shows that $P_\pm$ are the steady states of the nonlinear equation (25).

We now propose the following discrete model of traffic flow. Let $0 \leq P_1 \leq P_+$ be given and define

$$P_{n+1} = f(P_n),\quad n \geq 1. \tag{26}$$

We may restrict $\frac{\gamma}{\tau}$ in certain range, so $f$ maps the interval

$$[\delta, P_+]$$

to itself, for some $\delta > 0$. (23) shows that $P_\pm$ are the fixed points of the nonlinear map (26).

Traffic model (26) is a discretization of (25). In other words, we are considering the discretized traveling wave solutions. $\gamma > 0$ is the step size of the traveling wave variable in the discretization. From physical systems, values of output variables are often registered at discrete times. These values can be plotted against the values at the preceding time step yielding an iteration mapping of the system. One may study the power spectrum of the output signal to decide about the internal dynamics [54].

In [35], Li showed that the dynamics of map (26) are governed by the logistic map. The dynamics of the logistic map undergoes one stable steady state, a period-2 cycle, a period-4 cycle and further period-doublings to cycles of periods 8, 16, 32,. . ., $2^n$, . . ., as the bifurcation parameter

$$\alpha = \frac{\gamma}{\tau}$$

increases. The long-time behavior is aperiodic and exhibits sensitive dependence on initial data $P_1$. The results can explain the appearance of a phantom traffic jam, which is observed in real traffic flow.

It turns out that $P_+$ is an unstable fixed point of map (26) for all values of $\alpha > 0$.

If $\alpha > 0$ is small, then $P_-$ is a stable fixed point of map (26) and it is the only stable steady state. Thus the asymptotic behavior is that $P_n$ converges to $P_-$ as $n \to +\infty$.

When bifurcation parameter $\alpha > 0$ is beyond certain critical value, the fixed point $P_-$ loses its stability

$$f'(P_-) < -1.$$

In a certain bifurcation parameter range, there exist two stable fixed points of the second iteration map $f(f(P))$. There is a periodic asymptotic state which oscillates between these two states. This is a 2-cycle. The 2-cycle bifurcates continuously from steady state $P_-$.

The 2-cycle is stable for bifurcation parameter $\alpha > 0$ to be in certain range. Beyond that range, further period-doublings lead to cycles of periods 4, 8, 16, 32,..., $2^n$, ..., and then chaotic solutions as $\alpha > 0$ increases.

If we set the size of the discretization of the traveling wave variable $\gamma$ to be a constant, then increasing $\alpha > 0$ is equivalent to increasing $\frac{1}{\tau}$. The latter is the sensitivity to the stimulus. Therefore increasing the sensitivity to the stimulus results in instability of the traffic flow.

## 4. Comparison

The analysis of large sets of traffic data has revealed the existence of three traffic phases: free flow, synchronized flow and wide moving jams, Kerner [18] and Kerner and Rehborn [20]. Free traffic would correspond to the point attractor $P_-$. The bifurcation from steady state $P_-$. marks the transition from free flow to synchronized flow. Traffic jams are formed when further period-doublings lead to cycles of period 4, 8, 16, 32,..., $2^n$, ..., and to chaotic solutions. The dynamic behavior of map (26) correctly predicts the empirical findings for freeway traffic.

In particular, the dynamics of map (26) predicts that the instability of traffic occurs at some intermediate density range. This is because the oscillatory and the or chaotic traffic occurs around $P_-$. Therefore the oscillatory or chaotic traffic occurs at intermediate density. This is in agreement with the empirical findings that speeds, flows and densities exhibit greater variation when measured in moderately dense queues as compared with measurements in queues of higher density. Moreover, the oscillations did not affect freely flowing upstream of a queue's tail. It was also found that most oscillations formed in moderately dense queues while propagating upstream, against the flow of traffic, which agrees with our negative speed condition (16).

In summary, we proposed an innovative approach to model the nonlinear dynamics of traffic flow. A class of discrete models were derived from nonequilibrium continuum models. Discrete model (26) captures the essential features of traffic jams. We are able to identify and formulate the self-organized vehicle clustering and the transition to chaos in traffic. In particular, the dynamics of map (26) predicts that the traffic is stable at small traffic density, is unstable above a certain critical density, and becomes stable again at very high density. This is identical to the empirical findings for freeway traffic in Daganzo et al. [6], Helbing [13], Kerner, [18] Kerner and Rehborn [20], Knospe, Santen, Schadschneider and Schreckenberg [22], Mauch and Cassidy [40], Treiterer and Myers [50], and to numerical simulations including Gray and Griffeath's [10] and Nagel's [43] results on the emergence of traffic jams by using a probabilistic cellular automaton (CA), Kerner and Konhäuser's [19], Jin and Zhang's [16] results by numerically solving certain nonequilibrium continuum traffic models. Moreover, our clustering solutions propagate with a speed that is less than the vehicle speed. This reflects the physical setting that the vehicle clustering travels against the traffic flow.

We found a unified condition in obtaining clustering solutions: the fundamental diagrams change concavity. The condition is suggested from the experiment data (see Figure 1) and is used in the previous work of Bando et al. [1], Kerner and Konhaüser [19], Jin and Zhang [16], and Lee et al. [27].

In global optimization, many nonconvex minimization problems are NP-hard. Due to the nonconvexity of the total potential energy of the system concerned, application of local analytic methods and the standard optimization procedures cannot guarantee the identification of the global minima. In nonconvex dynamical systems, numerical methods may produce the so-called chaotic solutions [7].

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